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On the Non-Real Roots of the Riemann Zeta Function ς(s)

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Abstract

This paper aims to solve a very difficult problem closely related to analytic number theory and specifically to prime numbers. This problem is the Riemann hypothesis according to which the prime numbers follow a very regular distribution among natural numbers. The statement of the Riemann hypothesis is: "All the non-real roots of the complex function ς(s) have real part equal exactly to 1/2". In this paper, we give an original proof to this hypothesis proceeding by contradiction. This proof is exceptional and more special because it is based on the relevant property that when η(s)=0 where η(s) is the Dirichlet eta function, the sum of the first m-1 terms of η(s) converges to -0.5 multiplied by the m^{th} term with $m \to \infty$ and it is also based on the condition that when we want to replace the complex variable z by z' with z' ≠ z in the expression of η (z) which is defined on the half-plane $\Re$ (z)>0 by η (z)= $\sum(-1)^{n-1} n^{-z}$, we have to keep the same infinite number of η-terms begining from n=1 to n=∞ in order to converge exactly to the same function. After the exposure of the proof of the Riemann hypothesis, we give an original method combining two already existing for calculating the approximate value of the integral of any continuous real function f over a closed interval I=[a, b]⊂ℝ of course when the primitive of *f* is not obvious by algebraic computation. This new numerical method is called " The Method of trapezes and half-ellipses THE" and its general approximation formula is defined by the following expression: $\int_a^b f(x)dx \approx THE_n = (1-\gamma\pi/4)T_n + (\gamma\pi/4)M_n$ where n is the number of subintervals obtained by regular subdivision of the interval [a, b] and γ ∈ [O, 4/π] and T_n is the approximate value of $\int_a^b f(x)dx$ using the method of trapezes and M_n is the approximate value of $\int_a^b f(x)dx$ using the method of rectangles with midpoint and comparing the margins of error, we note that The Method of trapezes and half-ellipses THE is more accurate than the following three known methods: The method of rectangles on the left $R^{(l)}_n$, The method of rectangles on the right $R^{(r)}_n$ and The method of trapezes T_n.

Keywords: Riemann 's hypothesis; Riemann Zeta function; Non-real roots of Riemann Zeta function; Dirichlet Eta function; Analytic number theory; Prime numbers; Numerical integration; The method of trapezes and half-ellipses THE

Introduction

The Riemann zeta function ς(s) plays a very important role in analytic number theory; its importance comes essentially from the very close connection it has with prime numbers; this connection that the great German mathematician Georg Friedrich Bernhard Riemann (1826-1866) had shown in his famous manuscript published in 1859(it is the same date when Charles Darwin (1809-1882) published his work "On the Origin of Species") when he gave an explicit formula linking the function counting the prime numbers with the roots of zeta function ς(s), namely the solutions s ∈ℂ of the equation ς(s)=0, this explicit formula given by Riemann is $\pi(x) = Li(x) - \sum Li(x^\rho) - \log 2 + \int_x^\infty dt/t(t^2-1)\log t$ where π(x) is the function counting the prime numbers and Li is the Logarithmic integral defined by $Li(x) = \int_x dt/\log t$ *and* ρ represents all the non-real roots of ς(s) and in his manuscript, Riemann claimed that it is very likely that all these non-real roots have real part equal exactly to 1/2. The connection between the prime numbers and zeta function ς(s) which is defined on the half-plane $\Re(s) \geq 1 \, by \, \varsigma(s) = \sum n^{-s}$ was established before Riemann by the celebrated Euler (1707-1783) product: $\sum n^{-s} = \Pi(1-p^{-s})^{-1}$ where the series $\sum n^{-s}$ is absolutely convergent for $\Re$ (s)>1 and the product $\Pi(1-p^{-s})^{-1}$ is being over the prime numbers p ∈ P={2, 3, 5, 7, 11,...}. So, in simple terms, the Riemann hypothesis says: "The non-real roots of ς(s) have all real part equal exactly to 1/2". This has been checked for the first 10,000,000,000,000 roots by experts and no counter-examples have been found. In practical terms, the Riemann hypothesis seems true, but theoretically, no proof to this moment has confirmed it. The validity of the Riemann hypothesis is equivalent to saying that the deviation of the number of the prime numbers from the mean Li(x) is π (x)=Li(x)+O(√xlogx). In this modest paper, we prove the absolute validity of the great Riemann hypothesis giving a very simple and rigorous proof which does not appeal to any complex theory and is very easy to understand.

Some Proven Results about the Non-Real Roots of ς(s)

- The non-real roots of the Riemann zeta function ς(s) have all real part belonging to the critical strip] O, 1 [and they are symmetric with respect to $\Re$ (s)=1/2;
- In 1914, the British mathematician Godfrey Harold Hardy (1877-1947) proved that there are in finitely many roots of ς(s) on the critical line $\Re$ (s)=1/2;
- The number of roots s=σ+it of ς(s) in the critical strip]O, 1[with 0<t≤T is asymptotically equal to N(T)=(T/2π)log(T/2π)-(T/2π)+O(logT);

On the other hand, Hardy and Littlewood (1885-1977) had proved in the 1920s a region without root of form $\Re$ (s)>1-k(log(log τ)) (log

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τ)-1 where τ=|t| +3, t= $\mathfrak{I}$ (s) and k is a positive constant;

And the Russian mathematician Ivan Vinogradov (1891-1983), in the thirties, had obtained a region without root of form $\mathfrak{R}$ (s)>1-k (logτ)$^{-\gamma}$ where τ=|t|+3, t= $\mathfrak{I}$ (s), 3/4<γ< 1 and k is a certain positive constant;

The largest known region that contains no roots of the function ς(s) is given asymptotically by $\mathfrak{R}$ (s)>1-k(logτ) $^{-2/3}$ (1oglogτ)$^{-1/3}$ where τ=|t| +3, t=$\mathfrak{I}$ (s) and k is a certain positive constant;

In 1942, the Norwegian mathematician AtleSelberg (1917-2007) demonstrates that at least some fraction of roots of ς(s) is on the critical line $\mathfrak{R}$ (s)=1/2, but its fraction is less than 1%;

In 1974, the American mathematician Norman Levinson (1912-1975) increases this fraction to 1/3;

In 2004, the two mathematicians Xavier Gourdon and Patrick Demichel use the Odlyzko-Schonhage algorithm and find that the first 10,000,000,000,000 roots of ς(s) are on the critical line $\mathfrak{R}$ (s)=1/2;.

In 2011, an other American mathematician John Brian Conrey (1955) proves that at least 41.05% of the non-real roots of ς(s) are aligned on the critical line $\mathfrak{R}$ (s)=1/2 and are simple;

According to the Vinogradov-Korobov method, we have the following property: there are two constants c and C strictly positive such that for all 1/2≤σ≤1and |t| ≥ 2, we have: |ς(σ+it)|≤ $C|t|c(1-\sigma)^{3/2}(\log|t|)^{2/3}$. In the current state of knowledge, according to the American mathematic cian Kevin Ford, we can take c=4.45 and C=76.2;

Littlewood also has proved the following theorem: "Either the function ς(s) or the function ς'(s) has an infinity of roots in the strip 1-δ<σ<1, σ= $\mathfrak{R}$ (s) and o is an arbitrarily small positive quantity";

In 1934, the Swiss mathematician Andreas Speiser (1885-1970) has proved the following theorem: '∀s ∈ C with 0< $\mathfrak{R}$ (s)<1/2: ς'(s) ≠ O";

The Turkish mathematician Gem Yalpn Yildirzm (1961) has proved that:

"The Riemann hypothesis implies that ς"(s) and ς'" (s) do not vanish in the strip 0< $\mathfrak{R}$ (s)<1/2"; There are still more proven results about the non-real roots of ((s) apart from these results just mentioned, but all these results are not enough to say that the Riemann hypothesis is true. Fortunately, after several attempts we were able to solve this great problem and we believe that if there are many proofs to the Riemann hypothesis, our proof would probably be the simplest and the most beautiful.

On the Non-Real Roots of ς(s)

Key-question

There is an important question we have to ask ourself, we wonder if any of the mathematicians who tried to prove the Riemann hypothesis and who failed if asked himself this question, it is the following key-question: "If ŋ(s)=0 and ŋ (l-s)=0 and 0< $\mathfrak{R}$ (s)<1 and $\mathfrak{R}$ (s)≠ 1/2, then the limit $\lim_{z\to\delta}\eta(1-z)/\eta(z)$ has to be finite or infinite? it could be equal to 0? why not?". It is from this intelligent and relevant question and based on the explicit expression of η(s) which is described as an infinite complex series for $\mathfrak{R}$ (s)>0 and based on the special form of its general term (-1)$^{n-1}$n^{-s} and referring to universally known formulas and proven theorems that we started our reflection and our reasoning and carefully following the logical and rational implications we were able to find the right answer to the previous question and then to prove the absolute validity of the Riemann hypothesis and it should be noted here that the key result of the concise proof below lies in the expression of ŋ(s).

The proof of the Riemann hypothesis

We know that the Riemann zeta function ς(s) is the analytic function of the complex variable s, defined on the half-plane $\mathfrak{R}$ (s)>1 by ς(s)= $\sum n^{-s}=\Pi\left(1-p^{-s}\right)^{-1}$ [l] where the series $\sum n^{-s}$ is absolutely convergent for $\mathfrak{R}$ (s)>1 and the product $\Pi\left(1-p^{-s}\right)^{-1}$ is being over the prime numbers p ∈ P={2, 3, 5, 7, 11,...} and the function ς(s) is defined in the complex plane C\{l} by analytic continuation. As shown by Riemann, ς(s) can be continued analytically to C\{l} as a meromorphic function and has a first order pole at s=1 with residue 1. On the other hand, we know that the Riemann zeta function ς(s) is defined for any complex number *s* different from 1 and with real part strictly greater than 0 by ς(s)= η (s)/(1-2^{1-s}) [2] where η is the Dirichl et eta function which is defined on the half-plane $\mathfrak{R}$ (s)>0 by η (s)= $\sum(-1)^{n-1}n^{-s}$ [3]. Noticing that $\left(1-2^{1-s}\right)\neq\tilde{\infty}$ (remark: we denote 0={0, i0, 0+i0} and ∞={∞, i∞, ∞+i∞} where i ∈ C, i^2=-1) at least for $\mathfrak{R}$ (s)>0, then we have to note the following result: if ς(s)=0, then η (s)=0, we thus have the Riemann hypothesis is equivalent to the following statement: "The non-real roots of the Dirichet eta function η (s) belonging to the critical strip]0, 1[have all real part equal exactly to 1/2".

