Linear Programming

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1 Introduction

A general optimization problem is of the form: choose $\mathbf{x}$ to

$$\begin{aligned} &\text{maximise} \quad f(\mathbf{x}) \\ &\text{subject to} \quad \mathbf{x} \in S \end{aligned}$$

where

$$\begin{aligned} &\mathbf{x} = (x_1, \dots, x_n)^T, \\ &f : \mathbb{R}^n \to \mathbb{R} \quad \text{is the } \textit{objective function}, \\ &S \subset \mathbb{R}^n \quad \text{is the } \textit{feasible set}. \end{aligned}$$

We might write this problem:

$$\max_{\mathbf{x}} f(\mathbf{x}) \quad \text{subject to } \mathbf{x} \in S.$$

For example

- $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ for some vector $\mathbf{c} \in \mathbb{R}^n$,
- $S = \{\mathbf{x} : A\mathbf{x} \leqslant \mathbf{b}\}$ for some $m \times n$ matrix A and some vector $\mathbf{b} \in \mathbb{R}^m$.

If f is linear and $S \subset \mathbb{R}^n$ can be described by linear equalities/inequalities then we have a *linear programming* (LP) problem.

If $\mathbf{x} \in S$ then $\mathbf{x}$ is called a *feasible solution*.

If the maximum of $f(\mathbf{x})$ over $\mathbf{x} \in S$ occurs at $\mathbf{x} = \mathbf{x}^*$ then

- $\mathbf{x}^*$ is an *optimal solution*, and
- $f(\mathbf{x}^*)$ is the *optimal value*.

Questions

In general:

- does a feasible solution $\mathbf{x} \in S$ exist?
- if so, does an optimal solution exist?
- if so, is it unique?

Example

A company produces drugs A and B using machines M_1 and M_2.

- 1 ton of drug A requires 1 hour of processing on M_1 and 2 hours on M_2
- 1 ton of drug B requires 3 hours of processing on M_1 and 1 hour on M_2
- 9 hours of processing on M_1 and 8 hours on M_2 are available each day
- Each ton of drug produced (of either type) yields £1 million profit

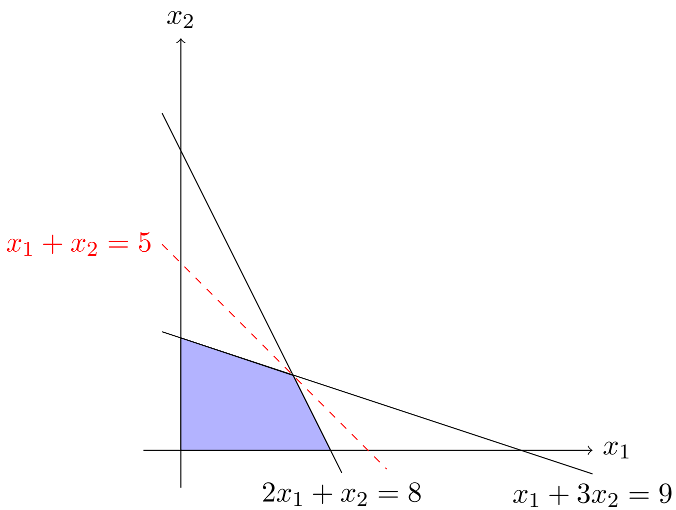
To maximise its profit, how much of each drug should the company make per day?

Solution

Let

- x_1 = number of tons of A produced
- x_2 = number of tons of B produced

$$\begin{array}{llll} P1: \text{maximise} & x_1 + x_2 & & \text{(profit in £ million)} \\ \quad\text{subject to} & x_1 + 3x_2 \leqslant 9 & & (M_1 \text{ processing}) \\ & 2x_1 + x_2 \leqslant 8 & & (M_2 \text{ processing}) \\ & x_1, x_2 \geqslant 0 & & \end{array}$$



The shaded region is the feasible set for $P1$. The maximum occurs at $\mathbf{x}^* = (3, 2)$ with value 5.

Diet problem

A pig-farmer can choose between four different varieties of food, providing different quantities of various nutrients.

		food				required
		1	2	3	4	amount/wk
	A	1.5	2.0	1.0	4.1	4.0
nutrient	B	1.0	3.1	0	2.0	8.0
	C	4.2	1.5	5.6	1.1	9.5
cost/kg		£5	£7	£7	£9	

The (i, j) entry is the amount of nutrient i per kg of food j.

Problem $P2$:

$$\begin{array}{lrl} \text{minimise} & 5x_1 + 7x_2 + 7x_3 + 9x_4 & \\ \text{subject to} & 1.5x_1 + 2x_2 + x_3 + 4.1x_4 & \geqslant 4 \\ & x_1 + 3.1x_2 + 2x_4 & \geqslant 8 \\ & 4.2x_1 + 1.5x_2 + 5.6x_3 + 1.1x_4 & \geqslant 9.5 \\ & x_1,\, x_2,\, x_3,\, x_4 & \geqslant 0 \end{array}$$

General form of the diet problem

Foods $j = 1, \dots, n$, nutrients $i = 1, \dots, m$.

Data:

- a_{ij} = amount of nutrient i in one unit of food j
- b_i = required amount of nutrient i
- c_j = cost per unit of food j

Let x_j = number of units of food j in the diet.

The diet problem is

$$\begin{array}{lll} \text{minimise} & c_1x_1 + \cdots + c_nx_n & \\ \text{subject to} & a_{i1}x_1 + \cdots + a_{in}x_n \geqslant b_i & \text{for } i = 1, \dots, m \\ & x_1, \dots, x_n \geqslant 0. & \end{array}$$

In matrix notation the diet problem is

$$\min_{\mathbf{x}} \mathbf{c}^T\mathbf{x} \quad \text{subject to } A\mathbf{x} \geqslant \mathbf{b},\ \mathbf{x} \geqslant \mathbf{0}.$$

Note that our vectors are always column vectors.

We write $\mathbf{x} \geqslant \mathbf{0}$ to mean $x_i \geqslant 0$ for all i. ($\mathbf{0}$ is a vector of zeros.)

Similarly $A\mathbf{x} \geqslant \mathbf{b}$ means $(A\mathbf{x})_i \geqslant b_i$ for all i.

Real applications

"Programming" = "planning"

Maybe many thousands of variables or constraints

- Production management: realistic versions of $P1$, large manufacturing plants, farms, etc
- Scheduling, e.g. airline crews:
  - need all flights covered
  - restrictions on working hours and patterns
  - minimise costs: wages, accommodation, use of seats by non-working staff

  shift workers (call centres, factories, etc)
- Yield management (airline ticket pricing: multihops, business/economy mix, discounts, etc)
- Network problems: transportation capacity planning in telecoms networks
- Game theory: economics, evolution, animal behaviour

Free variables

Some variables may be positive or negative, e.g. omit the constraint $x_1 \geqslant 0$.

Such a *free variable* can be replaced by

$$x_1 = u_1 - v_1$$

where $u_1, v_1 \geqslant 0$.

Slack variables

In $P1$ we had

$$\begin{array}{ll} \text{maximise} & x_1 + x_2 \\ \text{subject to} & x_1 + 3x_2 \leqslant 9 \\ & 2x_1 + x_2 \leqslant 8 \\ & x_1, x_2 \geqslant 0. \end{array}$$

We can rewrite by

$$\begin{array}{ll} \text{maximise} & x_1 + x_2 \\ \text{subject to} & x_1 + 3x_2 + x_3 = 9 \\ & 2x_1 + x_2 + x_4 = 8 \\ & x_1, \dots, x_4 \geqslant 0. \end{array}$$

- x_3 = unused time on machine M_1
- x_4 = unused time on machine M_2

x_3 and x_4 are called *slack variables*.

With the slack variables included, the problem has the form

$$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x} \quad \text{subject to} \quad A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geqslant \mathbf{0}.$$

Two standard forms

In fact any LP (with equality constraints, weak inequality constraints, or a mixture) can be converted to the form

$$\max_{\mathbf{x}} \mathbf{c}^T\mathbf{x} \quad \text{subject to} \quad A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geqslant \mathbf{0}$$

since:

- minimising $\mathbf{c}^T\mathbf{x}$ is equivalent to maximising $-\mathbf{c}^T\mathbf{x}$,
- inequalities can be converted to equalities by adding slack variables,
- free variables can be replaced as above.

Similarly, any LP can be put into the form

$$\max_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \quad \text{subject to} \quad A\mathbf{x} \leqslant \mathbf{b}$$
$$\mathbf{x} \geqslant \mathbf{0}$$

since e.g.

$$A\mathbf{x} = \mathbf{b} \iff \begin{cases} A\mathbf{x} \leqslant \mathbf{b} \\ -A\mathbf{x} \leqslant -\mathbf{b} \end{cases}$$

(more efficient rewriting may be possible!).

So it is OK for us to concentrate on LPs in these forms.

Remark

We always assume that the underlying space is $\mathbb{R}^n$.

