COMPLEX ANALYSIS

PIOTR HAJŁASZ

Complex numbers

Complex numbers are ordered pairs of real numbers, (a, b), so they can be identified with elements of $\mathbb{R}^2$. Real numbers are identified with complex numbers of the form $(a, 0)$. With this identification and with the notation $i = (0, 1)$ we can write

$$(a, b) = a + bi.$$

Addition is defined by

$$(a+bi) + (c+di) = (a+c) + (b+d)i$$

and the multiplication is defined according to the rule $i^2 = -1$, i.e.

$$(a+bi)(c+di) = (ac - bd) + (ad + bc)i.$$

Hence we can also get a formula for the division by $c+di \neq 0$.

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i$$

Complex number form a field. It will be denoted by $\mathbb{C}$.

<u>conjugate</u> of $z = a+bi$ is $\bar{z} = a-bi$ and the <u>absolute value</u> or <u>modulus</u> is $|z| = \sqrt{a^2+b^2}$.

$\operatorname{re} z = a$ is the real part of z

$\operatorname{im} z = b$ is the imaginary part of z.

It is easy to check that

$$|z|^2 = z\bar{z}, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \quad \overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \quad |z_1 z_2| = |z_1||z_2|, \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Complex numbers can also be written in the <u>polar form</u>

$$z = |z|(\cos\alpha + i\sin\alpha). \qquad (*)$$

Every angle α satisfying $(*)$ is called <u>argument</u> of z and denoted by $\arg z$.

$\arg z$ is defined up to a multiplicity of 2π. If α is an argument such that

$$0 \leq \alpha < 2\pi,$$

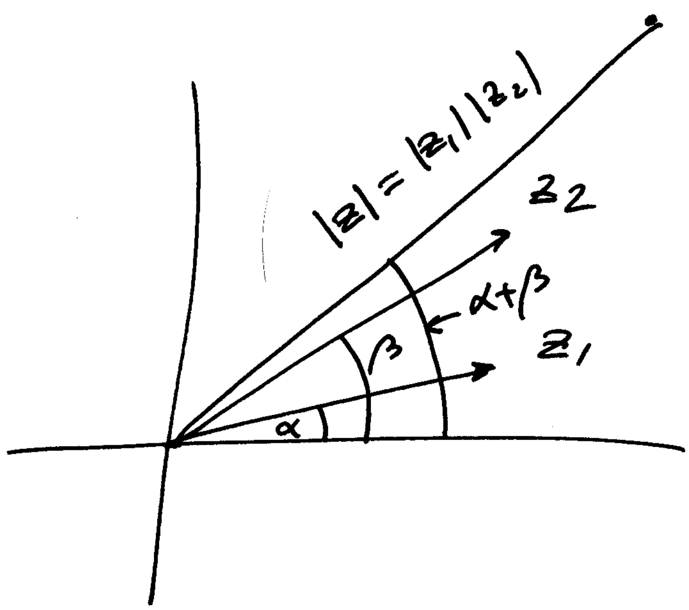
then we denote it by $\operatorname{Arg} z$ and call <u>principal value of the argument</u>.

Formulas for $\sin(\alpha \pm \beta)$ and $\cos(\alpha \pm \beta)$ imply

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$\frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta} = \cos(\alpha-\beta) + i\sin(\alpha-\beta).$$

This leads to the following geometric interpretation of the complex multiplication



Note that $\arg(z_1 z_2) = \arg z_1 + \arg z_2$.
This formula is not true if we replace arg with Arg.
The two formulas from the beginning of the page immediately give de Moivre's formula

$$(\cos\alpha + i\sin\alpha)^n = \cos(n\alpha) + i\sin(n\alpha), \quad n \in \mathbb{Z}.$$

Example 1 Find the formulas for

$$1+\cos\alpha+\cos 2\alpha+\ldots+\cos n\alpha$$

and

$$\sin\alpha+\sin 2\alpha+\ldots+\sin n\alpha .$$

Solution. If $\alpha=2k\pi$, then the first sum equals $n+1$ and the second one is 0. Therefore, we can assume $\alpha\neq 2k\pi$. Let

$z=\cos\alpha+i\sin\alpha$. Obviously $z\neq 1$ and hence

$$1+z+\ldots+z^n=\frac{1-z^{n+1}}{1-z}=\frac{(1-z^{n+1})(1-\bar{z})}{|1-z|^2}=$$

$$=\frac{(1-\cos(n+1)\alpha-i\sin(n+1)\alpha)(1-\cos\alpha+i\sin\alpha)}{(1-\cos\alpha)^2+\sin^2\alpha}=$$

$$=\frac{1+\cos\alpha\cos(n+1)\alpha+\sin\alpha\sin(n+1)\alpha-\cos\alpha-\cos(n+1)\alpha}{2-2\cos\alpha}$$

$$+\frac{\cos\alpha\sin(n+1)\alpha-\sin\alpha\cos(n+1)\alpha+\sin\alpha-\sin(n+1)\alpha}{2-2\cos\alpha}\,i$$

$$=\frac{1+\cos n\alpha-\cos\alpha-\cos(n+1)\alpha}{2-2\cos\alpha}+$$

$$+\frac{\sin n\alpha+\sin\alpha-\sin(n+1)\alpha}{2-2\cos\alpha}\,i$$

On the other hand

$$1+z+z^2+\ldots+z^n =$$

$$1+\cos\alpha+i\sin\alpha+\cos 2\alpha+i\sin 2\alpha+\ldots+\cos n\alpha+i\sin n\alpha$$

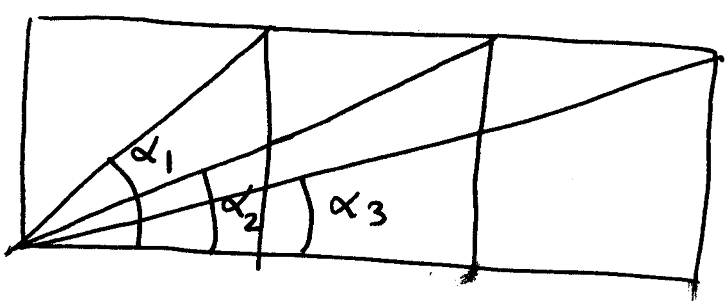
$$=(1+\cos\alpha+\cos 2\alpha+\ldots+\cos n\alpha)+i(\sin\alpha+\sin 2\alpha+\ldots+\sin n\alpha).$$

Comparing the real and imaginay parts we obtain

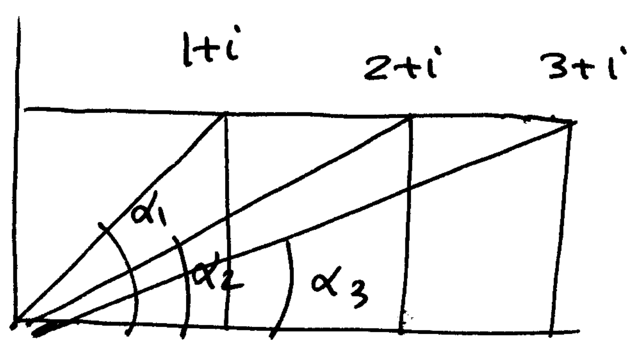
$$1+\cos\alpha+\cos 2\alpha+\ldots+\cos n\alpha=\frac{1+\cos n\alpha-\text{[illegible]}\,\alpha-\cos(n+1)\alpha}{2-2\cos\alpha}$$

$$\sin\alpha+\sin 2\alpha+\ldots+\sin n\alpha=\frac{\sin n\alpha+\sin\alpha-\sin(n+1)\alpha}{2-2\cos\alpha}.$$

<u>Example 2</u> Three squares are shown on the picture. Prove that $\alpha_1+\alpha_2+\alpha_3=\frac{\pi}{2}$



<u>Proof</u>.



$\alpha_1=\operatorname{Arg}(1+i)$
$\alpha_2=\operatorname{Arg}(2+i)$
$\alpha_3=\operatorname{Arg}(3+i)$.

Now

$$\alpha_1+\alpha_2+\alpha_3 = \mathrm{Arg}\,(1+i) + \mathrm{Arg}\,(2+i) + \mathrm{Arg}\,(3+i) =$$
$$= \mathrm{Arg}\,(1+i)(2+i)(3+i) = \mathrm{Arg}\,(10i) = \frac{\pi}{2}. \qquad \square$$

If $z \neq 0$, then there are exactly n distinct solutions to the equation

$$\omega^n = z,$$

namely

$$\omega_k = |z|^{1/n} \cdot \left(\cos \frac{\mathrm{Arg}\, z + 2k\pi}{n} + i \sin \frac{\mathrm{Arg}\, z + 2k\pi}{n} \right),$$

for $k = 0, 1, 2, \ldots, n-1$.

Each such solution is called <u>n-th root of z</u>.

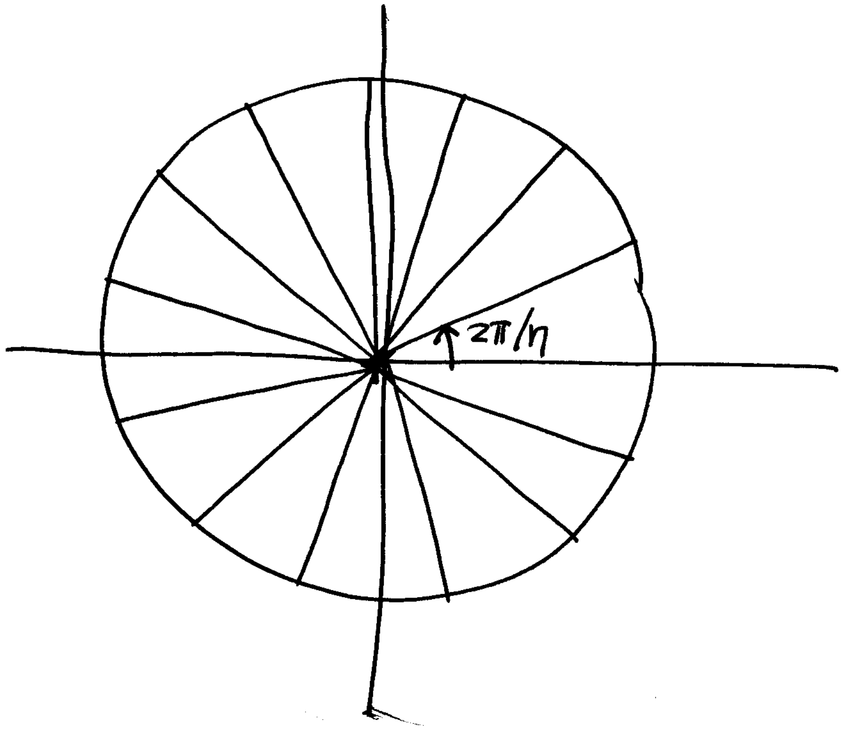
n-th rooths of unity are

$$\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$$

or

$$\zeta^k = \left(\cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n} \right)^k$$

for $k = 0, 1, 2, \ldots, n-1$. These numbers are nicely located on the unit circle



<u>Riemann splere</u> It is often that we add $+\infty$ and $-\infty$ to the set of real numbers.

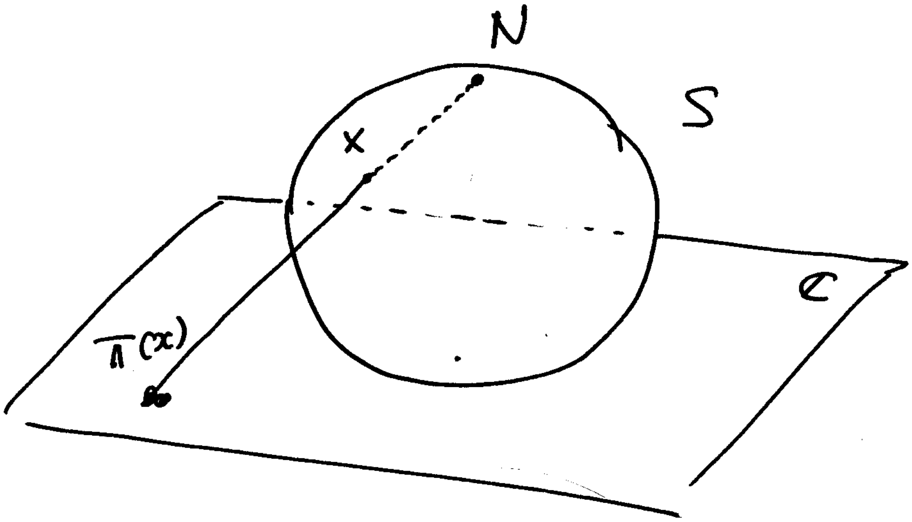
$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

However in the case of complex number we just add one point at infinity

$$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

Such a completed set of complex numbers is called <u>Riemann sphere</u> because of the interpretation that we are going to describe next.

Let S be a sphere that is tangent to the complex plane at the origin



and let N be its North Pole. Finally let $\pi : S \setminus \{N\} \to \mathbb{C}$ be the stereographic projection defined as on the picture.

$\pi : S \setminus \{N\} \to \mathbb{C}$ is a homeomorphism.

Note that

$$x_n \to N \text{ in } S \iff \pi(x_n) \to \infty \text{ i.e. } |\pi(x_n)| \to \infty .$$

Therefore we identify N with ∞, so

$\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = S$

and we also identify the topology in $\overline{\mathbb{C}}$ with the topology in S. With this identification

$$\{z : |z| > r\} \cup \{\infty\}$$

becomes a basis of neighborhoods of ∞.

<u>Complex derivative</u> Let $\Omega \subset \mathbb{C}$ be an open set and let $f : \Omega \to \mathbb{C}$ be a function. For $z_0 \in \Omega$ we define the <u>(complex) derivative</u> of f at z_0 as the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided the limit exists.

If $f'(z)$ exists for every $z \in \Omega$, then we say that f is <u>holomorphic</u> in Ω. The class of all holomorphic functions in Ω is denoted by $H(\Omega)$.

The following formulas are easy to prove

$$(f \pm g)' = f' \pm g' \quad (z_1 f)' = z_1 f'$$

$$(fg)' = f'g + fg', \quad (f/g)' = \frac{f'g - fg'}{g^2},$$

provided $g \neq 0$

$$(f \circ g)' = (f' \circ g) g'.$$

Examples

1. Constant function is holomorphic with derivative equal 0.
2. $z' = 1$.
3. The product rule gives
$$(z^n)' = n z^{n-1}, \quad n = 1, 2, 3, \ldots$$
4. The formula for $(1/g)'$ implies now
$$(z^n)' = n z^{n-1} \text{ for } n = -1, -2, \ldots, z \neq 0.$$
Hence z^n, $n = -1, -2, \ldots$ is holomorphic in $\mathbb{C} \setminus \{0\}$.
5. Formula 3 implies the familiar formula for the derivative of a polynomial. Polynomials are holomorphic.
6. The function $\bar{z}$ is not ~~holomorphic.~~ ~~the~~ differentiable (in the complex plane)

Functions $f: \mathbb{C} \to \mathbb{C}$ which are holomorphic in $\mathbb{C}$ are called <u>entire functions</u>.

Cauchy - Riemann equations

Complex functions $f: \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{C}$ can also be regarded as mappings from a subset of $\mathbb{R}^2$ into $\mathbb{R}^2$ and hence beside the complex derivative of f we can talk also about the derivative of a mapping f as was discussed in Advanced Calculus.

A mapping $f: \Omega \to \mathbb{R}^2$, $\Omega \subset \mathbb{R}^2$ is differentiable in the real sense at $z_0 \in \Omega$ if there is a linear transformation $L: \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{|f(z_0+z) - f(z_0) - Lz|}{|z|} \xrightarrow[|z| \to 0]{} 0$$

L is called derivative of f at z_0 and denoted by $Df(z_0)$.

We will show now that if f is differentiable in the complex sense at z_0, then it is also differentiable in the real sense.

f is differentiable in the complex sense at z_0 if there is a constant number, denoted by $f'(z_0)$ such that

$$\frac{f(z+z_0)-f(z_0)}{z} \longrightarrow f'(z_0) \text{ as } z \to 0.$$

This condition is equivalent to

$$\frac{|f(z+z_0)-f(z_0)-f'(z_0)z|}{|z|} \xrightarrow[|z|\to 0]{} 0$$

Therefor f is differentiable in the real sense and the linear mapping L is given by the multiplication by the complex number $f'(z_0)$

$$z \longmapsto f'(z_0)z.$$

Let us characterize now those linear transformations of $\mathbb{R}^2$ that are given by the multiplication by a complex number.

Let $z_1 = (a,b) \in \mathbb{C}$ and consider the linear mapping $A: \mathbb{C} \to \mathbb{C}$ given by

$$z \overset{A}{\longmapsto} z_1 z$$

We can regard it as a linear mapping $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$. It is given by a multiplication of a vector by a matrix. We will find the matrix.

$z = x + iy$

$$Az = (a+bi)(x+iy) = (ax - by,\ ay + bx)$$

$$= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

On the other hand if $A : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\alpha = \delta$ and $\beta = -\gamma$, then this transformation is equivalent with the multiplication by $z_1 = \alpha + i\gamma$.

We proved

<u>Proposition</u> Linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ with the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is equivalent with the transformation of the form

$$z \longmapsto z_1 z$$

for some $z_1 \in \mathbb{C}$ if and only if $\alpha = \delta$ and $\beta = -\gamma$. Then $z_1 = \alpha + i\gamma$.

As we already observed, function $f: \Omega \to \mathbb{C}$ is differentiable in the complex sense if and only if it is differentiable in the real sense and the real derivative is a linear mapping of $\mathbb{R}^2$ given by the multiplication by a complex number. According to the proposition it happens if and only if

$$f = u + iv = (u, v), \quad Df(z_0) = \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix}$$

satisfies

$$u_x(z_0) = v_y(z_0)$$
$$u_y(z_0) = -v_x(z_0)$$

Then it is multiplication by $u_x(z_0) + i v_x(z_0)$.

We proved

Theorem A function $f: \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{C}$ is complex differentiable at z_0 if and only if it is real differentiable at z_0 and

$$(*) \quad \begin{cases} u_x(z_0) = v_y(z_0) \\ u_y(z_0) = -v_x(z_0) \end{cases}$$

Then $f'(z_0) = u_x(z_0) + i v_x(z_0)$

$(*)$ are called Cauchy–Riemann equations.

Example The function $f(z) = \bar{z}$ is differentiable in the real sense but it is not differentiable in the complex sense. Indeed,

$$f(x,y) = (x,-y), \quad \text{so}$$

$$Df(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and therefore the Cauchy-Riemann equations are not satisfied.

Another approach to Cauchy-Riemann equations

Let us introduce the following notation

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

It is eachy to check by a direct computation that if $f: \Omega \to \mathbb{C}$ is differentiable in the real sense, then

$$Df(z_0)z = \frac{\partial f}{\partial z} z + \frac{\partial f}{\partial \bar{z}} \bar{z}$$

matrix multiplication

complex numbers multiplication

If f is differentiable in the complex sense, then

$$Df(z_0)\, z = z_1 \, z$$

matrix multiplication — complex numbers multiplication

for some $z_1 \in \mathbb{C}$. Comparing the two formulas we get

$$\left(z_1 - \frac{\partial f}{\partial z}\right) z = \frac{\partial f}{\partial \bar{z}}\, \bar{z}$$

for erey $z \in \mathbb{C}$. Taking $z = 1$ and $z = i$ we obtain

$$z_1 - \frac{\partial f}{\partial z} = \frac{\partial f}{\partial \bar{z}}$$

and

$$\left(z_1 - \frac{\partial f}{\partial z}\right) i = \frac{\partial f}{\partial \bar{z}}(-i)$$

or

$$z - \frac{\partial f}{\partial z} = - \frac{\partial f}{\partial \bar{z}}$$

This, however, implies that

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad \text{and} \quad z_1 = \frac{\partial f}{\partial z}$$

If we substitute f by $f = u + iv$ in the equation

$$(*) \qquad \frac{\partial f}{\partial \bar{z}} = 0$$

then after simple calculations we arrive at

$$u_x(z_0) = v_y(z_0) \quad \text{and} \quad u_y(z_0) = -v_x(z_0)$$

which is the familiar system of Cauchy-Riemann equations. Accordingly, (*) is an equivalent way to write the Cauchy-Riemann equations.