We also know that there is an important relationship between ς(s) and ς(1-s). This relationship is defined for any s in the complex plane ℂ by the following Riemann functional equation: ξ(s)=ξ (1-s) [4] where ξ (s)=π$^{-s/2T}$ Γ (s/2)ς(s), Γ being the Euler gamma function which is defined for any complex number s such that $\mathfrak{R}$ (s)>0 by: Γ(s)= $\int_0^\infty x^{s-1}e^{-x}dx$ [5]. So, the two equations [2] and [4] imply: η (1-s)/ η (s)= $\Lambda(s)/\Lambda(1-s)$ where: $\Lambda(s)=\pi^{-3/2}\Gamma(s/2)\left(1-2^{s}\right)$. We also know that in 1896, the two mathematicians Jacques Salomon Hadamard (1865-1963) and Charles-Jean de LaVallee Poussin (1866-1962) independently proved that no root of ς(s) could be on the line $\mathfrak{R}$ (s)=1, and that all the non-real roots have to be in the interior of the critical strip 0 < $\mathfrak{R}$ (s)<1, and it is known that this demonstration was a key result in the first complete proof of the prime number theorem $\left(PNT:\lim_{x\to\infty}\eta(1-z)/\eta(z)=0,\tilde{\infty}(^*)\right)$. Now, let 's assume that there is a complex number s such that 0< $\mathfrak{R}$ (s)<1 verifying $\lim_{x\to s}\Lambda(z)/\Lambda(1-z)\neq0,\tilde{\infty}$. According to this assumption, we have to have: $\lim_{x\to s}\Lambda(z)/\Lambda(1-z)=0,\infty$ So, we have if $\mathfrak{R}$ (s), $\mathfrak{R}$ (s)) ≠ (0, 1) and knowing that Γ(s) does not vanish for any *s* in ℂ and has an infinity of simple poles with residue (-l)^k/k! at s=-k where k=0, 1, 2, 3, 4, 5,... etc. then $\lim_{x\to s}\Lambda(z)/\Lambda(1-z)=0\Rightarrow s$ =3, 5, 7, 9, 11,... etc. and $\lim_{x\to s}\Lambda(z)/\Lambda(1-z)=\tilde{\infty}\Rightarrow s$ =-2, -4, -6, -8, -10,... etc. then we have for 0< $\mathfrak{R}$ (s)<1: $\lim_{x\to s}\Lambda(z)/\Lambda(1-z)\neq0,\tilde{\infty}$, that is to say the assumption (*) is false. Thus, for every complex number s such that 0<$\mathfrak{R}$(s)<1, we have to be sure that $\lim_{x\to\infty}\eta(1-z)/\eta(z)=0,\tilde{\infty}$. Therefore, we have to mention the following property: if η (s)=0 and 0<$\mathfrak{R}$(s)<1, then η (1-s)=0.

Important remark: This proof is mainly based on the following known data:

- If ς(s)=0 and s ≠ -2, -4, -6,... etc. then 0< $\mathfrak{R}$ (s)<1;

ς(s)= η (s)/(l-2 $^{1-s}$), $\eta$is defined for $\mathfrak{R}$ (s)>0 by η (s)= $\sum(-1)^{n-1}n^{-s}$;

For any s in C: ξ(s)=ξ(1-s) where ξ(s)=π$^{-s/2}$ Γ(s/2)ς (s);

$\forall|x|\leq1:\lim_{t\to x}\sum_{n\geq0}(-1)^n t^n=\lim_{t\to x}\frac{1}{1+t}$

In general $\lim_{x\to x_0}\frac{f(x)}{g(x)}\neq\frac{\lim_{x\to x_0}f(x)}{\lim_{x\to x_0}g(x)}$,

If $\eta_m(z)=\sum_{n=1}^{m}(-1)^{n-1}n^{-z}$ then $\{\eta(s)=\lim_{z\to s}\eta_m(z)\,and\,\eta(s')=\lim_{m'\to s'}\eta_{m*}(z')\}\Rightarrow m*=m$, applying this condition we keep the same infinite number of η- terms in order to converge exactly to the same function in numerator and denominator,

To prove the Riemann hypothesis, it seemed to us that it is more convincing to prove the conjecture:

If η (s)=0 and 0< $\Re$ (s)<1, then: $\lim_{z\to s}\frac{\eta(1-z)}{\eta(z)}=0,\tilde{\infty}$ or $\Re$ (s)= $\frac{1}{2}$[C]

In the conjecture above, we denote: 0={0, i0, 0+i0} and $\tilde{\infty}$ ={ ∞, i∞, ∞+i∞} where i ∈ ℂ,

The proof of the conjecture [C] [algebraic proof]: We have previously shown that for every complex number s such that 0<$\Re$ (s)<1, we have $\lim_{z\to s}\eta(1-z)/\eta(z)\neq 0,\tilde{\infty}$ and we have deduced according to this result that if we have η (s)=0 and 0< $\Re$ (s) <1, then we also must have η (1-s)=0, and we know that η-function is defined on the half-plane η (s)>0 by η (s)= $\sum\alpha_n(s)$ where $\alpha_n(s)=(-1)^{n-1}n^{-s}$.

To prove the conjecture [C], we have to prove the following proposition:

If η (s)=0, and η (1-s)=0 and 0<$\Re$(s)<1, then $\lim_{z\to s}\frac{\eta(1-z)}{\eta(z)}=\lim_{\substack{z\to s\\ m\to\infty}}\frac{\alpha_m(1-z)}{\alpha_m(z)}$

So, let's prove the proposition [P],

Proof: We have:

$$\eta(z)=\sum_{n=1}^{m-1}\alpha_n(z)+\sum_{n\geq m}\alpha_n(z),m\geq 2$$

We denote:

$$\eta_m(z)=\sum_{n=1}^{m-1}\alpha_n(z)\,and\,\eta_{\geq m}(z)=\sum_{n\geq m}\alpha_n(z)$$

So, we have

$$\lim_{z\to s}\frac{\eta(1-z)}{\eta(z)}=\lim_{\substack{z\to s\\ m\to\infty}}\frac{\eta_m(1-z)+\eta_{\geq m}(1-z)}{\eta_m(z)+\eta_{\geq m}(z)}$$

and for any s in the complex plane ℂ with $\Re$ (s)>0, we have: $\lim_{\substack{z'\to s\\ m_*\to\infty}}\eta_{\geq m}(z)=0$ that is to say if then

$\lim_{\substack{z'\to s\\ m_*\to\infty}}\eta_{\geq m}(z)\neq 0$ we have:

$$\lim_{\substack{z\to s\\ m\to\infty}}q_m(z)=0\;where\,q_m(z)=\frac{\eta_{\geq m}(z)}{\eta_m(z)}$$

but if $\lim_{\substack{z\to s\\ m\to\infty}}\eta_m(z)=0$, it will still remain $\lim_{\substack{z\to s\\ m\to\infty}}q_m(z)=0$? in the following, we are going to prove $\lim_{\substack{z\to s\\ m\to\infty}}\eta_m(z)=0$ then $\lim_{\substack{z\to s\\ m\to\infty}}q_m(z)\neq 0$, so in general we can simply write:

$$\lim_{z\to s}\frac{\eta(1-z)}{\eta(z)}=\lim_{\substack{z\to s\\ m\to\infty}}\frac{\eta_m(1-z)}{\eta_m(z)}$$

If m*=m (m_*=m is a necessary condition in the following calculation of limits), then we have:

$$\eta(s)=\lim_{\substack{z\to s\\ m\to\infty}}(\eta_m(z)+\eta_{\geq m}(z))=\lim_{\substack{z\to s\\ m\to\infty}}\eta_m(z)+\lim_{\substack{z\to s\\ m_*\to\infty}}\eta_{m\geq m}(z)$$

And $\forall_n\geq 1$ and $\forall_j\geq 0$, we have: $\alpha_{n+j}(z)=(1)^j\left(\frac{n}{n+j}\right)^z\alpha_n(z)$,

So, for m_*=m, we have

$$\lim_{\substack{z\to s\\ m_*\to\infty}}\sum_{n\geq m_*}\alpha_n(z)=\lim_{\substack{z\to s\\ m\to\infty}}\left(\alpha_m(z)+\sum_{j\geq 1}\alpha_{m+j}(z)\right)$$

$$=\lim_{\substack{z\to s\\ m_*\to\infty}}\left(\alpha_{m_*}(z)+\sum_{j\geq 1}(-1)^j\left(\frac{m_*}{m_*+j}\right)^z\alpha_{m*}(z)\right)$$

$$=\lim_{\substack{z\to s\\ m_*\to\infty}}\left(\alpha_m(z)+\left(1+\sum_{j\geq 1}(-1)^j\left(\frac{m}{m+j}\right)^z\right)\right)$$

It's trivial that if $\lim_{x\to x_0}f(x)$ exists and $\lim_{x\to x_0}g(x)$ exists, then we can write:

$$\lim_{x\to x_0}f(x)g(x)=\lim_{x\to x_0}f(x)\lim_{x\to x_0}g(x)$$

We have $\lim_{\substack{z\to s\\ m\to\infty}}\alpha_m(z)$ exists for $\Re$ (s)>0 and is equal to 0, but we have to prove that L also exists,

where:

$$\lim_{\substack{z\to s\\ m\to\infty}}\left(1+\sum_{j\geq 1}(-1)^j\left(\frac{m}{m+j}\right)^z\right)$$

We have

$$L=\lim_{\substack{z\to s\\ m\to\infty}}\left(1+\sum_{j\geq 1}(-1)^j\left(\frac{m}{m+j}\right)^z\right)=\lim_{\substack{z\to s\\ m\to\infty}}\left(1+\sum_{j\geq 1}(-1)^j\left(\left(\frac{m}{m+j}\right)^{z/j}\right)^j\right)$$

Denoting $\left(\frac{m}{m+j}\right)^{z/j}$ by x_j(z, m), we note that ∀j, m≥1

$$|x_j(z,m)|\leq 1$$

That means ∀j, m≥1:∃ϵj(z, m)>0 verifying $|x_j(z,m)|\leq 1-\in j(z,m)$ and when m → ∞:ϵj(z,m) → 0,

That is to say $\forall j,j'\geq 1$ with $j\neq j'$ when z,m → s,∞

$$x_j(z,m)=x_{j'}(z,m)=1^-+i0$$

And we know that $\forall|x|<1$: $\lim_{t\to x}\sum_{n\geq 0}(-1)^n t^n=\lim_{t\to x}\frac{1}{1+t}$

We thus get,

$$L=\lim_{\substack{z\to s\\ m\to\infty}}\left(1+\sum_{j\geq 1}(-1)^j x_j(z,m)^j\right)=1+\lim_{\substack{z\to s\\ m\to\infty}}\sum_{j\geq 1}(-1)^j x_j(z,m)^j$$

$$=1+\lim_{\substack{z\to s\\ m\to\infty}}\frac{1}{1+x_j(z,m)}-1=1+\frac{1}{1+1}-1=\frac{1}{2}=0.5$$

So, for m_*=m (we must not forg et that m_*=m is a necessary condition in calculation of limits):

$$L=\lim_{\substack{z\to s\\ m_*\to\infty}}\eta_{\geq m*}(z)=0.5\lim_{\substack{z\to s\\ m\to\infty}}\sum_{j\geq 1}(-1)^j x_j(z,m)^j$$

