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Langmuir waves in semi-relativistic spinless quantum plasmas

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Many-particle quantum hydrodynamics based on the Darwin Hamiltonian (the Hamiltonian corresponding to the Darwin Lagrangian) is considered. A force field appearing in the corresponding Euler equation is considered in detail. Contributions from different terms of the Darwin Hamiltonian in the Euler equation are traced. For example, the relativistic correction to the kinetic energy of particles leads to several terms in the Euler equation; these terms have different form. One of them has a form similar to a term appearing from the Darwin term. Hence, the two different mechanisms give analogous contributions in wave dispersion. A microscopic analog of the Biot–Savart law, called the current–current interaction, describing an interaction of moving charges via the magnetic field, is also included in our description. The semi-relativistic generalization of the quantum Bohm potential is obtained. Contribution of the relativistic effects in the spectrum of plasma collective excitations is considered. The contributions of the spin–spin, spin–current, and spin–orbit interactions in this model are considered. The contribution of the spin evolution in the Langmuir wave spectrum is calculated at the propagation of wave perpendicular to the external magnetic field.

Subject Index A63, I23, I40

1. Introduction

There is fast-growing interest in the theory of the relativistic [1–8] and the semi-relativistic (weakly relativistic) [9,10] quantum plasmas. In this paper we develop many-particle quantum hydrodynamics (QHD) [11–18] in the semi-relativistic approximation. In this way we are going to discuss relations between the quantum, thermal, and semi-relativistic effects in a system of many charged particles. As a result, we a present complete theory including the effects mentioned above for spinless charged particles.

Spin leads to effects appearing in the semi-relativistic approximation. However, it plays a significant role in non-relativistic physical systems, for example, in ferromagnetic materials. Spin dynamics is also very important in the physics of the quantum plasma, where electrons and positrons are the most widespread objects, and their spin is an inherent dynamical property. Separate evolution of electrons in the spin-up and spin-down states can be considered along with the single fluid model of electrons [19]. Over the last decade a lot of papers have been dedicated to studies of spin dynamics in quantum plasmas, especially by means of quantum hydrodynamics and Vlasov-like kinetic equations. However, it is very interesting and important to understand the quantum many-particle physics appearing from consideration of the Darwin Hamiltonian, the Hamiltonian corresponding to the Darwin Lagrangian, which is the spinless analog of the Breit Hamiltonian [22].

The Darwin Hamiltonian contains both the non-relativistic terms, which describe the kinetic energy of the particles and the Coulomb interaction, and semi-relativistic terms. They describe the relativistic correction to the kinetic energy of particle (RCKE), the interaction energy of moving charges (which is also called the current–current interaction), the Darwin term proportional to $\nabla \mathbf{E}$, and the term, describing inter-particle interaction, proportional to the Dirac delta function, corresponding to the Darwin term, which we call the Darwin interaction. The current–current interaction presents the Biot–Savart–Laplace law. The RCKE and the current–current interaction should be important when studying relativistic beams in the plasmas.

The suggestion was made in Ref. [10] that in some cases the contribution of the RCKE is much smaller than the Darwin term. However, our studies of the semi-relativistic effects in quantum plasmas based on the quantum hydrodynamics method show that the RCKE leads to the existence of terms in the semi-relativistic Euler equation. One of these terms has a form close to the only term brought by the Darwin term and Darwin interaction. Thus the Darwin term and the RCKE must be considered together. In this paper we develop the hydrodynamic model of the semi-relativistic effects in quantum plasmas, while Ref. [10] is dedicated to the quantum kinetic model.

Moreover, the consideration of the Darwin interaction in Refs. [10,20,21] was based on the single-particle Hamiltonian, which does not describe the interaction of two particles of comparable mass, while the Breit Hamiltonian should be considered in the many-particle regime as presented below.

The contribution of the RCKE and the current–current interaction in the plasma wave dispersion have been considered recently [23] in terms of the many-particle quantum hydrodynamics developed in Refs. [11–18], but the Darwin term was not considered there. Another derivation of the QHD equations for systems of charged spinning particles suggested later can be found in Refs. [24,25]. Some aspects of quantum plasma physics were reviewed in Ref. [26].

The Darwin term is the semi-relativistic trace of the Zitterbewegung contribution in the Langmuir wave dispersion. The Zitterbewegung effect has been actively studied [27–35]. It has been considered for electrons in semiconductors [30,31], ions [27–29], and the quantum gases of neutral atoms [33–35]. Consequently it is worthwhile to point out that the RCKE gives the contribution in the equations of collective motion counteractive to the Darwin term and Darwin interaction contributions.

This paper is dedicated to the comparison of the RCKE, the Darwin interaction, and the current–current interaction contributions in the Euler equation, obtaining the explicit form of the semi-relativistic pressure tensor, and its influence on the dispersion properties of the longitudinal waves.

Our paper is organized as follows. In Sect. 2 we discuss the basic Hamiltonian and compare the contributions of different terms. In Sect. 3, a set of QHD equations is presented in a semi-relativistic approximation. Different contributions in the Euler equation are discussed. In Sect. 4, the method of obtaining the dispersion equation is described, and a linearized set of the semi-relativistic Euler equations is presented. The dispersion relation for quantum semi-relativistic Langmuir waves is also calculated and discussed. In Sect. 5, brief summary of the results obtained is presented.

2. The model description

The equations of quantum hydrodynamics are derived from the non-stationary Schrödinger equation for a system of N particles:

$$\imath\hbar\partial_t\psi(R,t) = \hat{H}\psi(R,t) \tag{1}$$

with Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_{\mathrm{Rel}} + \hat{H}_{\mathrm{D}}, \tag{2}$$

where

$$\hat{H}_0 = \sum_i \left(\frac{1}{2m_i} \mathbf{D}_i^2 + e_i \varphi_{i,\mathrm{ext}} \right) + \frac{1}{2} \sum_{i,j\neq i} e_i e_j G_{ij}, \tag{3}$$

$$\hat{H}_{\mathrm{Rel}} = -\sum_i \frac{1}{8m_i^3 c^2} \mathbf{D}_i^4 - \frac{1}{2} \sum_{i,j\neq i} \frac{e_i e_j}{2m_i m_j c^2} G_{ij}^{\alpha\beta} D_i^\alpha D_j^\beta, \tag{4}$$

and

$$\hat{H}_{\mathrm{D}} = -\sum_i \frac{e_i \hbar^2}{8m_i^2 c^2} \nabla_i \mathbf{E}_{i,\mathrm{ext}} - \frac{1}{2} \sum_{i,j\neq i} \frac{\pi e_i e_j \hbar^2}{2c^2} \left(\frac{1}{m_i^2} + \frac{1}{m_j^2} \right) \delta(\mathbf{r}_i - \mathbf{r}_j). \tag{5}$$