Also note that

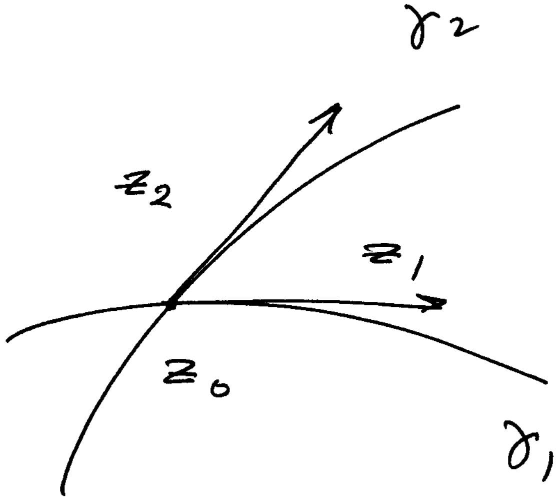
$$f'(z_0) = \frac{\partial f}{\partial z}(z_0).$$

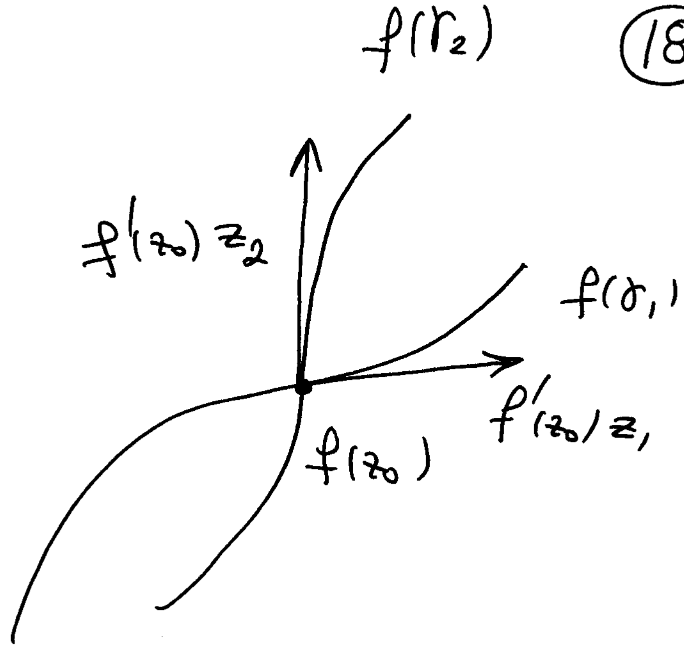
Conformal mappings

Suppose that $f: \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{C}$ is complex differentiable at $z_0 \in \Omega$. Assume also that $f'(z_0) \neq 0$.

Consider two smooth curves that intersect at z_0. Denote them by γ_1 and γ_2.

We claim that the angle at which γ_1 and γ_2 intersect at z_0 is the same as the angle at which the curves $f(\gamma_1)$ and $f(\gamma_2)$ intersect at $f(z_0)$. Moreover the two angles have the same orientation.





By the definition, the angle between curves is the angle between the tangent vectors to the curves. If the tangent vectors to γ_1 and γ_2 at z_0 are z_1 and z_2 respectively, then

$f'(z_0)z_1$ and $f'(z_0)z_2$ are tangent vectors

to $f(\gamma_1)$ and $f(\gamma_2)$ at $f(z_0)$.

Both vectors $f'(z_0)z_1$ and $f'(z_0)z_2$ are obtained from z_1 and z_2 by rotation by the same angle $(\operatorname{Arg} f'(z_0))$ and then by dilation by $|f'(z_0)|$. Therefore

$$\measuredangle z_1 z_2 = \measuredangle f'(z_0)z_1,\ f'(z_0)z_2$$

and the orientation is preserved.

This geometric interpretation also easily implies that

$$\det Df(z_0) = |f'(z_0)|^2 \neq 0.$$

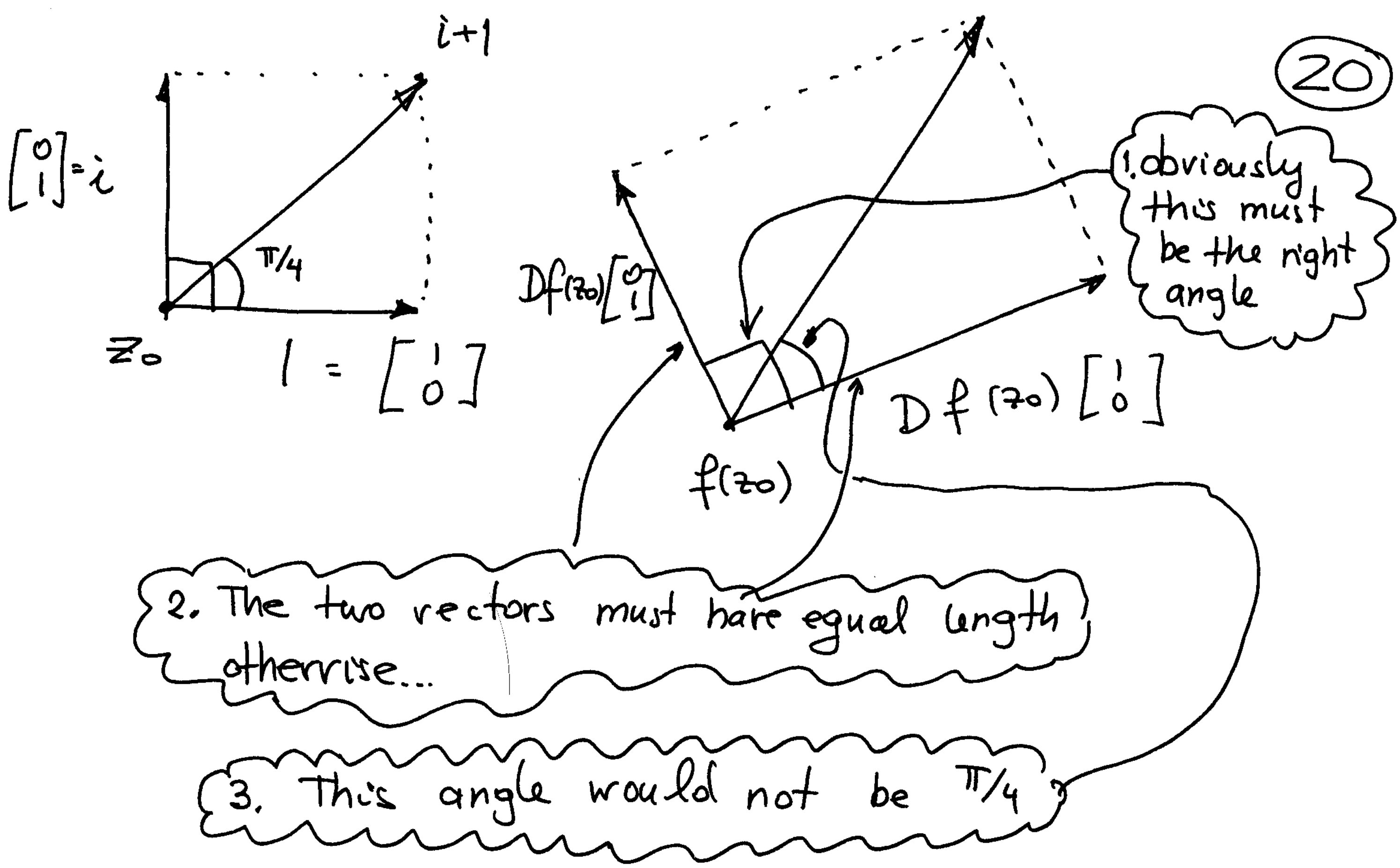
Actually a converse statement is also true. Namely, we have

<u>Theorem</u> Let $f: \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{C}$ be differentiable in the real sense and let $\det Df(z_0) \neq 0$ Then the following conditions are equivalent

(1) f preserves angles and orientation at z_0

(2) f is complex differentiable at z_0

We have already proved the implication (2) $\Rightarrow$ (1) and therefore we are left with the proof of (1) $\Rightarrow$ (2).



Denote the common length of the vectors

$$Df(z_0)\begin{bmatrix}0\\1\end{bmatrix} = Df(z_0)\, i$$

and

$$Df(z_0)\begin{bmatrix}1\\0\end{bmatrix} = Df(z_0)\cdot 1$$

by λ. Since they form the right angle, the same angle as the one between the vectors 1 and i, they were obtained from 1 and i by rotation by the same angle α and then by dilation by λ.

We can write it a

$$Df(z_0)\,1 = \lambda\,(\cos\alpha + i\sin\alpha)\,1$$

$$Df(z_0)\,i = \lambda\,(\cos\alpha + i\sin\alpha)\,i$$

This, however, easily implies that

$$Df(z_0)\,z = \lambda\,(\cos\alpha + i\sin\alpha)\,z$$

for evey z. Now complex differentiability follows with

$$f'(z_0) = \lambda\,(\cos\alpha + i\sin\alpha).$$

Function series

Consider the series

$$\sum_{n=1}^{\infty} f_n = f_1 + f_2 + \ldots$$

where the f_i's are complex valued functions defined on $A \subset \mathbb{C}$. We say that the series is convergent pointwise if for evey $z \in A$ the series of complex numbers

$$\sum_{n=1}^{\infty} f_n(z)$$

is convergent. Denote the sum by

$g(z) = \sum_{n=1}^{\infty} f_n(z)$. We say that the

series is convergent <u>uniformly</u> if

$$\sup_{z \in A} \left| g(z) - \sum_{n=1}^{k} f_n(z) \right| \longrightarrow 0$$

as $k \to \infty$.

<u>Theorem</u> If $|f_n(z)| \le a_n$ for all $z \in A$ and all $n = 1, 2, 3, \dots$ and if $\sum_{n=1}^{\infty} a_n < \infty$, then the series $f_1 + f_2 + \dots$ is uniformly convergent on A.

<u>Theorem</u> If the series $f_1 + f_2 + \dots$ of continuous functions is uniformly convergent on A, then the sum of the series is a continuous function on A.

Power series

Series of the form $\sum_{n=0}^{\infty} a_n z^n$ are called power series. They will play central role in the theory of holomorphic functions.

Let

$$(*) \qquad \sum_{n=0}^{\infty} a_n z^n$$

be a power series. It is easy to see that there is $R \in [0, \infty]$ such that

$$|z| < R \implies (*) \text{ is convergent}$$
$$|z| > R \implies (*) \text{ is divergent.}$$

Number R is called <u>radius of convergence</u>.

The disc $D(0,R) = \{z : |z| < R\}$ will be called <u>disc of convergence</u>

Nothing can be said about the convergence of $(*)$ for $|z| = R$ for a general series,

<u>Proposition</u> If the series (*) has the radius of convergence $R > 0$ and if $0 < \varrho < R$, then the series converges uniformly in $\overline{D}(0, \varrho) = \{z : |z| \leq \varrho\}$.
In particular the sum of a power series is a continuous function inside the disc of convergence.

The following theorem provides a formula for the radius of convergence.

<u>Theorem (Cauchy - Hadamard)</u> The radius of convergence of (*) is given by

$$R = \frac{1}{\lambda}, \quad \text{where } \lambda = \limsup_{n \to \infty} \sqrt[n]{|c_n|}$$

If $\lambda = 0$, then $R = \infty$.
If $\lambda = \infty$, then $R = 0$.

<u>Proof</u> Let $|z| < R = \frac{1}{\lambda}$. Then there is γ such that

$$\frac{1}{|z|} > \gamma > \lambda = \limsup_{n \to \infty} \sqrt[n]{|c_n|}.$$

Hence
$$\frac{1}{|z|} > \gamma > \sqrt[n]{|c_n|} \quad \text{for all } n \geq n_0,$$
provided n_0 is sufficiently large. Hence
$$\gamma^n |z|^n \geq |c_n| |z|^n$$
and
$$\sum |c_n z|^n \leq \sum (\gamma |z|)^n \underset{\substack{\text{because} \\ \gamma |z| < 1}}{<} \infty$$

Therefore the series $\sum c_n z^n$ is convergent.

Let us now suppose that $|z| > R$. Then
$$\frac{1}{|z|} < \gamma_1 < \lambda$$
for some constant γ_1. That means there is a sequence of indices $n_1 \leq n_2 \leq n_3 \leq \dots$ such that
$$\gamma_1 < \sqrt[n_k]{|c_{n_k}|} \qquad k = 1, 2, 3, \dots$$

This, however, implies
$$|z|^{n_k} |c_{n_k}| > (|z| \gamma_1)^{n_k} \longrightarrow \infty,$$
so
$$z^{n_k} c_{n_k} \not\to 0$$
and hence the series (*) cannot converge.

<u>Theorem</u> The power series $\sum_{n=0}^{\infty} c_n z^n$ has the same radius of convergence R as $\sum_{n=1}^{\infty} n c_n z^{n-1}$. The sum of the series $\sum_{n=0}^{\infty} c_n z^n$ is a holomorphic function in the disc of convergence and

$$\left(\sum_{n=0}^{\infty} c_n z^n \right)' = \sum_{n=1}^{\infty} n c_n z^{n-1} \quad \text{for } |z|<R.$$

This theorem says that the power series can be differentiated term by term.

<u>Proof.</u> The equality of radii of convergence follows from the easy observation:

$$\limsup \sqrt[n-1]{n|c_n|} = \limsup \sqrt[n]{|c_n|}$$

Now let z_0 be an element of the disc of convergence. Then $|z_0| < \rho < R$ for some ρ.

Denote $f(z) = \sum_{n=0}^{\infty} c_n z^n$.

Note that for $z \neq z_0$

$$\frac{z^n - z_0^n}{z - z_0} = z^{n-1} + z_0 z^{n-2} + \dots + z_0^{n-1}.$$

For $|z| \leq \rho$ consider an auxiliary series

$$G(z) = \sum_{n=1}^{\infty} c_n \left(z^{n-1} + z_0 z^{n-2} + \dots + z_0^{n-1}\right).$$

Note that $\left|c_n\left(z^{n-1} + z_0 z^{n-2} + \dots + z_0^{n-1}\right)\right| \leq n|c_n|\rho^{n-1}$

Since the radius of convergence of $\sum n c_n z^{n-1}$ is $R > \rho$, we conclude that

$$\sum n |c_n| \rho^{n-1} < \infty$$

and therefore the series G converges uniformly in $\overline{D}(0, \rho) = \{z : |z| \leq \rho\}$.

In particular the function G is continuous at z_0. We have for $z \neq z_0$

$$\frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \frac{z^n - z_0^n}{z - z_0} = G(z) \longrightarrow$$

$$\xrightarrow[\text{as } z \to z_0]{} G(z_0) = \sum_{n=1}^{\infty} n c_n z_0^{n-1},$$

$$f'(z_0) = \sum_{n=1}^{\infty} n c_n z^{n-1} .$$

The proof is complete.

Let us recall the Abel theorem from advanced calculus

<u>Theorem</u> Suppose that $0 < R < \infty$ is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$, $a_n \in \mathbb{R}$ and that the series $\sum_{n=0}^{\infty} a_n R^n$ (or $\sum_{n=0}^{\infty} a_n (-R)^n$) is convergent as well. Then

$$\lim_{x \to R-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n$$

$$\left(\text{or} \quad \lim_{x \to -R+} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n (-R)^n \right).$$

Now we will formulate the counterpart of this theorem for power series in the complex plane

Theorem (Abel) Suppose that $\infty > R > 0$ is the radius of convergence of

$$f(z) = \sum_{n=0}^{\infty} c_n z^n.$$

Suppose also that the series

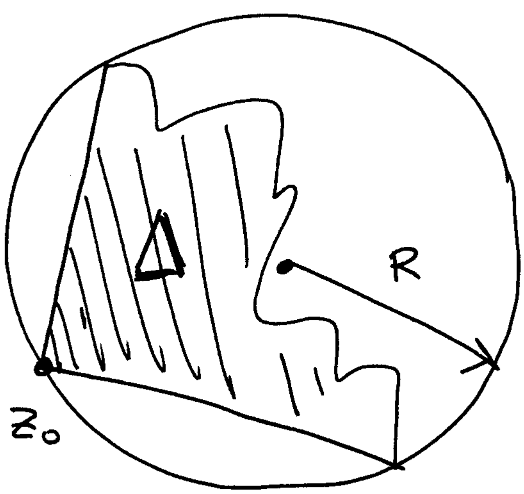
$$f(z_0) = \sum_{n=0}^{\infty} c_n z_0^n$$

is convergent for some z_0 with $|z_0| = R$.

Fix an arbitrary angle Δ at z_0 formed by two chords

Then

$$\lim_{\Delta \ni z \to z_0} f(z) = f(z_0).$$



We omit the proof.

Power series can be used to define holomorphic functions.

$$\boxed{e^{z} \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{z^n}{n!}}$$

Properties

① $e^{z_1} e^{z_2} = e^{z_1+z_2}$, $\quad z_1, z_2 \in \mathbb{C}$.

This follows from the formula for the multiplication of two series.

② $e^{z} \neq 0$ for all $z \neq 0$.

Indeed, $e^{z} e^{-z} = 1$, so $e^{z} \neq 0$.

③ $\overline{e^{z}} = e^{\bar{z}}$

④ $(e^{z})' = e^{z}$

$$\boxed{\begin{aligned} \cos z &\overset{\text{def}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \\ \sin z &\overset{\text{def}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \end{aligned}}$$

5. $\cos(-z) = \cos z$, $\sin(-z) = -\sin z$

6. $\sin' z = \cos z$, $\cos' z = -\sin z$

7. $e^{iz} = \cos z + i\sin z$,

$e^{-iz} = \cos z - i\sin z$

Therefore we have

8. Euler's formulas

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}, \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

Hence

9. $$\sin^2 z + \cos^2 z = 1.$$

there are, however, surprises

$$\cos i = \frac{e^{-1}+e}{2} > 1$$

$$|\sin i| = \left|\frac{e^{-1}-e}{2i}\right| > 1$$

10. If $z = x+iy$, then

$$e^z = e^x e^{iy} = e^x(\cos y + i\sin y).$$

(11) $e^z = 1 \iff z = 2k\pi i\,, \quad k \in \mathbb{Z}.$

It follows easily from (10). As a consequence of (11) we also have

(12) $e^{z_1} = e^{z_2} \iff z_2 = z_1 + 2k\pi i$

Therefore e^z is periodic with the period $2\pi i$.

(13) $\cos z = 0 \iff z = \frac{1}{2}(2k+1)\pi\,, \quad k \in \mathbb{Z}$

$\sin z = 0 \iff z = k\pi\,, \quad k \in \mathbb{Z}.$

$$\boxed{\tan z = \frac{\sin z}{\cos z}\,, \quad \operatorname{ctg} z = \frac{\cos z}{\sin z}}$$

<u>Logarithm.</u> For $a \in \mathbb{C} \setminus \{0\}$, every number $z \in \mathbb{C}$ such that

$$e^z = a \qquad (*)$$

is called <u>logarithm</u> of a and is denoted by $\log a$.

If $z = x+iy$, then $(*)$ implies

$$e^z = e^x e^{iy} = |a| e^{i \arg a}$$

and hence

$$e^x = |a| \quad \text{and} \quad y = \arg a.$$

Let $\mathrm{Log}\,|a|$ be a real number such that

$$e^{\mathrm{Log}\,|a|} = |a|$$

i.e. $\mathrm{Log}\,|a|$ is the standard real logarithm. Then ~~the~~ solutions z of $(*)$ are given by

$$z = \underbrace{\mathrm{Log}\,|a| + i \arg a}_{\log a} = \mathrm{Log}\,|a| + i\,\mathrm{Arg}\,a + 2k\pi i$$

We see that there are infinitely many solutions to $(*)$. The <u>principal value</u> of the logarithm is defined by

$$\mathrm{Log}\, a = \mathrm{Log}\,|a| + i\,\mathrm{Arg}\,a.$$

and hence

$$\log a = \mathrm{Log}\, a + 2k\pi i \qquad k \in \mathbb{Z}.$$

If $a \in \mathbb{R}$, then $\operatorname{Arg} a = 0$ and hence $\operatorname{Log} a$ is the standard real logarithm.