That is to say if z,m → s,∞ with ℜ (s)>0, then:

$$\eta(z)=\eta_m(z)+0.5\alpha_m(z) \qquad (*)$$

So, if ŋ(z) → 0when z,m → s,∞, then we have:

$$\eta_m(z)\to -0.5\alpha_m(z)$$

We have to note that if $\lim_{\substack{z\to s\\ m\to\infty}}\eta_m(1-z)=0$ and $\lim_{\substack{z\to s\\ m\to\infty}}\eta_m(z)=0$ then,

$$\lim_{z\to s}\frac{\eta(1-z)}{\eta(z)}\neq\frac{\lim_{\substack{z\to s\\ m\to\infty}}\eta_m(1-z)}{\lim_{\substack{z\to s\\ m\to\infty}}\eta_m(z)}=\frac{0}{0}$$

So, according to (*), it follows that if ŋ (s)=0 and ŋ(l-s)=0, then we must have:

$$\lim_{z\to s}\frac{\eta(1-z)}{\eta(z)}=\lim_{\substack{z\to s\\ m\to\infty}}\frac{\eta_m(1-z)}{\eta_m(z)}=\lim_{\substack{z\to s\\ m\to\infty}}\frac{-0.5\alpha_m(1-z)}{\alpha(1-z)}=\lim_{\substack{z\to s\\ m\to\infty}}\frac{\alpha_m(1-z)}{\alpha_m(z)}\equiv \textbf{[P]}$$

So, the proposition **[P]** implies the following result:

If ŋ(s)=0 and 0<ℜ(s)<1and $\Re(s)\neq\frac{1}{2}$, then $\lim_{z\to s}\frac{\eta(1-z)}{\eta(z)}=\lim_{\substack{z\to s\\ m\to\infty}}\frac{\alpha_m(1-z)}{\alpha_m(z)}=0,\tilde{\infty}$

Thus it has been shown that:

If ŋ(s)=0 and 0< ℜ (s)<1 [, then $\lim_{z\to s}\frac{\eta(1-z)}{\eta(z)}=0,\tilde{\infty}\, or\, \Re(s)=\frac{1}{2}\equiv$ **[C]**

So, if ŋ(s)=0 and ℜ (s)∈]0,1/2[∪]1/2,1 [, then $\lim_{z\to s}\frac{\eta(1-z)}{\eta(z)}=0,\tilde{\infty}$, that is equivalent to say that $\lim_{z\to s}\frac{\Lambda(1-z)}{\Lambda(z)}=0,\tilde{\infty}$ [we remind here that A(z)=π $^{-z/2Γ}$ (z/2) (1-2$^{z)}$J. However, for any complex s with number s with 0 < ℜ (s) <1, we have shown that $\lim_{z\to s}\frac{\Lambda(1-z)}{\Lambda(z)}\neq 0,\tilde{\infty}$. Therefore, we deduce that:

if ŋ(s)=0 and 0 < ℜ (s) <1, then ℜ (s)=1/2, and according to **[P]**, we also note that:

if ŋ(s)=0 and 0 < ℜ (s) <1, then ℑ(s)=πx/logy where x ∈ Z* and y ∈] 0, l[∪]1+∞[.

The proposition **[P]** can be generalized by the following statement:

$\forall s,s'\in\mathbb{C}\, with\, \Re(s),\Re(s')\geq 0, if\ \eta(s)=0\, and\ \eta(s')=0$, *then*

$$\lim_{\substack{z\to s\\ z'\to s'}}\frac{\eta(z)}{\eta(z')}=\lim_{\substack{z\to s\\ z'\to s'\\ m\to\infty}}\frac{\alpha_m(z)}{\alpha_m(z')}$$

▪ For example, for t being the imaginary part of a non-real root of ς(s) and belonging to the critical line ℜ (s)=σ=1/2 and k E ℤ*, we have:

$$\lim_{\substack{z\to 1/2+it\\ z'\to i2k\pi/\log 2}}\frac{\eta(z)}{\eta(z')}=\lim_{\substack{z\to 1/2+it\\ z'\to i2k\pi/\log 2\\ m\to\infty}}\frac{\alpha_m(z)}{\alpha_m(z')}=\lim_{m\to\infty}m^{1/2+i((2k\pi/\log 2)-t)}=\tilde{\infty}\ (**)$$

If we look for more precision and rigor, we have to take into consideration in our calculation of limits the following obvious condition (note: this condition is respected in the previous calculation of limits) [6]:

$$*If\,\eta_m(z)=\sum_{n=1}^{m}(-1)^{n-1}n^{-z}, then\left\{\eta(s)=\lim_{\substack{z\to s\\ m\to\infty}}\eta_m(z)\, and\, \eta(s')=\lim_{\substack{z\to s'\\ m_*\to\infty}}\eta_{m_*}(z')\right\}$$

$$\Rightarrow \lim_{\substack{z\to s'\\ z'\to s'}}\frac{\eta(z)}{\eta(z')}=\lim_{\substack{z\to s'\\ z'\to s'\\ m\to\infty}}\frac{\eta_m(z)}{\eta_m(z')}*$$

By this condition, we want to say that if someone wants to replace z by z' with z'≠ z in the expression of ŋ, he has to keep the same number of ŋ-terms beginning from n=1 to n=m=∞ to keep exactly the same function and to change only the complex variable z in the expression of ŋ and this condition is valid in the general case.

In short, the most interesting result proved in this paper is the following theorem:

If ς(s)=0 and ℜ (s)>0, then ℜ (s)=1/2 and ℑ(s)=πx/logy where x ∈ ℤ* and y ∈]0, l[∪] l, +∞ [.

Numerical verification: We have made the numerical verification for (*) and (**) using the first 5 roots of the Riemann zeta function ς(s) whose real parts are equal to 1/2 and the imaginary parts are positive and we should note that this numerical verification was done using Microsoft Office Excel 2007,

The first 5 roots of the Riemann zeta function ς(s) are:

S1=1/2+i14,134 725 141…

S2=1/2+i21,022 039 638….

S3=1/2+i25,010 857 580…

S4=1/2+i30,424 876 125…

S5=1/2+i32,935 061 587…

and using the first root of the Dirichlet eta function ŋ(s) whose real part is equal to 1 and the imaginary part is equal to 2π/log2 in order to show that $\lim_{\substack{z\to s\\ z'\to s'}}\frac{\eta_m(z)}{\eta_m(z)}=\tilde{\infty}$ where sj represents all the non-real roots of ((s) aligned on the critical line at ℜ (s)=σ=1/2 and $s_1'\, is: s'=1+i2\pi/\log 2$,

In this verification we have taken: $\eta_m(s)=\sum_{n=1}^{m-1}\alpha_n(s)$ with $\alpha_n(s)=(-1)^{n-1}n^{-s}=\alpha n(s)+ib_n(s)$,

See the calculation tables in the next pages: from [09/26] to [21/26] and tables [1-13]. The paces of the two quotients $\left|\eta_m(s_1)/\eta_m(s_1')\right|\xi\left|\alpha_m(s_1)/\alpha_m(s_1')\right|$ for m ranging from 10 to 1000000 on the page [22/26] (Figure 1 and Tables 1-13).

Conclusion

Based on the following condition:

$$*If\,\eta_m(z)=\sum_{n=1}^{m}(-1)^{n-1}n^{-z}, then\left\{\eta(s)=\lim_{\substack{z\to s\\ m\to\infty}}\eta_m(z)\, and\, \eta(s')=\lim_{\substack{z\to s'\\ m_*\to\infty}}\eta_{m_*}(z')\right\}\Rightarrow m_*=m$$

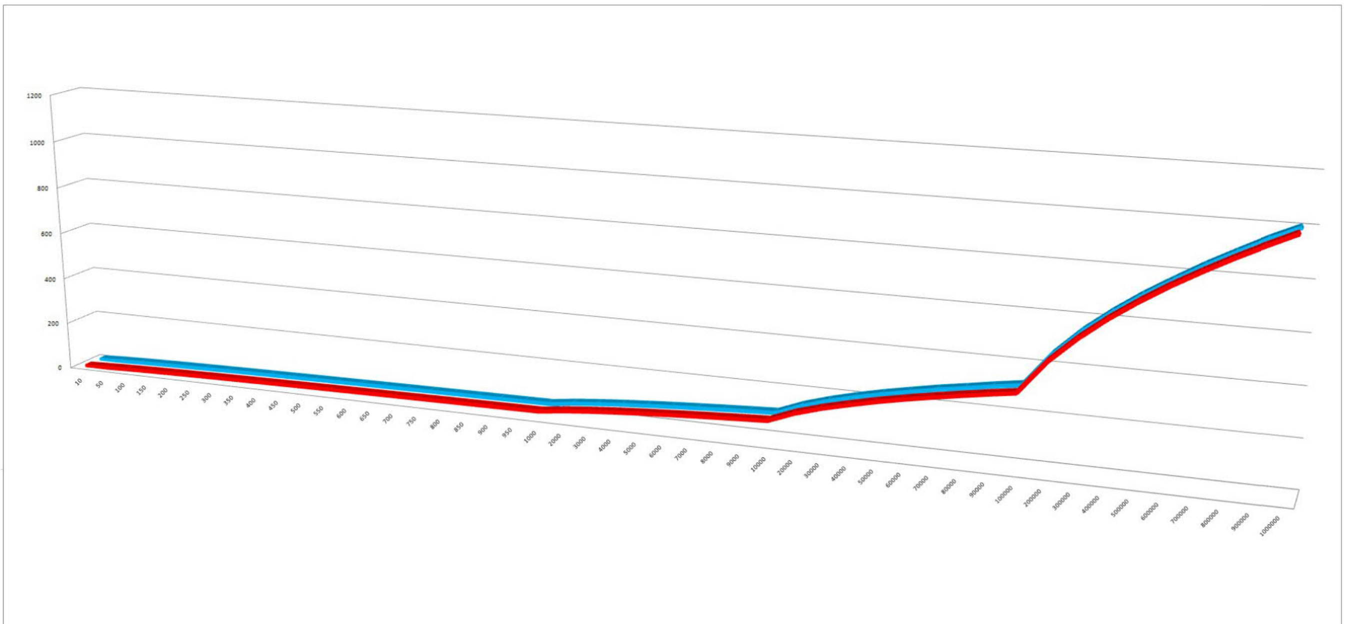
$$\Rightarrow \lim_{\substack{z\to s'\\ z'\to s'}}\frac{\eta(z)}{\eta(z')}=\lim_{\substack{z\to s'\\ z'\to s'\\ m\to\infty}}\frac{\eta_m(z)}{\eta_m(z')}*,$$

and on the special property that when ŋ (s)=0 for ℜ (s)>0 and z, m → s,∞:

$$\eta_m(z)\to -0.5\alpha_m(z)\, where\, \eta_m(z)=\sum_{n=1}^{m-1}\alpha_n(z)\, and\, \alpha_n=(-1)^{n-1}n^{-z}$$

it has been shown that if ŋ(s)=0 and ŋ(l-s)=0 with 0< ℜ (s) <1, then we get:

Citation: Azkour M (2018) On the Non-Real Roots of the Riemann Zeta Function ϛ(s). J Phys Math 9: 279. doi: 10.4172/2090-0902.1000279



m	10	50	100	150	200	250	300	350	400	450	500	550
■ (red)	3,6897	7,0786	9,9899	12,235	14,129	15,799	17,308	18,697	19,989	21,202	22,350	23,442
■ (blue)	3,1622	7,0710	10	12,247	14,142	15,811	17,320	18,708	20	21,213	22,360	23,452

m	600	650	700	750	800	850	900	950	1000	2000	3000	4000
■ (red)	24,485	25,486	26,448	27,377	28,276	29,146	29,992	30,814	31,615	44,716	54,767	63,241
■ (blue)	24,494	25,495	26,457	27,386	28,284	29,154	30	30,822	31,622	44,721	54,772	63,245

m	5000	6000	7000	8000	9000	10000	20000	30000	40000	50000	60000	70000
■ (red)	70,707	77,456	83,663	89,44	94,865	99,997	141,42	173,20	199,24	223,55	244,91	264,51
■ (blue)	70,710	77,459	83,666	89,442	94,868	100	141,42	173,20	200,01	223,70	244,48	264,62

m	80000	90000	100000	200000	300000	400000	500000	600000	700000	800000	900000	1000000
■ (red)	282,64	300,02	316,15	447,12	546,70	632,32	707,06	774,91	836,57	894,70	948,68	1000,3
■ (blue)	281,76	300,02	316,16	447,12	548,25	632,33	707,05	773,16	836,57	894,24	950,35	999,93

Figure 1: The paces of the two quotients 11'/rn(s1)/77rn(sDI 8/arn(s))/l for m ranging from 10 to 1000000.

