LARGE TIME SOLUTION FOR QUADRATIC SCHRÖDINGER EQUATIONS IN 2D AND 3D

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Abstract. The aim of this paper is to establish the large time well-posedness of the quadratic Schrödinger equation in 2D and 3D. Let $\epsilon > 0$ be the size of an initial datum, which is sufficiently small. Then, by using the operator $\mathscr{L} = x - 2it\nabla$ and the associated weighted Sobolev spaces, we show that there exists a solution whose life-span T is given by $T = e^{1/\epsilon}$ (almost global) in 2D and $T = \infty$ in 3D.

1. Introduction

In this paper, we are concerned with the well-posedness of the Schrödinger equations with the polynomial nonlinearity;

$$i\partial_t u - \Delta u = N_\alpha(u, \bar{u}), \tag{1.1}$$

where $N_\alpha(u)$ is a homogeneous function of u and $\bar{u}$ of order α. This problem is easily solvable if α is large enough, which is the case when the linear term is dominant. The difficulty in the study of the large time behavior of solutions for small α's lies in the fact that the structure of the nonlinearity plays a significant role. This can be understood, for example, in terms of the Strauss exponents; $\alpha_0 = \sqrt{2}+1$ in 2D and $\alpha_0 = 2$ in 3D ([7]). For α larger than the Strauss exponent, there exists global-in-time solutions in $L^{\alpha+1}$ for small data by using the Strichartz estimates ([1, 8]). For detailed list of known results for the solvability of (1.1), see [5].

In this paper, we consider the Schrödinger equation with quadratic nonlinearity, $\alpha = 2$, which is equal to or less than the Strauss exponent in 3D or 2D. Therefore, small initial data do not guarantee a global existence of a solution in L^3. (See Appendix for details.) To establish the existence of a solution whose life span is beyond the existence time in L^3, we now use the maximal decay rate of the linear part in L^∞;

$$\left\|e^{it\Delta}u_0\right\|_{L^\infty} \lesssim t^{-\frac{d}{2}}\|u_0\|_{L^1},$$

which will lead to a global existence in 3D (relatively) easily. However, the structure of the nonlinearity plays a significant role in 2D; the quadratic nonlinear terms decay in time with the same speed as the linear term. Thus, the special oscillating structure of nonlinear terms must be taken into account. Along this direction, the role of the nonlinearity is well understood in [3, 5], where they establish the

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existence of a global solution of the Schrödinger equation

$$i\partial_t u - \Delta u = N_2(u), \quad \widehat{N_2(u)}(\xi) = \int_{\mathbb{R}^2} Q(\xi,\eta)\hat{u}(\xi-\eta)\hat{u}(\eta)d\xi, \tag{1.2}$$

where

(1) $Q = 1$ in 3D ($N(u) = u^2$),
(2) Q is linear for $|(\xi,\eta)| \leq 1$ and $Q = 1$ for $|(\xi,\eta)| \geq 2$ in 2D. ($N_2(u)$ behaves like $u\nabla u$ for low frequency part and u^2 for large frequency part.)

The main idea in [3, 5] is the so called *space-time resonance*, where the particular structure of $N_2(u)$ in fact cancels the *space-time resonant set*, which is equal to the zero frequencies of the interacting waves.

The goal of this paper is to provide a different approach of the work [3, 5] to the equation

$$i\partial_t u - \Delta u = u^2 + \bar{u}^2 + |u|^2. \tag{1.3}$$

We observe that the (classical) weighted L^2 estimates and L^∞ decay of solutions are enough to prove the almost global existence of a solution in 2D (which is weaker than the result in [5], but without the derivative structure in $N(u)$) and the global existence in 3D. Moreover, our method covers the nonlinearity $|u|^2$, which cannot be addressed by the space-time resonance method due to the large resonance set.