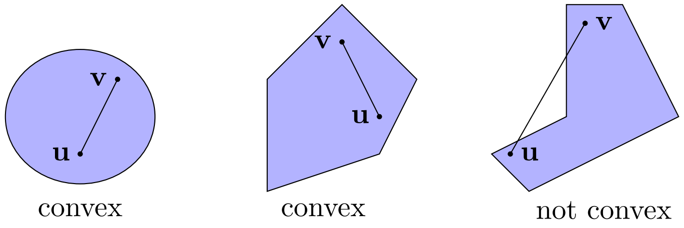
In particular $x_1, \ldots, x_n$ need not be integers. If we restrict to $\mathbf{x} \in \mathbb{Z}^n$ we have an *integer linear program* (ILP).

ILPs are in some sense *harder* than LPs. Note that the optimal value of an LP gives a *bound* on the optimal value of the associated ILP.

2 Geometry of linear programming

Definition 2.1

A set $S \subset \mathbb{R}^n$ is called *convex* if for all $\mathbf{u}, \mathbf{v} \in S$ and all $\lambda \in (0, 1)$, we have $\lambda\mathbf{u} + (1-\lambda)\mathbf{v} \in S$.



That is, a set is convex if all points on the line segment joining $\mathbf{u}$ and $\mathbf{v}$ are in S, for all possible line segments.

For now we will consider LPs in the form

$$\max \mathbf{c}^T\mathbf{x} \quad \text{subject to} \quad A\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \geqslant \mathbf{0}.$$

Theorem 2.2

The feasible set

$$S = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geqslant \mathbf{0}\}$$

is convex.

Proof.

Suppose $\mathbf{u}, \mathbf{v} \in S$, $\lambda \in (0, 1)$. Let $\mathbf{w} = \lambda\mathbf{u} + (1-\lambda)\mathbf{v}$. Then

$$\begin{aligned} A\mathbf{w} &= A[\lambda\mathbf{u} + (1-\lambda)\mathbf{v}] \\ &= \lambda A\mathbf{u} + (1-\lambda)A\mathbf{v} \\ &= [\lambda + (1-\lambda)]\mathbf{b} \\ &= \mathbf{b} \end{aligned}$$

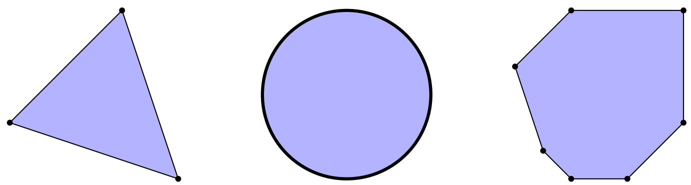
and $\mathbf{w} \geqslant \lambda\mathbf{0} + (1-\lambda)\mathbf{0} = \mathbf{0}$. So $\mathbf{w} \in S$. □

Extreme points

Definition 2.3

A point $\mathbf{x}$ in a convex set S is called an *extreme point* of S if there are no two distinct points $\mathbf{u}, \mathbf{v} \in S$, and $\lambda \in (0, 1)$, such that $\mathbf{x} = \lambda\mathbf{u} + (1-\lambda)\mathbf{v}$.

That is, an extreme point $\mathbf{x}$ is not in the interior of any line segment lying in S.



Theorem 2.4

If an LP has an optimal solution, then it has an optimal solution at an extreme point of the feasible set.

Proof.

Idea: If the optimum is not extremal, it's on some line within S all of which is optimal: go to the end of that line, repeat if necessary.

Since there exists an optimal solution, there exists an optimal solution $\mathbf{x}$ with a minimal number of non-zero components.

Suppose $\mathbf{x}$ is not extremal, so that

$$\mathbf{x} = \lambda\mathbf{u} + (1-\lambda)\mathbf{v}$$

for some $\mathbf{u} \neq \mathbf{v} \in S$, $\lambda \in (0, 1)$.

Since $\mathbf{x}$ is optimal, $\mathbf{c}^T\mathbf{u} \leqslant \mathbf{c}^T\mathbf{x}$ and $\mathbf{c}^T\mathbf{v} \leqslant \mathbf{c}^T\mathbf{x}$.

But also $\mathbf{c}^T\mathbf{x} = \lambda\mathbf{c}^T\mathbf{u} + (1-\lambda)\mathbf{c}^T\mathbf{v}$ so in fact $\mathbf{c}^T\mathbf{u} = \mathbf{c}^T\mathbf{v} = \mathbf{c}^T\mathbf{x}$.

Now consider the line defined by

$$\mathbf{x}(\varepsilon) = \mathbf{x} + \varepsilon(\mathbf{u} - \mathbf{v}), \quad \varepsilon \in \mathbb{R}.$$

Then

(a) $A\mathbf{x} = A\mathbf{u} = A\mathbf{v} = \mathbf{b}$ so $A\mathbf{x}(\varepsilon) = \mathbf{b}$ for all ε,

(b) $\mathbf{c}^T\mathbf{x}(\varepsilon) = \mathbf{c}^T\mathbf{x}$ for all ε,

(c) if $x_i = 0$ then $u_i = v_i = 0$, which implies $\mathbf{x}(\varepsilon)_i = 0$ for all ε,

(d) if $x_i > 0$ then $\mathbf{x}(0)_i > 0$, and $\mathbf{x}(\varepsilon)_i$ is continuous in ε.

So we can increase ε from zero, in a positive or a negative direction as appropriate, until at least one extra component of $\mathbf{x}(\varepsilon)$ becomes zero.

This gives an optimal solution with fewer non-zero components than $\mathbf{x}$.

So $\mathbf{x}$ must be extreme. □

Basic solutions

Let $\mathbf{a}_i$ be the ith column of A, so that

$$A\mathbf{x} = \mathbf{b} \iff \sum_{i=1}^{n} x_i\mathbf{a}_i = \mathbf{b}.$$

Definition 2.5

(1) A solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ is called a *basic solution* if the vectors $\{\mathbf{a}_i : x_i \neq 0\}$ are linearly independent.
(That is, columns of A corresponding to non-zero variables x_i are linearly independent.)

(2) A basic solution satisfying $\mathbf{x} \geqslant \mathbf{0}$ is called a *basic feasible solution* (BFS).

Note: If A has m rows, then at most m columns can be linearly independent. So any basic solution $\mathbf{x}$ has at least $n - m$ zero components. More later.

Theorem 2.6

$\mathbf{x}$ is an extreme point of

$$S = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geqslant \mathbf{0}\}$$

if and only if $\mathbf{x}$ is a BFS.

Proof.

(1) Let $\mathbf{x}$ be a BFS. Suppose $\mathbf{x} = \lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ for $\mathbf{u}, \mathbf{v} \in S$, $\lambda \in (0,1)$. To show $\mathbf{x}$ is extreme we need to show $\mathbf{u} = \mathbf{v}$.

Let $I = \{i : x_i > 0\}$. Then

(a) if $i \notin I$ then $x_i = 0$, which implies $u_i = v_i = 0$.

(b) $A\mathbf{u} = A\mathbf{v} = \mathbf{b}$, so $A(\mathbf{u} - \mathbf{v}) = \mathbf{0}$

$$\implies \sum_{i=1}^{n} (u_i - v_i)\mathbf{a}_i = \mathbf{0}$$

$$\implies \sum_{i \in I} (u_i - v_i)\mathbf{a}_i = \mathbf{0} \quad \text{since } u_i = v_i = 0 \text{ for } i \notin I$$

which implies $u_i = v_i$ for $i \in I$ since $\{\mathbf{a}_i : i \in I\}$ are linearly independent.

Hence $\mathbf{u} = \mathbf{v}$, so $\mathbf{x}$ is an extreme point.

(2) Suppose $\mathbf{x}$ is *not* a BFS, i.e. $\{\mathbf{a}_i : i \in I\}$ are linearly dependent.

Then there exists $\mathbf{u} \neq \mathbf{0}$ with $u_i = 0$ for $i \notin I$ such that $A\mathbf{u} = \mathbf{0}$.
For small enough ε, $\mathbf{x} \pm \varepsilon\mathbf{u}$ are feasible, and

$$\mathbf{x} = \frac{1}{2}(\mathbf{x} + \varepsilon\mathbf{u}) + \frac{1}{2}(\mathbf{x} - \varepsilon\mathbf{u})$$

so $\mathbf{x}$ is not extreme. □

Corollary 2.7

If there is an optimal solution, then there is an optimal BFS.

Proof.

This is immediate from Theorems 2.4 and 2.6. □

Discussion

Typically we may *assume*:

- $n > m$ (more variables than constraints),
- A has rank m (its rows are linearly independent; if not, either we have a contradiction, or redundancy).

Then: $\mathbf{x}$ is a basic solution $\iff$ there is a set $B \subset \{1, \ldots, n\}$ of size m such that

- $x_i = 0$ if $i \notin B$,
- $\{\mathbf{a}_i : i \in B\}$ are linearly independent.

Proof.
Simple exercise. Take $I = \{i : x_i \neq 0\}$ and augment it to a larger linearly independent set B if necessary. □

Then to look for basic solutions:

- choose $n - m$ of the n variables to be 0 ($x_i = 0$ for $i \notin B$),
- look at remaining m columns $\{\mathbf{a}_i : i \in B\}$.
  Are they linearly independent? If so we have an invertible $m \times m$ matrix.
  Solve for $\{x_i : i \in B\}$ to give $\sum_{i \in B} x_i \mathbf{a}_i = \mathbf{b}$.
  So also

$$A\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{a}_i = \sum_{i \notin B}^{n} 0\mathbf{a}_i + \sum_{i \in B}^{n} x_i \mathbf{a}_i$$
$$= \mathbf{b} \quad \text{as required.}$$

This way we obtain all basic solutions (at most $\binom{n}{m}$ of them).