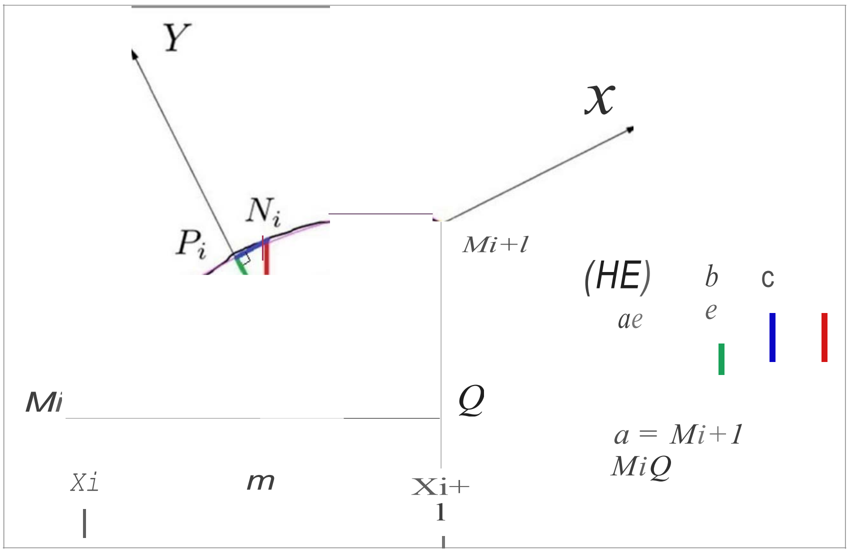


Figure 2: Chart explaining the existence and purpose of method of Trapezes and Half-ellipses T.H.E.

$$\lim_{z\to s}\frac{\eta(1-z)}{\eta(z)}=\lim_{\substack{z\to s\\ m\to\infty}}\frac{\eta_m(1-z)}{\eta_m(z)}=\lim_{\substack{z\to s\\ m\to\infty}}\frac{-0.5\alpha_m(1-z)}{-0.5\alpha_m(z)}=\lim_{\substack{z\to s\\ m\to\infty}}\frac{\alpha_m(1-z)}{\alpha_m(z)}$$

and refering to this result, it has been deduced that:

$$If\ \eta(s)=0\ and\ 0\le\Re(s)\le 1, then: \lim_{z\to s}\frac{\eta(1-z)}{\eta(z)}=0.\tilde{\infty}\ or\ \Re(s)=\frac{1}{2}$$

and it is also proved that if ŋ(s)=0 and 0< $\Re$ (s) <1, then $\Im$ (s)=πx/logy where x∈ $\mathbb{Z}^*$ and

y ∈]0, l[U] l, +∞ [.

The Method of Trapezes and Half-Ellipses T.H.E

Purpose

This method is original and its main purpose is the approximate calculation of the integral of any continuous real function f on a closed interval [a, b] with a<b replacing each arc of f -curve (M_iM_{i+1}) by an elliptical half-arc (H E) and we remind that this method can be used

m	$\Re$ ($\eta_m(s_1)$)	$a_m(s_1)$	$\frac{\Re(\eta_m(s_1))}{am(s_1)}$
10	0,192867143	-0, 134784134	-1,43,09,33,577
50	0,01263537	-0,044150483	-0,286188715
100	-0,029184638	0,063666475	-0,458398835
150	-0,003723617	0,011241463	-0,331239537
200	0,030322087	-0,06178251	-0,490787555
250	-0,027422519	0,055639564	-0,492860063
300	0,013530651	-0,028224466	-0,479394402
350	0,01216527	-0,023342749	-0,521158498
400	-0,024727443	0,049543065	-0,499110077
450	-0,001342723	0,001944584	-0,690493699
500	0,022164627	-0,044384495	-0,499377699
550	0,007500051	-0,014478126	-0,518026366
600	-0,015631457	0,031554882	-0,495373648
650	-0,017808192	0,035419543	-0,502778706
700	-0,001687111	0,002992738	-0,563734948
750	0,0141525	-0,028510481	-0,496396395
800	0,017224078	-0,034363969	-0,501224931
850	0,007995398	-0,015732746	-0,508201048
900	-0,005300603	0,010845709	-0,488728123
950	-0,014373582	0,028849992	-0,498217885
1000	-0,015351898	0,030640875	-0,501026749
2000	0,009107231	-0,018166118	-0,501330609
3000	0,009108325	-0,018212105	-0,50012478
4000	-0,004315441	0,008606909	-0,50139266
5000	0,003783442	-0,007549603	-0,501144497
6000	-0,005834998	0,01166299	-0,500300352
7000	0,005184651	-0,010374924	-0,499729058
8000	0,001131601	-0,002253457	-0,502162233
9000	-0,005238931	0,010478471	-0,49997094
10000	-0,000950073	0,00189316	-0,501845063
20000	-0,0006394	0,001281242	-0,499047018
30000	0,001044122	-0,002086959	-0,500307864
40000	0,001316711	-0,002634157	-0,499860487
50000	-0,001201397	0,002403316	-0,4998914
60000	5.52E-01	-l,15162E-05	-0,47912419
70000	0,001548127	-0,003096024	-0,500037144
80000	-0,001414379	0,002828936	-0,499968539
90000	-0,00087016	0,001740092	-0,500065514
100000	0,001276813	-0,002553752	-0,49997533
200000	-0,001080988	0,002161994	-0,499995837
300000	-0,000629301	0,001258633	-0,499987685
400000	0,00078539	-0,001570776	-0,500001273
500000	-0,000701411	0,001402818	-0,500001426
600000	0,000584699	-0,001169403	-0,499997862
700000	-0,000101445	0,000202903	-0,499967965
800000	-0,000493971	0,000987937	-0,500002531
900000	0,000289383	-0,000578773	-0,499993953
1000000	0,000438865	-0,000877726	-0,500002279

Table 1: Calculation of numerical verification 1.1.

m	$\Re\ (\eta_m(s_1))$	$b_m(s_1)$	$\frac{\Im(\eta_m(s_1))}{b_m(s_1)}$
10	0,09,94,02,356	-0,28,60,65,093	-0,34,74,81,599
50	-00,70,67,349	0,13,43,53,023	-0,52,60,28,283
100	0,04,09,08,291	-0,07,71,14,071	-0,53,04,90,616
150	0,04,07,68,907	-0'080872098	-0,50,41,15,857
200	-0,01,83,10,429	0,03,43,93,626	-0,53,23,78,558
250	0,01,58,37,008	-0'030070566	-0,52,66,61,454
300	-0'02553643	00,50,36,579	-0,50,70,19,348
350	0'023824466	-0,04,80,85,954	-0,49,54,55,825
400	0,00,38,11,944	-0,00,67,44,234	-0,56,52,15,264
450	-0,02,35,47,985	0,04,71,00,327	-0,49,99,53,748
500	-0'003054444	0,00,54,78,744	-0,55,75,08,071
550	0,01,99,69,559	-0,04,01,06,928	-0,49,79,07,967
600	0,01,31,42,658	-00,25,90,282	-0,50,73,83,289
650	-0,00,82,35,604	0,01,68,49,851	-0,48,87,64,203
700	-0,01,88,30,511	0,03,76,77,778	-0,49,97,77,641
750	-0,01,15,45,216	0,02,28,14,158	-05,06,05,488
800	0,00,40,06,358	-0,00,83,13,703	-0,48,18,98,139
850	0,01,51,78,428	-0,03,04,78,702	-0,49,80,01,129
900	0,01,58,06,735	-0,03,15,19,545	-0,50,14,89,949
950	0,00,75,30,681	-0,01,48,42,827	-0,50,73,61,637
1000	-0'003802202	0,00,78,19,003	-0,48,62,77,087
2000	00,06,48,777	-0,01,30,38,104	-0,49,76,00,725
3000	0,00,06,21,364	-0,00,12,85,525	-0,48,33,54,272
4000	-0,00,66,24,574	0,01,32,63,526	-0,49,94,57,987
5000	0,00,59,74,167	-0,01,19,58,407	-0,49,95,78,832
6000	-0'002760977	0,00,55,35,461	-0,49,87,79,957
7000	-0,00,29,72,586	0,00,59,34,484	-0,50,09,005
8000	0,00,54,74,619	-0,01,09,50,887	-0,49,99,24,709
9000	0,00,05,77,008	-0,00,11,45,756	-0,50,36,04,607
10000	-0,004909035	0,009819162	-0,499944394
20000	0,003477281	-0,006954022	-0,500038826
30000	0,00269 1335	-0,00,53,83,116	-0,49,99,58,574
40000	-0,00,21,25,171	0,00,42,49,849	-0,50,00,58,002
50000	00,01,88,592	-0,00,37,71,481	-0,50,00,47,594
60000	-0,00,20,41,243	0,00,40,82,467	-0,50,00,02,327
70000	0,00,10,83,862	-0,00,21,68,029	-04,99,92,966
80000	0,00,10,60,449	-0,00,21,20,641	-0,50,00,60,595
90000	-0,00,14,21,483	0'002843095	-0,49,99,77,313
100000	-00,00,93,261	0,00,18,65,034	-0,50,00,49,865
200000	0'000285425	-0,00,05,70,773	-0,50,00,67,452
300000	0,00,06,61,298	-0,00,13,22,565	-05,00,01,172
400000	9,03502E-05	-0,00,01,80,728	-0,49,99,23,642
500000	-8,95746E-05	0,00,01,79,169	-0,49,99,44,745
600000	-0,00,02,73,486	0,00,05,46,957	-0,50,00,13,712
700000	0,00,05,88,941	-00,01,17,788	-0,50,00,00,849
800000	-0,00,02,61,711	0,00,05,23,431	-0,49,99,91,403
900000	-0'000440494	0'000880984	-05,00,00,227
1000000	0,00,02,39,578	-0,00,04,79,162	-0,49,99,93,739