The weighted L^2 spaces are defined by the operator $\mathscr{L} = (\mathscr{L}_1, \cdots, \mathscr{L}_d)$, $\mathscr{L}_j = x_j - 2it\nabla_j$, which is unitarily equivalent to x and $2it\nabla$ such as

$$\mathscr{L}_j = U(t)x_jU(-t) = e^{ix^2/t}(2it\nabla_j)e^{-ix^2/t}, \quad U(t) = e^{it\Delta} \text{ is the free Schrödinger operator.}$$

This method was developed by Klainerman([2]) and was used in [6] to establish the global well-posedness for the nonlinear Schrödinger equation of the form $i\partial_t u - \Delta u = |u|^{2q}u$. In this paper, we follow the same approach in [6].

Notation and main result. Let $\mathbb{H}_{(0,n)}$ be the space of complex valued functions with

$$\mathbb{H}_{(0,n)}[f] = \sum_{m=0}^{n} \left(\|\nabla^m f\|_{L^2} + \|X^m f\|_{L^2}\right), \ n \in \mathbb{N}, \tag{1.4}$$

where X is the operator $f \mapsto xf$, $x = (x_1, \cdots, x_d)$. And the fact that the linear operator $(i\partial_t - \Delta)$ commutes with the operator $\mathscr{L} = (x - 2it\nabla)$ leads to the time-dependent norm;

$$\mathbb{H}_{(t,n)}[f] = \sum_{m=0}^{n} \left(\|\nabla^m f\|_{L^2} + \|\mathscr{L}^m f\|_{L^2}\right). \tag{1.5}$$

The first term in (1.5) is the usual energy norm and it is easy to show that u satisfies

$$\|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s} + \int_0^t \|u(s)\|_{L^\infty}\|u(s)\|_{H^s}ds. \tag{1.6}$$

Therefore, it is natural to obtain decay estimates of $\|u(t)\|_{L^\infty}$ and the bound of second term in (1.5) similar to (1.6). It is shown in [6] that $\|u(t)\|_{L^\infty}$ decays as

$$\|u(t)\|_{L^\infty} \lesssim (1+t)^{-d/2}\mathbb{H}_{(t,n)}[u], \quad n > \frac{d}{2}. \tag{1.7}$$

The main contribution of the paper is to show that a solution of (1.3) satisfies the estimate

$$\frac{d}{dt}\|\mathscr{L}^k u\|_{L^2} \lesssim \|u\|_{L^\infty}\|\mathscr{L}^k u\|_{L^2}, \quad k = 1, 2 \tag{1.8}$$

which is enough to show the following result.

Theorem 1.1. *Let u_0 be an initial datum such that $\mathbb{H}_{(0,2)}[u_0] = \epsilon$ which is sufficiently small. Then, there exists a solution u of (1.3) such that $\sup_{0<t\le T} \mathbb{H}_{(t,2)}[u] \lesssim \epsilon$ with $T = e^{1/\epsilon}$ in 2D and $T = \infty$ in 3D.*

2. Proof of Theorem

To prove Theorem 1.1, we need the following lemma. For detailed proofs, see [6].

Lemma 2.1. *(1) $\mathbb{H}_{(0,n)} = \mathbb{H}_{(t,n)}$, (2) $\left\|(\mathscr{L}f)^2\right\|_{L^2} \lesssim \|f\|_{L^\infty}\|\mathscr{L}^2 f\|_{L^2}$.*

The first statement of Lemma 2.1 implies that $\|\mathscr{L}^k\bar{u}\|_{L^2} \lesssim \mathbb{H}_{(t,k)}[u]$ because $\mathbb{H}_{(0,n)}$ norm is independent of the complex conjugate. This property is the main ingredient of the proof of Theorem 1.1.