Bad algorithm:

- look through all basic solutions,
- which are feasible?
- what is the value of the objective function?

We can do much better!

Simplex algorithm:

- will move from one BFS to another, improving the value of the objective function at each step.

3 The simplex algorithm (1)

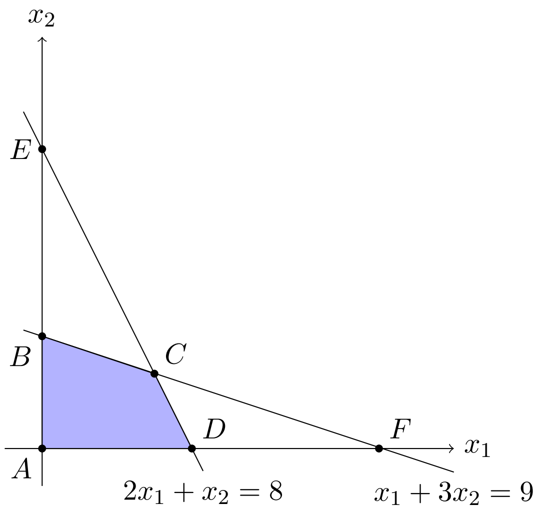
The simplex algorithm works as follows.

1. Start with an initial BFS.
2. Is the current BFS optimal?
3. If YES, stop.
   If NO, move to a new and improved BFS, then return to 2.

From Corollary 2.7, it is sufficient to consider only BFSs when searching for an optimal solution.

Recall $P1$, expressed without slack variables:

$$\begin{array}{lrl} \text{maximise} & x_1 + x_2 & \\ \text{subject to} & x_1 + 3x_2 & \leqslant 9 \\ & 2x_1 + x_2 & \leqslant 8 \\ & x_1, x_2 & \geqslant 0 \end{array}$$



Rewrite:

$$\begin{array}{rrrrcrl} x_1 & +3x_2 & +x_3 & & = & 9 & (1) \\ 2x_1 & +\ x_2 & & +x_4 & = & 8 & (2) \\ x_1 & +\ x_2 & & & = & f(\mathbf{x}) & (3) \end{array}$$

Put $x_1, x_2 = 0$, giving $x_3 = 9$, $x_4 = 8$, $f = 0$ (we're at the BFS $\mathbf{x} = (0, 0, 9, 8)$).

Note: In writing the three equations as (1)–(3) we are effectively expressing x_3, x_4, f in terms of x_1, x_2.

1. Start at the initial BFS $\mathbf{x} = (0,0,9,8)$, vertex A, where $f = 0$.

2. From (3), increasing x_1 or x_2 will increase $f(\mathbf{x})$. Let's increase x_1.
   (a) From (1): we can increase x_1 to 9, if we decrease x_3 to 0.
   (b) From (2): we can increase x_1 to 4, if we decrease x_4 to 0.

   The *stricter* restriction on x_1 is (b).

3. So (keeping $x_2 = 0$),
   (a) increase x_1 to 4, decrease x_4 to 0 – using (2), this maintains equality in (2),
   (b) and, using (1), decreasing x_3 to 5 maintains equality in (1).

   With these changes we move to a new and improved BFS $\mathbf{x} = (4,0,5,0)$, $f(\mathbf{x}) = 4$, vertex D.

To see if this new BFS is optimal, rewrite (1)–(3) so that

- each non-zero variable appears in exactly one constraint,
- f is in terms of variables which are zero at vertex D.

Alternatively, we want to express x_1, x_3, f in terms of x_2, x_4.

How? Add multiples of (2) to the other equations.

$$\begin{array}{rcccccl} (1) - \frac{1}{2}\times(2): & & \frac{5}{2}x_2 & +x_3 & -\frac{1}{2}x_4 & = & 5 \qquad (4) \\ \frac{1}{2}\times(2): & x_1 & +\frac{1}{2}x_2 & & +\frac{1}{2}x_4 & = & 4 \qquad (5) \\ (3) - \frac{1}{2}\times(2): & & \frac{1}{2}x_2 & & -\frac{1}{2}x_4 & = & f-4 \qquad (6) \end{array}$$

Now $f = 4 + \frac{1}{2}x_2 - \frac{1}{2}x_4$.

So we should increase x_2 to increase f.

1′. We are at vertex D, $\mathbf{x} = (4,0,5,0)$ and $f = 4$.

2′. From (6), increasing x_2 will increase f (increasing x_4 would decrease f).
   (a) From (4): we can increase x_2 to 2, if we decrease x_3 to 0.
   (b) From (5): we can increase x_2 to 8, if we decrease x_1 to 0.

   The *stricter* restriction on x_2 is (a).

3′. So increase x_2 to 2, decrease x_3 to 0 (x_4 stays at 0, and from (5) x_1 decreases to 3).
   With these changes we move to the BFS $\mathbf{x} = (3,2,0,0)$, vertex C.

Rewrite (4)–(6) so that they correspond to vertex C:

$$\begin{array}{rllll} \frac{2}{5}\times(4): & & x_2 & +\frac{2}{5}x_3 & -\frac{1}{5}x_4 & = & 2 & (7)\\ (5)-\frac{1}{5}\times(4): & x_1 & & -\frac{1}{5}x_3 & +\frac{3}{5}x_4 & = & 3 & (8)\\ (6)-\frac{1}{5}\times(4): & & & -\frac{1}{5}x_3 & -\frac{2}{5}x_4 & = & f-5 & (9)\end{array}$$

1″. We are at vertex C, $\mathbf{x} = (3, 2, 0, 0)$ and $f = 5$.

2″. We have deduced $f = 5 - \frac{1}{5}x_3 - \frac{2}{5}x_4 \leqslant 5$.

So $x_3 = x_4 = 0$ is the best we can do!

In that case we can read off $x_1 = 3$ and $x_2 = 2$.

So $\mathbf{x} = (3, 2, 0, 0)$, which has $f = 5$, is optimal.

Summary

At each stage:

- B = 'basic variables',
- we express x_i, $i \in B$ and f in terms of x_i, $i \notin B$,
- setting $x_i = 0$, $i \notin B$, we can read off f and x_i, $i \in B$ (gives a BFS!).

At each update:

- look at f in terms of x_i, $i \notin B$,
- which x_i, $i \notin B$, would we like to increase?
- if none, STOP!
- otherwise, choose one and increase it as much as possible, i.e. until one variable x_i, $i \in B$, becomes 0.

Summary continued

So

- one new variable *enters* B (becomes non-zero, becomes basic),
- another one *leaves* B (becomes 0, becomes non-basic).

This gives a new BFS.

We update our expressions to correspond to the new B.

Simplex algorithm

We can write equations

$$\begin{array}{rrrrcrr} x_1 & +3x_2 & +x_3 & & = & 9 & (1)\\ 2x_1 & +\ x_2 & & +x_4 & = & 8 & (2)\\ x_1 & +\ x_2 & & & = & f-0 & (3)\end{array}$$

as a 'tableau'

x_1	x_2	x_3	x_4		
1	3	1	0	9	ρ_1
2	1	0	1	8	ρ_2
1	1	0	0	0	ρ_3

This initial tableau represents the BFS $\mathbf{x} = (0, 0, 9, 8)$ at which $f = 0$.

Note the identity matrix in the x_3, x_4 columns (first two rows), and the zeros in the bottom row below it.

The tableau

At a given stage, the tableau has the form

$(\overline{a}_{ij})$	$\overline{\mathbf{b}}$
$\overline{\mathbf{c}}^T$	$\overline{f}$

which means

$$\overline{A}\mathbf{x} = \mathbf{b}$$
$$f(\mathbf{x}) = \overline{\mathbf{c}}^T\mathbf{x} - \overline{f}$$

We start from $\overline{A} = A$, $\overline{\mathbf{b}} = \mathbf{b}$, $\overline{\mathbf{c}} = \mathbf{c}$ and $\overline{f} = 0$.

Updating the tableau is called *pivoting*.

To update ('pivot')

1. *Choose a pivot column*
   Choose a j such that $\overline{c}_j > 0$ (corresponds to variable $x_j = 0$ that we want to increase).
   Here we can take $j = 1$.

2. *Choose a pivot row*
   Among the i's with $\overline{a}_{ij} > 0$, choose i to minimize $\overline{b}_i/\overline{a}_{ij}$ (strictest limit on how much we can increase x_j).
   Here take $i = 2$ since $8/2 < 9/1$.

3. Do row operations so that column j gets a 1 in row i and 0s elsewhere:
   - multiply row i by $\dfrac{1}{\overline{a}_{ij}}$,
   - for $i' \neq i$,
     add $-\dfrac{\overline{a}_{i'j}}{\overline{a}_{ij}} \times$ (row i) to row i',
   - add $-\dfrac{\overline{c}_j}{\overline{a}_{ij}} \times$ (row i) to objective function row.