This Hamiltonian corresponds to the spin independent part of the Breit Hamiltonian [see [22], Sects. 33 and 83, and [36], Eq. (4.74b)]. All terms except the fourth and sixth also correspond to the classic Hamiltonian derived from the Darwin Lagrangian (see [37], Sect. 65). The following designations are used in Eqs. (2)–(5): e_i, m_i are the charge and the mass of particle, $\hbar$ is the Planck constant and c is the speed of light, $D_i^\alpha = -\imath\hbar\partial_i^\alpha - e_i A_{i,\mathrm{ext}}^\alpha / c$ is the covariant derivative, $\varphi_{i,\mathrm{ext}}$, $A_{i,\mathrm{ext}}^\alpha$ the potentials of an external electromagnetic field, $\partial_i^\alpha = \nabla_i^\alpha$ are the spatial derivatives, $G_{ij} = 1/r_{ij}$ are the Green functions of the Coulomb interaction, $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$,

$$G_{ij}^{\alpha\beta} = \frac{\delta^{\alpha\beta}}{r_{ij}} + \frac{r_{ij}^\alpha r_{ij}^\beta}{r_{ij}^3} \tag{6}$$

are the Green functions of the current–current interaction, $\psi(R, t)$ is the psi-function of the N-particle system, $R = (\mathbf{r}_1, \ldots, \mathbf{r}_N)$. Let us consider the physical meaning of the terms in the Hamiltonian (2).

We consider the Hamiltonian as the sum of three parts: the non-relativistic part H_0, the relativistic part H_{Rel}, and the quantum-relativistic terms H_{D}. The first term in the non-relativistic part of the Hamiltonian H_0 is the kinetic energy, the first term in H_{Rel} is the RCKE, the second term in H_0 is the potential energy of the classic charge in the external electric field, the first term in H_{D} is the quantum contribution in the energy of the charge being in the external electric field, which is called the Darwin term. All these terms are valid for each particle, as they describe kinematic properties and interaction with the external field. They present the first groups of terms in the Hamiltonians (3), (4), and (5). The second groups of terms in H_0, H_{Rel}, H_{D} describe inter-particle interactions. First of all, the Coulomb interaction is presented by the third term in H_0. The second term in H_{D} describes a quantum contribution to the interaction of charges. It is the Darwin interaction. The second term in H_{Rel} describes the current–current interaction, which is the microscopic analog of the Biot–Savart law.

The first term in H_{D} shows a semi-relativistic contribution to the force acting from the external electric field on a charged particle (the Darwin term). The second term presents the interaction between two particles, which can be considered as a semi-relativistic addition to the Coulomb interaction (the Darwin interaction). If we deal with the interaction of two electrons, the Darwin interaction is

$$H_{\mathrm{D}} = -\pi \left(\frac{e\hbar}{mc} \right)^2 \delta(\mathbf{r}_i - \mathbf{r}_j), \tag{7}$$

where $\mathbf{r}_i$ and $\mathbf{r}_j$ are the coordinates of the two electrons. The explicit form of the Darwin interaction was derived from the scattering amplitude in quantum electrodynamics. Now we have to compare the Darwin term describing interaction with the external field, the first term in Eq. (4), which appears in the semi-relativistic limit of the Dirac equation [22], and the Darwin interaction presented by the second term in H_{D} [22]. Admitting that $\triangle_i(1/|\mathbf{r}_i - \mathbf{r}_j|) = -4\pi\delta(\mathbf{r}_i - \mathbf{r}_j)$ and introducing the microscopic electric field caused by particle j acting on particle i as $\mathbf{E}_{ij} = -\nabla_i(e_j/r_{ij})$, we see that the second term in Hamiltonian (5) can be represented as

$$H_{\mathrm{D}} = -\frac{e_i\hbar^2}{8c^2}\left(\frac{1}{m_i^2} + \frac{1}{m_j^2}\right)\nabla_i\mathbf{E}_{ij}. \tag{8}$$

In Eq. (8) we used the general dependence of masses for interacting particles obtained in Ref. [22]. In Eq. (7) we have assumed $m_i = m_j = m$. Comparing the first term in the Hamiltonian (5) and formula (8) we get that these terms coincide if $m_j \to \infty$, corresponding to the Dirac equation. The Dirac equation describes the motion of an electron in an external field, so the motion of the electron has no influence on the external field. Consequently, the mass of the source of the external field can be considered as equal to infinity. However, if we consider the interaction of two electrons, we have $m_i = m_j$ and from (7) we find

$$H_{\mathrm{D}} = -\frac{e\hbar^2}{4m^2c^2}\nabla_i\mathbf{E}_{ij}, \tag{9}$$

which differs in two ways from the first term in (5). It was expected that the terms discussed should coincide due to the superposition principle, so we have to put an additional factor of two in the first term in the Hamiltonian (5), but we keep in mind that we can make another choice and accept the consequence of the Dirac equation. In our discussion of wave dispersion we consider the consequences of both choices.

We have dealt with the methods based on a certain equation, in our case it is the Schrödinger equation (1), describing the system evolution in terms of the Hamiltonian for particles. In the relativistic case the Dirac equation is the corresponding equation. However, the Dirac equation describes the quantum motion of one relativistic electron in an external electromagnetic field. There is no proper equation describing the quantum or classic motion of many relativistic electrons in terms of a Hamiltonian, since the Hamiltonian of the electromagnetic field has to be included and the field should be considered as an independent variable, as it is in quantum electrodynamics. Thus there is no proper many-particle generalization of the Dirac equation. Consequently the Dirac equation does not allow the derivation of many-particle relativistic hydrodynamics directly. Even semi-relativistic hydrodynamics cannot be derived by means of the Dirac equation. However, the Breit Hamiltonian obtained from the quantum electrodynamic scattering amplitude of two charged spinning particles describes the semi-relativistic system of two particles (see Ref. [22], Sect. 83). It is easy to generalize the Breit Hamiltonian on a system of N particles, where $N > 2$. Including the fact that we consider spinless particles, we see that the many-particle Breit Hamiltonian corresponds to the classic Hamiltonian obtained from the Darwin Lagrangian (see Ref. [37], Sect. 65). But the Breit Hamiltonian contains the Darwin term and the Darwin interaction has a quantum semi-relativistic nature, so it does not appear in the classic semi-relativistic theory.

For short references below we introduce a new function $\tilde{G}_{ij}$, which is defined as $\tilde{G}_{ij} = G_{ij} - (\hbar^2/4m^2c^2)\delta(\mathbf{r}_i - \mathbf{r}_j)$.

$\tilde{G}_{ij}$ leads to the existence of two force field terms in the Euler equation. Let us consider how they emerge during the derivation of the semi-relativistic Euler equation. We differentiate the current $\mathbf{j}$ appearing in the continuity equation with respect to time and use the Schrödinger equation. One of these terms appears due to commutation of $\tilde{G}_{ij}$ with the momentum operator $\hat{p}_i^\alpha$ in the current $\mathbf{j}$.