Table 2: Calculation of numerical verification 1.2.

m	$\Re(\eta_m(s_2))$	$a_m(s_1)$	$\frac{\Re(\eta_m(s_2))}{am(s_2)}$
10	-0,298803655	0,090320283	-3,30826748
50	0,06833117	-0,120022828	-0,569318113
100	-0,039062783	0,083684493	-0,466786397
150	0,000838921	-0,007372696	-0,113787548
200	-0,006958822	0,01022426	-0,680618646
250	-0,030995955	0,062370315	-0,496966466
300	0,025512949	-0,049970481	-0,510560405
350	-0,02218031	0,043391872	-0,511162782
400	0,024164864	-0,047925056	-0,504221925
450	-0,021729465	0,043839399	-0,49566065
500	0,005463041	-0,011826955	-0,461914415
550	0,016570237	-0,032600321	-0,508284474
600	-0,01650938	0,033415728	-0,494060162
650	-0,009686153	0,018808426	-0,51499009
700	0,016332111	-0,032931023	-0,495949093
750	0,011016676	-0,021613651	-0,509709165
800	-0,011529711	0,023400347	-0,492715386
850	-0,015702739	0,031220641	-0,502960173
900	0,000767843	-0,001923818	-0,399124553
950	0,015024203	-0,030172433	-0,497944697
1000	0,012184744	-0,024148842	-0,504568459
2000	-0,0101168	0,020280562	-0,498842192
3000	0,002094356	-0,004250569	-0,49272368
4000	-2,51383E-05	8.73E-01	-2,8803618
5000	-0,007069382	0,014138711	-0,500001874
6000	0,005070098	-0,010125745	-0,500713577
7000	-0,00430446	0,008596143	-0,500743182
8000	0,005075981	-0,010145473	-0,500319798
9000	-0,005128103	0,01025875	-0,499876008
10000	0,001998329	-0,004006189	-0,498810466
20000	0,00234438	-0,004685918	-0,500303249
30000	-0,002882405	0,005764873	-0,499994536
40000	-0,002395292	0,004790929	-0,499963994
50000	0,000686115	-0,001371329	-0,500328513
60000	0,000755897	-0,001512452	-0,499782472
70000	-0,000869896	0,00174029	-0,49985692
80000	0,000253446	-0,000507351	-0,49954765
90000	0,000830553	-0,001660765	-0,500102664
100000	-0,001569306	0,003138564	-0,500007647
200000	0,000590745	-0,001181589	-0,499958107
300000	0,000308187	-0,000616313	-0,500049488
400000	0,000433172	-0,000866309	-0,500020201
500000	0,000583069	-0,001166153	-0,499993569
600000	-0,000642894	0,001285785	-0,500001167
700000	0,000586993	-0,001173981	-0,50000213
800000	-0,000553097	0,001106195	-0,499999548
900000	0,000362932	-0,000725874	-0,499993112
1000000	8.3178E-05	-0,000166333	-0,500031864

Table 3: Calculation of numerical verification 1.3.

m	$\Im(\eta_m(s_2))$	$b_m(s_2)$	$\frac{\Im(\eta_m(s_2))}{b_m(s_2)}$
10	-0,1292896	0,303054857	-0,426621112
50	0,024809322	-0,074796528	-0,331690824
100	0,031856819	-0,054744001	-0,58192347
150	-0,040985555	0,081316112	-0,50402748
200	-0,034758978	0,069967596	-0,496786798
250	0,006559889	-0,010485409	-0,625620708
300	0,01359563	-0,028918581	-0,470134755
350	-0,014966219	0,031213591	-0,479477642
400	0,006501864	-0,014254439	-0,45612907
450	0,009181916	-0,017330011	-0,529827477
500	-0,0216997	0,043129145	-0,50313309
550	0,013436983	-0,027484557	-0,488892108
600	0,012024249	-0,023453269	-0,512689681
650	-0,017064307	0,034419539	-0,495773839
700	-0,0095258	0,018550449	-0,513507786
750	0,014568941	-0,029430994	-0,495020352
800	0,013409517	-0,026503279	-0,505956904
850	-0,006910842	0,014203596	-0,486555799
900	-0,016654744	0,033277771	-0,500476549
950	-0,006132047	0,011927105	-0,514127024
1000	0,010083883	-0,020416498	-0,493908554
2000	0,004762884	-0,009418005	-0,505721116
3000	-0,008886052	0,017755731	-0,500461062
4000	-0,007906176	0,015811386	-0,500030548
5000	0,000170479	-0,00031122	-0,547776493
6000	0,003995544	-0,008008493	-0,498913341
7000	-0,004145907	0,008304425	-0,499240706
8000	0,002342311	-0,004697805	-0,498596898
9000	0,001217342	-0,002422635	-0,502486755
10000	-0,004583444	0,009162448	-0,500242293
20000	0,002646545	-0,005295486	-0,49977377
30000	0,000158785	-0,000315547	-0,503205545
40000	0,000716001	-0,001430734	-0,500443129
50000	0,002128214	-0,004256695	-0,499968638
60000	-0,001896133	0,003791986	-0,50003692
70000	0,001677717	-0,003355161	-0,500040684
80000	-0,00174951	0,003498942	-0,500011146
90000	0,001444982	-0,002890151	-0,499967649
100000	-0,000193106	0,00038654	-0,499575723
200000	-0,000949222	0,00189838	-0,500016856
300000	0,000859276	-0,001718573	-0,49999389
400000	0,000661334	-0,00132269	-0,499991684
500000	-0,000400039	0,000800054	-0,500014999
600000	-5.79E+00	0,000115863	-0,499902471
700000	0,000112173	-0,000224364	-0,499959887
800000	8.11E+00	-0,00016227	-0,500046219
900000	-0,000382175	0,000764342	-0,500005233
1000000	0,000493034	-0,00098607	-0,499998986

Table 4: Calculation of numerical verification 1.4.

m	$\Re(\eta_m(s_3))$	$a_m(s_3)$	$\frac{\Re(\eta_m(s_3))}{am(s_3)}$
10	0,448284561	-0,159819003	-2,804951555
50	-0,071854242	0,127115345	-0,565268041
100	-0,01904003	0,048915773	-0,389241115
150	0,037357774	-0,076882011	-0,485910469
200	0,031022457	-0,059585908	-0,520634124
250	0,031161654	-0,062681924	-0,497139399
300	-0,009305261	0,016286624	-0,57134376
350	-0,010228407	0,02217931	-0,461168855
400	0,014023613	-0,029296023	-0,478686578
450	-0,009241699	0,019663369	-0,469995706
500	-0,002259912	0,003402611	-0,664169957
550	0,016122602	-0,031579255	-0,51054409
600	-0,019793606	0,039763414	-0,49778437
650	0,003578293	-0,007892821	-0,453360465
700	0,016879458	-0,033431943	-0,504890129
750	-0,010664078	0,021809242	-0,488970593
800	-0,013885918	0,027414235	-0,506522177
850	0,009887957	-0,020178073	-0,490034752
900	0,014835989	-0,029446734	-0,503824601
950	-0,004109635	0,008628723	-0,476273836
1000	-0,015808874	0,03161724	-0,500008034
2000	-0,000363142	0,000865888	-0,419386803
3000	0,006228512	-0,012511409	-0,497826584
4000	0,007872022	-0,015738314	-0,50018204
5000	0,005802183	-0,011623931	-0,499158417
6000	-0,004449265	0,008878626	-0,501120894
7000	0,000276627	-0,000531904	-0,520069411
8000	0,000846807	-0,001710832	-0,494967945
9000	0,000229212	-0,000443778	-0,516501494
10000	-0,002612648	0,005214494	-0,501035767
20000	-0,003116677	0,006235361	-0,499839063
30000	0,002814184	-0,005627784	-0,500051885
40000	0,001051196	-0,002100961	-0,500340558
50000	0,002028101	-0,004055711	-0,500060532
60000	0,000568791	-0,001138395	-0,499642918
70000	-0,001586411	0,003173177	-0,499944062
80000	0,001644039	-0,00328827	-0,499970805
90000	-0,001401291	0,002802825	-0,499956651
100000	0,000747314	-0,001494973	-0,499884613
200000	-0,000953273	0,00190647	-0,500019932
300000	0,000273893	-0,000547713	-0,500066641
400000	-0,000451122	0,000902284	-0,499977834
500000	6,70006E-05	-0,000133966	-0,500131377
600000	0,000625865	-0,001251737	-0,499997204
700000	-0,000533734	0,001067459	-0,500004216
800000	0,000439992	-0,000879973	-0,50000625
900000	-0,000470137	0,000940267	-0,500003722
1000000	0,00049965	-0,0009993	-0,5

Table 5: Calculation of numerical verification 1.5.

m	$\Im(\eta_m(s_3))$	$b_m(s_3)$	$\frac{\Im(\eta_m(s_3))}{b_m(s_3)}$
10	0,063087093	-0,272869724	-0,231198581
50	-0,014944382	0,061981361	-0,241110904
100	0,046798444	-0,087219534	-0,536559207
150	-0,016981693	0,027492236	-0,6176905
200	0,017195077	-0,038072556	-0,45163968
250	-0,005785644	0,00842475	-0,686743702
300	-0,027378675	0,055390245	-0,494286945
350	0,024730603	-0,048633538	-0,508509231
400	-0,02073001	0,040518429	-0,511619293
450	0,02170702	-0,042843601	-0,506657225
500	-0,022264473	0,044591728	-0,499296035
550	0,01397341	-0,02865192	-0,487695414
600	0,005040697	-0,009248653	-0,545019583
650	-0,019293777	0,038420892	-0,502168898
700	0,008520338	-0,017631694	-0,483239897
750	0,0148299	-0,02928635	-0,506375837
800	-0,010952313	0,022326212	-0,490558497
850	-0,014020797	0,027736545	-0,505499045
900	0,007607843	-0,01562053	-0,487041285
950	0,015698822	-0,031275817	-0,501947623
1000	0,000493631	-0,000591706	-0,834250455
2000	0,011176058	-0,022343908	-0,500183674
3000	-0,006674901	0,01329654	-0,502002852
4000	0,000734637	-0,001518383	-0,48382852
5000	-0,004042275	0,008055075	-0,501829592
6000	-0,004677003	0,009372121	-0,49903357
7000	0,005969961	-0,011940445	-0,4999781
8000	-0,005525843	0,011048668	-0,500136578
9000	0,005265628	-0,01053158	-0,499984618
10000	-0,004263257	0,008532822	-0,499630368
20000	0,001669324	-0,003334707	-0,500590906
30000	0,000643305	-0,001288945	-0,49909422
40000	0,002268274	-0,004537176	-0,499930794
50000	0,00094173	-0,001884465	-0,499733346
60000	-0,001960402	0,003920551	-0,500032266
70000	0,001027013	-0,002053451	-0,500140008
80000	-0,000649735	0,001298953	-0,500199006
90000	0,000902318	-0,00180424	-0,500109741
100000	-0,001393389	0,002786585	-0,50003463
200000	-0,000584187	0,001168492	-0,499949508
300000	0,000870814	-0,00174165	-0,499993684
400000	0,000649223	-0,001298416	-0,500011553
500000	0,000703926	-0,001407854	-0,49999929
600000	-0,000157985	0,000315945	-0,500039564
700000	-0,000268832	0,000537684	-0,499981402
800000	0,00034483	-0,000689672	-0,4999913
900000	-0,00023822	0,000476454	-0,499985308
1000000	-l,87141E-05	3,74162E-05	-0,500160358