In our example we pivot on $j = 1$, $i = 2$. The updated tableau is

x_1	x_2	x_3	x_4		
0	$\frac{5}{2}$	1	$-\frac{1}{2}$	5	$\rho_1' = \rho_1 - \rho_2'$
1	$\frac{1}{2}$	0	$\frac{1}{2}$	4	$\rho_2' = \frac{1}{2}\rho_2$
0	$\frac{1}{2}$	0	$-\frac{1}{2}$	-4	$\rho_3' = \rho_3 - \rho_2'$

which means

$$\begin{array}{rrrrcl} & \frac{5}{2}x_2 & +x_3 & -\frac{1}{2}x_4 & = & 5 \\ x_1 & +\frac{1}{2}x_2 & & +\frac{1}{2}x_4 & = & 4 \\ & \frac{1}{2}x_2 & & -\frac{1}{2}x_4 & = & f-4 \end{array}$$

'non-basic variables' x_2, x_4 are 0, $\mathbf{x} = (4, 0, 5, 0)$.

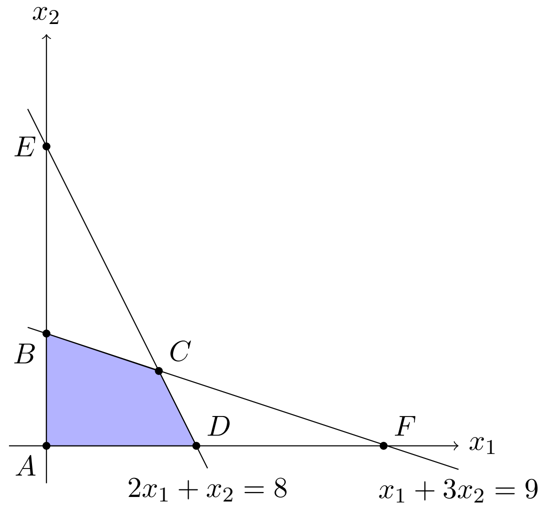
Note the identity matrix inside $(\overline{a}_{ij})$ telling us this.

Next pivot: column 2, row 1 since $\dfrac{5}{5/2} < \dfrac{4}{1/2}$.

x_1	x_2	x_3	x_4		
0	1	$\frac{2}{5}$	$-\frac{1}{5}$	2	$\rho_1'' = \frac{2}{5}\rho_1'$
1	0	$-\frac{1}{5}$	$\frac{3}{5}$	3	$\rho_2'' = \rho_2' - \frac{1}{2}\rho_1''$
0	0	$-\frac{1}{5}$	$-\frac{2}{5}$	-5	$\rho_3'' = \rho_3' - \frac{1}{2}\rho_1''$

Now we have only non-positive entries in the bottom row: STOP.
$\mathbf{x} = (3, 2, 0, 0)$, $f(\mathbf{x}) = 5$ optimal.

Geometric picture for $P1$



	x_1	x_2	x_3	x_4
A	0	0	9	8
B	4	0	5	0
C	3	2	0	0
D	0	3	0	5

A, B, C, D are BFSs. E, F are basic solutions but not feasible.

Simplex algorithm: $A \to D \to C$ (or $A \to B \to C$ is we choose a different column for the first pivot).

Higher-dimensional problems less trivial!

Comments on simplex tableaux

- We always find an $m \times m$ identity matrix embedded in $(\overline{a}_{ij})$, in the columns corresponding to *basic variables* x_i, $i \in B$,
- in the objective function row (bottom row) we find *zeros* in these columns.

Hence f and x_i, $i \in B$, are all written in terms of x_i, $i \notin B$.
Since we set $x_i = 0$ for $i \notin B$, it's then trivial to read off f and x_i, $i \in B$.

Comments on simplex algorithm

- *Choosing a pivot column*
  We may choose any j such that $\bar{c}_j > 0$. In general, there is no way to tell which such j result in fewest pivot steps.

- *Choosing a pivot row*
  Having chosen pivot column j (which variable x_j to increase), we look for rows with $\bar{a}_{ij} > 0$.

  If $\bar{a}_{ij} \leqslant 0$, constraint i places no restriction on the increase of x_j.

  If $\bar{a}_{ij} \leqslant 0$ for all i, x_j can be increased *without limit*: the objective function is *unbounded*.

  Otherwise, the most stringent limit comes from the i that minimises $\bar{b}_i/\bar{a}_{ij}$.

4 The simplex algorithm (2)

Two issues to consider:

- can we always find a BFS from which to *start* the simplex algorithm?
- does the simplex algorithm always *terminate*, i.e. find an optimal BFS or prove the problem is unbounded?

Initialisation

To start the simplex algorithm, we need to start from a BFS, with basic variables x_i, $i \in B$, written in terms of non-basic variables x_i, $i \notin B$.

If A already contains I_m as an $m \times m$ submatrix, this is usually easy!

This *always* happens if $\mathbf{b} \geqslant \mathbf{0}$ and A is created by adding slack variables to make inequalities into equalities.

Suppose the constraints are

$$A\mathbf{x} \leqslant \mathbf{b}, \quad \mathbf{x} \geqslant \mathbf{0}$$

where $\mathbf{b} \geqslant \mathbf{0}$. Then an initial BFS is immediate: introducing slack variables $\mathbf{z} = (z_1, \ldots, z_m)$,

$$A\mathbf{x} + \mathbf{z} = \mathbf{b}, \quad \mathbf{x}, \mathbf{z} \geqslant \mathbf{0}$$

so the initial tableau is

$x_1 \cdots x_n$	$z_1 \cdots z_m$	
A	I_m	$\mathbf{b}$
$\mathbf{c}^T$	$\mathbf{0}^T$	0

and an initial BFS is $\binom{\mathbf{x}}{\mathbf{z}} = \binom{\mathbf{0}}{\mathbf{b}}$.

Example

$$\begin{array}{llcl} \max & 6x_1 + x_2 + x_3 & & \\ \text{s.t.} & 9x_1 + x_2 + x_3 & \leqslant & 18 \\ & 24x_1 + x_2 + 4x_3 & \leqslant & 42 \\ & 12x_1 + 3x_2 + 4x_3 & \leqslant & 96 \\ & x_1, x_2, x_3 & \geqslant & 0. \end{array}$$

Add slack variables:

$$\begin{array}{l} 9x_1 + x_2 + x_3 + w_1 = 18 \\ 24x_1 + x_2 + 4x_3 + w_2 = 42 \\ 12x_1 + 3x_2 + 4x_3 + w_3 = 96 \end{array}$$

Solve by simplex

$$\begin{array}{cccccc|c} 9 & 1 & 1 & 1 & 0 & 0 & 18 \\ 24 & 1 & 4 & 0 & 1 & 0 & 42 \\ 12 & 3 & 4 & 0 & 0 & 1 & 96 \\ \hline 6 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}$$

This initial tableau is already in the form we need.

$$\begin{array}{cccccc|c} 9 & 1 & 1 & 1 & 0 & 0 & 18 \\ 15 & 0 & 3 & -1 & 1 & 0 & 24 \\ -15 & 0 & 1 & -3 & 0 & 1 & 42 \\ \hline -3 & 0 & 0 & -1 & 0 & 0 & -18 \end{array} \qquad \begin{array}{l} \rho_1' = \rho_1 \\ \rho_2' = \rho_2 - \rho_1 \\ \rho_3' = \rho_3 - 3\rho_1 \\ \rho_4' = \rho_4 - \rho_1 \end{array}$$

This solution is optimal: $x_2 = 18$, $x_1 = 0$, $x_3 = 0$, $f = 18$.

What if A doesn't have this form?

In general we can write the constraints as

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geqslant \mathbf{0} \tag{4.1}$$

where $\mathbf{b} \geqslant \mathbf{0}$ (if necessary, multiply rows by -1 to get $\mathbf{b} \geqslant \mathbf{0}$).

If there is no obvious initial BFS and we need to find one, we can introduce *artificial variables* $w_1, \dots, w_m$ and solve the LP problem

$$\begin{aligned} \min_{\mathbf{x},\mathbf{w}} \quad & \sum_{i=1}^{m} w_i \\ \text{subject to} \quad & A\mathbf{x} + \mathbf{w} = \mathbf{b} \\ & \mathbf{x}, \mathbf{w} \geqslant \mathbf{0}. \end{aligned} \tag{4.2}$$

Two cases arise:

(1) if (4.1) has a feasible solution, then (4.2) has optimal value 0 with $\mathbf{w} = \mathbf{0}$.

(2) if (4.1) has no feasible solution, then the optimal value of (4.2) is > 0.

We can apply the simplex algorithm to determine whether it's case (1) or (2).

- In case (2), give up.
- In case (1), the optimal BFS for (4.2) with $w_i \equiv 0$ gives a BFS for (4.1)!

This leads us to the *two-phase simplex algorithm*.