Let us point out that the operator $\hat{p}_i^{\alpha}$ exists in the current **j** due to the presence of the kinetic energy operator in the Hamiltonian (2). In the self-consistent field approximation this term has the following form:

$$\mathbf{F}_{\mathrm{C}} = -e^2 n \nabla \int d\mathbf{r}' \left(\frac{1}{\mid \mathbf{r} - \mathbf{r}' \mid} - \frac{\pi \hbar^2}{m^2 c^2} \delta(\mathbf{r} - \mathbf{r}') \right) n(\mathbf{r}', t). \tag{10}$$

The self-consistent field approximation allows the introduction of the electric field **E** caused by the charges. It has the following explicit form:

$$\mathbf{E} = -e \nabla \int d\mathbf{r}' \frac{1}{\mid \mathbf{r} - \mathbf{r}' \mid} n(\mathbf{r}', t), \tag{11}$$

where n is the particle concentration, and field **E** satisfies the quasi-electrostatic Maxwell equations:

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = 0, \tag{12}$$

and

$$\nabla \mathbf{E}(\mathbf{r}, t) = 4\pi \sum_a e_a n_a(\mathbf{r}, t), \tag{13}$$

where subscript a describes the species of particles. We are interested in the dispersion of the Langmuir waves, which are the high-frequency oscillations. Consequently, electrons give the main contribution. Thus we can neglect the motion of ions and consider ions as motionless. Having a mixture of electrons and ions we work with a stable system. The presence of motionless ions is revealed in the Poisson equation (13), where the ions cancel the equilibrium charge density of the electrons. So perturbations of the electric field are caused by the perturbation of the electron density

$$\nabla \mathbf{E}(\mathbf{r}, t) = 4\pi e \delta n(\mathbf{r}, t), \tag{14}$$

where δn is the perturbation of the electron concentration.

Now force field $\mathbf{F}_{\mathrm{C}}$ (10) takes the form

$$\mathbf{F}_{\mathrm{C}} = en\mathbf{E} + \frac{\pi e^2 \hbar^2}{m^2 c^2} n \nabla n, \tag{15}$$

where the concentration under the space derivative represents the source of the field. So, using Eq. (14) for this concentration we have come to

$$\mathbf{F}_{\mathrm{C}} = en\mathbf{E} + \frac{e\hbar^2}{4m^2 c^2} n \nabla (\nabla \mathbf{E}). \tag{16}$$

As presented here, the form of the second term corresponds to the semi-relativistic contribution in the force acting on the charged particle from the external electric field obtained from the Dirac equation [22]. The second term in Eq. (15) gives the general form of the Darwin interaction force field. In some cases it can be rewritten in terms of the self-consistent electric field. We have done this representation for electron–ion plasmas with motionless ions. Equation (16) is useful for comparing the Darwin interaction with the RCKE.

The second term associated with G_{ij} appears due to the RCKE. Or, more precisely, it exists due to simultaneous account of the RCKE and interaction of charges with the electric field, which is the sum of the external field and inter-particle electric field. Hence let us call it the RCKE–electric field (RCKE–EF) interaction. In the self-consistent field approximation it appears as

$$F_{sr}^{\alpha} = \frac{e\hbar^2}{4m^2 c^2} \partial_\beta (\partial_\alpha E_\beta \cdot n). \tag{17}$$

As the RCKE–EF interaction has a semi-relativistic origin, we can write G_{ij} instead of $\tilde{G}_{ij}$ in this term. The RCKE also gives other terms in the force field, all of which are presented below in the Euler equation.

The force caused by the Darwin term was accounted for in Ref. [10], when the force field presented by Eq. (17) was not considered in the derivation of the kinetic equation and the calculation of the Langmuir wave dispersion. The inter-particle Darwin interaction was not considered in Ref. [10] either. Thus, we are going to generalize the result of Ref. [10] by calculating the contribution of the force field (17) in the Langmuir wave dispersion. We also present equations of semi-relativistic collective motion of plasma. But we consider the hydrodynamic equations, while the kinetic equation was considered in Ref. [10].

Equations (16) and (17) are very similar, but they have two differences. The first difference is the distinction in the tensor structure and the second one is the fact that Eq. (17) contains the concentration under the spatial derivative, whereas in Eq. (16) the concentration is contained as an external multiplier.

3. Equations of quantum hydrodynamics

In the previous section we have shown the similarity of the RCKE–EF and the Darwin interactions. One of the aims of the paper is to compare the contributions of these terms in the QHD equations and the Langmuir wave dispersion. We want to trace the separate contribution of the each term. Thus we need to mark them.

The first equation of the QHD set is the continuity equation

$$\partial_t n + \nabla \mathbf{j} = 0. \tag{18}$$

In that equation a function of current $\mathbf{j}(\mathbf{r}, t) = n(\mathbf{r}, t)\mathbf{v}(\mathbf{r}, t)$ arises, where $\mathbf{v}(\mathbf{r}, t)$ is the velocity field.

The second equation of the QHD set is the Euler equation, but in the semi-relativistic approximation the function $\mathbf{j}(\mathbf{r}, t)$ appearing in the continuity equation is the particle current. However, in contrast to the non-relativistic case we cannot call it momentum density, thus the Euler equation is the equation of particle current evolution [38]. This equation has the form

$$\begin{aligned}
&mn\left(\partial_t + v^\beta \nabla^\beta\right)v^\alpha + \partial_\beta P_{\alpha\beta} \\
&\quad = enE^\alpha + \frac{e}{c}\varepsilon^{\alpha\beta\gamma} nv^\beta B^\gamma + \frac{e\hbar^2}{8m^2c^2} n\partial^\alpha\left(\partial^\beta E^\beta_{\text{ext}}\right) + \frac{e\hbar^2}{4m^2c^2} n\partial^\alpha\left(\partial^\beta E^\beta_{\text{int}}\right) \\
&\qquad + \frac{e\hbar^2}{4m^2c^2}\partial_\beta\left(\partial_\alpha E_\beta \cdot n\right) - \frac{e}{mc^2}\left[E_\beta(mnv_\alpha v_\beta + P_{\alpha\beta}) + E_\alpha\left(\frac{1}{2}mnv^2 + n\varepsilon\right)\right] \\
&\qquad - \frac{e^2\hbar^2}{8m^2c^2}\partial_\beta n \int d\mathbf{r}' \partial_\alpha G_{\beta\gamma}(\mathbf{r} - \mathbf{r}')\partial'_\gamma n(\mathbf{r}', t) - \frac{e^3}{2mc^2} n \int d\mathbf{r}' G_{\alpha\beta}(\mathbf{r} - \mathbf{r}') E_\beta(\mathbf{r}', t) n(\mathbf{r}', t) \\
&\qquad + \frac{e^2}{2c^2}\int d\mathbf{r}'\left[\partial_\alpha G_{\beta\gamma}(\mathbf{r} - \mathbf{r}') - \partial_\beta G_{\alpha\gamma}(\mathbf{r} - \mathbf{r}')\right]\pi_{\beta\gamma}(\mathbf{r}, \mathbf{r}', t) + \frac{e^2}{2mc^2} n \int d\mathbf{r}' \partial_\gamma G_{\alpha\beta}(\mathbf{r} - \mathbf{r}') \\
&\qquad \times \left[mn(\mathbf{r}', t)v_\beta(\mathbf{r}', t)v_\gamma(\mathbf{r}', t) + P_{\beta\gamma}(\mathbf{r}', t)\right],
\end{aligned} \tag{19}$$