Proof of Theorem. We now take $\mathscr{L}$ to the equation (1.3).

$$(i\partial_t - \Delta)\mathscr{L}u = 2u\mathscr{L}u + 2\bar{u}\mathscr{L}\bar{u} + \bar{u}\mathscr{L}u + u\mathscr{L}\bar{u} - xu^2 - x\bar{u}^2 - x|u|^2. \tag{2.1}$$

Then, by Lemma 2.1, we have

$$\frac{d}{dt}\|\mathscr{L}u\|_{L^2}^2 \lesssim \|u\|_{L^\infty}\left(\|\mathscr{L}u\|_{L^2}^2 + \|\mathscr{L}u\|_{L^2}\mathbb{H}_{(t,1)}[u]\right). \tag{2.2}$$

We take $\mathscr{L}$ one more time to the equation (1.3). The direct computation yields that

$$\begin{aligned}\mathscr{L}^2\left(u^2+\bar{u}^2+|u|^2\right) &= x^2\left(u^2+\bar{u}^2+|u|^2\right) - 4itx\left(u\nabla u + \bar{u}\nabla\bar{u} + u\nabla\bar{u} + \bar{u}\nabla u\right)\\ &\quad - 2it\delta_{ij}\left(u^2+\bar{u}^2+|u|^2\right) + 8t^2\left((\nabla u)^2 + (\nabla\bar{u})^2 + |\nabla u|^2\right)\\ &\quad - 4t^2\left(2u\nabla^2 u + 2\bar{u}\nabla^2\bar{u} + u\nabla^2\bar{u} + \bar{u}\nabla^2 u\right).\end{aligned} \tag{2.3}$$

In (2.3), we can estimate x^2u^2 and $xut\nabla u$ (and variant in $\bar{u}$) in terms of $\mathbb{H}_{(t,2)}[u]$. However, terms $t\delta_{ij}\left(u^2+\bar{u}^2+|u|^2\right)$ and $t^2\left(u\nabla^2 u + \bar{u}\nabla^2\bar{u} + u\nabla^2\bar{u} + \bar{u}\nabla^2 u\right)$ cannot be estimated directly in $\mathbb{H}_{(t,2)}[u]$. We thus compute further. Since

$$\mathscr{L}^2 u = x^2u - 4itx\nabla u - 2it\delta_{ij}u - 4t^2\nabla^2 u, \quad \mathscr{L}^2\bar{u} = x^2\bar{u} - 4itx\nabla\bar{u} - 2it\delta_{ij}\bar{u} - 4t^2\nabla^2\bar{u},$$

by properly multiplying these terms and conjugates with u and $\bar{u}$, we have

$$u\mathscr{L}^2u = x^2u^2 - 4itxu\nabla u - 2it\delta_{ij}u^2 - 4t^2u\nabla^2u, \tag{2.4a}$$

$$\bar{u}\mathscr{L}^2\bar{u} = x^2\bar{u}^2 - 4itx\bar{u}\nabla\bar{u} - 2it\delta_{ij}\bar{u}^2 - 4t^2\bar{u}\nabla^2\bar{u}, \tag{2.4b}$$

$$\bar{u}\mathscr{L}^2u = x^2|u|^2 - 4itx\bar{u}\nabla u - 2it\delta_{ij}|u|^2 - 4t^2\bar{u}\nabla^2u, \tag{2.4c}$$

$$u\mathscr{L}^2\bar{u} = x^2|u|^2 - 4itxu\nabla\bar{u} - 2it\delta_{ij}|u|^2 - 4t^2u\nabla^2\bar{u}. \tag{2.4d}$$

Therefore, we can express $t\delta_{ij}\left(u^2+\bar{u}^2+|u|^2\right)$ and $t^2\left(u\nabla^2u+\bar{u}\nabla^2\bar{u}+u\nabla^2\bar{u}+\bar{u}\nabla^2u\right)$ in terms of $\mathscr{L}^2$ and x^2u^2 and $xut\nabla u$ (and variants in $\bar{u}$) as follows.