Two-phase simplex algorithm

Example:

$$\begin{array}{lrcrcrcl} \text{maximize} & 3x_1 & & & + & x_3 & & \\ \text{subject to} & x_1 & + & 2x_2 & + & x_3 & = & 30 \\ & x_1 & - & 2x_2 & + & 2x_3 & = & 18 \\ & & & & & \mathbf{x} & \geqslant & \mathbf{0} \end{array}$$

With artificial variables:

$$\begin{array}{lrcrcrcrcrl} \text{minimize} & & & & & & & w_1 & + & w_2 & \\ \text{subject to} & x_1 & + & 2x_2 & + & x_3 & + & w_1 & & & = 30 \\ & x_1 & - & 2x_2 & + & 2x_3 & & & + & w_2 & = 18 \\ & & & & & & & & & \mathbf{x}, \mathbf{w} & \geqslant \mathbf{0}. \end{array}$$

Note: To minimise $w_1 + w_2$, we can maximise $-w_1 - w_2$. So start from the simple tableau

x_1	x_2	x_3	w_1	w_2	
1	2	1	1	0	30
1	-2	2	0	1	18
0	0	0	-1	-1	0

The objective function row should be expressed in terms of non-basic variables (the entries under the 'identity matrix' should be 0).

So start by adding row 1 + row 2 to objective row:

x_1	x_2	x_3	w_1	w_2	
1	2	1	1	0	30
1	-2	2	0	1	18
2	0	3	0	0	48

Now start with simplex – pivot on $\overline{a}_{23}$:

x_1	x_2	x_3	w_1	w_2	
$\frac{1}{2}$	3	0	1	$-\frac{1}{2}$	21
$\frac{1}{2}$	-1	1	0	$\frac{1}{2}$	9
$\frac{1}{2}$	3	0	0	$-\frac{3}{2}$	21

Pivot on $\overline{a}_{12}$:

x_1	x_2	x_3	w_1	w_2	
$\frac{1}{6}$	1	0	$\frac{1}{3}$	$-\frac{1}{6}$	7
$\frac{2}{3}$	0	1	$\frac{1}{3}$	$\frac{1}{3}$	16
0	0	0	-1	-1	0

So we have found a point with $-w_1 - w_2 = 0$, i.e. $\mathbf{w} = \mathbf{0}$.

Phase I is finished. BFS of the original problem is $\mathbf{x} = (0, 7, 16)$.

Deleting the **w** columns and replacing the objective row by the original objective function $3x_1 + x_3$:

x_1	x_2	x_3	
$\frac{1}{6}$	1	0	7
$\frac{2}{3}$	0	1	16
3	0	1	0

Again we want zeros below the identity matrix – subtract row 2 from row 3:

x_1	x_2	x_3	
$\frac{1}{6}$	1	0	7
$\frac{2}{3}$	0	1	16
$\frac{7}{3}$	0	0	-16

Now do simplex.

Pivot on $\overline{a}_{21}$:

x_1	x_2	x_3	
0	1	$-\frac{1}{4}$	3
1	0	$\frac{3}{2}$	24
0	0	$-\frac{7}{2}$	-72

Done! Maximum at $x_1 = 24$, $x_2 = 3$, $x_3 = 0$, $f = 72$.

Shadow prices

Recall $P1$:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 3x_2 \leqslant 9 \\ & 2x_1 + x_2 \leqslant 8 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

We had initial tableau

1	3	1	0	9	ρ_1
2	1	0	1	8	ρ_2
1	1	0	0	0	ρ_3

and final tableau

0	1	$\frac{2}{5}$	$-\frac{1}{5}$	2	ρ_1'
1	0	$-\frac{1}{5}$	$\frac{3}{5}$	3	ρ_2'
0	0	$-\frac{1}{5}$	$-\frac{2}{5}$	-5	ρ_3'

From the mechanics of the simplex algorithm:

- ρ_1', ρ_2' are created by taking linear combinations of ρ_1, ρ_2
- ρ_3' is $\rho_3 -$ (a linear combination of ρ_1, ρ_2).

Directly from the tableaux (look at columns 3 and 4):

$$\begin{aligned} \rho_1' &= \tfrac{2}{5}\rho_1 - \tfrac{1}{5}\rho_2 \\ \rho_2' &= -\tfrac{1}{5}\rho_1 + \tfrac{3}{5}\rho_2 \\ \rho_3' &= \rho_3 - \tfrac{1}{5}\rho_1 - \tfrac{2}{5}\rho_2 \end{aligned}$$

Suppose we change the constraints to

$$\begin{aligned} x_1 + 3x_2 &\leqslant 9 + \varepsilon_1 \\ 2x_1 + x_2 &\leqslant 8 + \varepsilon_2. \end{aligned}$$

Then the final tableau will change to

$$\begin{array}{cccc|c} 0 & 1 & \frac{2}{5} & -\frac{1}{5} & 2 + \frac{2}{5}\varepsilon_1 - \frac{1}{5}\varepsilon_2 \\ 1 & 0 & -\frac{1}{5} & \frac{3}{5} & 3 - \frac{1}{5}\varepsilon_1 + \frac{3}{5}\varepsilon_2 \\ \hline 0 & 0 & -\frac{1}{5} & -\frac{2}{5} & -5 - \frac{1}{5}\varepsilon_1 - \frac{2}{5}\varepsilon_2 \end{array}$$

This is still a valid tableau as long as

$$\begin{aligned} 2 + \tfrac{2}{5}\varepsilon_1 - \tfrac{1}{5}\varepsilon_2 &\geqslant 0 \\ 3 - \tfrac{1}{5}\varepsilon_1 + \tfrac{3}{5}\varepsilon_2 &\geqslant 0. \end{aligned}$$

In that case we still get an optimal BFS from it, with optimal value

$$5 + \tfrac{1}{5}\varepsilon_1 + \tfrac{2}{5}\varepsilon_2.$$

The objective function increases by $\frac{1}{5}$ per extra hour on M_1 and by $\frac{2}{5}$ per extra hour on M_2 (if the changes are 'small enough').

These *shadow prices* can always be read off from the initial and final tableaux.

Digression: Termination of simplex algorithm

'Typical situation': each BFS has exactly m non-zero and $n - m$ zero variables.

Then each pivoting operation (moving from one BFS to another) strictly increases the new variable 'entering the basis' and strictly increases the objective function.

Since there are only finitely many BFSs, we have the following theorem.

Theorem 4.1

If each BFS has exactly m non-zero variables, then the simplex algorithm terminates (i.e. finds an optimal solution or proves that the objective function is unbounded).

What is some BFSs have extra zero variables?

Then the problem is 'degenerate'.

Almost always: this is no problem.

In rare cases, some choices of pivot columns/rows may cause the algorithm to *cycle* (repeat itself). There are various ways to avoid this (e.g. always choosing the leftmost column, and then the highest row, can be proved to work always.)
See e.g. Chvátal book for a nice discussion.

5 Duality: Introduction

Recall $P1$:

$$\begin{aligned} \text{maximise} \quad & x_1 + x_2 \\ \text{subject to} \quad & x_1 + 3x_2 \leqslant 9 \qquad (5.1) \\ & 2x_1 + x_2 \leqslant 8 \qquad (5.2) \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

'Obvious' bounds on $f(\mathbf{x}) = x_1 + x_2$:

$$\begin{aligned} x_1 + x_2 \leqslant x_1 + 3x_2 \leqslant 9 \quad & \text{from } (5.1) \\ x_1 + x_2 \leqslant 2x_1 + x_2 \leqslant 8 \quad & \text{from } (5.2). \end{aligned}$$

By combining the constraints we can improve the bound, e.g. $\frac{1}{3}[(5.1) + (5.2)]$:

$$x_1 + x_2 \leqslant x_1 + \tfrac{4}{3}x_2 \leqslant \tfrac{17}{3}.$$

More systematically?

For $y_1, y_2 \geqslant 0$, consider $y_1 \times (5.1) + y_2 \times (5.2)$. We obtain

$$(y_1 + 2y_2)x_1 + (3y_1 + y_2)x_2 \leqslant 9y_1 + 8y_2$$

Since we want an upper bound for $x_1 + x_2$, we need coefficients $\geqslant 1$:

$$\begin{aligned} y_1 + 2y_2 &\geqslant 1 \\ 3y_1 + y_2 &\geqslant 1. \end{aligned}$$

How to get the best bound by this method?

$$\begin{aligned} D1: \text{minimise} \quad & 9y_1 + 8y_2 \\ \text{subject to} \quad & y_1 + 2y_2 \geqslant 1 \\ & 3y_1 + y_2 \geqslant 1 \\ & y_1, y_2 \geqslant 0. \end{aligned}$$

$P1$ = 'primal problem', $D1$ = 'dual of $P1$'.

Duality: General

In general, given a primal problem

$$P: \quad \text{maximise } \mathbf{c}^T\mathbf{x} \quad \text{subject to } A\mathbf{x} \leqslant \mathbf{b},\ \mathbf{x} \geqslant \mathbf{0}$$

the dual of P is defined by

$$D: \quad \text{minimise } \mathbf{b}^T\mathbf{y} \quad \text{subject to } A^T\mathbf{y} \geqslant \mathbf{c},\ \mathbf{y} \geqslant \mathbf{0}.$$

Exercise

The dual of the dual is the primal.