where $\mathbf{E} = \mathbf{E}_{\text{ext}} + \mathbf{E}_{\text{int}}$ and $\mathbf{B}$ are the electric and magnetic fields, $n\varepsilon$ is the density of thermal energy including the quantum part (which is an analog of the quantum Bohm potential), $\varepsilon^{\alpha\beta\gamma}$ is the antisymmetric symbol (the Levi–Civita symbol), $P^{\alpha\beta}$ is the pressure tensor, which is the semi-relativistic generalization of the sum of non-relativistic thermal pressure $p^{\alpha\beta}$ and the quantum Bohm potential $T^{\alpha\beta}$. The right-hand side of Eq. (19) contains a force field. The force field consists of the Lorentz force and specific quantum semi-relativistic terms, which are discussed below. $\pi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t)$ is presented explicitly and considered below after analysis of the $P^{\alpha\beta}$ structure. The vector potential appears in the form

$$A_{\alpha}^{\mathrm{int}}(\mathbf{r},t)=\frac{e}{2c}\int d\mathbf{r}' G_{\alpha\beta}(\mathbf{r}-\mathbf{r}')n(\mathbf{r}',t)v_{\beta}(\mathbf{r}',t), \tag{20}$$

which gives the contribution in the Lorentz force, the second term on the right-hand side of Eq. (19), along with external magnetic field. The magnetic field $\mathbf{B}=\nabla\times\mathbf{A}^{\mathrm{int}}$ satisfies the quasi-magnetostatic Maxwell equation [37]:

$$\nabla\times\mathbf{B}=\frac{4\pi}{c}\mathbf{j}, \tag{21}$$

and

$$\nabla\mathbf{B}=0. \tag{22}$$

We believe that it is worthwhile noting that we have not neglected time derivatives in the Maxwell equations (12) and (21). We do not present these terms, because they do not appear in the semi-relativistic approximation. The Hamiltonian (2) contains the Coulomb interaction and the current–current interaction (the Biot–Savart law). The Maxwell equations obtained correspond to the Hamiltonian. We can put these well-known time derivatives back in the Maxwell equations. However, this step breaks the logic of the semi-relativistic description. So, we keep working with the electromagnetic fields appearing in the semi-relativistic approximation and described by Eqs. (12), (14), (21), and (22). The dropped terms are essential for the description of the transverse waves in plasmas, even in the regime of classical plasmas. If one needs to consider the transverse waves, these terms need to be reconstructed. In the Hamiltonian (2) we consider the semi-relativistic part of the full retarding potentials [37]. The full retarding potentials allow consideration of the relativistic effects at the inter-particle interaction. These effects arise at large enough particle velocities if the terms proportional to $(v/c)^n$, with $n\geq 3$, are not negligible. The full retarding potentials also describe the propagation of electromagnetic waves in plasmas, which are transverse waves. The spectrum of the electromagnetic waves in unmagnetized plasmas, $\omega^2=\omega_{Le}^2+k^2c^2$, consists of two parts. The non-relativistic charge–charge interaction is shown via the square of the Langmuir frequency ω_{Le}^2. The second part is the spectrum of the electromagnetic wave in a vacuum, $\omega_0^2=k^2c^2$, following from the Maxwell equations with the time derivatives of the fields. The electromagnetic waves can propagate even in the absence of particles. This part of the electromagnetic field is independent of particles. The semi-relativistic approximation, considered in this paper, includes terms up to $(v/c)^2$, which contains the part of the electromagnetic field bound to the particles creating it.

Before discussion of the pressure tensor $P^{\alpha\beta}$ we explain the physical meaning of the force field terms presented in the right-hand side of the Euler equation (19). This is especially important as some of these terms are presented for the first time.

The first two terms present the density of the Lorentz force. The self-consistent part of the Coulomb interaction gives the contribution in the first term. The second term contains the contribution of the current–current interaction in the self-consistent field approximation. We should admit that all terms in the Euler equation are presented in the self-consistent field approximation. Actually, only a part of the whole contribution from the current–current interaction came in the Lorentz force, it also leads to several other terms—the seventh–tenth terms of the Euler equation. In fact, the eighth–tenth terms already appear in classical semi-relativistic hydrodynamics, but in quantum theory these terms have a richer structure. First of all, they contain the contribution of exchange interactions via quantum correlations, which is not considered in this paper, but they naturally appear in the many-particle QHD. We neglect them here, considering the self-consistent field approximation only. However, they contain the contribution of the quantum Bohm potential along with the thermal pressure.

The third term corresponds to the Darwin term. The fourth term corresponds to the Darwin interaction. Terms three and four have the same nature, but they have different coefficients. The Darwin term appears from the Dirac equation describing the motion of an electron in an external field. Then the Darwin interaction comes from the quantum electrodynamic scattering amplitude of two particles. The third and fourth terms contain the electric field. The third (fourth) term includes the external (inter-particle Coulomb) electric field $\mathbf{E}_{\text{ext}}$ ($\mathbf{E}_{\text{int}}$). Due to different coefficients before these terms we cannot combine them together to obtain the full electric field $\mathbf{E} = \mathbf{E}_{\text{ext}} + \mathbf{E}_{\text{int}}$. We should not expect additivity of electric fields in these terms, since Eq. (16) used in the fourth term is asymptotic. The original force field is presented by Eq. (15). The original formula does not contain the electromagnetic field. It is presented in terms of the particle concentration of interacting species $\mathbf{F}_{\text{D}} = \frac{\pi e^2 \hbar^2}{m^2 c^2} n \nabla n$. In our case we have one species, so it describes an electron–electron interaction. In this paper we consider plasmas without an external electric field. Consequently, the third term equals zero. In Ref. [10], the authors dealt with the Darwin term appearing from the Dirac equation, which corresponds to the interaction of electrons with the external field. As a consequence they get a coefficient two times less then we get from the inter-particle interaction.

Terms five and six present the contribution of the RCKE. The fifth term has a simple structure, containing the divergence ∇^β of the tensor which is the product of the particle concentration and $\nabla^\alpha E^\beta$ and represents the RCKE–EF interaction. In the sixth term, which contains a number of terms in square brackets, the first set is the convolution of E^β with the tensor which is the particle current $\mathbf{j}$, and as a part of this current we have the pressure tensor $P^{\alpha\beta}$. As the sixth term of the Euler equation has a semi-relativistic nature, we should consider only the non-relativistic part of $P^{\alpha\beta}$. The second set of terms in the sixth term is the product of the electric field E^α and the energy density. The energy density was separated into two parts here. First is the kinetic energy density of a local ordered motion. We need to say that $n\varepsilon$ is the energy density which consists of two parts: thermal energy and quantum contribution—an analog of the quantum Bohm potential. In the one-particle case we lose the contribution of thermal motion and quantum-thermal terms, and get quantum terms arising for non-interacting particles. ε gives no contribution in the problem considered below, and therefore we do not present its explicit form.