<u>Example</u> $\operatorname{Log} i = \operatorname{Log}|i| + i \operatorname{Arg} i = \frac{\pi}{2} i$, hence

$$\log i = \left(2k + \tfrac{1}{2}\right)\pi i, \quad k \in \mathbb{Z}$$

<u>Power</u> For $a, b \in \mathbb{C}$, $a \neq 0$ we define

$$a^b \overset{\text{def}}{=\!=} e^{b \log a} = e^{b(\operatorname{Log} a + 2k\pi i)}, \quad k \in \mathbb{Z}.$$

That shows that, in general, a^b is not uniquely defined. The number

$$e^{b \operatorname{Log} a}$$

is called <u>the principal value</u> of the power a^b

<u>Example</u> For $b = 1/n$ we have

$$a^{1/n} = e^{\frac{1}{n}(\operatorname{Log}|a| + (\operatorname{Arg} a + 2k\pi)i)}$$

$$= e^{\frac{1}{n}\operatorname{Log}|a|} \, e^{\frac{\operatorname{Arg} a + 2k\pi}{n} i}$$

$$= \sqrt[n]{|a|}\left(\cos \frac{\operatorname{Arg} a + 2k\pi}{n} + i \sin \frac{\operatorname{Arg} a + 2k\pi}{n}\right) \quad (*)$$

$k \in \mathbb{Z}$

Note that although $k \in \mathbb{Z}$ we only have n distinct values for $k = 0, 1, 2, \ldots, n-1$, because the values for k and $k+n$ are the same.

(*) are all nth roots of a.

<u>Example</u> $i^i = e^{i \log i} \underset{\text{(previous examples)}}{\overset{\text{one of the}}{=}} e^{-(2k+\frac{1}{2})\pi}$, $k \in \mathbb{Z}$. The principal value of i^i is $e^{-\pi/2}$.

<u>Branch of the logarithm</u> Let $E \subset \mathbb{C} \setminus \{0\}$.

Every function $\ell(z)$, continuous on E and such that

$$e^{\ell(z)} = z \quad \text{for all } z \in E$$

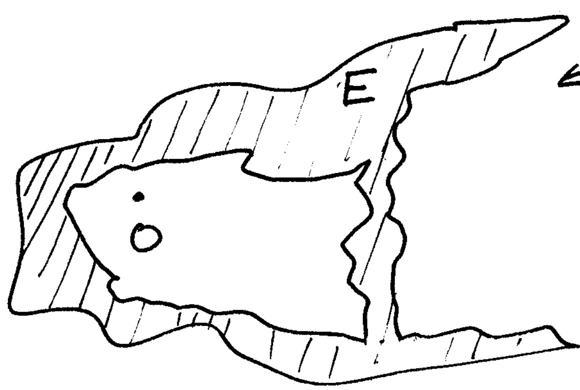
will be called a <u>branch of the logarithm</u>.

The branch of the logarithm does not always exist. For example if E contains a circle of radius r, $|z| = r$, then the branch does not exist on E. Indeed, along the circle we have

$$\log z = \mathrm{Log}\, r + i \arg z.$$

If we move along the circle (one revolution), then the value of arg z, assuming it changes in a continuous way, will increase by 2π, so it will ~~to~~ not be equal to the initial value – – contradiction.

This example can be generalized: the branch of logarithm does not exist if we can go around the origin along the set E (not necessarily along a circle)



← no branch of logarithm in E.

Properties:

① If in a connected set there is a branch of the logarithm, then there are infinitely many branches. Every two such branches differ by a multiplicity of $2\pi i$

(2) Any branch of logarithm is a one-to-one function. Indeed, suppose that $\log z_1 = \log z_2$. Then

$$z_1 = e^{\log z_1} = e^{\log z_2} = z_2.$$

(3) If there is a branch of logarithm in a set E, then it is the inverse function to e^z defined on the set $E' = \log(E)$.

Branch of power function

Let $\mu \in \mathbb{C}$. Suppose that there is a branch of logarithm in the set E. By z^μ we will denote any function

$$z^\mu = e^{\mu \log z},$$

where $\log z$ is an arbitrary branch of logarithm in E.

If $\mu = 1/n$, then we often write $\sqrt[n]{z}$ instead of $z^{1/n}$.

Differentiability of the logarithm.

It is easy to see that e^z maps the strip

$$\Delta = \{ z : 0 < \operatorname{im} z < 2\pi \}$$

in a diffeomorphic way onto $\mathbb{C} \setminus \mathbb{R}_+$

It follows directly from the definition that the principal value of the Logarithm

$$\operatorname{Log} : \mathbb{C} \setminus \mathbb{R}_+ \longrightarrow \Delta$$

is the invers mapping to e^z defined in Δ. Hence $\operatorname{Log} z$ is differentiable in the complex sense and

$$e^{\operatorname{Log} z} = z , \quad \left(e^{\operatorname{Log} z} \right)' = 1$$

$$\underbrace{e^{\operatorname{Log} z}}_{z} \left(\operatorname{Log} z \right)' = 1 , \quad \left(\operatorname{Log} z \right)' = \frac{1}{z} .$$

Therefore we proved the following result.

Theorem The function $z \mapsto \operatorname{Log} z$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}_+$. It maps this set in a diffeomorphic way onto the strip $\Delta = \{ z : 0 < \operatorname{im} z < 2\pi \}$.
Moreover

$$\operatorname{Log}' z = \frac{1}{z}$$

We will show now how to expand Log as a power series.

The function $z \mapsto \mathrm{Log}(1-z)$ is holomorphic in the unit disc $D(0,1)$. This easily follows from the above theorem. Moreover

$$\mathrm{Log}'(1-z) = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n = \left(-\sum_{n=..}^{\infty} \frac{z^n}{n}\right)'$$

Therefore

$$\mathrm{Log}(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{for} \quad |z|<1.$$

Holomorphic functions $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$

We know that

$D(z_0, r)$ is a neighborhood of z_0 and

$\{|z|>r\} \cup \{\infty\}$ is a neighborhood of ∞

The sets $D(z_0, r) \setminus \{z_0\}$ and $\{|z|>r\}$ will be called punctured neighborhoods of z_0 and ∞ respectively.

Suppose we are given a continuous function on the Riemann sphere

$$f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}.$$

Suppose that $f(\infty) = z_0 \in \mathbb{C}$. Next, suppose that f is holomorphic in a punctured neighborhood of ∞ i.e. it is holomorphic in the set $\{z : |z| > r\}$, for some $r > 0$.

Consider the function

$$g(z) = \begin{cases} f(1/z) & \text{for } z \neq 0 \\ f(\infty) & \text{for } z = 0 \end{cases}$$

Then the function g is holomorphic in some punctured neighborhood of 0 and continuous at 0.

Indeed, the transformation $z \mapsto 1/z$ maps punctured neighborhoods of ∞ onto punctured neighborhoods of 0 and maps holomorphic functions onto holomorphic functions in the punctured neighborhoods.

Therefore the following definition is very natural

<u>Definition</u> The function f as above is holomorphic at if the function g is holomorphic at 0.

There are other situations when the definition of a holomorphic function at z_1 needs to be clarified. Namely: (1) $z_1 \in \mathbb{C}$ and $f(z_1) = \infty$, (2) $z_1 = \infty$ and $f(z_1) = \infty$.

Let $f : \Omega \to \overline{\mathbb{C}}$, where $\Omega \subset \overline{\mathbb{C}}$ is open, be a continuous mapping. We say that f is complex differentiable at $z_0 \in \Omega$ if one of the following conditions is satisfied

(1) $z_0 \neq \infty$, $f(z_0) \neq \infty$ – classical definition in this case

(2) $z_0 = \infty$, $f(z_0) \neq \infty$: The function

$$g(z) = \begin{cases} f(1/z), & \text{for } z \neq 0 \\ f(\infty) & \text{for } z = 0 \end{cases}$$

is differentiable at 0

(3) $z_0 \neq \infty$, $f(z_0) = \infty$: the function

$$g(z) = \begin{cases} \frac{1}{f(z)} & \text{for } z \neq z_0 \\ 0 & \text{for } z = z_0 \end{cases}$$

is defferentiable at

(4) $z_0 = \infty$, $f(z_0) = \infty$: the function

$$g(z) = \begin{cases} 1/f(1/z) & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$$

is differentiable at 0.

Functions of the form $\omega(z) = P(z)/Q(z)$, where P and Q are polynomials are called rational functions.

<u>Theorem</u> Rational functions define holomorphic functions on the Riemann sphere

$$\omega: \overline{\mathbb{C}} \to \overline{\mathbb{C}}.$$

Later we will prove a much stronger result. $\omega: \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is holomorphic if and only if ω is rational.

<u>Proof of the theorem</u> Let

$$\omega(z) = \frac{P(z)}{Q(z)} = A\,\frac{(z-z_1)\dots(z-z_k)}{(z-\omega_1)\dots(z-\omega_n)}$$

We can assume that $A \neq 0$ and $\omega_i \neq z_j$. We employed here the fundamental theorem of algebra. Actually, the function is defined for $z \in \mathbb{C} \setminus \{\omega_1, \dots, \omega_n\}$. However

$$\lim_{z \to \omega_i} \omega(z) = \infty$$

and therefore it is natural to define $\omega(\omega_i) = \infty$. Moreover we set

$$\omega(\infty) = \lim_{z \to \infty} \omega(z) = \begin{cases} A & \text{if } k = n \\ \infty & \text{if } k > n \\ 0 & \text{if } k < n \end{cases}$$

In this way
$$\omega : \bar{\mathbb{C}} \longrightarrow \bar{\mathbb{C}}$$
is a continuous function. It is holomorphic in $\mathbb{C} \setminus \{\omega_1, \dots, \omega_k\}$. We need to prove that ω is complex differentiable at the ω_i's and at ∞.

To prove differentiability at ω_i we need to prove differentiability of
$$\frac{1}{\omega(z)} = \frac{1}{A} \frac{(z-\omega_1)\dots(z-\omega_n)}{(z-z_1)\dots(z-z_k)}$$
at ω_i, and it is obvious.

We are left with the proof of differentiability at ∞. To this end we need prove differentiability of
$$\omega\left(\tfrac{1}{z}\right) = A \frac{\left(\frac{1}{z}-z_1\right)\dots\left(\frac{1}{z}-z_k\right)}{\left(\frac{1}{z}-\omega_1\right)\dots\left(\frac{1}{z}-\omega_n\right)}$$
at 0. Note that the function $z \mapsto \omega\left(\frac{1}{z}\right)$ is rational and therefore it is differentiable at any finite point by previous arguments.

Fractional Linear Transformation

It is a transformation of the form
$$L(z) = \frac{az+b}{cz+d}, \text{ where } ad-bc \neq 0.$$

Theorem Fractional linear transformation is the one-to-one holomorphic transformation of $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$.

Proof It is holomorphic, because fractional linear transformations are rational functions. It is one-to-one and onto because L has the inverse transformation (Exercise: find it).

Fractional linear transformations (FLT) form a group: If L_1 and L_2 are FLT, then $L_1 \circ L_2$ is FLT and L_1^{-1} is FLT.

The coefficients a, b, c, d are not uniquely defined because

$$\frac{az+b}{cz+d} = \frac{(Aa)z+(Ab)}{(Ac)z+(Ad)}.$$

Choosing suitable A we can always find coefficients a, b, c, d for a given FLT, such that

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc = 1.$$

$SL(2, \mathbb{C})$ denotes the group of all complex 2×2 matrices with the determinant 1.

$\mathbb{Z}_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$ is a subgroup of $SL(2, \mathbb{C})$

We set $PSL(2, \mathbb{C}) = SL(2, \mathbb{C}) / \mathbb{Z}_2$.

Theorem The mapping

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \frac{az+b}{cz+d}$$

is a homomorphism of the group $SL(2,\mathbb{C})$ onto the group of fractional linear transformations. The kernel is

$$\mathbb{Z}_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

and therefore the group of fractional linear transformations is isomorphic to $PSL(2,\mathbb{C})$

Proof To see that it is homomorphism note that (1)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longmapsto z$$

so the identity of $SL(2,\mathbb{C})$ is mapped onto the identity transformation, and

(2)

$$A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \quad A_1 \circ A_2 = \begin{bmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{bmatrix}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$L_1 = \frac{a_1 z + b_1}{c_1 z + d_1}, \quad L_2 = \frac{a_2 z + b_2}{c_2 z + d_2}, \quad L_1 \circ L_2 = \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)}.$$

It is obvious that the mapping is onto the group of FLT and that the kernel is $\mathbb{Z}_2$.

Therefore the theorem follows.

One can prove that stereographic projection maps circles on the sphere onto the circles or lines on the plane. Lines correspond to the circles that pass through the North Pole on the sprere. Therefore it is natural to regard lines as circles passing through infinity. Thus

By a circle on $\overline{\mathbb{C}}$ we mean a circle or a line on $\mathbb{C}$.

<u>Theorem</u> Fractional linear transformations map circles on $\overline{\mathbb{C}}$ onto circles on $\overline{\mathbb{C}}$.

<u>Proof</u> Every fractional linear transformation

$$L(z) = \frac{az+b}{cz+d}$$

is a composition of

(a) translations $z \longmapsto z+b$

(b) rotations $z \longmapsto az, \quad |a|=1$

(c) dilations $z \longmapsto rz, \quad r>0$

(d) inversions $z \longmapsto 1/z$.

Indeed, it is obrious when $c=0$, and when $c\neq 0$ it follows from the formula

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{\lambda}{cz+d}, \quad \lambda = \frac{bc-ad}{c}.$$

Obriously transformations (a), (b) and (c) preserre circles and lines. Therefre it remainds to prove that inversion transforms circles on $\overline{\mathbb{C}}$ onto circles on $\overline{\mathbb{C}}$.

Easy algebraic calculation shows that every circle or line on $\mathbb{C}$ is a set of points satisfying the equation

$$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0 \qquad (*)$$

where $\alpha, \gamma \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $\beta\bar{\beta} > \alpha\gamma$. If $\alpha = 0$, then it is a line and it is a circle when $\alpha \neq 0$.

Now note that the inversion transform the equation $(*)$ onto

$$\alpha \frac{1}{z}\frac{1}{\bar{z}} + \beta\frac{1}{z} + \bar{\beta}\frac{1}{\bar{z}} + \gamma = 0$$

which is equivalent to

$$\gamma z\bar{z} + \bar{\beta} z + \beta\bar{z} + \alpha = 0,$$

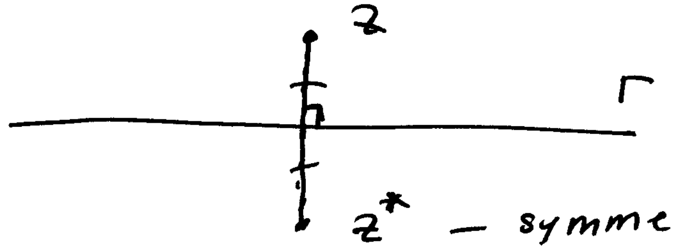
and it is an equation of the same type as $(*)$, so it describes a circle on $\overline{\mathbb{C}}$.

The proof is complete.

Symmetry with respect to a circle

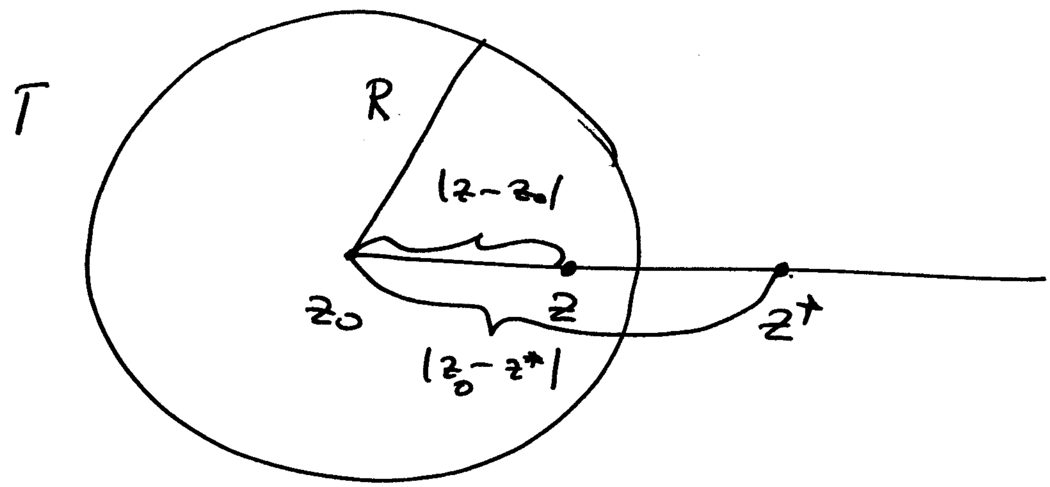
Let Γ be a circle on $\overline{\mathbb{C}}$ i.e. circle or a line on $\mathbb{C}$.

Symmetric points are understood in an standard way



z^* – symmetric with respect to z

In the case of T being a circle we define

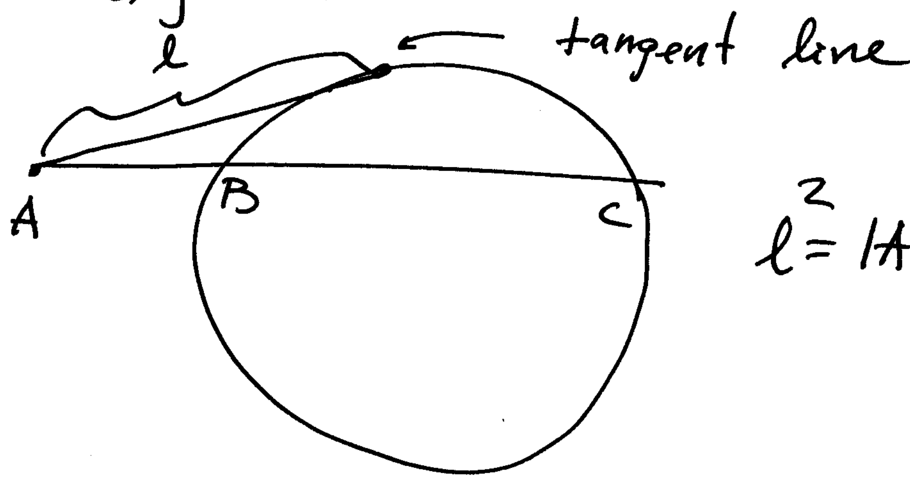


Points z and z^* are symmetric with respect to a circle T on $\mathbb{C}$ if they are on the same half line starting at the center of the circle and

$$|z-z_0|\,|z^*-z_0| = R^2.$$

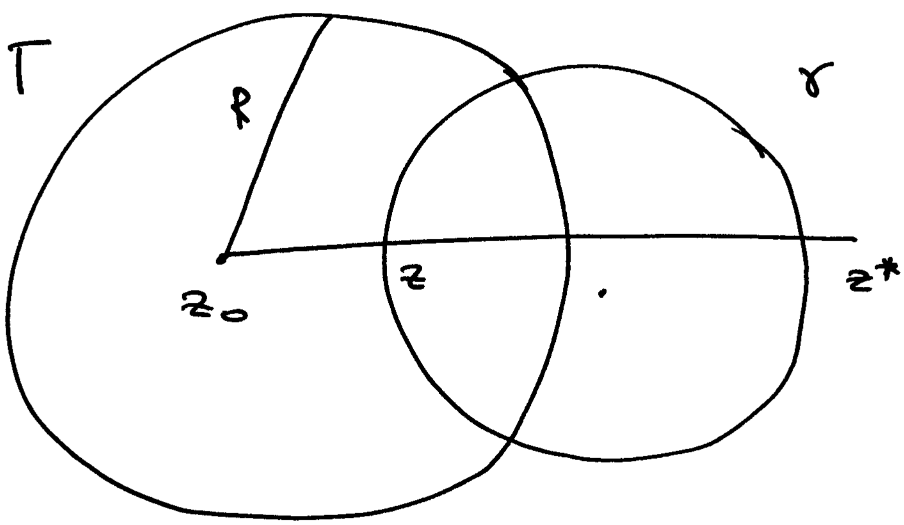
Lemma Points z and z^* are symmetric with respect to a circle Γ on $\overline{\mathbb{C}}$ if and only if every circle on $\overline{\mathbb{C}}$ passing through z and z^* is orthogonal to Γ.

Proof. It is known from an elementary geometry that



$$l^2 = |AB|\,|AC|$$

In our situation



Circle γ passing through z and z^* will be orthogonal to Γ iff the tangent line from z_0 to γ will have the length $l = R$.

i.e.

$$|z_0 - z|\,|z_0 - z^*| = l^2 = R^2.$$

<u>Theorem</u> If L is a fractional linear transformation, T a circle on $\overline{\mathbb{C}}$ and z, z^* are points symmetric with respect to T, then $L(z)$ and $L(z^*)$ are symmetric with respect to $L(T)$.