Weak duality

Theorem 5.1 (Weak duality theorem)

If $\mathbf{x}$ is feasible for P, and $\mathbf{y}$ is feasible for D, then

$$\mathbf{c}^T\mathbf{x} \leqslant \mathbf{b}^T\mathbf{y}.$$

Proof.

Since $\mathbf{x} \geqslant \mathbf{0}$ and $A^T\mathbf{y} \geqslant \mathbf{c}$: $\mathbf{c}^T\mathbf{x} \leqslant (A^T\mathbf{y})^T\mathbf{x} = \mathbf{y}^T A\mathbf{x}$.

Since $\mathbf{y} \geqslant \mathbf{0}$ and $A\mathbf{x} \leqslant \mathbf{b}$: $\mathbf{y}^T A\mathbf{x} \leqslant \mathbf{y}^T\mathbf{b} = \mathbf{b}^T\mathbf{y}$.

Hence $\mathbf{c}^T\mathbf{x} \leqslant \mathbf{y}^T A\mathbf{x} \leqslant \mathbf{b}^T\mathbf{y}$. □

Comments

Suppose $\mathbf{y}$ is a feasible solution to D. Then any feasible solution $\mathbf{x}$ to P has value bounded above by $\mathbf{b}^T\mathbf{y}$.

So D feasible $\implies$ P bounded.

Similarly P feasible $\implies$ D bounded.

Corollary 5.2

If $\mathbf{x}^*$ is feasible for P, $\mathbf{y}^*$ is feasible for D, and $\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$, then $\mathbf{x}^*$ is optimal for P and $\mathbf{y}^*$ is optimal for D.

Proof.

For all $\mathbf{x}$ feasible for P,

$$\begin{aligned} \mathbf{c}^T\mathbf{x} &\leqslant \mathbf{b}^T\mathbf{y}^* \quad \text{by Theorem 5.1} \\ &= \mathbf{c}^T\mathbf{x}^* \end{aligned}$$

and so $\mathbf{x}^*$ is optimal for P.

Similarly, for all $\mathbf{y}$ feasible for D,

$$\mathbf{b}^T\mathbf{y} \geqslant \mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$$

and so $\mathbf{y}^*$ is optimal for D. □

As an example of applying this result, look at $\mathbf{x}^* = (3, 2)$, $\mathbf{y}^* = (\frac{1}{5}, \frac{2}{5})$ for $P1$ and $D1$ above.

Both are feasible, both have value 5. So both are optimal.

Does this nice situation always occur?

Strong duality

Theorem 5.3 (Strong duality theorem)

If P has an optimal solution $\mathbf{x}^*$, then D has an optimal solution $\mathbf{y}^*$ such that

$$\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*.$$

Proof.

Write the constraints of P as $A\mathbf{x} + \mathbf{z} = \mathbf{b}$, $\mathbf{x}, \mathbf{z} \geqslant \mathbf{0}$.

Consider the bottom row in the final tableau of the simplex algorithm applied to P.

$\mathbf{x}$-columns			$\mathbf{z}$-columns			
c_1^*	$\cdots$	c_n^*	$-y_1^*$	$\cdots$	$-y_m^*$	$-f^*$

Here f^* is the optimal value of $\mathbf{c}^T\mathbf{x}$,

$$c_j^* \leqslant 0 \quad \text{for all } j \tag{5.3}$$

$$-y_i^* \leqslant 0 \quad \text{for all } i \tag{5.4}$$

and the bottom row tells us that

$$\mathbf{c}^T\mathbf{x} - f^* = \mathbf{c}^{*T}\mathbf{x} - \mathbf{y}^{*T}\mathbf{z}.$$

Thus

$$\begin{aligned} \mathbf{c}^T\mathbf{x} &= f^* + \mathbf{c}^{*T}\mathbf{x} - \mathbf{y}^{*T}\mathbf{z} \\ &= f^* + \mathbf{c}^{*T}\mathbf{x} - \mathbf{y}^{*T}(\mathbf{b} - A\mathbf{x}) \\ &= f^* - \mathbf{b}^T\mathbf{y}^* + (\mathbf{c}^{*T} + \mathbf{y}^{*T}A)\mathbf{x}. \end{aligned}$$

This is true for all $\mathbf{x}$, so

$$f^* = \mathbf{b}^T\mathbf{y}^* \tag{5.5}$$

$$\mathbf{c} = A^T\mathbf{y}^* + \mathbf{c}^*.$$

From (5.3) $\mathbf{c}^* \leqslant \mathbf{0}$, hence

$$A^T\mathbf{y}^* \geqslant \mathbf{c}. \tag{5.6}$$

So (5.4) and (5.6) show that $\mathbf{y}^*$ is feasible for D.

And (5.5) shows that the objective function of D at $\mathbf{y}^*$ is $\mathbf{b}^T\mathbf{y}^* = f^* =$ optimal value of P.

So from the weak duality theorem (Theorem 5.1), $\mathbf{y}^*$ is optimal for D. □

Comments

Note:

- the coefficients $\mathbf{y}^*$ from the bottom row in the columns corresponding to slack variables give us the optimal solution to D,
- comparing with the shadow prices discussion: these optimal dual variables are the shadow prices!

Example

It is possible that neither P nor D has a feasible solution: consider the problem

$$\begin{aligned} \text{maximise} \quad & 2x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 \leqslant 1 \\ & -x_1 + x_2 \leqslant -2 \\ & x_1, x_2 \geqslant 0. \end{aligned}$$

Example

Consider the problem

$$\begin{aligned} \text{maximise} \quad & 2x_1 + 4x_2 + x_3 + x_4 \\ \text{subject to} \quad & x_1 + 3x_2 + x_4 \leqslant 4 \\ & 2x_1 + x_2 \leqslant 3 \\ & x_1 + 4x_3 + x_4 \leqslant 3 \\ & x_1, \ldots, x_4 \geqslant 0 \end{aligned}$$

The final tableau is

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
0	1	0	$\frac{2}{5}$	$\frac{2}{5}$	·	·	1
1	0	0	$-\frac{1}{5}$	$-\frac{1}{5}$	·	·	1
0	0	1	$\frac{3}{10}$	$\frac{1}{20}$	·	·	$\frac{1}{2}$
0	0	0	$-\frac{1}{2}$	$-\frac{5}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{13}{2}$

(1) By the proof of Theorem 5.3, $\mathbf{y}^* = (\frac{5}{4}, \frac{1}{4}, \frac{1}{4})$ is optimal for the dual.

(2) Suppose the RHSs of the original constraints become $4 + \varepsilon_1$, $3 + \varepsilon_2$, $3 + \varepsilon_3$. Then the objective function becomes $\frac{13}{2} + \frac{5}{4}\varepsilon_1 + \frac{1}{4}\varepsilon_2 + \frac{1}{4}\varepsilon_3$.

If the original RHSs of 4, 3, 3 correspond to the amount of raw material i available, then 'the most you'd be prepared to pay per additional unit of raw material i' is y_i^* (with $\mathbf{y}^*$ as in (1)).

(3) Suppose raw material 1 is available at a price $< \frac{5}{4}$ per unit. How much should you buy? With $\varepsilon_1 > 0$, $\varepsilon_2 = \varepsilon_3 = 0$, the final tableau would be

	$1 + \frac{2}{5}\varepsilon_1$
$\cdots$	$1 - \frac{1}{5}\varepsilon_1$
	$\frac{1}{2} + \frac{1}{20}\varepsilon_1$

For this tableau to represent a BFS, the three entries in the final column must be $\geqslant 0$, giving $\varepsilon_1 \leqslant 5$. So we should buy 5 additional units of raw material 1.

(4) The optimal solution $\mathbf{x}^* = (1, 1, \frac{1}{2}, 0)$ is *unique* as the entries in the bottom row corresponding to non-basic variables (i.e. the $-\frac{1}{2}$, $-\frac{5}{4}$, $-\frac{1}{4}$, $-\frac{1}{4}$) are < 0.

(5) If say the $-\frac{5}{4}$ was zero, we could pivot in that column (observe that there would somewhere to pivot) to get a second optimal BFS $\mathbf{x}^{**}$. Then $\lambda\mathbf{x}^* + (1-\lambda)\mathbf{x}^{**}$ would be optimal for all $\lambda \in [0, 1]$.

6 Duality: Complementary slackness

Recall

$$P: \quad \text{maximise } \mathbf{c}^T\mathbf{x} \quad \text{subject to } A\mathbf{x} \leqslant \mathbf{b},\ \mathbf{x} \geqslant \mathbf{0},$$
$$D: \quad \text{minimise } \mathbf{b}^T\mathbf{y} \quad \text{subject to } A^T\mathbf{y} \geqslant \mathbf{c},\ \mathbf{y} \geqslant \mathbf{0}.$$

The optimal solutions to P and D satisfy 'complementary slackness conditions' that we can use to solve one problem when we know a solution of the other.

Theorem 6.1 (Complementary slackness theorem)

Suppose $\mathbf{x}$ is feasible for P and $\mathbf{y}$ is feasible for D. Then $\mathbf{x}$ and $\mathbf{y}$ are optimal (for P and D) if and only if

$$(A^T\mathbf{y} - \mathbf{c})_j x_j = 0 \quad \textit{for all } j \tag{6.1}$$

and

$$(A\mathbf{x} - \mathbf{b})_i y_i = 0 \quad \textit{for all } i. \tag{6.2}$$

Conditions (6.1) and (6.2) are called the *complementary slackness conditions* for P and D.