Table 6: Calculation of numerical verification 1.6.

m	$\Re(\eta_m(s_4))$	$a_m(s_4)$	$\frac{\Re(\eta_m(s_4))}{am(s_4)}$
10	0,743881692	-0,186295877	-3,993012105
50	0,058896264	-0,13246976	-0,444601575
100	-0,008037138	0,030587692	-0,262757255
150	0,000843704	0,006583153	0, 12816108
200	-0,021972495	0,039421248	-0,557376951
250	-0,004620945	0,005397286	-0,856160856
300	-0,022151837	0,042272429	-0,524025648
350	-0,016905337	0,035522259	-0,475908275
400	0,025014555	-0,049851106	-0,501785357
450	-0,020871699	0,040929649	-0,509940826
500	0,019049601	-0,037331041	-0,510288502
550	-0,020296719	0,040180244	-0,505141756
600	0,020103667	-0,040348224	-0,498254074
650	-0,01246367	0,02561283	-0,48661823
700	-0,003702041	0,006592795	-0,561528305
750	0,017265557	-0,03426361	-0,503903617
800	-0,011751598	0,023989606	-0,489862068
850	-0,009247696	0,017967146	-0,514700331
900	0,015357509	-0,030917047	-0,496732725
950	0,005180575	-0,009863568	-0,525223226
1000	-0,014941446	0,03002631	-0,497611794
2000	0,003748574	-0,007655995	-0,489625973
3000	0,001040701	-0,002173143	-0,478892093
4000	0,004176745	-0,008301795	-0,503113483
5000	0,000352638	-0,000662262	-0,532475063
6000	0,004564823	-0,009106065	-0,501294796
7000	0,004131647	-0,008281728	-0,498887068
8000	-0,005554139	0,011105475	-0,500126199
9000	0,004476607	-0,008943536	-0,500541061
10000	-0,004068925	0,008128786	-0,500557525
20000	0,003396641	-0,006794685	-0,499896758
30000	0,002517775	-0,005036938	-0,499862218
40000	-0,000945091	0,00189193	-0,499538038
50000	-0,001742758	0,003486351	-0,499880247
60000	-0,000320479	0,000641978	-0,49920558
70000	0,001872554	-0,003744983	-0,500016689
80000	-0,000869719	0,001738848	-0,500169653
90000	0,000120573	-0,000240583	-0,501170074
100000	-I,32844E-05	2,60877E-05	-0,509220821
200000	0,000883075	-0,001766044	-0,500030011
300000	0,000829724	-0,001659407	-0,500012354
400000	-0,000767542	0,001535097	-0,499995766
500000	-0,000682612	0,001365212	-0,500004395
600000	-0,000574894	0,001149802	-0,499993912
700000	0,0002836	-0,000567177	-0,500020276
800000	0,000231324	-0,000462667	-0,499979467
900000	-0,000402352	0,000804714	-0,499993787
1000000	0,000401575	-0,000803159	-0,499994397

Table 7: Calculation of numerical verification 1.7.

m	$\Im\left(\eta_m\left(s_4\right)\right)$	$b_m(s_4)$	$\frac{\Im\left(\eta_m\left(s_4\right)\right)}{b_m\left(s_4\right)}$
10	0, 16006872	-0,255526606	-0 '626426823
50	-0,0457116 11	0,049515278	-0,923181952
100	0,050075594	-0,095207106	-0,525964879
150	0,0410968 14	-0,081383836	-0,50497514
200	-0,027886389	0,058702344	-0,475047283
250	-0,031374863	0,063014834	-0,497896464
300	-0,018605809	0,039323976	-0,473141602
350	0,020757479	-0,03994 1357	-0,519698893
400	0,000980712	-0,003855806	-0,254346821
450	-0,011008302	0,023387733	-0,470687005
500	0,011750631	-0,024625056	-0,4771819
550	-0,006584136	0,014273394	-0,461287343
600	-0,003622867	0,006219928	-0,58246 1244
650	0,015158478	-0,029705967	-0,510283944
700	-0,018543516	0,037217019	-0,498253662
750	0,005966035	-0,012622931	-0,472634684
800	0,013217734	-0,025971115	-0,508939797
850	-0,014452169	0,029217328	-0,494643761
900	-0,006492962	0,012459829	-0,521111646
950	0,01537939	-0,030908601	-0,497576387
1000	0,005190006	-0,009920722	-0,523148013
2000	-0,010535022	0,021009 182	-0,501448462
3000	-0,009070078	0,018127625	-0,500345633
4000	0,006712936	-0,013456605	-0,49885807
5000	0,007062656	-0,014126621	-0,499953669
6000	0,004564295	-0,00915 1298	-0,498759302
7000	-0,004318159	0,008618012	-0,501062078
8000	-0,000635297	0,001291671	-0,491841189
9000	0,00278 1975	-0,005578913	-0,498658968
10000	-0,002906056	0,005824331	-0,498951038
20000	-0,000981404	0,0019576 14	-0,501326615
30000	-0,001412192	0,002821806	-0,5004568
40000	0,002314493	-0,004628239	-0,5000807
50000	0,00140 1015	-0,002800957	-0,50019 1542
60000	0,002015935	-0,004031691	-0,500022199
70000	0,000254944	-0,000510702	-0,499203058
80000	-0,001539029	0,003078377	-0,499948187
90000	0,001662304	-0,00332464	-0,499995187
100000	-0,001581087	0,00316217	-0,500000632
200000	0,000685697	-0,001371528	-0,499951149
300000	0,000380648	-0,00076 1381	-0,49994418
400000	0,000189421	-0,000378783	-0,500077881
500000	-0,000184504	0,000369049	-0,499944452
600000	0,000293537	-0,000587045	-0,5000247
700000	0,000526036	-0,001052084	-0,499994297
800000	-0,00050891	0,001017811	-0,500004421
900000	0,000340428	-0,000680842	-0,500010281
1000000	-0,000297889	0,000595765	-0,50001091

Table 8: Calculation of numerical verification 1.8.

m	$\Re(\eta_m(s_5))$	$a_m(s_5)$	$\frac{\Re(\eta_m(s_5))}{a_m(s_5)}$
10	0,96,21,75,573	-0,28,64,35,644	-33,59,13,352
50	-0'07204829	0,14,13,22,356	-0,50,98,15,234
100	0,03,85,10,014	-0,06,40,90,757	-0,60,08,66,893
150	0,000725 118	00,07,49,855	0,09,67,01,096
200	0,00,21,22,301	-0,01,00,08,362	-0,21,20,52,781
250	0,02,88,55,067	-0,05,91,30,526	-0,48,79,89,351
300	0,02,22,05,736	-0'046269956	-0,47,99,16,947
350	-0,00,85,13,308	0,01,45,94,976	-0,58,33,04,008
400	-00,20,19,761	0,04,15,16,874	-0,48,64,91,589
450	0,02,34,56,676	-0,04,66,35,379	-05,02,98,028
500	-0'020232304	00,39,77,083	-0,50,87,22,197
550	0,01,92,83,981	-0,03,79,69,523	-0,50,78,80,518
600	-0,02,01,37,142	0,04,00,38,679	-0,50,29,42,217
650	0,01,85,40,022	-0,03,73,67,727	-0,49,61,50,649
700	-0'009683806	0,02,01,13,366	-0,48,14,61,233
750	-0,00,59,21,664	0,01,10,75,533	-0,53,46,61,763
800	0,01,72,37,036	-0'034285655	-0,50,27,47,753
850	-0,01,04,26,495	0,02,13,66,588	-0,48,79,81,282
900	-0,00,94,81,895	00,18,45,065	-0,51,39,05,743
950	0,01,49,85,393	-0,03,01,69,727	-04,96,70,297
1000	0,00,42,88,876	-00,08,07,219	-0,53,13,15,046
2000	0,00,60,46,856	-0,01,22,46,231	-0,49,37,72,819
3000	0,00,89,30,818	-0'017880408	-0,49,94,75,068
4000	-0'00780822	0,01,56,25,417	-0,49,97,12,744
5000	-0'004344305	0,00,86,51,331	-0,50,21,54,524
6000	-0,00,52,10,786	0,01,04,00,146	-0,50,10,30,082
7000	-0,00,50,17,103	00,10,04,907	-0,49,92,60,429
8000	0,00,43,39,471	-0,00,86,64,127	-0,50,08,54,962
9000	-0,00,07,92,091	0,00,15,65,065	-0,50,61,07,414
10000	-0,000884296	0,00178475	-0,495473316
20000	0,003005846	-0,006014678	-0,499751774
30000	0,002808479	-0,005616177	-0,500069531
40000	-0,002400136	0,004799665	-500063234
50000	-0,000490229	0,000979016	-0,500736454
60000	-0,000977535	0,001954077	-0,500254084
70000	-0,001872691	0,003745487	-0,499985983
80000	0,000767762	-0,001534864	-0,500215003
90000	0,000473951	0,000948484	-0,499693195
100000	-0,000914615	0,001829651	-0,499884951
200000	0,001110497	-0,002221013	-0,499995723
300000	0,000714685	-0,001429306	-0,50022388
400000	-0,000593546	0,001187049	-0,500018112
500000	0,000152119	-0,000304283	-0,499926056
600000	-3,79E-05	7,94E-05	-0,500222865
700000	-0,000570392	0,0011400775	-0,500003945
800000	6,44E-06	-1,29E-05	-0,500888333
900000	0,000349926	-0,000699866	-0,499989998
1000000	-0,000434836	0,000869679	-0,499995976

Table 9: Calculation of numerical verification 1.9.