<u>Proof</u> L is a conformal mapping because

$$\left(\frac{az+b}{cz+d}\right)' = \frac{ad-bc}{(cz+d)^2} \neq 0$$

and therefore it maps circles perpendicular to T onto circles perpendicular to $L(T)$.

<u>Theorem</u> For every distinct $z_1, z_2, z_3 \in \overline{\mathbb{C}}$ and distinct $\omega_1, \omega_2, \omega_3 \in \overline{\mathbb{C}}$ there exists exactly one fractional linear transformation L such that

$$L(z_i) = \omega_i, \quad i = 1, 2, 3.$$

<u>Proof</u> First we prove existence.

$$L_1(z) = \frac{z - z_1}{z - z_2} \; \frac{z_3 - z_2}{z_3 - z_1}$$

is a FLT such that $L_1(z_1) = 0$, $L_1(z_2) = \infty$, $L_1(z_3) = 1$.

Similarly we define a FLT L_2 such that

$L_2(\omega_1) = 0$, $L_2(\omega_2) = \infty$ and $L_2(\omega_3) = 1$.

Now $L = L_2^{-1} \circ L_1$ satisfies $L(z_i) = \omega_i$, $i = 1, 2, 3$.

To prove uniqueness let λ be a FLT such that

$$\lambda(z_i) = \omega_i, \quad i = 1, 2, 3.$$

Then $\mu = L_2 \circ \lambda \circ L_1^{-1}$ satisfies

$$\mu(0) = 0, \quad \mu(\infty) = \infty \text{ and } \mu(1) = 1.$$

It easy follows that $\mu(z) \equiv z$ and therefore

$$\mu = L_2^{-1} \circ L_1.$$

Since three points on $\overline{\mathbb{C}}$ uniquely define a circle on $\overline{\mathbb{C}}$ we obtain as a corollary

<u>Theorem</u> Every circle on $\overline{\mathbb{C}}$ can be mapped onto another arbitrary circle on $\overline{\mathbb{C}}$ by a fractional linear transformation.

<u>Example</u> Find all fractional linear transformations L such that

$$L(D(0,1)) = D(0,1).$$

<u>Solution</u> Let a, $|a|<1$ be such that $L(a) = 0$. Since $a^* = \frac{1}{\bar{a}}$, is a symmetric point, we conclude that $L(1/\bar{a}) = \infty$. Therefore L must be of the form

$$L(z) = k\,\frac{z-a}{z-1/\bar{a}} = k_1\,\frac{z-a}{1-\bar{a}z},$$

where k and k_1 are some constants.
Since $|L(1)| = 1$ we have

$$|k_1|\left|\frac{1-a}{1-\bar{a}}\right| = |k_1| = 1, \text{ so } k_1 = e^{i\theta}$$

and finally

$$L(z) = e^{i\theta}\,\frac{z-a}{1-\bar{a}z} \qquad (*)$$

for some $\theta \in \mathbb{R}$ and $|a|<1$. It is easy to see that mappings of the form $(*)$ satisfy $L(D(0,1)) = D(0,1)$.

Later we will prove that if $f: D(0,1) \to D(0,1)$ is a holomorphic diffeomorphism of $D(0,1)$ onto $D(0,1)$, then f must be of the form $(*)$ which is a much stronger statement from what we have just proved.

Line integrals

By a curve we mean any continuous mapping $\gamma : [a, b] \to \mathbb{C}$. We say that a curve is closed if $\gamma(a) = \gamma(b)$.

Jordan curve is a closed curve $\gamma : [a, b] \to \mathbb{C}$ such that it is one-to-one on $[a, b)$, i.e. Jordan curve is a closed curve with no self-intersections. Often a Jordan curve is identified with its image $\Gamma = \gamma([a, b])$.

Theorem (Jordan) A Jordan curve Γ divides $\mathbb{C}$ into two disjoint domains: Ω (bounded) and Ω' (unbounded) and Γ is the common boundary of Ω and Ω'. Any curve connecting $z_1 \in \Omega$ with $z_2 \in \Omega'$ intersects with Γ.

Piecewise smooth curve is a continuous curve $\gamma : [a, b] \to \mathbb{C}$ such that for some partition $a = t_0 < t_1 < \ldots < t_n < t_{n+1} = b$ the mappings $\gamma|_{[t_i, t_{i+1}]} : [t_i, t_{i+1}] \to \mathbb{C}$

are of class C^1 and $\gamma'(t) \neq 0$ for all $t \in [t_i, t_{i+1}]$ (at the endpoints we consider one sided derivatives).

Unless otherwise assumed all the curves will be piecewise smooth.

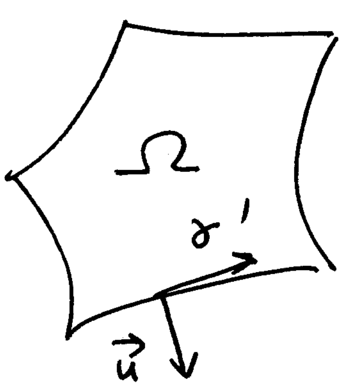
If γ is a (piecewise smooth) Jordan curve, then it divides $\mathbb{C}$ into a bounded domain Ω and unbounded domain Ω'. The domain Ω is called interior of the curve γ. We say that γ is positively oriented if the pairs of the vectors

$$(\vec{u}, \vec{\gamma})$$

normal outward vector

tangent vector

has positive orientation. That means the positive orientation of the boundary is counterclockwise.



Example $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$ is a positively oriented curve with the interior $D(a, r)$.

Definition. Let $\gamma : [a,b] \to \mathbb{C}$ be a (piecewise smooth) curve and let f be a continuous function defined on the set $\Gamma = \gamma([a,b])$. We define

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

Properties

① $\displaystyle\int_\gamma (af(z) + bg(z))\,dz = a\int_\gamma f(z)\,dz + b\int_\gamma g(z)\,dz$

② The curve $-\gamma$ is defined as $-\gamma(t) = \gamma(a+b-t)$, so it has opposite orientation to γ. We have

$$\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz.$$

③ If $f(z) = u(z) + i v(z)$, $dz = dx + i dy$ then

$$\int_\gamma f(z)\,dz = \int_\gamma (u + iv)(dx + i dy) =$$

$$= \int_\gamma (u\,dx - v\,dy) + i \int_\gamma (v\,dx + u\,dy)$$

which shows the connection between the complex line integral $\int_\gamma f(z)\,dz$ and the line integrals studied in Calculus III.

④ If a curve $\gamma : [a,b] \to \mathbb{C}$ has a finite number of self-intersections, then its length is given by

$$\ell(\gamma) = \int_a^b |\gamma'(t)|\,dt$$

⑤ If $\gamma : [a,b] \to \mathbb{C}$ has finite number of self-intersections, then

$$\left| \int_\gamma f(z)\,dz \right| \leq \ell(\gamma) \sup_\Gamma |f|$$

<u>Proof.</u> $\left| \int_\gamma f(z)\,dz \right| = \left| \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \right|$

$$\leq \int_a^b |f(\gamma(t))|\,|\gamma'(t)| \leq \sup_\Gamma |f| \underbrace{\int_a^b |\gamma'(t)|\,dt}_{\ell(\gamma)} \qquad \square$$

<u>Theorem</u> If $f(z) = \sum_{n=0}^{\infty} f_n(z)$ is a uniformly convergent series of continuous functions on $\Gamma = \gamma([a,b])$, then

$$\int_\gamma f(z)\,dz = \sum_{n=0}^{\infty} \int_\gamma f_n(z)\,dz.$$

<u>Proof.</u> Since f is continuous, the integral on the left hand side exists. It remains to prove that

$$\int_\gamma f(z)\,dz - \sum_{n=0}^{k} \int_\gamma f_n(z)\,dz =$$

$$= \int_\gamma \left(f(z) - \sum_{n=0}^{k} f_n(z) \right) dz \xrightarrow[k\to\infty]{} 0.$$

We have

$$\left| \int_\gamma \left(f(z) - \sum_{n=0}^{k} f_n(z) \right) dz \right| \leq$$

$$\sup_\Gamma \left| f(z) - \sum_{n=0}^{k} f_n(z) \right| \ell(\gamma) \xrightarrow[k\to\infty]{} 0$$

because of the uniform convergence of the series. $\square$

Example Let $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$ be a positively oriented boundary of $D(a,r)$. Then

$$\int_\gamma (z-a)^n dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = 1 \end{cases}$$

Proof

$$\int_\gamma (z-a)^n dz = \int_0^{2\pi} (re^{it})^n \, ire^{it} dt =$$

$$= i r^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt$$

If $n \neq -1$

$$\int_0^{2\pi} e^{i(n+1)t} dt = \left. \frac{e^{it(n+1)}}{i(n+1)} \right|_0^{2\pi} = 0$$

If $n = -1$

$$\int_0^{2\pi} e^{i(n+1)t} dt = \int_0^{2\pi} dt = 2\pi.$$

Theorem The integral $\int_\gamma f(z)dz$ does not depend on the parametrization of γ, provided a new parametrization has the same orientation as γ. This is to say that if $\varphi : [a_1, b_1] \to [a, b]$, $\varphi(a_1) = a$, $\varphi(b_1) = b$ is a strictly increasing function of class C^1 and $\gamma_1(t) = \gamma(\varphi(t))$, then

$$\int_{\gamma_1} f(z)dz = \int_\gamma f(z)dz$$

The theorem easily follows from the change of variables in the integral.

Antiderivative. Let $f : \Omega \to \mathbb{C}$ be a function defined on a domain $\Omega \subset \mathbb{C}$. Every function $F \in H(\Omega)$ such that $F'(z) = f(z)$ for $z \in \Omega$ is called antiderivative of f.

If $\gamma : [a, b] \to \mathbb{C}$ is a curve, then the points $z_0 = \gamma(a)$ and $z_1 = \gamma(b)$ are called the beginning and the end of the curve γ.

<u>Theorem</u> If a continuous function $f : \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{C}$ has an antiderivative F in Ω, then the integral $\int_\gamma f(z)\,dz$ along a curve that is contained in γ depends only on the position of endpoints z_0 and z_1 and

$$\int_\gamma f(z)\,dz = F(z_1) - F(z_0).$$

In particular the integral equals zero if γ is closed

<u>Remark</u> In the situation described in the above theorem it makes sense to write

$$\int_\gamma f(z)\,dz = \int_{z_0}^{z_1} f(z)\,dz$$

because we can replace γ by any curve that connects z_0 to z_1 and the value of the integral will not change.

<u>Proof</u> Let $\Phi(t) = F(\gamma(t))$. We have

$$\Phi'(t) = F'(\gamma(t))\,\gamma'(t) = f(\gamma(t))\,\gamma'(t)$$

and hence

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt =$$

$$= \int_a^b \Phi'(t)\,dt = \Phi(b) - \Phi(a) = F(z_1) - F(z_0).$$

Example Let γ be a closed curve that does not pass through a point $a \in \mathbb{C}$. Then for $n \neq -1$

$$\int_\gamma (z-a)^n\,dz = 0.$$

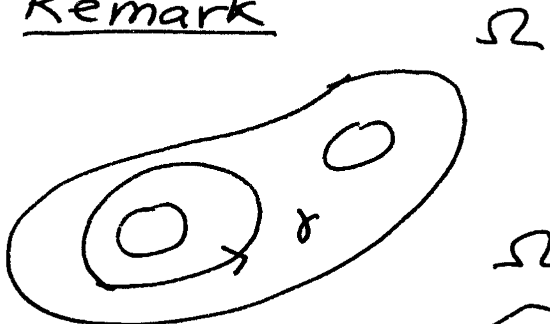
Indeed, $(z-a)^{n+1}/(n+1)$ is an antiderivative of $(z-a)^n$ in $\mathbb{C} \setminus \{a\}$.

Earier we proved this fact in a special case in which γ is a positively oriented boundary of $D(a,r)$.

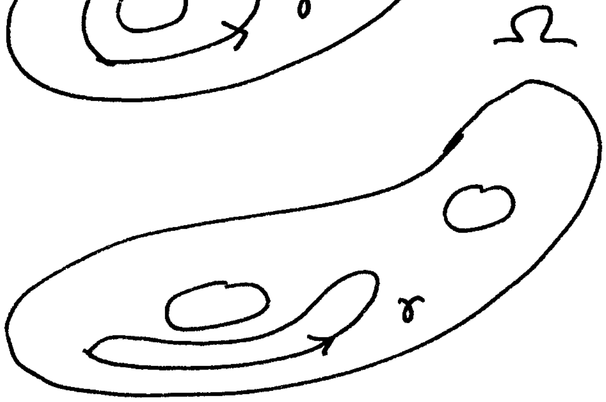
Theorem (Cauchy) Let $\gamma : [a,b] \to \Omega$ be a (piecewise smooth) Jordan curve such that the interior of γ is contained in Ω. Then for any $f \in H(\Omega)$

$$\int_\gamma f(z)\,dz = 0$$

Remark



γ is not good because there is a hole inside, so the interior of γ is not contained in Ω.



This curve γ is good.

First we prove the theorem under an additional condition that f has continuous partial derivatives and then we will provide another proof tha coveres the general case.

Proof under the assumption that f has continuous partial derivatives.

Let Δ be the interior of γ. We can assume that γ has positive orientation (if not we replace γ by $-\gamma$).

Let's recall that Green's theorem says that if P and Q have continuous partial derivatives in Ω, then

$$\int_\gamma P\,dx + Q\,dy = \iint_\Delta \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Let $f(z) = u(z) + i v(z)$. We assume that u and v have continuous partial derivatives.

We have

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+i\,dy) =$$

$$= \int_\gamma u\,dx - v\,dy + i\int_\gamma u\,dy + v\,dx =$$

$$= \iint_\Delta (-v_x - u_y)\,dx\,dy + i\iint_\Delta (u_x - v_y)\,dx\,dy$$

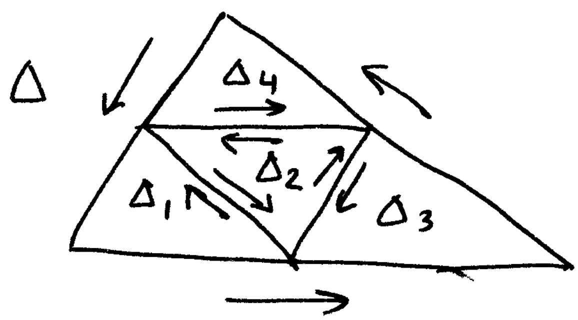
$= 0$ by the Cauchy-Riemann equations.

Proof in the general case

① First we will prove the theorem under the assumption that γ is a boundary of a triangle. Denote the interior of γ by Δ and write $\partial\Delta$ instead of γ. Let

$$J = \int_{\partial\Delta} f(z)\,dz.$$

We will prove that $J = 0$. Divide Δ into four congruent triangles $\Delta^1, \Delta^2, \Delta^3, \Delta^4$.



Then

$$J = \sum_{j=1}^{4} \int_{\partial\Delta^j} f(z)\,dz.$$

Thus there is a triangle among $\Delta^1, \Delta^2, \Delta^3, \Delta^4$, denote it by Δ_1 such that

$$\left| \int_{\partial\Delta_1} f(z)\,dz \right| \geq |J|/4$$

The perimeter of Δ_1 is $L/2$, where L stands for the perimeter of Δ.

Now we divide Δ_1 into four congruent triangles and proceed as above.

We obtain a sequence of triangles

$$\Delta \supset \Delta_1 \supset \Delta_2 \supset \Delta_3 \supset \dots$$

with the following properties

1) $$\left| \int_{\partial \Delta_n} f(z)\, dz \right| \geq |J| / 4^n$$

2) perimeter of $\Delta_n = L/2^n$.

Clearly

$$\bigcap_{n=1}^{\infty} \Delta_n = \{ z_0 \}.$$

The function f is differentiable at z_0. Hence for every $\varepsilon > 0$ there is $r > 0$ such that

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon |z - z_0| \qquad (*)$$

provided $|z - z_0| < r$.

The function $z \mapsto -f(z_0) - f'(z_0)(z - z_0)$ has an antiderivative, so

$$\int_{\partial \Delta_n} - f(z_0) - f'(z_0)(z - z_0)\, dz = 0$$

and thus

$$\int_{\partial \Delta_n} f(z)\, dz = \int_{\partial \Delta_n} \big(f(z) - f(z_0) - f'(z_0)(z - z_0) \big)\, dz \qquad (**)$$

There is n_0 such that $\Delta_n \subset D(z_0, r)$ for $n \geq n_0$ and hence (*) and (**) yields

$$\left| \int_{\partial \Delta_n} f(z)\,dz \right| \leq \varepsilon (2^{-n} L)^2 = \varepsilon 4^{-n} L^2$$

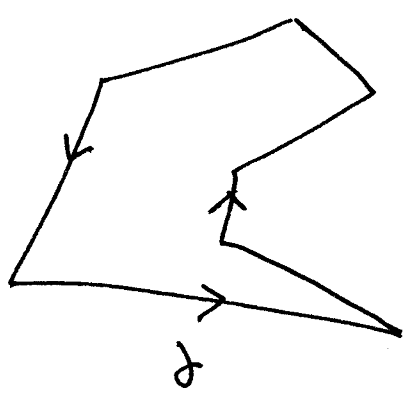
(observe that $|z - z_0| <$ perim. of $\Delta_n = 2^{-n} L$)

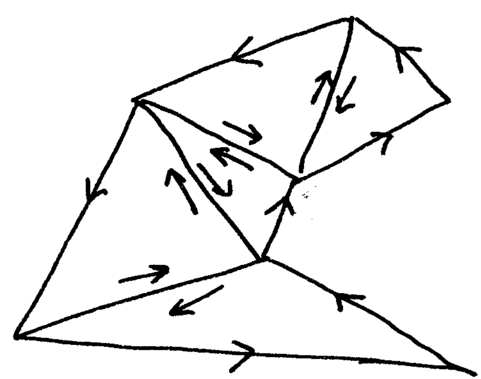
The above inequality together with the estimate 1) give

$$|J| \leq \varepsilon L^2.$$

Since $\varepsilon > 0$ was arbitrarily small we conclude that $J = 0$.

2 Assume now that γ is a boundary of a polygon. The integral along γ can be written as a sum of integrals along triangles, so $\int_\gamma f(z)\,dz = 0$





③ Suppose now that γ is a piecewise smooth curve. Then γ can be approximated by a sequence of polygons γ_n, $\gamma_n \to \gamma$. Since $\int_{\gamma_n} f(z)\,dz = 0$ we conclude that

$$0 = \int_{\gamma_n} f(z)\,dz \longrightarrow \int_{\gamma} f(z)\,dz$$

and hence

$$\int_{\gamma} f(z)\,dz = 0.$$

The proof is complete.

<u>The Cauchy transform</u> Let $\gamma : [a,b] \to \mathbb{C}$ be a (piecewise smooth) curve and f a continuous, complex valued function defined on $\Gamma = \gamma([a,b])$. The Cauchy transform is defined by the formula

$$(\Phi f)(z) = \int_{\gamma} \frac{f(\xi)}{\xi - z}\,d\xi \quad \text{for } z \in \mathbb{C} \setminus \Gamma$$

Theorem If $|z - z_0| < \operatorname{dist}(z_0, \Gamma)$, then

$$(\Phi f)(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

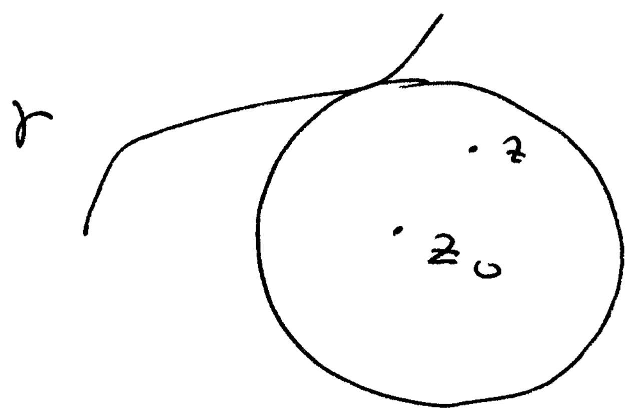
where

$$c_n = \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi .$$

Hence $\Phi f \in H(\mathbb{C} \setminus \Gamma)$ and

$$(\Phi f)^{(n)}(z_0) = n! \int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} \, d\xi .$$

Proof. Let $z_0 \in \mathbb{C} \setminus \Gamma$ and $|z - z_0| < \operatorname{dist}(z_0, \Gamma)$.