Interpretation

(1)
- If dual constraint j is slack, then primal variable j is zero.
- If primal variable j is > 0, then dual constraint j is tight.

(2) The same with 'primal' $\longleftrightarrow$ 'dual'.

Proof.

As in the proof of the weak duality theorem,

$$\mathbf{c}^T\mathbf{x} \leqslant (A^T\mathbf{y})^T\mathbf{x} = \mathbf{y}^T A\mathbf{x} \leqslant \mathbf{y}^T\mathbf{b}. \tag{6.3}$$

From the strong duality theorem,

$$\begin{aligned}
\mathbf{x}, \mathbf{y} \text{ both optimal } &\iff \mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y} \\
&\iff \mathbf{c}^T\mathbf{x} = \mathbf{y}^T A\mathbf{x} = \mathbf{b}^T\mathbf{y} \quad \text{from (6.3)} \\
&\iff (\mathbf{y}^T A - \mathbf{c}^T)\mathbf{x} = 0 \\
&\quad\ \text{and} \quad \mathbf{y}^T(A\mathbf{x} - \mathbf{b}) = 0 \\
&\iff \sum_{j=1}^{n} (A^T\mathbf{y} - \mathbf{c})_j x_j = 0 \\
&\quad\ \text{and} \quad \sum_{i=1}^{m} (A\mathbf{x} - \mathbf{b})_i y_i = 0.
\end{aligned}$$

But $A^T\mathbf{y} \geqslant \mathbf{c}$ and $\mathbf{x} \geqslant \mathbf{0}$, so $\sum_{j=1}^{n}(A^T\mathbf{y}-\mathbf{c})_j x_j$ is a sum of non-negative terms.

Also, $A\mathbf{x} \leqslant \mathbf{b}$ and $\mathbf{y} \geqslant \mathbf{0}$, so $\sum_{i=1}^{m}(A\mathbf{x}-\mathbf{b})_i y_i$ is a sum of non-positive terms.

Hence $\sum_{j=1}^{n}(A^T\mathbf{y}-\mathbf{c})_j x_j = 0$ and $\sum_{i=1}^{m}(A\mathbf{x}-\mathbf{b})_i y_i = 0$ is equivalent to (6.1) and (6.2). □

Comments

What's the use of complementary slackness?

Among other things, given an optimal solution of P (or D), it makes finding an optimal solution of D (or P) very easy, because we know which the non-zero variables can be and which constraints must be tight.

Sometimes one of P and D is much easier to solve than the other, e.g. with 2 variables, 5 constraints, we can solve graphically, but 5 variables and 2 constraints is not so easy.

Example

Consider P and D with

$$A = \begin{pmatrix} 1 & 4 & 0 \\ 3 & -1 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}.$$

Is $\mathbf{x} = (0, \frac{1}{4}, \frac{13}{4})$ optimal? It is feasible. If it is optimal, then

$$x_2 > 0 \implies (A^T\mathbf{y})_2 = c_2, \quad \text{that is } 4y_1 - y_2 = 1$$
$$x_3 > 0 \implies (A^T\mathbf{y})_3 = c_3, \quad \text{that is } 0y_1 + y_2 = 3$$

which gives $\mathbf{y} = (y_1, y_2) = (1, 3)$.

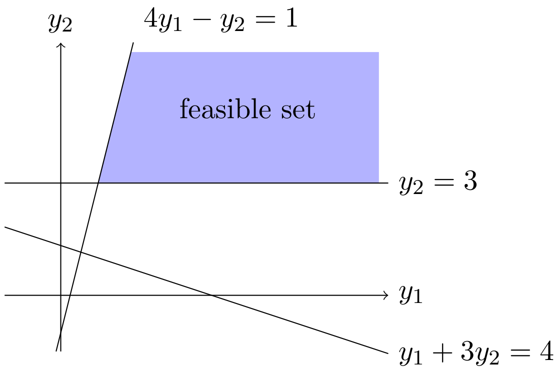
The remaining dual constraint $y_1 + 3y_2 \geqslant 4$ is also satisfied, so $\mathbf{y} = (1, 3)$ is feasible for D.

So $\mathbf{x} = (0, \frac{1}{4}, \frac{13}{4})$ and $\mathbf{y} = (1, 3)$ are feasible and satisfy complementary slackness, therefore they are optimal by Theorem 6.1.

Alternatively, we could note that this $\mathbf{x}$ and $\mathbf{y}$ are feasible and $\mathbf{c}^T\mathbf{x} = 10 = \mathbf{b}^T\mathbf{y}$, so they are optimal by Corollary 5.2.

Example continued

If we don't know the solution to P, we can first solve D graphically.



The optimal solution is at $\mathbf{y} = (1, 3)$, and we can use this to solve P:

$$\begin{aligned} y_1 > 0 &\implies (A\mathbf{x})_1 = b_1, \quad \text{that is } x_1 + 4x_2 = 1 \\ y_2 > 0 &\implies (A\mathbf{x})_2 = b_2, \quad \text{that is } 3x_1 - x_2 + x_3 = 3 \\ y_1 + 3y_2 > 4, &\quad \text{that is } (A^T\mathbf{y})_1 > c_1 \implies x_1 = 0 \end{aligned}$$

and so $\mathbf{x} = (0, \frac{1}{4}, \frac{13}{4})$.

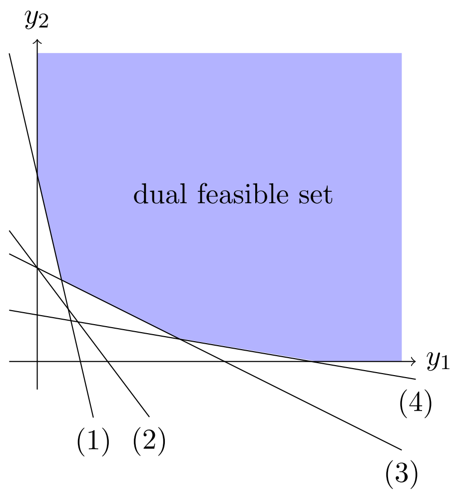
Example

Consider the primal problem

$$\begin{array}{lrcrcrcrl} \text{maximise} & 10x_1 & + & 10x_2 & + & 20x_3 & + & 20x_4 & \\ \text{subject to} & 12x_1 & + & 8x_2 & + & 6x_3 & + & 4x_4 & \leqslant 210 \\ & 3x_1 & + & 6x_2 & + & 12x_3 & + & 24x_4 & \leqslant 210 \\ & & & & & & & x_1, \dots, x_4 & \geqslant 0 \end{array}$$

with dual

$$\begin{aligned} \text{minimise} \quad & 210y_1 + 210y_2 & \\ \text{subject to} \quad & 13y_1 + 3y_2 \geqslant 10 & (1) \\ & 8y_1 + 6y_2 \geqslant 10 & (2) \\ & 6y_1 + 12y_2 \geqslant 20 & (3) \\ & 4y_1 + 24y_2 \geqslant 20 & (4) \\ & y_1, y_2 \geqslant 0. & \end{aligned}$$



The dual optimum is where (1) and (3) intersect.

Since the second and fourth dual constraints are slack at the optimum, the optimal $\mathbf{x}$ has $x_2 = x_4 = 0$.

Also, since $y_1, y_2 > 0$ at the optimum,

$$\left.\begin{aligned} 12x_1 + 6x_3 &= 210 \\ 3x_1 + 12x_3 &= 210 \end{aligned}\right\} \implies x_1 = 10, x_3 = 15.$$

Hence the optimal $\mathbf{x}$ is $(10, 0, 15, 0)$.

Example continued

Suppose the second 210 is replaced by 421.

The new dual optimum is where (3) and (4) intersect, at which point the first two constraints are slack, so $x_1 = x_2 = 0$.

Also, since $y_1, y_2 > 0$ at the new optimum,

$$\left.\begin{aligned} 6x_3 + 4x_4 &= 210 \\ 12x_3 + 24x_4 &= 421 \end{aligned}\right\} \implies x_3 = 35 - \tfrac{1}{24}, x_4 = \tfrac{1}{16}.$$

So the new optimum is at $\mathbf{x} = (0, 0, 35 - \frac{1}{24}, \frac{1}{16})$.

7 Two-player zero-sum games (1)

We consider games that are 'zero-sum' in the sense that one player wins what the other loses.

Each player has a choice of actions (the choices may be different for each player).

Players move *simultaneously*.

Payoff matrix

There is a *payoff matrix* $A = (a_{ij})$:

$$\begin{array}{cc} & \text{Player II plays } j \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \text{Player I plays } i \quad \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & 6 & 0 & -5 \end{pmatrix} \end{array}$$

If Player I plays i and Player II plays j, then Player I wins a_{ij} from Player II.

The game is defined by the payoff matrix.

Note that our convention is that I wins a_{ij} from II, so $a_{ij} > 0$ is good for Player I = row player.