Citation: Azkour M (2018) On the Non-Real Roots of the Riemann Zeta Function ς(s). J Phys Math 9: 279. doi: 10.4172/2090-0902.1000279

m	$\Im(\eta_m(s_5))$	$b_m(s_5)$	$\frac{\Im(\eta_m(s_5))}{b_m(s_5)}$
10	-0'099279099	-0,13,39,94,857	0,7409 17235
50	0,02,14,81,704	0,00,52,90,727	4,06,02,55,613
100	00,33,15,987	-0,07,67,61,806	-0,43,19,83,974
150	0,04,11,35,797	-0,08,13,04,603	-0,50,59,46,717
200	-00,35,45,701	0'069998805	-0,50,65,37,362
250	-0,01,31,81,946	0,02,24,40,609	-0,58,74,14,807
300	-00,18,55,146	0,03,45,31,501	-0,53,72,32,946
350	-0'025385438	0,05,14,211	-0,49,36,77,459
400	0,01,47,95,353	-0,02,78,63,043	-05,31,00,277
450	0,00,25,89,303	-0,00,68,82,125	-0,37,62,35,974
500	-0,00,95,75,928	0,02045 1922	-0,46,82,16,532
550	0,00,91,37,597	-0,01,94,03,534	-0,47,09,24,369
600	-0,00,34,38,645	0,00,79,73,131	-04,31,27,913
650	-0'006436337	0,01,19,21,178	-0,53,99,07,801
700	0,01,62,42,538	-0,03,20,00,374	-05,07,57,338
750	-00,17,28,151	0,03,47,94,624	-0,49,66,71,842
800	0,00,39,63,916	-0'008630982	-0,45,92,65,933
850	0,01,36,26,793	-0,02,68,31,688	-0,50,78,61,935
900	-0,01,37,15,647	0,02,77,61,207	-0,49,40,58,021
950	-0'006230058	00,11,93,395	-0,52,20,44,922
1000	0,01,52,24,928	-0,03,05,75,149	-0,49,79,51,065
2000	-00,09,40,613	0,01,87,09,084	-0,50,27,57,377
3000	-0,00,18,94,788	0,00,36,91,117	-0,51,33,37,291
4000	0,00,12,412	-0,00,24,17,921	-0,51,33,33,562
5000	-0,00,55,79,657	0,01,11,87,246	-0,49,87,51,614
6000	-0,00,38,10,271	0,00,76,48,766	-0,49,81,54,997
7000	0,00,32,47,415	-00,06,47,096	-0,50,18,44,394
8000	0,00,35,24,353	-0'007066322	-0,49,87,53,524
9000	-0,00,52,10,759	0'010424091	-0,49,98,76,584
100000	0,001289763	-0,000259037	-0,50002879
20000	-0,001861508	0,003718016	-0,500672402
30000	0,00,06,67,771	-0,00,13,38,615	-0,49,88,52,172
40000	-0,00,06,99,591	0,00,14,01,147	-0,49,92,98,789
50000	-00,02,18,168	00,04,36,366	-0,49,99,65,625
60000	-0,00,17,91,963	0,00,35,84,445	-0,49,99,27,604
70000	0,00,02,53,936	-00,00,50,699	-05,00,86,984
80000	0,00,15,92,344	-0,00,31,84,995	-0,49995 1805
90000	-0,00,15,97,863	0,00,31,95,542	-05,00,02,879
100000	0,00,12,89,763	-0'00257922	-05,00,05,932
200000	-0,00,01,29,611	0,00,02,59,037	-0,50,03,57,092
300000	0'000567943	-0,00,11,35,966	-0,49,99,64,788
400000	-0,00,05,22,211	0,00,10,44,469	-0,49,99,77,501
500000	-0'000690552	0,00,13,81,091	-0,50,00,04,706
600000	-0,00,06,44,275	0,00,12,88,549	-0,50,00,00,388
700000	-0,00,01,78,315	0'000356656	-04,99,96,355
800000	0,00,05,58,979	-00,01,11,796	-0,49,99,99,106
900000	-00,00,39,412	0'000788225	-0,50,00,09,515
1000000	0'000246815	-0,00,04,93,617	-0,50,00,13,168

Table 10: Calculation of numerical verification 1.10.

m	$\Re\left(\eta_m\left(s'\right)\right)$	$a_m(s'_1)$	$\frac{\Re\left(\eta_m\left(s'_1\right)\right)}{am\left(s'_1\right)}$
10	-0,001870233	0,043670938	-0,042825574
50	-0,006962498	0,012371397	-0,562789958
100	-0,003286542	0,006185698	-0,531313039
150	0,000543644	-0,000884624	-0,614548102
200	-0,001594816	0,003092849	-0,515646254
250	0,001950134	-0,00390792	-0,499020963
300	0,000246483	-0,000442312	-0,557260486
350	-0,001358336	0,002723944	-0,498665 171
400	-0,000785308	0,001546425	-0,507821589
450	0,000423631	-0,000866907	-0,488669488
500	0,000976023	-0,00195396	-0,499510225
550	0,000729462	-0,001448551	-0,503580475
600	0,00011691	-0,000221156	-0,528631373
650	-0,000425434	0,000859106	-0,495205481
700	-0,000680077	0,001361972	-0,499332585
750	-0,000634749	0,001266128	-0,501330829
800	-0,00038963	0,000773212	-0,503910958
850	-7,20E-05	0,000137773	-0,522902165
900	0,000214271	-0,000433453	-0,494335026
950	0,000407904	-0,000818535	-0,498334219
1000	0,000488251	-0,00097698	-0,499755369
2000	0,000244185	-0,00048849	-0,499877173
3000	-0,000158371	0,000316532	-0,50033172
4000	0,000122108	-0,000244245	-0,499940633
5000	-2,34E-05	4,69E-05	-0,498173531
6000	-7,92E-05	0,000158266	-0,500166176
7000	I,03E-05	-2,07E-05	-0,49782573
8000	6,IIE-05	-0,000122122	-0,49997134
9000	3,66E-05	-7,31E-05	-0,50031666
10000	-I,I7E-05	2,35E-05	-0,499086235
20000	-5,86E-06	I,I7E-05	-0,499545032
30000	I, 16E-05	-2,32E-05	-0,499931104
40000	-2,93E-06	5,87E-06	-0,499770813
50000	-7,72E-06	I,54E-05	-0,500043397
60000	5,81E-06	-I,16E-05	-0,499964691
70000	5,91E-06	-I,18E-05	-0,500027091
80000	-I,47E-06	2,93E-06	-0,499887536
90000	-5,36E-06	I,07E-05	-0,499997201
100000	-3,86E-06	7,72E-06	-0,500021374
200000	-I,93E-06	3,86E-06	-0,500010363
300000	5,68E-07	-I,14E-06	-0,50002023
400000	-9,65E-07	I,93E-06	-0,500004664
500000	9,09E-07	-I,82E-06	-0,4999978
600000	2,84E-07	-5,68E-07	-0,500011434
700000	-6,19E-07	I,24E-06	-0,499999193
800000	-4,82E-07	9,65E-07	-0,500002591
900000	I,03E-07	-2,05E-07	-0,499987818
1000000	4,54E-07	-9,09E-07	-0,49999945

Table 11: Calculation of numerical verification 1.11.

m	$\Im\left(\eta_m\left(s_1'\right)\right)$	$b_m(s'_1)$	$\frac{\Im\left(\eta_m\left(s_1'\right)\right)}{am\left(s_1'\right)}$
10	0,05,87,75,434	-0,08,99,60,265	-0,65,33,48,831
50	-0,00,73,75,109	0,01,57,14,596	-0,46,93,15,851
100	-00,03,80,811	0,00,78,57,298	-0,48,46,58,976
150	0,00,33,01,521	-0,0066077 14	-0,49,96,46,474
200	-00,01,93,419	0,00,39,28,649	-0,49,23,29,551
250	-0,00,04,62,944	0,00,08,53,326	-0,54,25,17,162
300	0,00,16,51,342	-0,00,33,03,857	-04,99,82,248
350	0,00,04,49,355	-0,00,08,62,203	-0,52,11,70,768
400	-0,00,09,74,629	0,00,19,64,325	-04,96,16,484
450	-00,01,02,858	0,00,20,46,153	-0,50,26,89,682
500	-0)000222401	0,00,04,26,663	-0,52,12,56,823
550	0,00,05,43,959	-0,00,10,98,857	-0,49,50,22,555
600	0,00,08,25,817	-0,00,16,51,929	-04,99,91,071
650	0,00,06,41,609	-0,00,12,76,245	-0,50,27,31,842
700	0,00,02,20,114	-0,00,04,31,101	-0,51,05,85,686
750	-0,00,02,05,298	0,00,04,17,968	-0,49,11,81,143
800	-0,00,04,89,198	0,00,09,82,162	-0,49,80,82,801
850	-5,84E-04	0,00,11,68,376	-0,49,99,79,459
900	-0,00,05,12,914	0,00,10,23,077	-0,50,13,44,474
950	-0,00,03,33,047	0,00,06,61,841	-0,50,32,13,007
1000	-0,00,01,08,933	0,00,02,13,331	-0,51,06,29,023
2000	-5,38997E-05	0,00,01,06,666	-0,50,53,12,846
3000	-5,20156E-05	0,00,01,04,492	-0,49,77,95,047
4000	-2,68081E-05	5,33329E-05	-0,50,26,55,959
5000	9,72E-05	-l,94E-04	-0,50,01,58,426
6000	-2) 61E-05	0,00,00,52,246	-0,49,88,97,523
7000	-7,07E-05	1,41E-04	-0,50,00,82,773
8000	-l,34E-05	2,66664E-05	-0,50,13,27,513
9000	4,18E-05	-8,37E-05	-04,99,80,818
10000	4,86E-05	-9,72E-05	-0,50,00,79,213
20000	2,43E-05	-4,86E-05	-0,50,00,40,121
30000	-l,20E-05	2,39E-05	-05,00,08,364
40000	l,22E-05	-2,43E-05	-0,50,00,20,575
50000	-6,36E-06	l,27E-05	-0,49,99,51,233
60000	-5,98E-06	1,20E-05	-05,00,04,182
70000	4,02E-06	-8,03E-06	-0,49,99,56,439
80000	6,08E-06	-1,22E-05	-0,50,00,10,699
90000	l,46E-06	-2,92E-06	-0,50,00,94,092
100000	-3,18E-06	6,36E-06	-04,99,97,483
200000	-1,59E-06	3,18E-06	-0,49,99,85,842
300000	1,57E-06	-3,13E-06	-05
400000	-7,95E-07	l,59E-06	-0,49,99,94,966
500000	-4,17E-07	8,34E-07	-0,50,00,10,795
600000	7,83E-07	-1,57E-06	-0,49,99,99,362
700000	3,56E-07	-7,12E-07	-0,50,00,06,323
800000	-3,97E-07	7,95E-07	-0,49,99,97,483
900000	-5,46E-07	1,09E-06	-0,50,00,02,747
1000000	-2,08E-07	4,17E-07	-0,50,00,05,997

Table 12: Calculation of numerical verification 1.12.

m	$\left.\dfrac{\eta_m(s_1)}{\eta_m(s_1')}\right\|$	$\left\|\dfrac{\alpha_m(s_1)}{\alpha_m(s_1')}\right\|$
10	3,689741821	3,162277858
50	7078600484	7,071076839
100	9,98959317	99,99995
150	12,23517792	12,24744686
200	14, 12972581	14, 142136
250	15'79934786	15,81130597
300	17,30893587	17,32048648
350	18,69717121	18,70825462
400	19,9893288	2,00,00,048
450	21,20294133	21,21320595
500	22,35083149	22,3605654
550	23,44257231	23,4520597
600	24,4857063	24,4948882
650	26,44885426	25,49512053
700	27,377337	26,45730969
750	28,27604478	27,38598194
800	28,27604478	28,2842539
850	29, 14678157	29,15478354
900	29,99220262	29,999915
950	30,8144809	30,82205544
1000	31,61533501	31,62277842
2000	44,71598015	44,72157446
3000	54,76782958	54,77287894
4000	63,24135589	63,24535762
5000	70,70715187	70,71052332
6000	77,45658376	77,4598321
7000	83,66307805	83,66605719
8000	89,44000195	89,44244237
9000	94,86559748	94,86848147
10000	99,99764024	99,99,995
20000	141,4195571	14,14,214
30000	173,2036131	173,2047748
40000	199,2429627	200,0112009
50000	223,5514486	223,7048923
60000	244,9163998	244,4861809
70000	264,5124602	264,6271471
80000	282,6427349	281,7625014
90000	300,0219743	300,024477
100000	316,1559247	316,1638253
200000	447,1197902	447,1210085
300000	546,7081404	548,252966
400000	632,3240857	632,3295467
500000	707,0654563	707,0558613
600000	774,9132585	773,1643492
700000	836,577005	836,5733484
800000	894,7034584	894,2423084
900000	948,6862736	950,3575028
1000000	1000,352186	999,9355062

Table 13: Calculation of numerical verification 1.13.

like all other numerical methods when the primitive off is not obvious by algebraic computation (Figure 2) [7,8].