$$8t^2u\nabla^2u = -u\left(\mathscr{L}^2u+\overline{\mathscr{L}^2\bar{u}}\right)+2x^2u^2, \tag{2.5a}$$

$$8t^2\bar{u}\nabla^2\bar{u} = -\bar{u}\left(\overline{\mathscr{L}^2u}+\mathscr{L}^2\bar{u}\right)+2x^2\bar{u}^2, \tag{2.5b}$$

$$2it\delta_{ij}u^2 = -\frac{1}{2}u\left(\mathscr{L}^2u-\overline{\mathscr{L}^2\bar{u}}\right)-4itxu\nabla u, \tag{2.5c}$$

$$2it\delta_{ij}\bar{u}^2 = \frac{1}{2}\bar{u}\left(\overline{\mathscr{L}^2u}-\mathscr{L}^2\bar{u}\right)-4itx\bar{u}\nabla\bar{u}, \tag{2.5d}$$

$$4t^2u\nabla^2\bar{u} = -\frac{1}{2}\left(u\overline{\mathscr{L}^2u}-\bar{u}\overline{\mathscr{L}^2u}\right)+x^2|u|^2-4itxu\nabla\bar{u}, \tag{2.5e}$$

$$4t^2\bar{u}\nabla^2u = -\frac{1}{2}\left(\bar{u}\mathscr{L}^2u-u\mathscr{L}^2u\right)+x^2|u|^2+4itx\bar{u}\nabla u, \tag{2.5f}$$

$$2it\delta_{ij}|u|^2 = -\frac{1}{2}\left(\bar{u}\mathscr{L}^2\bar{u}-u\mathscr{L}^2u\right)+x^2|u|^2-2itx\bar{u}\nabla u-2itxu\nabla\bar{u}-2t^2\bar{u}\nabla^2u+2t^2u\nabla^2\bar{u}, \tag{2.5g}$$

$$2t^2\bar{u}\nabla^2u+2t^2u\nabla^2\bar{u} = \frac{1}{4}\left(u\overline{\mathscr{L}^2u}+\bar{u}\overline{\mathscr{L}^2u}+\bar{u}\mathscr{L}^2u+u\mathscr{L}^2u\right)+x^2|u|^2+2itxu\nabla\bar{u}-2itx\bar{u}\nabla u. \tag{2.5h}$$

In sum, we have

$$\begin{aligned}(i\partial_t-\Delta)\mathscr{L}^2u &= \frac{11}{4}u\mathscr{L}^2u+2\bar{u}\mathscr{L}^2\bar{u}+\frac{5}{4}\bar{u}\overline{\mathscr{L}^2u}+\frac{1}{2}u\overline{\mathscr{L}^2\bar{u}}+\frac{3}{4}u\overline{\mathscr{L}^2u}+\frac{1}{2}\bar{u}\mathscr{L}^2u \\ &\quad +8t^2\left((\nabla u)^2+(\nabla\bar{u})^2+|\nabla u|^2\right)-x^2\left(u^2+\bar{u}^2+|u|^2\right)-4itx\bar{u}\nabla u-8itxu\nabla\bar{u}.\end{aligned} \tag{2.6}$$

By Lemma 2.1,

$$\frac{d}{dt}\|\mathscr{L}^2u\|_{L^2}^2 \lesssim \|u\|_{L^\infty}\left(\|\mathscr{L}^2u\|_{L^2}^2+\mathbb{H}^2_{(2,t)}[u]\right). \tag{2.7}$$

Combining (1.6), (2.2) and (2.7), we finally have

$$\mathbb{H}_{(t,2)}[u] \lesssim \epsilon+\int_0^t\|u(s)\|_{L^\infty}\mathbb{H}_{(s,2)}[u]ds \tag{2.8}$$

Let $\sup_{0<t<T}\mathbb{H}_{(t,2)}[u] := \|u\|_{X_T}$. By (1.7),

$$\|u\|_{X_T} \lesssim \epsilon+\ln T\|u\|_{X_T}^2 \quad \text{in 2D},$$

$$\|u\|_{X_T} \lesssim \epsilon+\|u\|_{X_T}^2 \quad \text{in 3D}.$$

In 3D, we can take $T = \infty$, while in 2D we choose T such that $\epsilon \times \ln T \simeq \frac{1}{2}$, which leads the existence of a solution u whose life span is given by $T \simeq e^{1/\epsilon}$ in 2D. This completes the proof. □

3. Appendix

As mentioned in Introduction, we show the existence of a solution for the Schrödinger equation (1.3) in $L^{\alpha+1}$. To this end, we begin with the Strichartz estimate.