What's the 'worst that can happen' to Player I if Player I chooses row 1? row 2? row 3? (We look at the smallest entry in the appropriate row.)

What's the 'worst that can happen' to Player II if Player II chooses a particular column? (We look at the largest entry in that column.)

The matrix above has a special property.

Entry $a_{23} = 4$ is both

- the smallest entry in row 2
- the largest entry in column 3

$(2, 3)$ is a 'saddle point' of A.

Thus:

- Player I can guarantee to win at least 4 by choosing row 2.
- Player II can guarantee to lose at most 4 by choosing column 3.
- The above is still true if either player announces their strategy in advance.

Hence the game is 'solved' and it has 'value' 4.

Mixed strategies

Consider the game of Scissors-Paper-Stone:

- Scissors beats Paper,
- Paper beats Stone,
- Stone beats Scissors.

	Scissors	Paper	Stone
Scissors	0	1	−1
Paper	−1	0	1
Stone	1	−1	0

No saddle point.

If either player announces a fixed action in advance (e.g. ‘play Paper’) the other player can take advantage.

So we consider a *mixed strategy*: each action is played with a certain probability. (This is in contrast with a *pure strategy* which is to select a single action with probability 1.)

Suppose Player I plays i with probability p_i, $i = 1, \ldots, m$.

Then Player I's *expected payoff* if Player II plays j is

$$\sum_{i=1}^{m} a_{ij} p_i .$$

Suppose Player I wishes to maximise (over $\mathbf{p}$) his minimal expected payoff

$$\min_j \sum_{i=1}^{m} a_{ij} p_i .$$

Similarly Player II plays j with probability q_j, $j = 1, \ldots, n$, and looks to minimise (over $\mathbf{q}$)

$$\max_i \sum_{j=1}^{n} a_{ij} q_j .$$

This aim for Player II may seem like only one of several sensible aims (and similarly for the earlier aim for Player I).

Soon we will see that they lead to a ‘solution’ in a very appropriate way, corresponding to the solution for the case of the saddle point above.

LP formulation

Consider Player II's problem ‘minimise maximal expected payout’:

$$\min_{\mathbf{q}} \left\{ \max_i \sum_{j=1}^{n} a_{ij} q_j \right\} \quad \text{subject to} \quad \sum_{j=1}^{n} q_j = 1, \ \mathbf{q} \geqslant \mathbf{0}.$$

This is not exactly an LP – look at the objective function.

Equivalent formulation

An equivalent formulation is:

$$\min_{\mathbf{q},v} v \quad \text{subject to} \sum_{j=1}^{m} a_{ij} q_j \leqslant v \quad \text{for } i = 1, \dots, m$$
$$\sum_{j=1}^{n} q_j = 1$$
$$\mathbf{q} \geqslant \mathbf{0}.$$

since v, on being minimised, will decrease until it takes the value of $\max_i \sum_{j=1}^{n} a_{ij} q_j$.

This is an LP but not exactly in a useful form for our methods. We will transform it!

First add a constant k to each a_{ij} so that $a_{ij} > 0$ for all i, j. This doesn't change the nature of the game, but guarantees $v > 0$.

So WLOG, assume $a_{ij} > 0$ for all i, j.

Now change variables to $x_j = q_j / v$. The problem becomes:

$$\min_{\mathbf{x},v} v \quad \text{subject to} \sum_{j=1}^{m} a_{ij} x_j \leqslant 1 \quad \text{for } i = 1, \dots, m$$
$$\sum_{j=1}^{n} x_j = 1/v$$
$$\mathbf{x} \geqslant \mathbf{0}.$$

This transformed problem for Player II is equivalent to

$$P : \max_{\mathbf{x}} \sum_{j=1}^{n} x_j \quad \text{subject to } A\mathbf{x} \leqslant \mathbf{1}, \ \mathbf{x} \geqslant \mathbf{0}$$

which is now in our 'standard form'. (**1** denotes a vector of 1s.)

Doing the same transformations for Player I's problem

$$\max_{\mathbf{p}} \left\{ \min_j \sum_{i=1}^{m} a_{ij} p_i \right\} \quad \text{subject to } \sum_{i=1}^{m} p_i = 1, \ \mathbf{p} \geqslant \mathbf{0}$$

turns into

$$D : \min_{\mathbf{y}} \sum_{i=1}^{m} y_i \quad \text{subject to } A^T \mathbf{y} \geqslant \mathbf{1}, \ \mathbf{y} \geqslant \mathbf{0}.$$

(Check: on problem sheet.)

P and D are *dual* and hence have the same optimal value.

Conclusion

Let $\mathbf{x}^*, \mathbf{y}^*$ be optimal for P, D. Then:

- Player I can guarantee an expected gain of at least $v = 1/\sum_{i=1}^{m} y_i^*$, by following strategy $\mathbf{p} = v\mathbf{y}^*$.
- Player II can guarantee an expected loss of at most $v = 1/\sum_{j=1}^{n} x_j^*$, by following strategy $\mathbf{q} = v\mathbf{x}^*$.
- The above is still true if a player announces his strategy in advance.

So the game is 'solved' as in the saddle point case (this was just a special case where the strategies were pure).

v is the *value* of the game (the amount that Player I should 'fairly' pay to Player II for the chance to play the game).

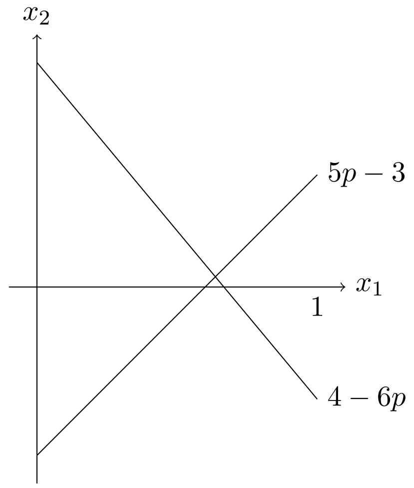
8 Two-player zero-sum games (2)

Some games are easy to solve without the LP formulation, e.g.

$$A = \begin{pmatrix} -2 & 2 \\ 4 & -3 \end{pmatrix}$$

Suppose Player I chooses row 1 with probability p, row 2 with probability $1-p$. The he should maximise

$$\begin{aligned} &\min(-2p + 4(1-p), 2p - 3(1-p)) \\ = &\min(4 - 6p, 5p - 3) \end{aligned}$$



So the min is maximised when

$$4 - 6p = 5p - 3$$

which occurs when $p = \frac{7}{11}$.

And then $v = \frac{35}{11} - 3 = \frac{2}{11}$.

(We could go on to find Player II's optimal strategy too.)

A useful trick: dominated actions

Consider

$$\begin{pmatrix} 4 & 2 & 2 \\ 1 & 3 & 4 \\ 3 & 0 & 5 \end{pmatrix}.$$

Player II should never play column 3, since column 2 is always at least as good as column 3 (column 2 'dominates' column 3.) So we reduce to

$$\begin{pmatrix} 4 & 2 \\ 1 & 3 \\ 3 & 0 \end{pmatrix}$$

Now Player I will never play row 3 since row 1 is always better, so

$$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

has the same value (and optimal strategies) as A.

Final example

Consider

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 3 & -1 \end{pmatrix},$$

add 1 to each entry

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 4 & 0 \end{pmatrix}.$$

This has value > 0 (e.g. strategy $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for Player I).

Solve the LP for Player II's optimal strategy:

$$\max_{x_1,x_2,x_3} x_1 + x_2 + x_3 \quad \text{subject to} \quad \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leqslant \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbf{x} \geqslant \mathbf{0}.$$

Initial simplex tableau:

0	1	2	1	0	0	1
2	0	1	0	1	0	1
0	4	0	0	0	1	1
1	1	1	0	0	0	0

final tableau:

0	0	1	$\frac{1}{2}$	0	$-\frac{1}{8}$	$\frac{3}{8}$
1	0	0	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{16}$	$\frac{5}{16}$
0	1	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$
0	0	0	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{3}{16}$	$-\frac{15}{16}$

Optimum: $x_1 = \frac{5}{16}$, $x_2 = \frac{1}{4}$, $x_3 = \frac{3}{8}$, and $x_1 + x_2 + x_3 = \frac{15}{16}$.

So value $v = 1/(x_1 + x_2 + x_3) = \frac{16}{15}$.

Player II's optimal strategy: $\mathbf{q} = v\mathbf{x} = \frac{16}{15}(\frac{5}{16}, \frac{1}{4}, \frac{3}{8}) = (\frac{1}{3}, \frac{4}{15}, \frac{2}{5})$.

Dual problem for Player I's strategy has solution $y_1 = \frac{1}{4}$, $y_2 = \frac{1}{2}$, $y_3 = \frac{3}{16}$ (from bottom row of final tableau).

So Player I's optmal strategy:
$\mathbf{p} = v\mathbf{y} = \frac{16}{15}(\frac{1}{4}, \frac{1}{2}, \frac{3}{16}) = (\frac{4}{15}, \frac{8}{15}, \frac{3}{15})$.

$\tilde{A}$ has value $\frac{16}{15}$, so the original game A has value $\frac{16}{15} - 1 = \frac{1}{15}$.