We have

$$(\Phi f)(z) = \int_\gamma \frac{f(\xi)}{\xi - z} \, d\xi = \int_\gamma \frac{f(\xi)}{(\xi - z_0) - (z - z_0)} \, d\xi$$

$$= \int_\gamma \frac{f(\xi)}{\xi - z_0} \, \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} \, d\xi = \clubsuit$$

Since $|z-z_0| < \operatorname{dist}(z_0, \Gamma)$, then

$\dfrac{|z-z_0|}{|\xi - z_0|} < 1$ and hence

$$\frac{1}{1-\dfrac{z-z_0}{\xi-z_0}} = \sum_{n=0}^{\infty} \left(\frac{z-z_0}{\xi-z_0}\right)^n .$$

It is easy to see that the series (as a function of ξ) converges uniformly on Γ and hence

$$\clubsuit = \int_\gamma \frac{f(\xi)}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{\xi-z_0}\right)^n d\xi =$$

$$= \sum_{n=0}^{\infty} (z-z_0)^n \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi$$

and the theorem follows.

Theorem (Cauchy Integral Formula)

Let $\gamma : [a,b] \to \Omega$ be a positively oriented Jordan curve such that the interior of γ is contained in Ω. Then for any $f \in H(\Omega)$ and any z in the interior of γ we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi .$$

According to the previous theorem the integral in the Cauchy formula defines a function that can be represented as a power series centered at any point $z_0 \in \mathbb{C} \setminus \Gamma$. Hence the Cauchy formula implies that $f \in H(\Omega)$ admits a power series expansion in a neighborhood of any point inside γ. Since any point $z_0 \in \Omega$ can be bounded by a Jordan curve we have

<u>Corollary</u> Let $\Omega \subset \mathbb{C}$ be open. Then $f \in H(\Omega)$ if and only if for every $z_0 \in \Omega$ and $r = \text{dist}(z_0, \partial\Omega)$ there are numbers $c_n \in \mathbb{C}$, $n = 0, 1, 2, \dots$ such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

for $|z - z_0| < r$. The numbers c_n are given by the formula

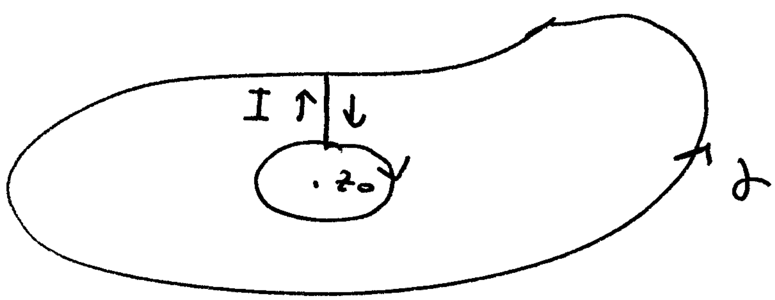
$$c_n = \frac{f^{(n)}(z_0)}{n!}.$$

Proof of the Cauchy Integral Formula.

Let z_0 be a point in the interior Δ of γ, and let $\eta_r(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. Choose r so small that the circle η_r is in the interior of γ. The function

$$\xi \longmapsto \frac{f(\xi)}{\xi - z_0}$$

is holomorphic in the domain $\Delta \setminus D(z_0, r)$. Let I be a segment that connects η_r to γ as on the picture



The function $f(\xi)/(\xi - z_0)$ is holomorphic in the interior of a domain with the boundary $\gamma + I - \eta_r - I$ and hence the Cauchy theorem yields

$$\int_\gamma \frac{f(\xi)}{\xi - z_0}\, d\xi + \int_I \frac{f(\xi)}{\xi - z_0}\, d\xi - \int_{\eta_r} \frac{f(\xi)}{\xi - z_0}\, d\xi - \int_I \frac{f(\xi)}{\xi - z_0}\, d\xi = 0.$$

Therefore

$$\int_{\gamma} \frac{f(\xi)}{\xi - z_0}\, d\xi = \int_{\eta_r} \frac{f(\xi)}{\xi - z_0}\, d\xi \qquad (*)$$

and it remains to prove that

$$\int_{\eta_r} \frac{f(\xi)}{\xi - z_0}\, d\xi = 2\pi i\, f(z_0).$$

The left hand side of $(*)$ does not depend on r so does the right hand side and hence it suffices to prove that

$$\lim_{r \to 0} \int_{\eta_r} \frac{f(\xi)}{\xi - z_0}\, d\xi = 2\pi i\, f(z_0). \quad (**)$$

Since

$$\int_{\eta_r} \frac{d\xi}{\xi - z_0} = 2\pi i$$

$(**)$ is equivalent to

$$\lim_{r \to 0} \int_{\eta_r} \frac{f(\xi) - f(z_0)}{\xi - z_0}\, d\xi = 0 \qquad (***)$$

Given $\varepsilon > 0$, there is r_0 such that for $r < r_0$ $|f(\xi) - f(z_0)| < \varepsilon$ whenever $|z_0 - \xi| = r$ and hence

$$\left| \int_{\eta_r} \frac{f(\xi) - f(z_0)}{\xi - z_0}\, d\xi \right| \le \frac{\varepsilon}{r} \cdot 2\pi r = 2\pi\varepsilon$$

and $(***)$ easily follows. The proof is complete. □

Applications of the Cauchy formula.

Theorem. Suppose $f \in H(\Omega)$ and Ω is connected. Let

$$Z(f) = \{a \in \Omega \mid f(a) = 0\}.$$

Then either $Z(f) = \Omega$ or $Z(f)$ consists of isolated points (i.e. for every $a \in Z(f)$ there is $\varepsilon > 0$ such that $D(a,\varepsilon) \cap Z(f) = \{a\}$).
In the later case for every $a \in Z(f)$ there is $m \in \{1, 2, 3, \dots\}$ such that

$$f(z) = (z-a)^m g(z), \quad z \in \Omega$$

where $g \in H(\Omega)$, $g(a) \neq 0$.

Observe that if $Z(f) \neq \Omega$ then the theorem implies that $Z(f)$ is at most countable.

Proof. Let $a \in Z(f)$. As we know, in a neighborhood of a the function f can be expanded as a power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \quad |z-a| < \operatorname{dist}(a, \partial\Omega).$$

If all the coefficients c_n are equal 0, then f equals zero on a disc centered at a. Since the function can be expanded in a neighborhood of any point in Ω, it easily follows from the connectivity of Ω that $f=0$ in Ω, so $Z(f)=\Omega$.

Suppose now that there is m such that $c_0 = c_1 = \dots = c_{m-1} = 0$, $c_m \neq 0$.

Then

$$f(z) = (z-a)^m \underbrace{\sum_{k=0}^{\infty} c_{m+k} (z-a)^k}_{g(z)} .$$

The function g is holomorphic in a neighborhood of a, $g(a) \neq 0$, so also $g(z) \neq 0$ in a neighborhood of a.

That implies $a \in Z(f)$ is isolated.

Moreover

$$f(z) = (z-a)^m \underbrace{\frac{f(z)}{(z-a)^m}}_{g(z)} \qquad z \in \Omega$$

$g \in H(\Omega)$, $g(a) \neq 0$. $\square$

<u>Corollary</u> If $f, g \in H(\Omega)$ where Ω is connected and $f(z) = g(z)$ for z belonging to a set that has an accumulation point in Ω, then $f = g$ in Ω.

<u>Corollary</u> If $f, g \in H(\Omega)$ where Ω is connected and $f^{(n)}(z_0) = g^{(n)}(z_0)$ for $n = 0, 1, 2, \dots$ then $f(z) = g(z)$ for all $z \in \Omega$.

<u>Proof.</u> In a neighborhood of z_0

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^n = g(z).$$

<u>Theorem (Weierstrass)</u> If a sequence $f_n \in H(\Omega)$ converges uniformly on compact sets to a function f, then $f \in H(\Omega)$. Moreover derivatives of f_n converge uniformly on compact sets to corresponding derivatives of f.

<u>Proof.</u> Let γ be an arbitrary Jordan curve with the interior Δ_γ contained in Ω. Then for $z \in \Delta_\gamma$ we have

$$\begin{array}{ccc} f_n(z) & = \frac{1}{2\pi i} \int_\gamma & \frac{f_n(\xi)}{\xi - z} d\xi \\ \downarrow & & \downarrow\downarrow \\ f(z) & & \frac{f(\xi)}{\xi - z} \end{array}$$

Hence

$$f(z) = \underbrace{\frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi}_{}$$

this function is holomorphic in Δ_γ as a Cauchy transform.

Therefore $f \in H(\Delta_\gamma)$. Since the curve γ was choosen arbitrarily it follows that $f \in H(\Omega)$.

We are left with the proof of convergence of derivatives.

$$f_n^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f_n(\xi)}{(\xi - z)^{n+1}}\, d\xi \to \frac{k!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi = f^{(k)}(z).$$

It is easy to see that the convergence is uniform on compact sets. $\square$

Theorem (The Cauchy inequality) If $f \in H(D(0,R))$, $f(z) = \sum_{n=0}^{\infty} c_n z^n$, then

$$|c_n| \leq \frac{M(f,r)}{r^n}, \quad n = 0, 1, 2, \ldots, \quad r < R,$$

where

$$M(f,r) = \sup_{|z|=r} |f(z)|.$$

Proof.

$$c_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\xi)}{(\xi - 0)^{n+1}}\, d\xi,$$

where
$$\gamma_n(t) = re^{it}, \quad t \in [0, 2\pi].$$

Hence
$$|c_n| \leq \frac{1}{2\pi} \sup_{|z|=r} |f(z)| \cdot \frac{1}{r^{n+1}} \cdot 2\pi r = \frac{M(f,r)}{r^n}.$$

Theorem (Liouville) If f is entire (i.e. $f \in H(\mathbb{C})$) and bounded, then f is constant.

Proof. Since f is bounded, $M(f,r) \leq C$ for some constant C independent of r, namely $C = \sup_{z \in \mathbb{C}} |f(z)|$. Hence if
$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$
it follows from the Cauchy inequality that
$$|c_n| \leq \frac{M(f,r)}{r^n} \leq \frac{C}{r^n} \longrightarrow 0 \quad \text{for } n \geq 1.$$
Thus $c_n = 0$ for $n = 1, 2, 3, \ldots$ and hence f is constant.

Theorem (Fundamental Theorem of Algebra) A complex polynomial of degree n has n complex roots (counted with multiplicity).

Proof. Let $P(z) = c_0 + c_1 z + \ldots + c_n z^n$, $n \geq 1$. It suffices to prove that P has one root.

(the existence of n roots will then follow from the induction argument).

Suppose that $P(z) \neq 0$ for all $z \in \mathbb{C}$.
Hence $f(z) = \frac{1}{P(z)}$ is an entire function.

Since

$$|f(z)| = \frac{1}{|z|^n \left| \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + \frac{a_{n-1}}{z} + a_n \right|} \xrightarrow{z \to \infty} 0$$

f is bounded and it follows from the Liouville theorem that f is constant.
Hence also P is constant, so $n = 0$.
Contradiction. $\square$

<u>Theorem (Morera)</u> If f is continuous in Ω and for every triangle $\Delta \subset \Omega$ we have

$$\int_{\partial \Delta} f(z)\, dz = 0,$$

then $f \in H(\Omega)$.

Compare this result with the Cauchy theorem.

<u>Proof.</u> It suffices to show that f is holomorphic in every disc $D(z_0, r) \subset \Omega$.

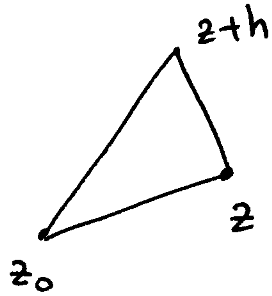
For every $z \in D(z_0, r)$ define

$$F(z) = \int_{\overline{z_0 z}} f(\xi)\, d\xi$$

where the integration is along the interval connecting z_0 to z. We will prove that

$$F'(z) = f(z) \quad \text{for } z \in D(z_0, r).$$

This will prove that F is holomorphic and hence f is holomorphic.



The assumption about vanishing of the integral along any triangle yields

$$\underbrace{\int_{\overline{z_0 (z+h)}} f(\xi)\, d\xi - \int_{\overline{z_0 z}} f(\xi)\, d\xi}_{F(z+h) - F(z)} = \int_{\overline{z(z+h)}} f(\xi)\, d\xi$$

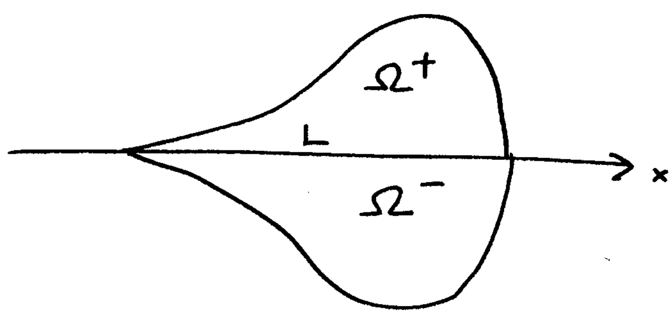
Hence

$$F'(z) = \lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{\overline{z(z+h)}} f(\xi)\, d\xi = f(z).$$

We employed here continuity of f. $\square$

Schwarz symmetry principle.

Let Ω^+ be a domain contained in the upper half-plane $\{\operatorname{im} z > 0\}$.
Assume that the intersection of the boundary $\partial\Omega^+$ with the line $\{\operatorname{im} z = 0\}$ is a segment L. Let Ω^- be the mirror reflection of Ω^+ with respect to the line $\{\operatorname{Im} z = 0\}$.



$$\Omega = \Omega^+ \cup L \cup \Omega^-$$

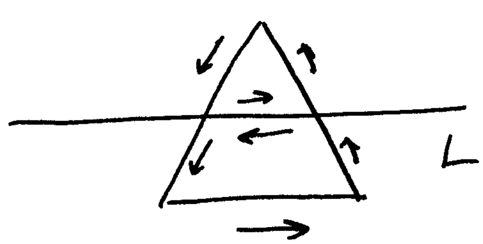
Theorem (The Schwarz symmetry principle)

If $f \in H(\Omega^+)$ is continuous in $\Omega^+ \cup L$ and has real values along L, then f can be extended to a holomorphic function in $\Omega = \Omega^+ \cup L \cup \Omega^-$ by the formula

$$(*) \quad f(z) = \overline{f(\bar{z})} \quad \text{for } z \in \Omega^-.$$

<u>Proof.</u> Since f attains real values along L, the formula $(*)$ defines a continuous function in Ω. It directly follows from the Cauchy-Riemann equations that f is holomorphic in Ω^-.

According to the Morrera theorem it suffices to show that for every triangle $\Delta \subset \Omega$, $\int_{\partial \Delta} f(z)\,dz = 0$.
If $\Delta \subset \Omega^+$ or $\Delta \subset \Omega^-$ then the integral equals zero because f is holomorphic both in Ω^+ and in Ω^-. If Δ intersects with L, then we split the integral along the boundary of Δ into two integrals as the picture shows



and each integral is zero because f is holomorphic in Ω^+ and in Ω^-.

□

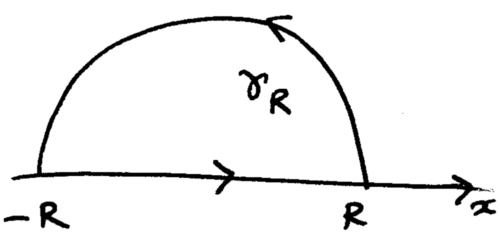
The following example shows a typical application of the Cauchy theorem in the computation of integrals

<u>Example</u> Prove that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

<u>Proof</u> Let

$$f(z) = \begin{cases} \frac{e^{iz}-1}{z} & \text{for } z \neq 0 \\ i & \text{for } z = 0 \end{cases}$$

The function f is continuous. It is easy to see that it is entire (look at the power series expansion). Let γ be a curve as on the picture



According to the Cauchy theorem

$$\int_{\gamma_R} \frac{e^{iz}-1}{z}\,dz = 0.$$

We have

$$\int_{\gamma_R} \frac{e^{iz}-1}{z}\,dz = \underbrace{\int_{-R}^{R} \frac{e^{it}-1}{t}\,dt}_{A} + \underbrace{\int_0^\pi \frac{e^{iRe^{i\theta}}-1}{\cancel{Re^{i\theta}}}\, i\cancel{Re^{i\theta}}\,d\theta}_{B}$$

$$B = i\int_0^\pi e^{iRe^{i\theta}}\,d\theta - i\int_0^\pi d\theta$$

$$\left| \int_0^{\pi} e^{iRe^{i\theta}} d\theta \right| \leq \int_0^{\pi} \underbrace{e^{-R\cos\theta}}_{\substack{\downarrow R\to\infty \\ 0}} d\theta \xrightarrow[R\to\infty]{} 0$$

Hence

$$B \longrightarrow -i\pi$$

Since $A+B=0$, we conclude $A \xrightarrow[R\to\infty]{} i\pi$, i.e.

$$\int_{-R}^{R} \frac{e^{it}-1}{t} dt = \int_{-R}^{R} \frac{\cos t - 1}{t} dt + i \int_{-R}^{R} \frac{\sin t}{t} dt \to i\pi.$$

Thus

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\cos t - 1}{t} dt = 0, \quad \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin t}{t} dt = \pi.$$

Since the function $\sin t / t$ is even we have

$$\int_0^{\infty} \frac{\sin t}{t} dt = \frac{1}{2} \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin t}{t} dt = \frac{\pi}{2}.$$

<u>Homotopy</u> Let $\gamma_0, \gamma_1 : [0,1] \to \Omega$ be two closed curves. We say that they are <u>homotopic in Ω as closed curves</u> if there is a continuous mapping

$$H : [0,1] \times [0,1] \longrightarrow \Omega$$

such that

$$H(t,0) = \gamma_0(t), \quad H(t,1) = \gamma_1(t),$$
$$H(0,s) = H(1,s)$$

<u>Remark</u> If we fix s, the curve $\gamma_s(t) = H(t,s)$ is closed. If s changes from 0 to 1, γ_s is a family of closed curves that shows a continuous deformation of γ_0 to γ_1.

We say that a curve $\gamma : [0,1] \to \Omega$ is <u>contractible</u> if it is a closed curve homotopic to a constant one.

We say that a domain Ω is simply connected if every closed curve in Ω is contractible.

<u>Example</u> Every convex domain is simply connected. Every domain homeomorphic with the unit disc is simply connected. Later we prove a converse result that every simply connected domain is homeomorphic with the unit disc.

We say that two curves $\gamma_0, \gamma_1 : [0,1] \to \Omega$ with the same ends

$$\gamma_0(0) = \gamma_1(0) = z_0, \qquad \gamma_0(1) = \gamma_1(1) = z_1$$

are <u>homotopic as curves with fixed ends</u> if there is a continuous mapping

$$H : [0,1] \times [0,1] \to \Omega$$

such that

$$H(t,0) = \gamma_0(t), \qquad H(t,1) = \gamma_1(t)$$
$$H(0,s) = z_0, \qquad H(1,s) = z_1.$$

<u>Remark</u> If we fix s, then the curve $\gamma_s(t) = H(t,s)$ has the same endpoints as γ_0 and γ_1. If s changes from 0 to 1, γ_s is a family of curves with fixed endpoints that shows a continuous deformation of γ_0 to γ_1.

<u>Theorem</u> If $f \in H(\Omega)$ and the curves γ_0 and γ_1 are homotopic in Ω as closed curves or as curves with fixed ends, then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

<u>Proof</u> We will need the following lemma

Lemma If U is a disc and $f \in H(U)$, then f has an antiderivative in U.

Proof We define the antiderivative by the formula

$$F(z) = \int_{\overline{z_0 z}} f(\xi)\, d\xi \,, \quad z \in U$$

where z_0 is the center of the disc U.