Lemma 3.1 ($L^q - L^{q'}$ Estimate)**.** *Let $U(t) = e^{it\Delta}$ be the Schrödinger operator. For $1 \le q \le 2$,*

$$\|U(t)f\|_{L^{q'}} \lesssim t^{-\frac{d}{2}\left(\frac{2}{q}-1\right)}\|f\|_{L^q}.$$

3.1. **3D case.** Since $\alpha = 2$, we use $L^3 - L^{3/2}$ estimate.

$$\begin{aligned}
\|u(t)\|_{L^3} &\lesssim \frac{1}{\sqrt{t}}\|u(1)\|_{L^{\frac{3}{2}}} + \int_1^t \frac{1}{\sqrt{t-s}}\|u(s)\|_{L^3}^2 ds \\
&\lesssim \frac{1}{\sqrt{t}}\|u(1)\|_{L^{\frac{3}{2}}} + \left(\sup_{1\le s\le t}\sqrt{s}\|u(s)\|_{L^3}\right)^2 \int_1^t \frac{1}{\sqrt{t-s}}\frac{1}{s}ds \\
&\lesssim \frac{1}{\sqrt{t}}\|u(1)\|_{L^{\frac{3}{2}}} + \left(\sup_{1\le s\le t}\sqrt{s}\|u(s)\|_{L^3}\right)^2 \times \frac{\ln t}{\sqrt{t}},
\end{aligned} \tag{3.1}$$

where we take the initial data at $t = 1$ for the convenience when performing estimates since $\frac{1}{s}$ is not integrable at 0. We set $\|u\|_{X_T} := \sup_{0\le t\le T}\sqrt{t}\|u(t)\|_{L^3}$. Then,

$$\|u\|_{X_T} \lesssim \|u_0\|_{L^{\frac{3}{2}}} + \ln T\|u\|_{X_T}^2 \tag{3.2}$$

which implies that there exists a unique solution whose life span T is given by

$$T \sim e^{\frac{1}{\epsilon}}, \text{ with } \epsilon = \|u(1)\|_{L^{\frac{3}{2}}} << 1.$$

3.2. **2D case.** We now apply the same argument to the 2D case.

$$\begin{aligned}
\|u(t)\|_{L^3} &\lesssim t^{-\frac{1}{3}}\|u_0\|_{L^{\frac{3}{2}}} + \int_0^t (t-s)^{-\frac{1}{3}}\|u(s)\|_{L^3}^2 ds \\
&\lesssim t^{-\frac{1}{3}}\|u_0\|_{L^{\frac{3}{2}}} + \left(\sup_{0\le s\le t} s^\beta\|u(s)\|_{L^3}\right)^2 \int_0^t (t-s)^{-\frac{1}{3}}s^{-2\beta}ds \\
&\lesssim t^{-\frac{1}{3}}\|u_0\|_{L^{\frac{3}{2}}} + t^{\frac{2}{3}-2\beta}\cdot\left(\sup_{0\le s\le t} s^\beta\|u(s)\|_{L^3}\right)^2.
\end{aligned} \tag{3.3}$$

We choose β such that $\frac{2}{3} - 2\beta = -\beta$, i.e. $\beta = \frac{2}{3}$. Then,

$$\sup_{0\le t\le T} t^{2/3}\|u(t)\|_{L^3} \lesssim T^{1/3}\cdot\|u_0\|_{L^{\frac{3}{2}}} + \left(\sup_{0\le t\le T} t^{2/3}\|u(t)\|_{L^3}\right)^2. \tag{3.4}$$

We set $\|u\|_{X_T} := \sup_{0\leq t\leq T} t^{2/3}\|u(t)\|_{L^3}$. Then,

$$\|u\|_{X_T} \lesssim T^{1/3}\cdot\|u_0\|_{L^{\frac{3}{2}}} + \|u\|_{X_T}^2 \tag{3.5}$$

which implies the existence of a solution whose life span is given by $T\sim 1/\epsilon$, $\epsilon = \|u_0\|_{L^{\frac{3}{2}}}$.

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