The explicit form of the tensor $P^{\alpha\beta}$ is

$$\begin{aligned}
P_{\alpha\beta}(\mathbf{r},t) = \int dR \sum_{i=1}^{N} \delta(\mathbf{r}-\mathbf{r}_i) a^2 \Bigg[& m u_{i\alpha} u_{i\beta} - \frac{\hbar^2}{2m}\left(1 - \frac{v_i^2}{c^2}\right) \partial_{i\alpha}\partial_{i\beta} \ln a \\
& + \frac{\hbar^2}{2mc^2}\left(\partial_{i\alpha} v_{i\gamma} \partial_{i\beta} v_{i\gamma} + v_{i\gamma}\partial_{i\alpha}\partial_{i\beta} v_{i\gamma}\right) \\
& + \frac{\hbar^2}{4mc^2}\left(v_{i\alpha}\partial_{i\beta} + v_{i\beta}\partial_{i\alpha}\right)\left(\partial_{i\gamma} v_{i\gamma} + 2 v_{i\gamma}\partial_{i\gamma} \ln a\right) \\
& - \frac{\hbar^4}{4m^3c^2a^2}\left(a\partial_{i\alpha}\partial_{i\beta}\Delta_i a + \partial_{i\alpha}\partial_{i\beta} a \Delta_i a - \partial_{i\alpha} a \partial_{i\beta}\Delta_i a - \partial_{i\beta} a \partial_{i\alpha}\Delta_i a\right)\Bigg] \\
& + \int dR \sum_{i=1, j=1, i\neq j}^{N} \delta(\mathbf{r}-\mathbf{r}_i) a^2 \frac{\hbar^2 e^2}{4m^2c^2}\left(G_{ij}^{\beta\gamma}\partial_{i\alpha}\partial_{j\gamma}\ln a + G_{ij}^{\alpha\gamma}\partial_{i\beta}\partial_{j\gamma}\ln a\right),
\end{aligned} \tag{23}$$

where $v_{i\alpha}$ is the velocity of the ith particle, and is the sum of the velocity field $v^\alpha(\mathbf{r},t)$ and the thermal velocity u_i^α, a is the amplitude of the wave function $\psi(R,t) = a\exp(iS/\hbar)$; the velocity of the ith particle v_i^α connects with the phase of the wave function as

$$v_i^\alpha = \frac{s_i^\alpha}{m_i} - \frac{s_i^\alpha s_i^2}{2m_i^3c^2} + \frac{\hbar^2}{2m_i^3c^2}\left[s_i^\alpha a^{-1}\Delta_i a + \partial_i^\alpha\left(s_i^\beta \partial_i^\beta \ln a\right) + \frac{1}{2}\partial_i^\alpha \partial_i^\beta s_i^\beta\right] - \sum_{j=1, j\neq i}^{N} \frac{e_i e_j}{2m_i m_j c^2} G_{ij}^{\alpha\beta} s_j^\beta, \tag{24}$$

where $s_i^\alpha = \partial_i^\alpha S - \frac{e_i}{c}A_i^\alpha$. The first term in Eq. (23) is the non-relativistic thermal pressure; the second term consists of two parts, the first being the non-relativistic quantum Bohm potential, the other terms representing semi-relativistic effects. Neglecting thermal velocities in the semi-relativistic terms of the pressure tensor $P_{\alpha\beta}$ we get purely quantum semi-relativistic pressure, which is the semi-relativistic generalization of the quantum Bohm potential $T_{\alpha\beta}$, whose explicit form for an ideal gas is

$$T_{\alpha\beta} = -\frac{\hbar^2}{4m}\partial^\alpha \partial^\beta n + \frac{\hbar^2}{4m}\left(\frac{\partial^\alpha n \cdot \partial^\beta n}{n}\right). \tag{25}$$

We also drop the contribution of the current–current interaction. As a result we have

$$\begin{aligned} P_{\alpha\beta}(\mathbf{r}, t) = {} & p_{\alpha\beta} + T_{\alpha\beta} - \frac{v^2}{c^2}T_{\alpha\beta} + \frac{\hbar^2}{2mc^2}n\left(\partial^\alpha v^\gamma \partial^\beta v^\gamma + v^\gamma \partial^\alpha \partial^\beta v^\gamma\right) \\ & + \frac{\hbar^2}{4mc^2}n\left(v^\alpha \partial^\beta + v^\beta \partial^\alpha\right)(\nabla \mathbf{v}) + \frac{\hbar^2}{4mc^2}\left(\partial^\gamma n\right)\left(v^\alpha \partial^\beta v^\gamma + v^\beta \partial^\alpha v^\gamma\right) \\ & - \frac{1}{c^2}\left(v^\alpha v^\gamma T^{\beta\gamma} + v^\beta v^\gamma T^{\alpha\gamma}\right) - \frac{\hbar^4}{4m^3c^2}\left(\sqrt{n}\cdot \partial^\alpha \partial^\beta \triangle \sqrt{n} + \partial^\alpha \partial^\beta \sqrt{n}\right. \\ & \left. \times \triangle \sqrt{n} - \partial^\alpha \sqrt{n}\cdot \partial^\beta \triangle \sqrt{n} - \partial^\beta \sqrt{n}\cdot \partial^\alpha \triangle \sqrt{n}\right). \end{aligned} \tag{26}$$

The first two terms have a non-relativistic nature. The other terms are semi-relativistic, most of them are proportional to v^2/c^2, except for the four last terms. The thermal pressure $p_{\alpha\beta}$ does not depend on interaction, so we can use the equation of state for an ideal gas, and we write $p^{\alpha\beta} = nk_\mathrm{B}T\delta^{\alpha\beta}$, where k_B is the Boltzmann constant, T is the temperature, and $\delta^{\alpha\beta}$ is the Kronecker symbol. When $P^{\alpha\beta}$ stays in a semi-relativistic term we should neglect the semi-relativistic part and consider the non-relativistic one only.

Let us repeat part of the semi-relativistic quantum Bohm potential existing in linear approximation, assuming that an equilibrium condition is described by non-zero uniform concentration $n_0 \neq 0$ and zero velocity field $\mathbf{v}_0 = 0$. Hence we obtain

$$\partial_\beta T_{\text{lin}}^{\alpha\beta} = -\frac{\hbar^2}{4m}\partial^\alpha \triangle n - \frac{\hbar^4}{8m^3c^2}\partial^\alpha \triangle\triangle n. \tag{27}$$

We have two terms. The first appears from the first term in Eq. (25). The second term in Eq. (27) is the linear part of the first term in the last group of terms, which does not contain the velocity field, in Eq. (26).

Here we present the explicit form of $\pi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t)$, which is part of the seventh term in the force field:

$$\pi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t) = \int \prod_{j=1}^{N} d\mathbf{r}_j \sum_{i,j=1, i\neq j}^{N} \delta(\mathbf{r} - \mathbf{r}_i)\delta(\mathbf{r}' - \mathbf{r}_j)a^2\left(u_{i\alpha}u_{j\beta} - \frac{\hbar^2}{2m^2}\partial_{i\alpha}\partial_{j\beta}\ln a\right).$$

To close the QHD set of equations we should find an approximate connection between $\pi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t)$ and other hydrodynamic quantities. Calculating $\pi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t)$ for the system of independent particles we get $\pi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t) = 0$. Thus, in the first approximation we do not need to account for the contribution of $\pi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', t)$ in the QHD equations.