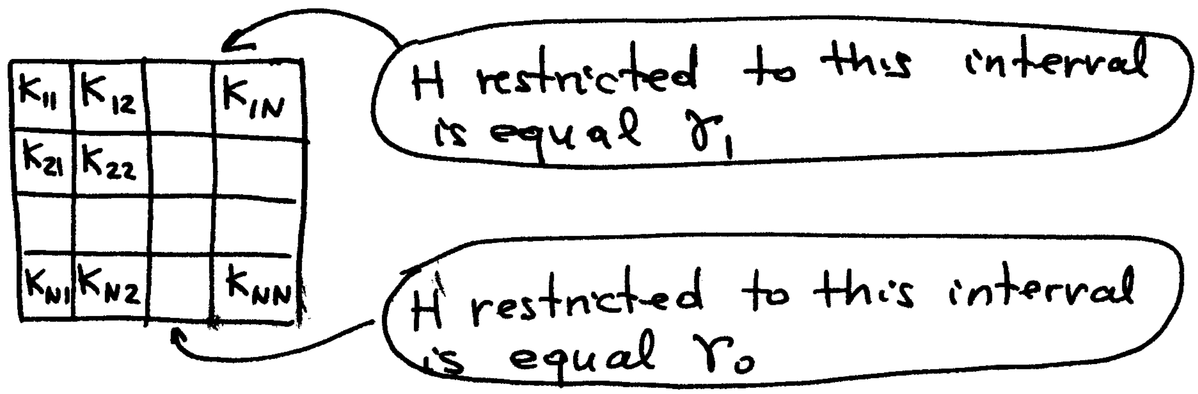
The proof that $F'(z) = f(z)$ follows from the same argument as the one used in the proof of the Morrera theorem. □

Suppose now that f has an antiderivative in a domain G. Then $\int_\gamma f(z)\, dz = 0$ for every closed curve in G. In particular we have

Corollary If $f \in H(\Omega)$, $\gamma : [a,b] \to \Omega$ is a closed curve and the image of γ is contained inside a disc $U \subset \Omega$, then $\int_\gamma f(z)\, dz = 0$.

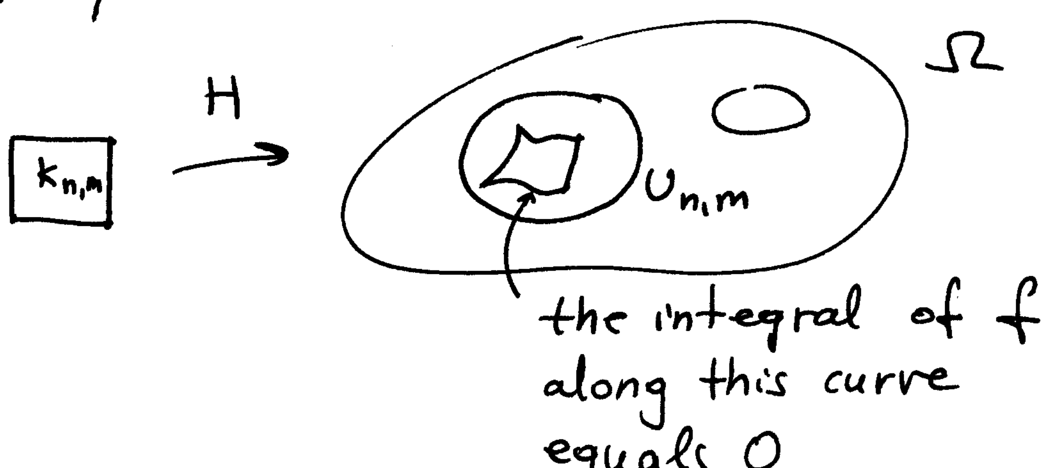
Now we can prove the theorem. Let $H : [0,1] \times [0,1] \to \Omega$ be a homotopy of the curves γ_0 and γ_1. Fix N and

and divide the square $[0,1]\times[0,1]$ into N^2 identical smaller squares $K_{n,m}$, $n,m=1,2,\dots,N$

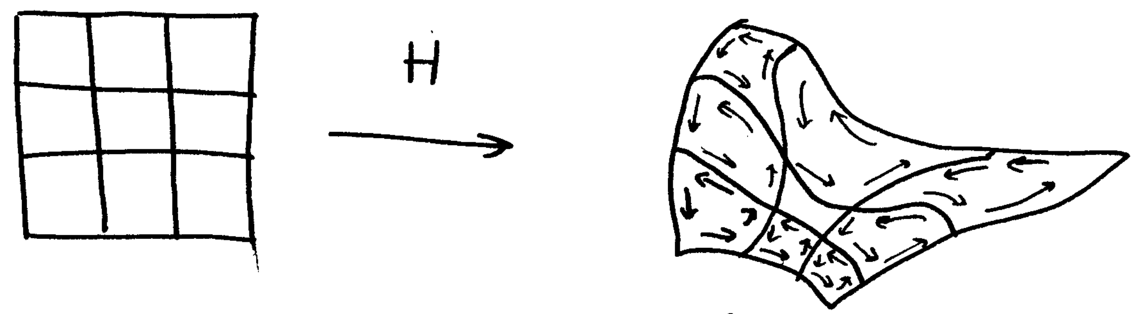


If N is sufficiently large, uniform continuity of H implies that the image of each square $H(K_{n,m})$ is contained in a certain disc $U_{n,m}$ inside Ω, $H(K_{n,m}) \subset U_{n,m} \subset \Omega$.

Accordingly, the corollary implies that the integral of f along a curve being the restriction of H to the boundary of $K_{n,m}$ equals 0

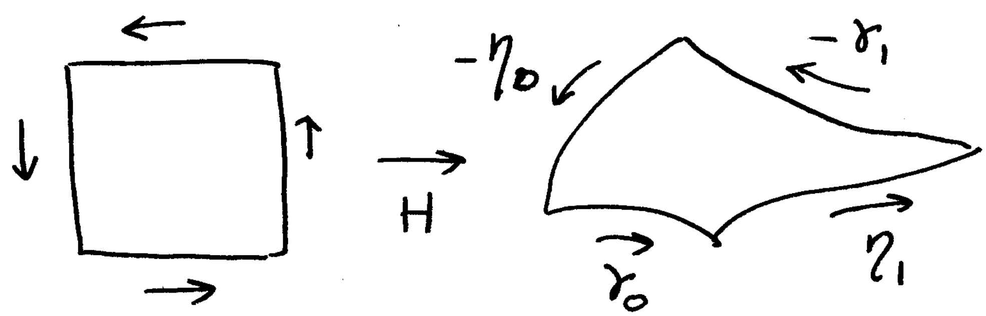


Now the integral over the curve being the restriction of H to the boundary of the big square $[0,1]\times[0,1]$ equals the sum of integrals over the curves being restrictions of H to the boundaries of $K_{n,m}$.



Hence the integral equals 0.

The curve being the restriction of H to the boundary of $[0,1]\times[0,1]$ consists of four curves



Hence

(*) $\int_{\gamma_0} f(z)dz + \int_{\eta_1} f(z)dz - \int_{\gamma_1} f(z)dz - \int_{\eta_0} f(z)dz = 0.$

We have two cases:

① γ_0 and γ_1 are homotopic as curves with fixed ends.

In this case the curves η_0 and η_1 are constant. η_0 maps the whole interval into the point z_0 and η_1 maps the whole interval into the point z_1.

Hence

$$\int_{\eta_0} f(z)\,dz = \int_{\eta_1} f(z)\,dz = 0$$

and (*) implies that

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz .$$

② γ_0 and γ_1 are homotopic as closed curves.

Then $\eta_0(s) = H(0,s) = H(1,s) = \eta_1(s)$

so $\eta_0 = \eta_1$ and hence

$$\int_{\eta_0} f(z)\,dz = \int_{\eta_1} f(z)\,dz .$$

Again (*) yields

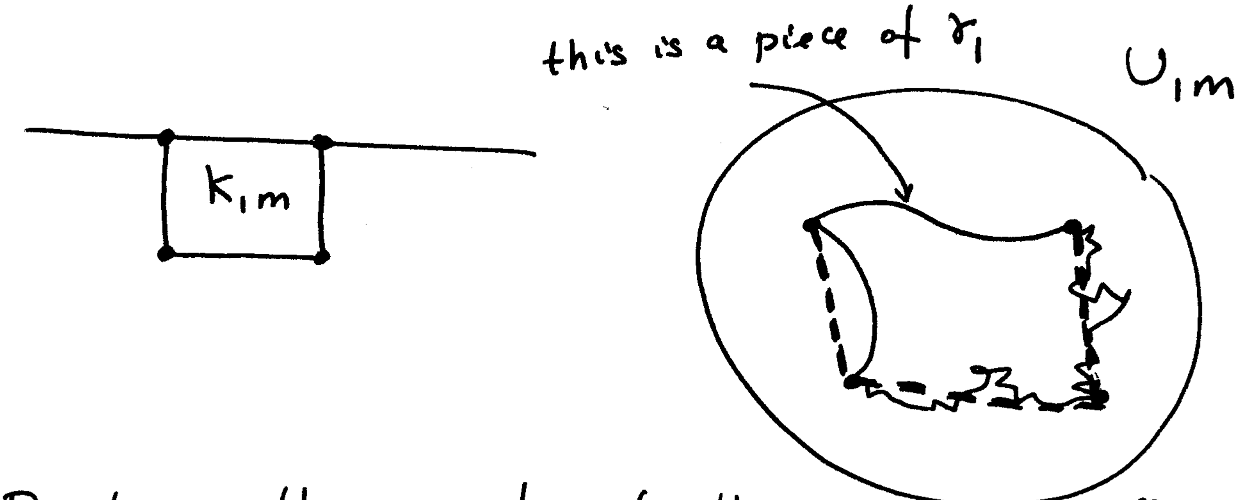
$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz .$$

$\square$

<u>Warning</u>! There is an error in the above reasoning. Indeed, in order to

the integral along a curve we need to know that the curve is piecewise smooth. However the homotopy H is only continuous, so the curves being the restriction of H to the boundaries of the squares $K_{n,m}$ are only continuous and, in general, we cannot talk about the integral along such curves.

This is how we can modify the argument to make it correct. Let us start with squares K_{1m} $m = 1, 2, \dots, N$.



Replace the parts of the curve $H(\partial K_{1m})$ which are not the part of γ_0 by intervals (broken lines on the picture). Thus we have a new closed and, this time, piecewise smooth curve: a piece of γ_1 and three intervals. This new curve is also contained in U_{1m} so the integral along that curve equals 0.

For the squares $K_{n,m}$, $1 < n < N$, i.e. such squares that they do not touch neither top nor bottom of $[0,1] \times [0,1]$ we replace the curve $H(\partial K_{n,m})$ by four intervals and finally for squares $K_{n,N}$ we leave one piece of γ_0 and replace three other pieces by intervals.

Now the correction of the proof consists of replacing the curves $H(\partial K_{n,m})$ by the above piecewise smooth curves. All the other arguments remain the same. $\square$

Now we can generalize the Cauchy theorem as follows.

<u>Theorem (Cauchy)</u> If $f \in H(\Omega)$, then for every contractible closed and contractible curve γ

$$\int_\gamma f(z)\, dz = 0.$$

<u>Theorem (Cauchy)</u> If Ω is simply connected and $f \in H(\Omega)$, then for every closed curve γ in Ω

$$\int_\gamma f(z)\,dz = 0.$$

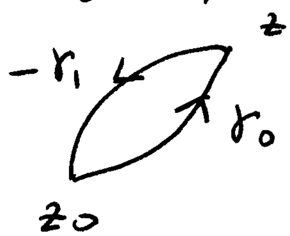
The antiderivative and a branch of logarithm.

<u>Theorem</u> Let Ω be a simply connected domain and $f \in H(\Omega)$. Then f has an antiderivative in Ω given by the formula

$$F(z) = \int_{z_0}^{z} f(\xi)\,d\xi$$

where $z_0 \in \Omega$ is fixed and the integral is along any curve connecting z_0 to z.

<u>Proof.</u> If γ_0 and γ_1 are two curves connecting z_0 to z_1, then the curve $\gamma_0 - \gamma_1$ is closed, so



$$\int_{\gamma_0 - \gamma_1} f(z)\,dz = 0$$

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

Thus the integral that defines F does not depend on the choice of a curve connecting z_0 to z, so F is correctly defined. The proof that $F' = f$ follows from the argument used in the proof of the Morrera theorem. $\square$

<u>Theorem</u> In any simply connected domain that does not contain 0 there is a holomorphic branch of the logarithm given by the integral

$$\log z = \int_{z_0}^{z} \frac{d\xi}{\xi} + \log z_0, \qquad (*)$$

where z_0 is any fixed point in Ω. Moreover

$$(\log z)' = \frac{1}{z}.$$

<u>Proof</u> Let $f(z)$ be the function defined by the right hand side of $(*)$. Then f is holomorphic and $f'(z) = \frac{1}{z}$, $z \in \Omega$. It suffices to prove that

$$e^{f(z)} = z \quad \text{for } z \in \Omega.$$

We have

$$\left(z e^{-f(z)}\right)' = e^{-f(z)} + z e^{-f(z)}\left(-\frac{1}{z}\right) = 0,$$

$$z e^{-f(z)} = c$$

$$z = c\, e^{f(z)}$$

Taking $z = z_0$ we have

$$z_0 = c\, e^{\log z_0} = c\, z_0,$$

so $c = 1$ and hence $e^{f(z)} = z$. $\square$

<u>Theorem</u> Let f be holomorphic and different than 0 in a simply connected domain. Then the function

$$\log f(z) = \int_{z_0}^{z} \frac{f'(\xi)}{f(\xi)}\, d\xi + \log f(z_0)$$

is holomorphic and satisfies

$$e^{\log f(z)} = f(z), \quad (\log f(z))' = \frac{f'(z)}{f(z)}.$$

Proof is very similar to the one presented above. The function $\log f(z)$ is called holomorphic branch of the logarithm of the function f.

Under the assumptions of the above theorem we can define the holomorphic branch of the n-th root of f.

$$\sqrt[n]{f(z)} = e^{\frac{1}{n}\log f(z)}.$$

Line integrals

The notion of the line integral was developed in the Calculus III course. Let's recall how it is connected with the complex integral.

Let $\gamma : [a,b] \to \Omega$ be a piecewise smooth curve and P, Q continuous curves defined in Ω. We define

$$\int_\gamma P\,dx + Q\,dy \overset{\text{def}}{=\!=} \int_a^b P(\gamma(t))\underbrace{\gamma_1'(t)}_{\text{the } x \text{ component of } \gamma'} + Q(\gamma(t))\underbrace{\gamma_2'(t)}_{\text{the } y \text{ comp. of } \gamma'.}\,dt$$

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)).$$

If $f(z) = u(z) + i\,v(z)$, $\quad dz = dx + i\,dy$,
then direct computation shows that

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+idy)$$

where the integral on the right hand side is understood as

$$\int_\gamma (u+iv)(dx+idy) = \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy$$

$$= \int_\gamma \underbrace{(u+iv)}_{P}\,dx + \underbrace{(-v+iu)}_{Q}\,dy$$

Now the Cauchy-Riemann equations can be stated as

$$P'_y = Q'_x .$$

Hence the Cauchy theorem is a special case of the following result.

<u>Theorem</u> If $P, Q \in C^1(\Omega)$ satisfy $P'_y = Q'_x$ and γ_0, γ_1 are homotopic in Ω as closed curves or as curves with fixed endpoints, then

$$\int_{\gamma_0} P\,dx + Q\,dy = \int_{\gamma_1} P\,dx + Q\,dy$$

In particular, if Ω is simply connected, then $\int_\gamma P\,dx + Q\,dy = 0$ for every closed curve γ.

Proof is very similar to the proof of a corresponding Cauchy theorem. We leave details to the reader. The reader should know the theorem from Calculus III.

Harmonic functions. A function $u \in C^2(\Omega)$ is called harmonic if

$$\Delta u \overset{\text{def}}{=} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } \Omega.$$

Theorem The real part and the imaginary part of a holomorphic function are harmonic functions.

Proof. Let $f(z) = u(z) + i v(z)$ be holomorphic. The Cauchy-Riemann equations yield

$$u_x = v_y, \qquad u_y = -v_x.$$

When we differentiate the first equation with respect to x, the second equation with respect to y and we add up the resulting equations, we obtain

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$

Similarly we prove that $\Delta v = 0$.

Another method: It is easy to see that

$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4} \Delta .$$

Hence

$$\Delta u + i \Delta v = \Delta f = 4 \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial \bar{z}} \right) = 0.$$

It turns out that in simply connected domains the converse result is also true.

<u>Theorem</u> In a simply connected domain every harmonic function is a real part of a holomorphic function.

<u>Proof.</u> Let u be a harmonic function in a simply connected domain Ω. We define a function v as the line integral

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -u_y \, dx + u_x \, dy$$

where the integration is along any curve connecting (x_0, y_0) to (x, y).

Note that the integral does not depend on the choice of the curve. Indeed,

$$\frac{\partial}{\partial y}\underbrace{(-u_y)}_{P} = \frac{\partial}{\partial x}\underbrace{(u_x)}_{Q}$$

because u is harmonic. Hence

$\int_\gamma -u_y\,dx + v\,dx = 0$ for every closed curve γ in Ω. If γ_0 and γ_1 are two curves connecting (x_0, y_0) to (x, y) then the curve $\gamma_0 - \gamma_1$ is closed and hence

$$\int_{\gamma_0-\gamma_1} -u_y\,dx + v\,dx = 0$$

$$\int_{\gamma_0} -u_y\,dx + v\,dx = \int_{\gamma_1} -u_y\,dx + v\,dx.$$

We will prove that the function $f = u + iv$ is holomorphic. To this end it suffices to show that

$$u_x = v_y, \quad u_y = -v_x.$$

We have

$$\frac{v(x,y+h)-v(x,y)}{h} = \frac{1}{h}\int_{(x,y)}^{(x,y+h)} -u_y\,dx + u_x\,dy = \clubsuit$$

Taking $\gamma(t) = (x, y+t)$, $t \in [0,h]$ we have

$$\clubsuit = \frac{1}{h}\int_0^h u_x(x,y+t)\,dt \xrightarrow[h\to 0]{} u_x(x,y),$$

so

$$v_y = u_x.$$

Using a similar argument we verify the second Cauchy-Riemann equation.

If the harmonic functions u and v are such that $u+iv$ is holomorphic, then v is called harmonic conjugate of u.

<u>Theorem</u> If u is harmonic in Ω and $D(z_0, r) \subset\subset \Omega$, then $u(z_0)$ equals the average value of u on the boundary of the disc

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta$$

<u>Proof.</u> Let v be the harmonic conjugate of u in the disc and let $f = u+iv$. If $\gamma(\theta) = z_0 + re^{i\theta}$, $\theta \in [0, 2\pi]$, then

the Cauchy formula yields

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z_0}\,d\xi = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta.$$

Now it remains to compare the real parts of both sides.

<u>Corollary</u> A nonconstant harmonic function does not attain neither maximum nor minimum in the interior of a domain.

<u>Corollary</u> If u and v are harmonic functions on a bounded domain Ω, continuous on the closure $\overline{\Omega}$ and $u = v$ on $\partial\Omega$, then $u = v$ in Ω.

<u>Proof.</u> Suppose $u - v \neq 0$. Since $u - v = 0$ on $\partial\Omega$ then $u - v$ attains maximum or minimum in Ω which is a contradiction. $\square$

<u>Theorem</u> Harmonic functions are infinitely differentiable.

<u>Proof</u> Harmonic functions are locally real parts of holomorphic functions and holomorphic functions are infinitely differentiable. $\square$

<u>Laurent series</u> is a series of the form

$$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n. \qquad (*)$$

Representing this series in the form

$$\underbrace{\sum_{n=0}^{\infty} c_n (z - z_0)^n}_{A} + \underbrace{\sum_{n=1}^{\infty} c_{-n} \left(\frac{1}{z - z_0}\right)^n}_{B}$$

and using the Cauchy - Hadamard formula for the radius of convergence we see that the series A converges for

$$|z - z_0| < R, \qquad R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$$

and the series B converges for

$$|z - z_0| > r, \qquad r = \limsup_{n\to\infty} \sqrt[n]{|c_{-n}|}$$

<u>Theorem</u> If $r < R$, then the Laurent series $(*)$ converges in the annulus

$$r < |z - z_0| < R$$

and defines a holomorphic function there.

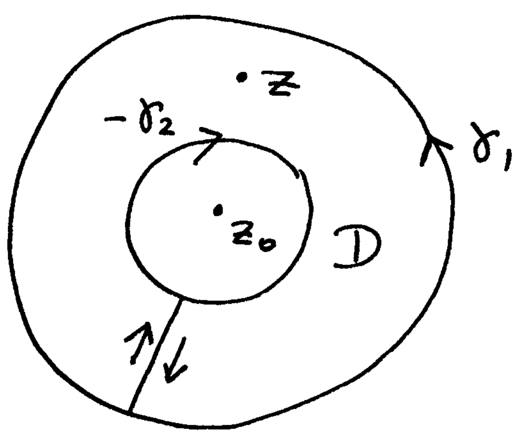
It turns out that also the converse theorem is true.