Hypothesis

- It is assumed here that the function- arc (M_iM_{i+1}) is concave on [x_i, X_{i+1}],
- (H E) is the elliptical half-arc passing through the points Mi, M;+1 and P_i; with: N_i is the point of intersection of the curve-arc (M_iM_{i+1}) and the vertical line whose equation is x=m_i where $m_i = \frac{x_i + x_{i+1}}{2}$
- P_i is the orthogonal projection of N_i on the axis Y,
- X is the axis passing through the points M_i and M_{i+1} and $X \perp Y$ where X and Y are the two axes of the half-ellipse (H E),
- a_e and b_e are the two parameters of the half-ellipse (H E),
- $O'\left(m_i \frac{f(x_i)+f(x_{i+1})}{2}\right)$ is the origin of the reference $\left(O', \vec{X}, \vec{Y}\right)$,
- Q is the point whose coordinates x=X_{i+i} and y=f(x_i),
- $\alpha = M_{i+1}\hat{M}_iQ \text{ and } \beta = Ni\hat{O}'P_i$,
- $c = \left|\overrightarrow{N_iP_i}\right|$ and $d = \left(f(m_i) - \frac{f(x_i)+f(x_{i+1})}{2}\right)$,

Calculation of the parameters a_e and b_e of the half-ellipse (HE): According to Pythagoras ' theorem, we have:

$$(2a_e)^2 = (x_{i+1} - x_i)^2 + \left(f(x_i+1) - f(x_i)\right)^2$$

we denote: $x_{i+1} = x_i = h = \frac{b-a}{n}$ with n is the number of subintervals obtained by regular subdivision of the interval [a, b], then we get:

$$a_e = \frac{\left(h^2 + \left(f(x_i+1) - f(x_i)\right)^2\right)^{0.5}}{2}$$

According to Pythagoras' theorem, we also have: $d^2=b_e^2+c^2$, that is to say be=$(d^2-c^2)^{0.5}$ with:

Noticingthatα=β,wecanwrite $\sin\beta = \frac{c}{d} = \sin\alpha = \frac{f(x_i+1) - f(x_i)}{2a_e}$

This implies $c = d\left(\frac{f(x_i+1) - f(x_i)}{2a_e}\right)$

So, $b_e = d\left(1 - \left(\frac{f(x_i+1)-f(x_i)}{2a_e}\right)^2\right)^{0.5} = \left(f(m_i) - \frac{f(x_i+1)-f(x_i)}{2}\right)\frac{h}{\left(h^2 + \left(f(x_i+1)-f(x_i)\right)^2\right)^{0.5}}$

Approximate calculation of $\int_x^{x_{i+1}} f(x)dx$:

Let, $s = \frac{\pi a_e b_e}{2} = \frac{\pi h}{4}\left(f(m_i) - \frac{f(x_i)+f(x_{i+1})}{2}\right)$

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx h\left(\frac{f(x_i)-f(x_{i+1})}{2}\right) + s = \left(1-\frac{\pi}{4}\right)h\left(\frac{f(x_i)-f(x_{i+1})}{2}\right) + \left(\frac{\pi}{4}\right)hf(m_i)$$

Appoximation formula

Doing the summation for i ranging from 0 to n-1 with x_0=a and x_n=b, we obtain the following approximation formula:

$$\int_a^b f(x)dx \approx THE = \left(1-\frac{\pi}{4}\right)h\left(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(x_i)\right) + \left(\frac{\pi}{4}\right)h\sum_{i=0}^{n-1} f(m_i)$$

Note: $h\left(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(x_i)\right) = T_n$ The approximate value of the integral of on [a, b] using the method of trapezes.

$h\sum_{i=0}^{n-1} f(m_i) = M_n$: The approximate value of the integral of f on [a,b] using the method of rectangles with midpoint.

So, the approximation formula of The Method of trapezes 8 half-ellipses is:

$$\int_a^b f(x)dx \approx THE_n = \left(1-\frac{\pi}{4}\right)T_n + \left(\frac{\pi}{4}\right)M_n$$

Note: Knowing that $2T_{2n}=T_n +M_n$, we can write: $THE_n = \left(1-\frac{\pi}{2}\right)T_n + \left(\frac{\pi}{2}\right)T_{2n}$

Calculation of the theoretical maximum error e(n)

Let's call e_i(n) the error committed in the subinterval [x_i, X_{i+1}]. Thus, according to the approximation formula previously established, we have:

$$e_i(n) = \left(1-\frac{\pi}{4}\right)h\left(\frac{f(x_i)+f(x_{i+1})}{2}\right) + \left(\frac{\pi}{4}\right)hf(m_i) - \int_{x_i}^{x_{i+1}} f(x)dx$$

$$\left(1-\frac{\pi}{4}\right)\left(h\left(\frac{f(x_i)+f(x_{i+1})}{2}\right) - \int_{x_i}^{x_{i+1}} f(x)dx\right) + \left(\frac{\pi}{4}\right)\left(hf(m_i) - \int_{x_i}^{x_{i+1}} f(x)dx\right)$$

Note: $\left(h\left(\frac{f(x_i)+f(x_{i+1})}{2}\right) - \int_{x_i}^{x_{i+1}} f(x)dx\right) = e_{i/trapeze}$ The error committed in [x_i,x_{i+1}] using the $\left(hf(m_i) - \int_x^{x_{i+1}} f(x)dx\right) = e_{i/rec\,\text{tangle}}$: The error committed in [x_i, x_{i+1}] using the method of rectangles with midpoint, so:

$$e_{i/trapeze} = \left(1-\frac{\pi}{4}\right)e_{i/trapeze} + \left(\frac{\pi}{4}\right)e_{i/rec\,\text{tangle}}$$

We assume that f is class C^2 on the interval [a, b], so we know that 003A.

$\left|e_{i/trapeze}\right| \le \frac{h^3\lambda_i}{12} \text{ and } \left|e_{i/rec\,\text{tangle}}\right| \le \frac{h^3\lambda_i}{12}$ where $\lambda_i = \sup\left|f''(u)\right|, u \in [x_i, x_{i+1}]$ Xi+i], then we have:

$$|e(n)| \le \left(1-\frac{\pi}{8}\right)\frac{h^3\lambda_i}{12},$$

and denoting $\lambda = \sup(\lambda_i) = \sup\left|f''(x)\right|, \text{x} \in [a,b]$, we get:

$$|e(n)| \le \left(1-\frac{\pi}{8}\right)\frac{(b-a)^3\lambda}{12}$$

Note: if someone replace the elliptical half-arc studied in the previous case by the quarter of the circumference of the ellipse whose parameters are $h = \frac{b-a}{n} \text{ and } \delta_i = f(x_{i+1}) - f(x_i)$ he gets:

$\int_a^b f(x)dx \approx RQE_n = \left(1-\frac{\pi}{4}\right)R_n^{(\ell)} + \left(\frac{\pi}{4}\right)R_n^{(r)}$, [RQE: Rectangles and Quarters of Ellipses].

where:

- $R_n^{(l)}$ is the approximate value of the integral of on [a, b] using the method of rectangles on the left,
- $R_n^{(r)}$ is the approximate value of the integral of on [a, b] using the method of rectangles on the right,

and assuming that f is class C^1 on the interval [a, b], we get:

$$\left|\int_a^b f(x)dx - RQE_n\right| \le \frac{(b-a)^2}{2}\sup_{[a,b]}|f'|$$

Conclusion 2

Comparing the margins of error, we note that The Method of trapezes and half-ellipses THE is more accurate and more precise than the following 3 methods:

- The method of rectangles on the left $R_n^{(l)}$,
- The method of rectangles on the right $R_n^{(r)}$ and
- The method of trapezes T_n.

The approximation formula of The Method of trapezes and half-ellipses given above can be considered as a special case of the following general formula:

$$\int_a^b f(x)dx \approx THE_n = \left(1-\frac{\gamma\pi}{4}\right)T_n + \left(\frac{\gamma\pi}{4}\right)M_n$$

$0 \le \gamma \le \frac{4}{\pi}$ & $|e(\gamma,n)| \le \frac{(b-a)^3}{12n^2}\left(1-\frac{\gamma\pi}{8}\right)\sup_{[a,b]}|f''|$ of course if f is C^2 on [a, b].

For $\gamma=0$: $THE_n=T_n$,

For $\gamma = \frac{4}{\pi}$: $THE_n=M_n$,

So, we can consider The method of trapezes and The method of rectangles (on the left, on the right and with midpoint) as special cases of the general method which is called The Method of trapezes and half-ellipses THE.

References

1. Riemann B, Wilkins R (1998) On the Number of Prime Numbers less than a Given Quantity. Public university in Gottingen, Germany.
2. Julio CB (2012) Andrade Random Matrix Theory and £ Functions In Function Fi elds. School of Mathematics, University of Bristol, England.
3. Marek W (2010) Two arguments that the nontrivial zeros of the Ri emann zeta function are irrational. Cardinal Stefan Wyszynski University, Warszawa, Poland.
4. Borwein P, Choi S, Rooney B, Weirathmueller A (2010) The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike. Springer Berlin Heidelberg, Springer New York, pp: 1-533.
5. Murad AAA (2016) A Proof of the Ri emann Hypoth esis and Determination of the Relationship Between Non-Trivial Zeros of Zeta Functions and Prime Numbers. Mutah University, Jordan.
6. Garaev MZ, Yildmm CY (2017) On small distances between ordinat es of zeros of ((s) and ('(s). Institute of Mathematics, Morelia, Mexico.
7. McPhedran RC (2018) Sum Rules for Functions of the Riemann Zeta Type. Univer sity of Sydney, Australia.
8. Vic Dannon H (2009) Riemann Zeta Function: The Ri emann Hypoth esis Origin, the Factorization Error, and the Count of the Primes. Gauge Institute Journal 5: 15-3.