4. Dispersion equation for quantum semi-relativistic Langmuir waves

To get semi-relativistic effects in the form of analytic simple formulas we consider quantum motion of electrons on the background of motionless ions. We consider small perturbations of the equilibrium state like

$$n = n_0 + \delta n, \quad \mathbf{v} = 0 + \delta\mathbf{v},$$
$$\mathbf{E} = 0 + \delta\mathbf{E}, \quad \mathbf{B} = 0 + \delta\mathbf{B}, \quad \delta p = 3mv_s^2\delta n,$$

where m is the mass of the electron. In Eqs. (18), (19) and the Maxwell equations (12), (13), (21), and (22), v_s^2 is the average thermal velocity (for the case of degenerate electrons we should write $\frac{v_{Fe}^2}{3}\left(1 - \frac{1}{10}\frac{v_{Fe}^2}{c^2}\right)$, instead of $3v_s^2$; see Appendix A, where $v_{Fe} = \sqrt[3]{3\pi^2 n_0}\hbar/m$ is the Fermi velocity). Substituting these relations into the set of equations and neglecting nonlinear terms, we obtain a system of linear homogeneous equations in partial derivatives with constant coefficients. Carrying the following representation for small perturbations δf,

$$\delta f = f(\omega, \mathbf{k})\exp(-\imath\omega t + \imath\mathbf{k}\mathbf{r}),$$

yields a homogeneous system of algebraic equations.

The Euler equation (19) is very complicated, thus we allow ourselves to present the algebraic form of the linearized Euler equation:

$$-\imath\omega m n_0 \delta v^\alpha + \imath k^\alpha\left(3mv_{se}^2 + \frac{\hbar^2 k^2}{4m} - \frac{\hbar^4 k^4}{8m^3c^2}\right)\delta n = en_0\delta E^\alpha - k^\alpha k^\beta 2\frac{en_0\hbar^2}{4m^2c^2}\delta E^\beta - \frac{e^3 n_0^2}{2mc^2}G^{\alpha\beta}(\mathbf{k})\delta E^\beta, \tag{28}$$

where $G^{\alpha\beta}(\mathbf{k})$ is the Fourier image of the current–current interaction Green function (6); its explicit form is

$$G^{\alpha\beta}(\mathbf{k}) = \frac{8\pi}{k^2}\left(\delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}\right).$$

The last term in the Euler equation (19) gives a linear term due to the linear part of the quantum Bohm potential, which is a part of $P^{\beta\gamma}$, but it is equal to zero because of the structure of $G^{\alpha\beta}(\mathbf{k})$. The last term in Eq. (28) gives no contribution in the dispersion of the Langmuir waves. We also admit that we do not consider the temperature-relativistic effects $\sim T/mc^2$.

The electric field is assumed to have a nonzero value. Expressing all quantities through the electric field, we come to the equation

$$\omega^2 = \omega_{Le}^2\left(1 - \frac{\hbar^2 k^2}{2m^2c^2}\right) + \left(\frac{1}{3}v_{Fe}^2\left(1 - \frac{1}{10}\frac{v_{Fe}^2}{c^2}\right) + \frac{\hbar^2 k^2}{4m^2} - \frac{\hbar^4 k^4}{8m^4c^2}\right)k^2 \tag{29}$$

for a degenerate electron gas, or

$$\omega^2 = \omega_{Le}^2\left(1 - \frac{\hbar^2 k^2}{2m^2c^2}\right) + \left(3v_{se}^2 + \frac{\hbar^2 k^2}{4m^2} - \frac{\hbar^4 k^4}{8m^4c^2}\right)k^2$$

for an electron gas with thermal distribution, where ω_{Le} is the Langmuir frequency, $\omega_{Le}^2 = 4\pi e^2 n_0/m$. The first group of terms on the right-hand side of (29) consists of three parts: the Langmuir frequency, the contribution of the RCKE–EF and Darwin interactions, where the RCKE–EF interaction leads to $-\frac{\hbar^2 k^2 \omega_{Le}^2}{4m^2c^2}$, and the Darwin interaction leads to the same structure $-\frac{\hbar^2 k^2 \omega_{Le}^2}{4m^2c^2}$. Together they give $-\frac{\hbar^2 k^2 \omega_{Le}^2}{2m^2c^2}$. The second group in Eq. (29) consists of three parts: the contribution of the pressure (thermal motion or Fermi pressure), the well-known quantum Bohm potential, and the contribution of the RCKE via the semi-relativistic part of the pressure tensor, or, speaking in other terms, it is the semi-relativistic part of the quantum Bohm potential.

Comparing Eq. (29) with the results of Ref. [10], we should mention several differences. In Ref. [10], the authors do not have the quantum Bohm potential contribution $\sim\hbar^2 k^4$. They also do not obtain the semi-relativistic part of the quantum Bohm potential. The most dramatic difference arises in comparing the second term in the first group of terms in Eq. (29) with the corresponding result of Ref. [10]. Here we have $-\frac{1}{8}\frac{\hbar^2 k^2 \omega_{Le}^2}{m^2c^2}$. In Ref. [10] it was found as $+\frac{1}{2}\frac{\hbar^2 k^2 \omega_{Le}^2}{m^2c^2}$. We have differences in the sign and magnitude of the coefficients. It looks like they chose another sign before the Darwin term (see Ref. [10], the fifth term in formula (5), and Eq. (5) of our paper). As we mentioned, they neglected the RCKE $\mathbf{D}^4$, because they did not want to consider the semi-relativistic part of the quantum Bohm potential. As we have shown, the RCKE gives several different contributions. Consequently the RCKE–EF interaction was also lost in Ref. [10], which gives half of the term under consideration. Moreover, they applied, for the electron–electron interaction, the Darwin term describing the interaction of the charges with the external electric field, which is two times smaller than the electron–electron Darwin interaction (see terms three and four in the right-hand side of Eq. (19) of our paper). Altogether we see why the coefficient obtained in Ref. [10] is four times smaller than our results.

4.1. Estimations

In this subsection we consider a system of degenerate electrons. Our aim is to find the parameters of the system when semi-relativistic effects are noticeable. To this end we represent the spectrum of the semi-relativistic Langmuir waves in terms of the Bohm velocity v_{B} defined as $v_{\mathrm{B}}^2 \equiv \hbar^2 k^2/(4m^2)$. The Bohm velocity is not a constant since it depends on the wave vector $v_{\mathrm{B}} = v_{\mathrm{B}}(k)$. The spectrum reappears in the following form:

$$\omega^2 = \omega_{Le}^2\left(1 - 2\frac{v_{\mathrm{B}}^2}{c^2}\right) + \frac{1}{3}v_{Fe}^2\left(1 - \frac{1}{10}\frac{v_{Fe}^2}{c^2}\right)k^2 + v_{\mathrm{B}}^2 k^2\left(1 - 2\frac{v_{\mathrm{B}}^2}{c^2}\right). \tag{30}$$