<u>Theorem (Laurent)</u> If f is a holomorphic function in the annulus $r < |z - z_0| < R$, then it can be represented there as a Laurent series $(*)$, where the coefficients are given by the formula

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi, \quad n \in \mathbb{Z},$$

where $\gamma(t) = z_0 + s e^{it}$, $r < s < R$, $t \in [0, 2\pi]$ is any positively oriented circle centered at z_0 and contained in the annulus.

<u>Proof.</u> Let z be a point in the annulus and let γ_1, γ_2 be two circles centered at z_0, contained in the annulus and such that the point z is between γ_1 and γ_2.



Let D be a domain obtained from the annulus between γ_1 and γ_2 by cutting it along a radius as is shown on the picture.

Since the function f is holomorphic in D, the Cauchy formula gives

$$(**) \quad f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\xi)}{\xi - z}\,d\xi - \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\xi)}{\xi - z}\,d\xi .$$

Now we follow the argument used in the proof that the Cauchy transform can be represented as a power series.

If ξ is on γ_1, then $|\xi - z_0| > |z - z_0|$ and hence

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0}\,\frac{1}{1 - \frac{z - z_0}{\xi - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}} .$$

If ξ is on γ_2, then $|\xi - z_0| < |z - z_0|$ and hence

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = -\frac{1}{z - z_0}\,\frac{1}{1 - \frac{\xi - z_0}{z - z_0}} = -\sum_{n=1}^{\infty} \frac{(\xi - z_0)^{n-1}}{(z - z_0)^n}$$

These two formulas together with $(**)$ give

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\,d\xi \right) (z - z_0)^n +$$

$$+ \sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\xi)}{(\xi - z_0)^{-n+1}}\,d\xi \right) (z - z_0)^{-n} .$$

Since in the last formula the integrals on the circles γ_1 and γ_2 can be replaced by the integral on γ, the Laurent theorem follows. □

<u>Classification of singularities</u> If $f \in H(\Omega \setminus \{a\})$, $a \in \Omega$, then we say that f has <u>isolated singularity</u> at a. If $D'(a,r) = D(a,r) \setminus \{a\} \subset \Omega$, then

the function f can be represented in $D'(a,r)$ as a Laurent series

$$f(z) = h(z) + \sum_{n=1}^{\infty} c_{-n} (z-a)^{-n}, \quad z \in D'(a,r),$$

where $h \in H(\Omega)$. The function h is called analytic part of f and the remaining series principal part of f.

If there is a holomorphic function $\tilde{f} \in H(\Omega)$ such that $\tilde{f}|\Omega \setminus \{a\} = f$, we say that f has removable singularity at a.

Theorem (Riemann) Suppose that $f \in H(\Omega \setminus \{a\})$ and that f is bounded in a punctured neighborhood of a, $D'(a,r) \subset \Omega$. Then f has a removable singularity at a.

Proof. Define a function h by the formula

$$h(z) = \begin{cases} 0 & \text{if } z = a \\ (z-a)^2 f(z) & \text{if } z \in \Omega \setminus \{a\} \end{cases}$$

Since f is bounded in a neighborhood of a it easily follows that $h'(a) = 0$ and hence $h \in H(\Omega)$. Accordingly

$$h(z) = \sum_{n=2}^{\infty} c_n (z-a)^n, \qquad z \in D(a,r).$$

Hence f has a holomorphic extension

$$\tilde{f}(z) = \frac{h(z)}{(z-a)^2} = \sum_{n=0}^{\infty} c_{n+2} (z-a)^n . \qquad \square$$

Since the function $\tilde{f}$ is obtained from f by adding its value at the point a, in what follows such holomorphic extension of f will be denoted by f without adding "$\sim$".

__Theorem__ If $f \in H(\Omega \setminus \{a\})$, $a \in \Omega$, then one of the conditions is satisfied:

(a) f has a removable singularity at a;

(b) There is a positive integer m and complex numbers $c_{-1}, c_{-2}, \dots, c_{-m}$, $c_{-m} \neq 0$ such that

$$f(z) - \sum_{k=1}^{m} c_{-k} (z-a)^{-k}$$

has a removable singularity at a.

(c) For every $r > 0$ such that $D(a,r) \subset \Omega$, the set $f(D'(a,r))$ is a dense subset of $\mathbb{C}$.

__Remark__ In the case (b) we say that f has a __pole of order m__ at a. The function

$$\sum_{k=1}^{m} c_{-k} (z-a)^{-k}$$

is called the <u>principal part</u> of the pole at a. It is clear that in this case $|f(z)| \to \infty$ as $z \to a$.

In the case (c) we say that f has the <u>essential singularity</u> at a. The part (c) of the theorem is known as the Casorati – – Weierstrass theorem. This result can be strenghten as follows.

<u>Theorem (Great Picard Theorem)</u> If a holomorphic function $f \in H(\Omega \setminus \{a\})$, $a \in \Omega$ has the essential singularity at a, then in any open set containing a, f takes on all possible complex values, with at most a single exception, infinitely often.

The theorem is difficult and we will not prove it.

<u>Proof of the theorem from p. 106</u> In the proof we will not use the Laurent expansion. Suppose that the case (c) does not hold.

Then there are positive numbers $r, \delta > 0$ and $\omega \in \mathbb{C}$ such that $|f(z) - \omega| > \delta$ for $z \in D'(a,r)$. This implies that the function

$$g(z) = \frac{1}{f(z) - \omega} \qquad (*)$$

is holomorphic in $D'(a,r)$ and bounded $|g(z)| < 1/\delta$. Hence due to Riemann's theorem it can be extended to a holomorphic function on $D(a,r)$. If $g(a) \neq 0$, then $(*)$ implies that f is bounded in a neighborhood of a, and hence a is a removable singularity of f, so the case (a) is satisfied. Suppose that $g(a) = 0$. Then there is $m \geq 1$ such that

$$g(z) = (z-a)^m g_1(z), \quad g_1(a) \neq 0.$$

The function g_1 has no roots in a neighborhood of a, so the function $h = 1/g_1$ is holomorphic and different than 0 in a neighborhood of a. Now it follows from $(*)$ that

$$f(z) - \omega = (z-a)^{-m} h(z).$$

Since
$$h(z) = \sum_{n=0}^{\infty} b_n (z-a)^n, \quad b_0 \neq 0$$
we have
$$f(z) - \omega = \sum_{n=0}^{\infty} b_{n+m} (z-a)^n + \underbrace{\sum_{k=1}^{m} b_{m-k} (z-a)^{-k}}_{\text{the principal part of the pole}}$$
and hence the case (b) is satisfied. $\square$

Meromorphic functions

<u>Definition.</u> We say that f is a <u>meromorphic function</u> in an open set Ω if there is a set $A \subset \Omega$ consisting of isolated points such that

(1) $f \in H(\Omega \setminus A)$

(2) f has a pole at every point of the set A.

Any rational function in $\mathbb{C}$ is meromorphic.

A <u>simple fraction</u> is a rational function of the form
$$\frac{c}{(z-a)^k}.$$

As an application of the theorem from p. 106 we will prove

Theorem Every rational function can be represented as a sum of a polynomial and simple fractions.

Proof. Let
$$w(z) = A\,\frac{(z-z_1)^{a_1}\cdots(z-z_m)^{a_m}}{(z-w_1)^{b_1}\cdots(z-w_n)^{b_n}},$$
$z_i \neq w_j$, be a rational function.

Clearly $w \in H(\mathbb{C}\setminus\{w_1,\dots,w_n\})$. Since $|w(z)| \to \infty$ as $z \to w_i$, the function w has poles at the points $w_1,\dots,w_n$. Let $P_1,\dots,P_n$ be the principal parts of the poles at $w_1,\dots,w_n$. Observe that each principal part is a rational function represented as a sum of simple fractions. Therefore
$$Q(z) = w(z) - P_1(z) - \dots - P_n(z)$$
is a rational function. It is easy to see that Q is holomorphic on $\mathbb{C}$ and hence Q is a polynomial. Thus
$$w = Q + P_1 + \dots + P_n$$
is the representation we were looking for. $\square$

If $a \in \Omega$ is an isolated singularity of $f \in H(\Omega \setminus \{a\})$ and if

$$f(z) = h(z) + \sum_{n=1}^{\infty} c_{-n} (z-a)^{-n}$$

is a representation of f as a sum of its analytic and principal part, then the coefficient c_{-1} is called <u>residuum</u> of f at a and is denoted by

$$\operatorname{res}_a f = c_{-1}.$$

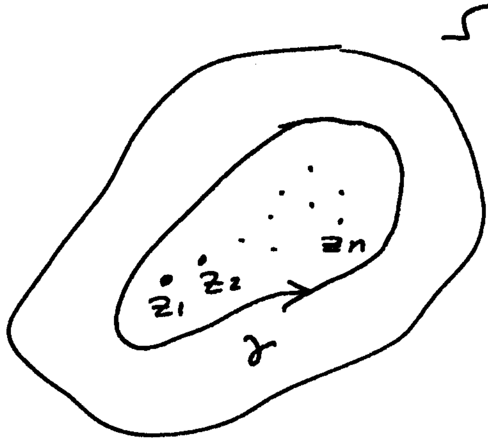
The importance of the notion of residuum comes from the fact that according to the formula for the coefficients in the Laurent series

$$\operatorname{res}_a f = \frac{1}{2\pi i} \int_\gamma f(\xi)\, d\xi \qquad (*)$$

where $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$, r - sufficiently small. The formula $(*)$ leads to the following generalization of the Cauchy theorem.

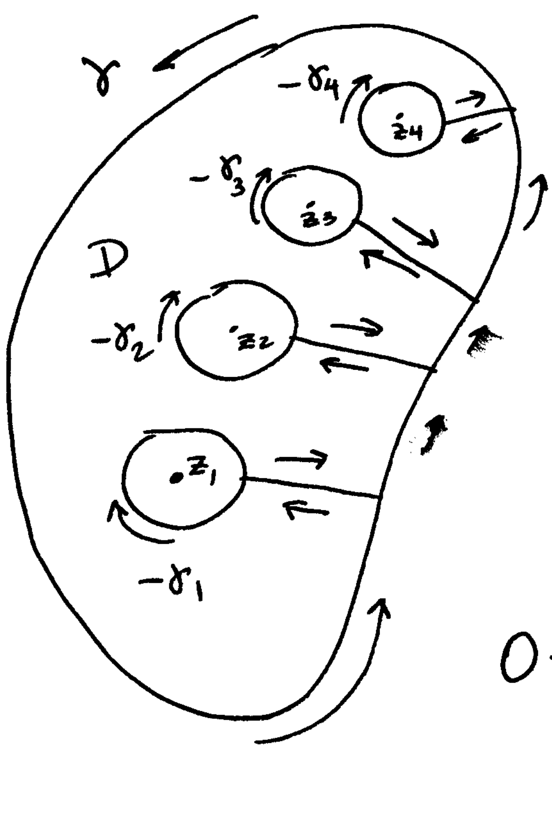
<u>Theorem (Cauchy)</u> Let $f \in H(\Omega \setminus \{z_1, \dots, z_n\})$ and let γ be a positively oriented Jordan curve such that $\{z_1, \dots, z_n\} \subset \Delta_\gamma \subset \Omega$. Then

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \sum_{k=1}^{n} \operatorname{res}_{z_k} f.$$



This generalizes the Cauchy theorem about vanishing of the integral of a holomorphic function. It also generalizes formula (*).

<u>Proof</u>. Let γ_k, $k=1,2,\dots,n$ be positively oriented circles centered at z_k of sufficiently small radii. Let D be a domain obtained from Δ_γ by removing discs with boundaries γ_k and by cutting the domain along intervals connecting γ_k to γ, as in the picture below. Since



the function f is holomorphic in D, the integral along its boundary equals zero. Since the integrals along the segments connecting γ_k to γ cancell out we have

$$0=\int_\gamma f(z)\,dz+\sum_{k=1}^{n}\int_{-\gamma_k} f(z)\,dz$$

and hence

$$\frac{1}{2\pi i}\int_{\gamma} f(z)dz = \sum_{k=1}^{n} \frac{1}{2\pi i}\int_{\gamma_k} f(z)dz = \sum_{k=1}^{n} \operatorname{res}_{z_k} f$$

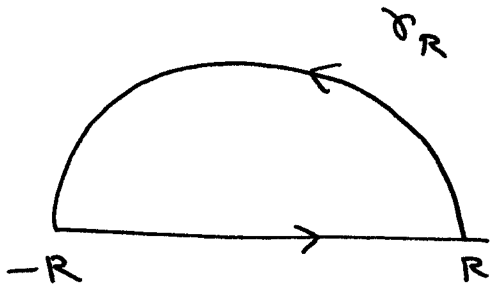
where the last equality follows from (*) p. 111. $\square$

This theorem plays an important role in the evaluation of integrals of functions of a real variable.

<u>Theorem</u> Suppose that f is a holomorphic function in the upper half-plane $\{\operatorname{im} z > 0\}$ except for a finite number of points $z_1, z_2, \dots, z_n$. Suppose also that f is continuous on the x-axis, takes on real values on the x-axis, and $|f(z)| \le M/|z|^{\alpha}$, $|z| \geq r$ for some $r>0$, $M>0$, $\alpha > 1$. Then

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{z_k} f.$$

<u>Proof.</u> Let γ be a contour that consists of an interval $[-R, R]$ and a semicircle γ_R as on the picture.



Let R be so large that all the points $z_1, \dots, z_n$ are inside the curve. According to the Cauchy residuum theorem

$$\int_{-R}^{R} f(z)\,dz + \int_{\gamma_R} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{res}_{z_k} f,$$

and it remains to prove that $\int_{\gamma_R} f(z)\,dz \to 0$ as $R \to \infty$. We have

$$\left| \int_{\gamma_R} f(z)\,dz \right| \leqslant \frac{M}{R^\alpha}\,\pi R = \frac{\pi M}{R^{\alpha-1}} \xrightarrow[R\to\infty]{} 0 \qquad \square.$$

<u>Example</u> As an application of the above theorem we will prove that

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2+2bx+c} = \frac{\pi}{\sqrt{ac-b^2}}$$

where $a, b, c \in \mathbb{R}$, $ac-b^2>0$.

<u>Proof</u>. Let $f(z) = \dfrac{1}{az^2+2bz+c}$. Then $|f(z)| \leqslant M/|z|^2$ for all sufficiently large $|z|$. We also have

$$f(z) = \frac{1}{a\left(z - \frac{-b+i\sqrt{ac-b^2}}{a}\right)\left(z - \frac{-b-i\sqrt{ac-b^2}}{a}\right)}.$$

Hence the function f has a simple pole (i.e. a pole of order 1) in the upper half plane at

$$z_0 = \frac{-b+i\sqrt{ac-b^2}}{a}.$$

It remains to compute the residuum. If

$$f(z) = a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + \dots$$

then

$$\operatorname{res}_{z_0} f = a_{-1} = \lim_{z\to z_0} f(z)(z-z_0)$$

and hence in our situation

$$\operatorname{res}_{z_0} f = \frac{1}{a\left(z_0 - \frac{-b-i\sqrt{ac-b^2}}{a}\right)} = \frac{1}{2i\sqrt{ac-b^2}}.$$

Therefore

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \frac{1}{2i\sqrt{ac-b^2}} = \frac{\pi}{\sqrt{ac-b^2}}$$

In particular

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

which can also easily be concluded from $(\arctan x)' = \frac{1}{1+x^2}$.

How to compute residuum.

The result below shows a method how to check if a function has a pole at an isolated singularity and of what order.

Theorem If $f(z) = g(z)/h(z)$ where g has zero at a of order p and h has zero at a of order $q > p$, then the function f has a pole at a of order $q - p$.

Proof. $g(z) = (z-a)^p g_1(z)$, $g_1(a) \neq 0$, $h(z) = (z-a)^q h_1(z)$, $h_1(a) \neq 0$. Hence

$$f(z) = \frac{s(z)}{(z-a)^{q-p}}$$

where $s = g_1/h_1$ is holomorphic and $s(a) \neq 0$. □

- If f has a simple pole at a (i.e. a pole of order 1), then

$$f(z) = g(z) + c_{-1}(z-a)^{-1}$$

and hence

(1) $$\operatorname{res}_a f = c_{-1} = \lim_{z \to a} f(z)(z-a).$$

- If a function f is of the form

$$f(z) = \frac{g(z)}{h(z)}, \quad g(a) \neq 0, \ h(a) = 0, \ h'(a) \neq 0$$

then h has zero at a of order 1, so f has a simple pole at a. Now

$$f(z)(z-a) = \frac{g(z)}{\frac{h(z)-h(a)}{z-a}} \xrightarrow[z\to a]{} \frac{g(a)}{h'(a)}$$

and therefore

$$(2) \qquad \operatorname{res}_a f = \frac{g(a)}{h'(a)}.$$

- If f has a pole of order m at a, then

$$f(z) = g(z) + \frac{c_{-1}}{z-a} + \frac{c_{-2}}{(z-a)^2} + \dots + \frac{c_{-m}}{(z-a)^m}$$

$$f(z)(z-a)^m = c_{-m} + c_{-(m-1)}(z-a) + \dots + c_{-1}(z-a)^{m-1} + \dots$$

and hence

$$(3) \qquad \operatorname{res}_a f = \frac{1}{(m-1)!} \lim_{z\to a} \left(f(z)(z-a)^m \right)^{(m-1)}$$

derivative

We proved

<u>Theorem</u> If a function f has a pole of order m at a then the resioluum at a can be computed from the formula (3). In particular, if $m=1$ we obtain (1). If $f = g/h$, $g(a) \neq 0$, $h(a) = 0$, $h'(a) \neq 0$, then f has a simple pole at a and (2) holds.

Index of a point with respect to a curve.

Let $\gamma : [a,b] \to \mathbb{C} \setminus \{0\}$ be a continuous curve. Then there is a continuous real valued function φ on $[a,b]$ such that

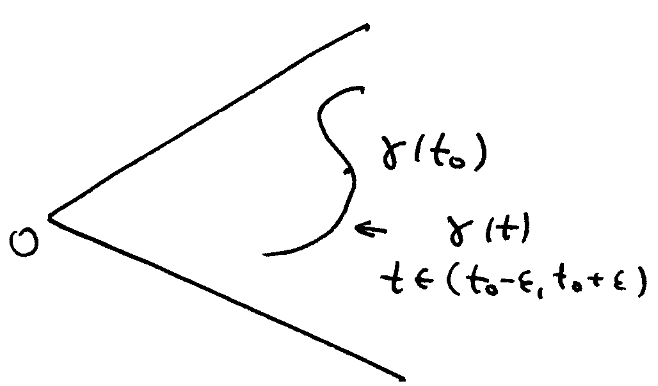
$$\gamma(t) = |\gamma(t)|\, e^{i\varphi(t)}, \quad t \in [a,b].$$

The function φ is called a <u>continuous branch of the argument</u> of the curve γ.

Indeed, we define the function φ as follows:

① For $t = a$, the argument $\varphi(a) = \arg \gamma(a)$ is choosen arbitrarily.

② If the argument $\varphi(t_0)$ is already defined, then for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, where $\varepsilon > 0$ is sufficiently small, values of $\gamma(t)$ belong to an angle with the vertex at 0, in which the argument has a continuous branch, see the picture.



This, however, implies that the argument $\varphi(t)$, $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ can be uniquely and continuously defined. Uniquely, because the value $\varphi(t_0)$ is already fixed.

It easily follows from ① and ② that the argument φ can be continuously defined on the whole interval.

If φ_1 and φ_2 are two continuous branches of the argument of the curve γ, then

$$\varphi_1(t) = \varphi_2(t) + 2k\pi \quad , \; t \in [0, 2\pi]$$

for some $k \in \mathbb{Z}$.

The number $\varphi(b) - \varphi(a)$ is called the increase of the argument along the curve. Note that this number does not depend on the choice of the branch φ.

If γ is a closed curve, then $\varphi(b) - \varphi(a)$ is a multiplicity of 2π (because $\gamma(a) = \gamma(b)$ and hence $e^{i\varphi(a)} = e^{i\varphi(b)}$).