The contribution of the semi-relativistic effects is noticeable at large Bohm velocity. The Bohm velocity v_{B} increases as the wave vector increases, which is bounded above. This limitation is related to the average inter-particle distance a giving minimal wavelength, and, consequently, maximal wave vector $k_{\max} \sim \frac{1}{a} \sim \sqrt[3]{n_0}$. For the parameters included in the spectrum (30), at the small wavelength limit, we find $\omega_{Le}^2 \sim n_0$, $v_{Fe}^2 \sim n_0^{2/3}$, $k^2 \sim \frac{1}{a^2} \sim n_0^{2/3}$, $v_{\mathrm{B}}^2 \sim k^2 \sim n_0^{2/3}$. We are interested in a regime when the Bohm velocity v_{B} is comparable with the speed of light c, but $v_{\mathrm{B}} \ll c$. Otherwise we do not get the semi-relativistic regime. Let us estimate the parameters of the system at $\frac{v_{\mathrm{B}}^2}{c^2} \approx 0.1$. In this case the semi-relativistic effects of the RCKE–EF+Darwin interactions and the semi-relativistic quantum Bohm potential decrease the corresponding terms by two percent. We obtain $k_{\max} = 1.2\ 10^{10}\ \mathrm{cm}^{-1}$. This corresponds to a particle concentration of $n_0 \approx 10^{30}\ \mathrm{cm}^{-3}$. Such huge densities correspond to astrophysical objects like white dwarfs and the atmosphere of neutron stars.

Similar to (30), solutions were found in Refs. [39,40], and reviewed in Ref. [41], where other methods of plasma description were applied. Let us mention that the solutions have similar structure, but instead of the Bohm velocities they have the Fermi velocities (see Ref. [41], formula 47).

4.2. Spin evolution contribution

The full semi-relativistic model requires consideration of the spin evolution. Following Refs. [12,13,15], we include the spin–spin, spin–current, and spin–orbit interactions correspondingly. In this case, the magnetic field in the Euler equation is caused by the magnetic moments as well as the electric currents. The contribution of the magnetic field of the magnetic moments in the Lorentz force is one of the traces of the spin–current interaction [13].

The spin–spin interaction causes the extra force field on the right-hand side of the Euler equation $F^{\alpha}_{ss} = M^{\beta}\partial^{\alpha}B^{\beta}$, where M^{α} is the vector of the density of the electron magnetic moments. The magnetic field in the spin–spin interaction force field F^{α}_{ss} is caused by the magnetic moments. However, the spin–current interaction also gives a contribution in this force field via the magnetic field caused by the electric currents acting on the magnetic moments. Since the spin of the electron is proportional to the electron magnetic moment, the spin–spin interaction force arises in terms of the magnetization M^{α} or, in other words, the electron magnetic moments density.

To include both sources of the magnetic field we need to rewrite Eq. (21) as follows:

$$\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j} + 4\pi\nabla\times\mathbf{M}. \tag{31}$$

The spin–orbit interaction leads to the extra force field $\mathbf{F}_{s-o}$ consisting of four terms [9,15]:

$$F^{\alpha}_{s-o} = \frac{1}{c}\frac{2\gamma}{\hbar}\varepsilon^{\alpha\beta\mu}\varepsilon^{\beta\gamma\delta}B^{\gamma}M^{\delta}E^{\mu} - \frac{1}{c}\varepsilon^{\alpha\beta\gamma}M^{\beta}\partial_t E^{\gamma} - \frac{1}{c}\varepsilon^{\alpha\beta\gamma}\partial^{\delta}E^{\gamma}J^{\beta\delta}_{M} - \frac{1}{c}\varepsilon^{\beta\gamma\mu}J^{\beta\gamma}_{M}\partial^{\alpha}E^{\mu}, \tag{32}$$

where $J^{\alpha\beta}_{M}$ is the spin current tensor. Its explicit form is $J^{\alpha\beta}_{M} = M^{\alpha}v^{\beta} - (\hbar/2m\mu_{\mathrm{B}})\varepsilon^{\alpha\mu\nu}M^{\mu}\partial^{\beta}(M^{\nu}/n)$, with $\mu_{\mathrm{B}} = e\hbar/2mc$ is the Bohr magneton.

As we explained, the spin evolution changes the force field in the Euler equation. Moreover, the spin evolution extends the quantum hydrodynamic equation set due to the appearance of the spin evolution equation

$$\partial_t M^{\alpha} + \nabla^{\beta}J^{\alpha\beta}_{M} = \frac{2\gamma}{\hbar}\varepsilon^{\alpha\beta\gamma}\left(M^{\beta}B^{\beta} + \frac{1}{c}\varepsilon^{\beta\mu\nu}J^{\gamma\nu}_{M}E^{\mu}\right). \tag{33}$$

This equation is the analog of the Bloch equation. The first term on the right-hand side of Eq. (33) describes the spin–spin and spin–current interactions, while the last term is the contribution of the spin–orbit interaction.

The spin dynamics are closely related to the magnetic field perturbations. In the quasi-static regime these perturbations can be described by the pair of Maxwell equations $\nabla\times\mathbf{B} = 4\pi\nabla\times\mathbf{M}$ and $\nabla\mathbf{B} = 0$. The quasi-static magnetic field does not relate to electromagnetic waves, but it can support matter wave propagation: the spin waves. It can also accompany charge excitations.

Let us consider the contribution of the quasi-static magnetic field in the spectrum of the Langmuir waves. If we have unmagnetized plasmas, the spin evolution does not give any contribution in the linear approximation. Therefore, we consider longitudinal perturbations in the magnetized quantum plasmas. After some calculations (see Appendix B) we find that there is a contribution of the spin evolution in the Langmuir wave spectrum at the propagation perpendicular to the external magnetic field of

$$\omega^2 = \omega_{Le}^2\left(1 - \frac{\hbar^2 k^2}{2m^2c^2} - \frac{M_0 B_0}{mc^2 n_0}\right) + \Omega_e^2 + \left(\frac{1}{3}v_{Fe}^2\left(1 - \frac{1}{10}\frac{v_{Fe}^2}{c^2}\right) + \frac{\hbar^2 k^2}{4m^2} - \frac{\hbar^4 k^4}{8m^4c^2}\right)k^2, \quad (34)$$

where $\Omega_e = eB_0/mc$ is the electron cyclotron frequency. The term caused by the spin evolution is proportional to the ratio of the equilibrium magnetization energy density $M_0 B_0/8\pi$ to the rest energy density mn_0c^2. It was mentioned in Ref. [10] that they indirectly found the contribution of parameter $\mu B_0/mc^2$, while we have found its explicit contribution in the spectrum.

For the parallel propagation we find no contribution of the spin evolution in the dispersion of the Langmuir waves, as well as at zero external magnetic field. In Ref. [10], the authors undertook quantum kinetic analysis of the spin contribution by a general dispersion equation (41). They did not find any explicit dependence $\omega(k)$. However, their dispersion equation is rather more complicated, containing more and different spin contributions, in comparison with our results.

5. Conclusion

We have given a derivation of the many-particle QHD equations for a semi-relativistic system of spinless charged particles. The contributions of the RCKE, Coulomb, Darwin, and current–current interactions in the Euler equation have been obtained. The contributions from different terms were compared. It is shown that simultaneous accounting of the RCKE and Darwin interactions is necessary, because the RCKE gives a number of terms having different structure, and one of them has the structure of the term connected with the Darwin interaction in the Hamiltonian. The RCKE leads to the complex structure of the pressure tensor. The semi-relativistic part of the pressure tensor contains terms proportional to the ratio of the velocity field to the square of the speed of light. It also includes several terms proportional to $\hbar^4/c^2$, and contains more higher spatial derivatives than in the non-relativistic quantum Bohm pressure.

Using the developed approximation of the QHD equations we studied the dispersion dependence $\omega(\mathbf{k})$ of the semi-relativistic Langmuir waves. We obtained the contribution of the RCKE, which gives two terms in $\omega(\mathbf{k})$, the RCKE–EF interaction and the semi-relativistic part of the quantum Bohm potential contributions, and the Darwin interaction, giving one term. We have found that the RCKE–EF and the Darwin interactions give an equal contribution in the dispersion dependence of the Langmuir waves.

We have developed the QHD method with all the semi-relativistic effects in spinless plasmas for further research of linear and nonlinear effects in semi-relativistic quantum plasmas.

The spinless semi-relativistic effects play an important role in plasmas of spinless particles and in plasmas of spinning particles. Spin-dependent semi-relativistic interactions were considered in earlier literature: the spin–spin interaction was studied in Refs. [12,13,15], the spin–current interaction is considered in [13], and the spin–orbit interaction was included in the QHD equations in Refs. [15] and [9].

We have discussed the spin evolution contribution in the semi-relativistic model considered. We have also considered the Langmuir waves propagating parallel and perpendicular to the external magnetic field in magnetized plasmas. We have found the spin–orbit interaction contribution in the spectrum at the wave propagation perpendicular to the external magnetic field.

We conclude that this paper fulfils the program of development of semi-relativistic hydrodynamics based on the Breit Hamiltonian.

Appendix A. Equation of state for semi-relativistic degenerate Fermi gas

For relativistic fermions, assuming a relativistic relation between the energy and momentum of each particle $\epsilon_i = \sqrt{p_i^2 c^2 + m^2 c^4}$ and applying the usual techniques (see [42], Sects. 56, 58), one can find the equation of state

$$p = \frac{\pi}{3} \frac{m^4 c^5}{(2\pi\hbar)^3} \Xi\left(\frac{p_{\rm F}}{mc}\right), \tag{A1}$$

where $p_{\rm F} = (3\pi^2 n_0)^{1/3}\hbar$ is the Fermi momentum, and

$$\Xi(x) = 8\int_0^x \frac{\xi^4}{\sqrt{1+\xi^2}} d\xi. \tag{A2}$$

We consider the semi-relativistic limit. Consequently we have $\frac{p_{\rm F}}{mc} \ll 1$, and $x \ll 1$, $\xi \ll 1$.

In the semi-relativistic approximation we can write equation of state in the following form:

$$p = p_{\rm NR}\left(1 - \frac{1}{14}\frac{v_{Fe}^2}{c^2}\right), \tag{A3}$$

where we have used $p_{\rm NR}$ for the non-relativistic Fermi pressure, $p_{\rm NR} = \frac{\hbar^2}{5m}(3\pi^2)^{\frac{2}{3}} n^{\frac{5}{3}}$.

For linear perturbation of the pressure (A3), we obtain

$$\delta p = \frac{\partial p}{\partial n}\delta n = \frac{1}{3} m v_{Fe}^2 \left(1 - \frac{1}{10}\frac{v_{Fe}^2}{c^2}\right). \tag{A4}$$

Appendix B. Calculation of the spin evolution contribution in the spectrum

For a semi-relativistic plasma with the spin evolution being in the external magnetic field we can consider two regimes of wave propagation: parallel with and perpendicular to the external magnetic field.

Let us start with the parallel propagation. In this regime we have $\mathbf{B}_0 = B_0\mathbf{e}_z$, $\mathbf{M}_0 \parallel \mathbf{B}_0$, $\mathbf{k} = k_z\mathbf{e}_z$. The equation $\nabla\mathbf{B} = 0$ gives $k_z\delta B_z = 0 \Rightarrow \delta B_z = 0$. In the spin–spin interaction force field we find $M_0^\beta \partial^\alpha \delta B^\beta = M_0^z \partial^\alpha \delta B^z = 0$. The third and fourth terms in the spin–orbit interaction force field are nonlinear. The first term is proportional to the vector product of the equilibrium magnetization and the external magnetic field. Hence, it is equal to zero. The second term in Eq. (32) arises as $\delta F^\alpha_{s-o} = -\varepsilon^{\alpha\beta\gamma} M_0^\beta (-\imath\omega)\delta E^\gamma/c$. In the longitudinal waves we have $\delta\mathbf{E} \parallel \mathbf{k}$, but $\mathbf{M}_0 \parallel \mathbf{B}_0 \parallel \mathbf{k}$. Consequently, the vector product of M_0^β and δE^γ is equal to zero. In the regime of parallel propagation we find no spin contribution in the Langmuir wave.

We now consider the perpendicular propagation. In this regime we have $\mathbf{B}_0 = B_0\mathbf{e}_z$, $\mathbf{M}_0 \parallel \mathbf{B}_0$, $\mathbf{k} = k_x\mathbf{e}_x$. The equation $\nabla\mathbf{B} = 0$ gives $k_x\delta B_x = 0 \Rightarrow \delta B_x = 0$. The equation $\nabla\times\mathbf{B} = 4\pi\nabla\times\mathbf{M}$ gives $\delta B_z = 4\pi\delta M_z$ and $\delta B_y = 4\pi\delta M_y$. In the spin–spin interaction force field we find $M_0^\beta\partial^\alpha\delta B^\beta = M_0^z\partial^\alpha\delta B^z = 4\pi M_0^z\partial^\alpha\delta M^z = 0$, since we find $\delta M^z = 0$ from the spin evolution equation (33). As in the previous regime, the first, third, and fourth terms in the spin–orbit interaction force field are equal to zero. The second term in Eq. (32) arises as $\delta F^\alpha_{s-o} = \imath\omega M_0 \delta E_x/c$ and is not equal to zero. Therefore, we find the following set of linearized equations for projections of the Euler equation:

$$-\imath\omega m n_0 \delta v_x + \imath k_x\left(3mv_{se}^2 + \frac{\hbar^2 k_x^2}{4m} - \frac{\hbar^4 k_x^4}{8m^3c^2}\right)\delta n = e n_0\delta E_x - k_x^2\frac{e n_0\hbar^2}{2m^2c^2}\delta E_x + m n_0\Omega_e\delta v_y, \tag{B1}$$

$$-\imath\omega m n_0\delta v_y = -m n_0\Omega_e\delta v_x + \imath\omega M_0\delta E_x/c, \tag{B2}$$

and $\delta v_z = 0$, where $G_{xx} = G_{yx} = G_{zx} = 0$, $\Omega_e = eB_0/mc$. Solving these equations together with the continuity equation we find the spectrum (34).

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