Geometrically, for a closed curve γ

$$\varphi(b) - \varphi(a) = 2\pi \cdot \begin{pmatrix} \text{The number of revolutions} \\ \text{of } \gamma \text{ around } O \text{ in the} \\ \text{counterclockwise direction} \end{pmatrix}$$

We call the number

$$\mathrm{Ind}_\gamma(0) = \frac{1}{2\pi}(\varphi(b) - \varphi(a))$$

the <u>index of 0 with respect to γ</u>. Clearly $\mathrm{Ind}_\gamma(0)$ is an integer.

Now we will find an integral formula for γ.

Let $\gamma : [a,b] \to \mathbb{C} \setminus \{0\}$ be a piecewise smooth curve. Denote

$$\ell(t) = \log \gamma(a) + \int_a^t \frac{\gamma'(s)}{\gamma(s)}\, ds, \quad t \in [a,b].$$

where $\log \gamma(a)$ is an arbitrarily choosen value of the logarithm of $\gamma(a)$.

We will prove that $\ell(t)$ is a continuous branch of the logarithm of the curve γ, i.e.

$$(*) \qquad e^{\ell(t)} = \gamma(t), \qquad t \in [a,b].$$

Indeed, $\ell'(t) = \gamma'(t)/\gamma(t)$, so

$$(\gamma(t)\, e^{-\ell(t)})' = \gamma'(t)\, e^{-\ell(t)} - \gamma(t)\, e^{-\ell(t)} \ell'(t) = 0.$$

Hence

$$\gamma(t) = C e^{\ell(t)}$$

Taking $t = a$ yields $C = 1$ and hence $(*)$ follows

Since $\ell(t) = \log \gamma(t)$ we have

$$\ell(t) = \operatorname{Log}|\gamma(t)| + i \underbrace{\arg \gamma(t)}_{\substack{\varphi(t) - \text{continuous branch} \\ \text{of the argument of } \gamma.}}$$

If γ is a closed curve, then

$$\ell(b) - \ell(a) = i(\varphi(b) - \varphi(a)) = 2\pi i \operatorname{Ind}_\gamma(0).$$

Accordingly

$$\operatorname{Ind}_\gamma(0) = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(s)}{\gamma(s)}\,ds = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}.$$

If γ is a closed curve in $\mathbb{C}\setminus\{z_0\}$, then we define the index of z_0 with respect to γ as

$$\operatorname{Ind}_\gamma(z_0) = \operatorname{Ind}_{\gamma - z_0}(0)$$

where $\gamma - z_0$ is the translation of the curve γ by $-z_0$. Hence we also obtain the integral formula for the index

$$\operatorname{Ind}_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0}.$$

Since the function $z \mapsto 1/(z - z_0)$ is holomorphic in $\mathbb{C}\setminus\{z_0\}$ the Cauchy theorem yields

Theorem If the closed curves γ_0 and γ_1 are homotopic in $\mathbb{C} \setminus \{z_0\}$, then

$$\operatorname{Ind}_{\gamma_0}(z_0) = \operatorname{Ind}_{\gamma_1}(z_0).$$

Corollary If Ω is simply connected and γ is a closed curve in Ω, then for every $z \in \mathbb{C} \setminus \Omega$, $\operatorname{Ind}_\gamma(z) = 0$.

The following result collects many important properties of the index

Theorem Let γ be a piecewise smooth curve in $\mathbb{C}$. Then the function $z \mapsto \operatorname{Ind}_\gamma(z)$, $z \in \mathbb{C} \setminus \Gamma$ has the following properties.

1. $\operatorname{Ind}_\gamma(z) \in \mathbb{Z}$
2. $\operatorname{Ind}_\gamma(z)$ is constant on every connected component of $\mathbb{C} \setminus \Gamma$
3. $\operatorname{Ind}_\gamma(z) = 0$ on the unbounded component of $\mathbb{C} \setminus \Gamma$.

Proof The property 1) has already been proved. The integral formula for the index shows that the index is a continuous function of z, so 2) follows from 1). The integral formula for the index shows that $\mathrm{Ind}_\gamma(z) \longrightarrow 0$ as $|z| \to \infty$, so 3) follows from 2). $\square$

Theorem Let $\gamma : [a,b] \longrightarrow \Omega$ be a Jordan curve such that $\Delta_\gamma \subset \Omega$. Let f be a meromorphic function in Ω such that neither zeroes nor poles of f belong to $\Gamma = \gamma([a,b])$. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = Z_f - B_f$$

where Z_f is the number of zeroes of f in Δ_γ (counted with the multiplicity) and B_f is the sum of orders of poles of f in Δ_γ.

Proof Singularities of the function $F = f'/f$

are located at zeroes and poles of f.
Let's compute the residua of F at these singularities.

1) $z_0 \in \Delta_\gamma$ is a zero of f of multiplicity k.
Then $f(z) = (z-z_0)^k g(z)$, $g(z_0) \neq 0$, and hence

$$F(z) = \frac{k(z-z_0)^{k-1} g(z) + (z-z_0)^k g'(z)}{(z-z_0)^k g(z)} = \frac{g'(z)}{g(z)} + \frac{k}{z-z_0}$$

Thus

$$\operatorname{res}_{z_0} F = k$$

2) $z_0 \in \Delta_\gamma$ is a pole of f of order k. Then
$f(z) = (z-z_0)^{-k} g(z)$, $g(z_0) \neq 0$, and hence

$$F(z) = \frac{-k(z-z_0)^{-k-1} g(z) + (z-z_0)^{-k} g'(z)}{(z-z_0)^{-k} g(z)} = \frac{g'(z)}{g(z)} - \frac{k}{z-z_0}$$

Thus

$$\operatorname{res}_{z_0} F = -k$$

Now the theorem follows immediately from the Cauchy residuum theorem.

Corollary (The argument principle)

Under the assumptions of the above theorem

$$\mathrm{Ind}_{f\circ\gamma}(0) = Z_f - B_f$$

Here $f\circ\gamma$ denotes the curve $t \mapsto f(\gamma(t))$. Observe that it takes on values in $\mathbb{C}\setminus\{0\}$. The name "argument principle" steams from the following geometric interpretation:

Let $\varphi(t)$ be a continuous branch of the argument of f on γ, i.e. $\varphi(t)$ is a continuous branch of the argument of the curve $f\circ\gamma$. The increase of the argument of f along γ equals the increase of the argument of the curve $f\circ\gamma$ and hence $\mathrm{Ind}_{f\circ\gamma}(0)$ equals the increase of the argument of f along γ divided by 2π.

Proof. We have

$$\mathrm{Ind}_{f\circ\gamma}(0) = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{dz}{z} = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\,\gamma'(t)}{f(\gamma(t))}\,dt.$$

On the other hand

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\,\gamma'(t)\,dt.$$

Hence

$$\mathrm{Ind}_{f\circ\gamma}(0) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = Z_f - B_f. \qquad \square$$

<u>Theorem (Rouché)</u> Let $f, g \in H(\Omega)$ and let γ be a Jordan curve in Ω such that $\Delta_\gamma \subset \Omega$. If

(*) $\qquad |f(z) - g(z)| < |f(z)|$ for $z \in \Gamma$

then the functions f and g have the same number of zeroes in Δ_γ.

Roughly speaking, the condition (*) says that $f \neq 0$ on Γ and g is sufficiently close to f on Γ in the supremum norm.

<u>Proof.</u> Let $h_\lambda(z) = f(z) + \lambda(g(z) - f(z))$, $\lambda \in [0,1]$. Since $|h_\lambda(z)| \geqslant |f(z)| - |g(z) - f(z)| > c > 0$ on Γ, the function

$$\lambda \mapsto Z_\lambda = \frac{1}{2\pi i}\int_\gamma \frac{h_\lambda'(z)}{h_\lambda(z)}\,dz$$

is a continuous function of λ. Since it attains integer values (the number of

zeroes of h_λ in Δ_γ) it is constant.
Accordingly

$$Z_f = Z_0 = Z_1 = Z_g. \qquad \square$$

Example Find the number of zeroes of the polynomial $z^7 + 3z^6 + 1$ in the disc $D(0,1)$.

Solution. For $|z| = 1$ we have

$$|3z^6 - (z^7 + 3z^6 + 1)| = |z^7 + 1| \leq 2 < 3 = |3z^6|.$$

Thus it follows from Rouché's theorem that the polynomials $3z^6$ and $z^7 + 3z^6 + 1$ have the same number of zeroes in the disc $D(0,1)$. Hence $z^7 + 3z^6 + 1$ has 6 zeroes.

Theorem (Hurwitz) Let $f_n, f \in H(\Omega)$, and f_n converges uniformly to f on compact subsets of Ω. Let γ be a Jordan curve in Ω such that $\Delta_\gamma \subset \Omega$. If $f \neq 0$ on Γ, then there is n_0 such that for $n \geq n_0$ the number of zeroes of f_n in Δ_γ equals the number of zeroes of f in Δ_γ

Proof It follows immediately from the Rouché theorem. $\square$

Theorem If $f_n \in H(\Omega)$ is a sequence of one-to-one functions that converge uniformly on compact sets to $f \in H(\Omega)$ then f is one-to-one or constant.

Proof. Suppose f is not constant. We will prove it is one-to-one. Let $z_1, z_2 \in \Omega$, $z_1 \neq z_2$. Let $\alpha = f(z_1)$, $\alpha_n = f_n(z_1)$. Let $\bar{D}$ be a closed disc centered at z_2 such that $z_1 \notin \bar{D}$ and $f-\alpha$ has no zeroes on the boundary of $\bar{D}$ (such $\bar{D}$ exists because $f-\alpha$ is holomorphic and non constant). Since $f_n - \alpha_n$ converges uniformly on compact sets to $f-\alpha$ it follows from the Hurwitz theorem that for $n \geq n_0$, $f-\alpha$ has the same number of zeroes in $\bar{D}$ as $f_n - \alpha_n$, However $f_n - \alpha_n$ has no zeroes in $\bar{D}$ because it is one-to-one and vanishes at $z_1 \notin \bar{D}$. Thus $f-\alpha$ has no zeroes in $\bar{D}$. In particular $f(z_2) - \alpha \neq 0$, $f(z_2) \neq f(z_1)$, so f is one-to-one. $\square$

Open mapping theorem

The following result follows directly from the Open Mapping Theorem in Advanced Calculus.

Theorem Suppose $\varphi \in H(\Omega)$, $z_0 \in \Omega$, $\varphi'(z_0) \neq 0$. Then there is a neighborhood V of z_0 such that

1) φ is one-to-one in V;
2) $W = \varphi(V)$ is open
3) The inverse mapping $\varphi^{-1}: W \to V$ is holomorphic.

Indeed, it follows from the Cauchy-Riemann equations that $J_\varphi = |\varphi'|^2 \neq 0$, where J_φ is the Jacobian of the mappin φ. Hence φ is a diffeomorphism in a neighborhood of z_0 and thus the properties 1) and 2) follow. Since φ is conformal, the inverse diffeomorphism is conformal too which is 3).

Theorem (Open mapping theorem)

If $\Omega \subset \mathbb{C}$ is a domain and $f \in H(\Omega)$ is not constant, then $f(\Omega)$ is open.

If we knew that $f' \neq 0$ in Ω then the result would follow from the previous theorem. Thus we have to investigate the behaviour of f at the set of isolated points where $f' = 0$.

Observe first that the function
$\pi_m(z) = z^m$, $m = 1, 2, 3, \dots$ is open
(i.e. it maps open sets into open sets).
Indeed, if $0 \notin \mathcal{U}$, then $\pi_m(\mathcal{U})$ is open, because π_m is locally a diffeomorphism in $\mathcal{U}$. If $0 \in \mathcal{U}$ it suffices to observe that

$$\pi_m(D(0,r)) = D(0, r^m).$$

Note also that the composition of open mappings is open. The two remarks show that the open mapping theorem follows from the following result.

<u>Theorem</u> Suppose $\Omega \subset \mathbb{C}$ is a domain, $f \in H(\Omega)$ is non constant and $f(z_0) = \omega_0$. Let m be the order of zero of $f - \omega_0$ at z_0. Then there is a neighborhood V of z_0 and $\varphi \in H(V)$ such that

1) $f(z) = \omega_0 + [\varphi(z)]^m$ for $z \in V$

2) φ' has no zeroes in V and φ is an invertible mapping of V onto a disc $D(0, r)$.

<u>Proof.</u> According to our assumptions

$$f(z) - \omega_0 = (z - z_0)^m g(z)$$

$g \in H(\Omega)$, $g \neq 0$ in a neighborhood of z_0. Thus there is a holomorphic branch of $\sqrt[m]{g}$ in a neighborhood of z_0. Now it suffices to take

$$\varphi(z) = (z - z_0) \sqrt[m]{g}.$$

The condition 1) is obviously satisfied.

Since $\varphi(z_0) = 0$, $\varphi'(z_0) = \sqrt[m]{g(z_0)} \neq 0$, φ is an invertible mapping of a neighborhood of z_0 onto a neighborhood of 0. $\square$

Corollary If $f \in H(\Omega)$ is one-to-one in Ω, then $f'(z) \neq 0$ in Ω and the inverse function is holomorphic too.

Thus conformal mappings are exactly one-to-one holomorphic functions.

Theorem (Maximum principle)
If $f \in H(\Omega)$, then the function $|f|$ does not attain the maximum in Ω unless f is constant.

Proof **1st method** We have
Suppose $|f(a)| = \max_{\Omega} |f|$, $a \in \Omega$.
Let $r > 0$ be such that $D(a,r) \subset\subset \Omega$.
Let $\gamma(t) = a + re^{i\theta}$, $\theta \in [0, 2\pi]$ be a parametrization of the boundary.

We have

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - a} d\xi = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta$$

and hence

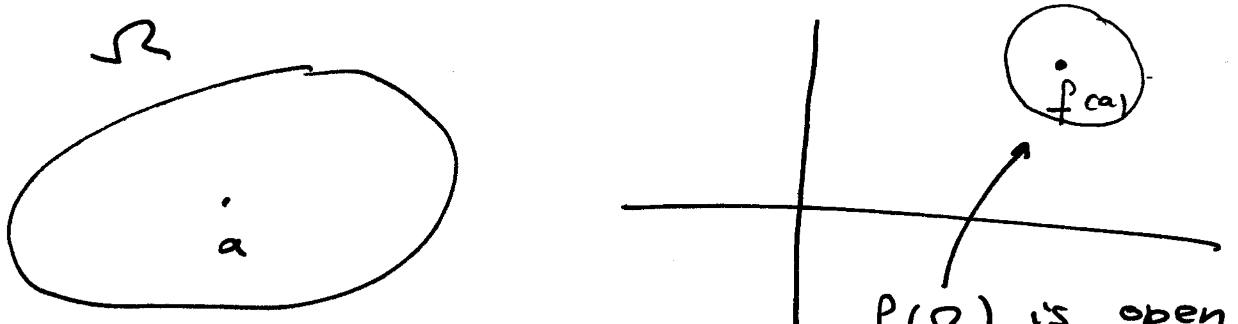
$$|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a+re^{i\theta})| d\theta \leq \max_\Omega |f|.$$

Since we have equality it follows that

$$|f(a+re^{i\theta})| = |f(a)| \text{ for all } r < r_0, \ \theta \in [0, 2\pi).$$

Thus f takes values into a circle of radius $|f(a)|$ and it easily follows that $f' = 0$ in the disc $D(a, r_0)$ so f is constant.

2nd method Suppose f is not constant. It suffices to look at the picture below and use the fact that f is open.



$f(\Omega)$ is open so certain disc around $f(a)$ is contained in $f(\Omega)$ and there are points $f(z)$ in that disc such that $|f(z)| > |f(a)|$.

Theorem (Schwarz lemma) If $f \in H(D(0,1))$, $|f(z)| \leq 1$ in $D(0,1)$ and $f(0) = 0$, then

1) $|f(z)| \leq |z|$ for $z \in D(0,1)$
2) $|f'(0)| \leq 1$.

If we have equality in 1) at some point $z \neq 0$ or if we have equality in 2), then f is a rotation, i.e.

$$f(z) = e^{i\theta} z \quad \text{for some } \theta \in \mathbb{R}.$$

Proof. Since $f(0) = 0$ we can write

$f(z) = z\, g(z)$, $g \in H(D(0,1))$.

For every $r < 1$ we have

$$|g(z)| \leq |f(z)| / |z| \leq 1/r$$

on the boundary of the disc $D(0,r)$. Thus the maximum principle yields $|g| \leq 1/r$ in the disc $D(0,r)$. Passing to the limit $r \to 1^-$ yields $|g| \leq 1$ in $D(0,1)$, so $|f(z)| \leq |z|$ and

$|f'(0)| = |g(0)| \leq 1$.

If the equality holds in 1) for some $z \neq 0$ or if the equality holds in 2), then $|g(z)| = 1$ for some $z \in D(0,1)$. Thus $g \equiv \text{const}$ according to the maximum principle. Hence $g(z) = e^{i\theta}$ for some $\theta \in \mathbb{R}$ and $f(z) = z e^{i\theta}$. $\square$

Conformal automorphisms of the unit disc

As an application of the Schwarz lemma we will find all conformal mappings of the unit disc $U = D(0,1)$ onto itself. As we remember all the fractional linear transformations preserving the unit disc are of the form

$$L(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha} z} \qquad \alpha \in U, \ \theta \in \mathbb{R}.$$

(see p. 52). Denote the boundary of the disc by T and let

$$\varphi_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha} z}, \qquad \alpha \in U.$$

The properties of the fractional linear transformation φ_α are summarized in the following theorem.

Theorem Fix $\alpha \in U$. Then φ_α is a one-to-one mapping of T onto T and U onto U. $\varphi_\alpha(\alpha) = 0$ and the inverse mapping to φ_α is $\varphi_{-\alpha}$.
Moreover
$$\varphi_\alpha'(0) = 1 - |\alpha|^2, \quad \varphi_\alpha'(\alpha) = \frac{1}{1-|\alpha|^2}.$$

Consider now the following extremal problem.

Let $\alpha, \beta \in U$. Find the upper estimate of $|f'(\alpha)|$ in the class of all holomorphic functions $f : U \to U$ such that $f(\alpha) = \beta$ (we do not assume f is one-to-one).

Let $g = \varphi_\beta \circ f \circ \varphi_{-\alpha}$. Then $g : U \to U$, $g(0) = 0$, so it follows from the Schwarz lemma that $|g'(0)| \leq 1$. Since $g'(0) = \varphi_\beta'(\beta) f'(\alpha) \varphi_{-\alpha}'(0)$ we conclude that

$$(*) \qquad |f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}.$$

If we have equality in $(*)$, then $|g'(0)|=1$ so $g(z) = e^{i\theta} z$ and hence

$$f(z) = \varphi_{-\beta}\left(e^{i\theta}\varphi_\alpha(z)\right)$$

is a fractional linear transformation preserving U and mapping α to β.

Although the extremal problem is posted in the class of all holomorphic functions $f: U \to U$, the extremal mapping turns out to be the conformal mapping of U onto U. This idea will be used in the proof of the Riemann mapping theorem.

<u>Theorem</u> If $f: U \to U$ is a conformal mapping of U onto U, then $f(z) = e^{i\theta}\varphi_\alpha(z)$ for some $\alpha \in U$ and $\theta \in \mathbb{R}$.

<u>Proof</u> Let $f(\alpha) = 0$. The inverse mapping $f^{-1}: U \to U$ is also conformal and it satisfies $f^{-1}(0) = \alpha$. Hence $(*)$ yields

$$(**) \quad |f'(\alpha)| \le \frac{1}{1-|\alpha|^2}, \quad |(f^{-1})'(0)| \le 1-|\alpha|^2.$$

the equality $f^{-1}(f(z)) = z$ and the chain rule gives $(f^{-1})'(0)\, f'(\alpha) = 1$ so we have equalities in $(**)$ and from the solution to the extremal problem it follows that

$$f(z) = e^{i\theta} \frac{z-\alpha}{1-\bar{\alpha} z} . \qquad \square$$

A conformal mapping of Ω onto Ω is called <u>conformal authomorphism</u> of Ω.

The above theorem gives a description of all conformal authomorphisms of the unit disc.

<u>Montel's theorem</u>

Let $\mathcal{F} \subset H(\Omega)$. We say that $\mathcal{F}$ is a normal family if every sequence of elements of $\mathcal{F}$ has a subsequence that converges uniformly on compact sets in Ω. We do not require that the limiting function belongs to $\mathcal{F}$.