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The Mathematics of Digital Photography

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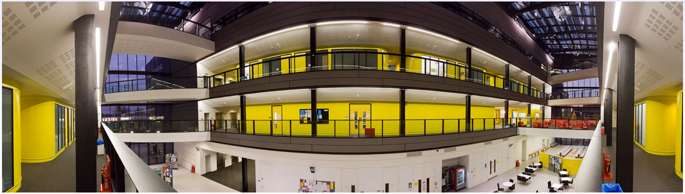

What is Color?

- Human perception; depends on light source.

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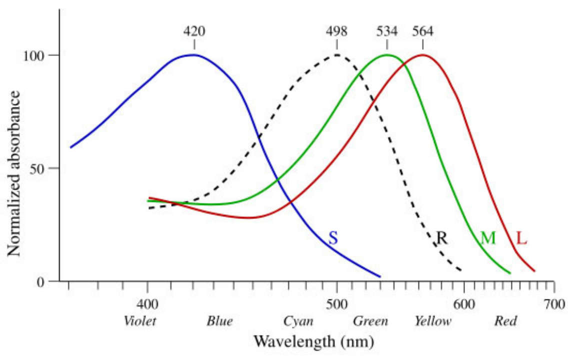
- Retina has 3 types of cones ⇒ trichromatic theory.



What is Color?

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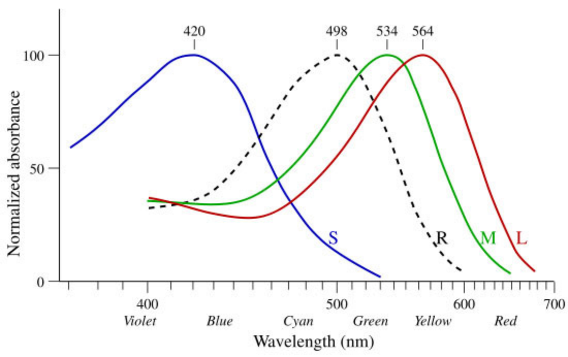
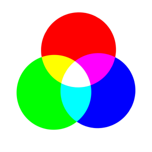
- Retina has 3 types of cones $\Rightarrow$ trichromatic theory.



- Why does yellow appear so bright?



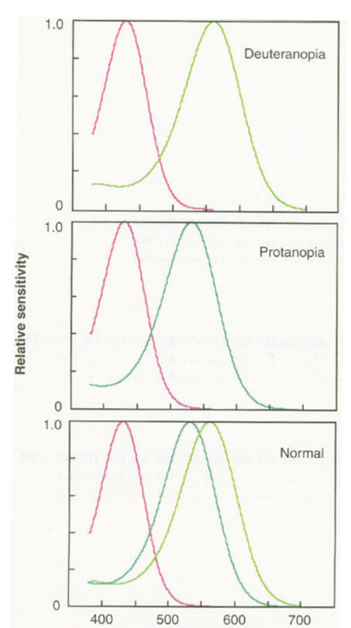
Colour Blindness

SCIENCE • VOL. 267 • 17 FEBRUARY 1995

The Chemistry of John Dalton's Color Blindness

David M. Hunt,* Kanwaljit S. Dulai, James K. Bowmaker, John D. Mollon

- John Dalton (1766–1844).
- Described his own c.b. in lecture to M/cr Lit & Phil Soc, 1794.
- He was a deuteranope.



Vector Space Model of Colour

- Model responses of the 3 cones as

$$c_i = \int_{\lambda_{\min}}^{\lambda_{\max}} s_i(\lambda) f(\lambda) d\lambda, \quad i = 1\colon 3,$$

where f = spectral distrib. of light, s_i = sensitivity of ith cone, $[\lambda_{\min}, \lambda_{\max}]$ = wavelengths of visible spectrum.

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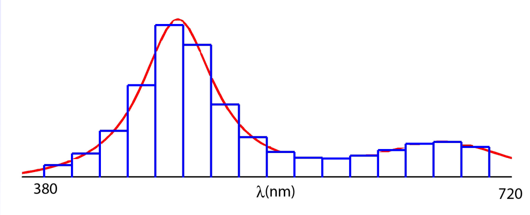
- **Discretizing** gives

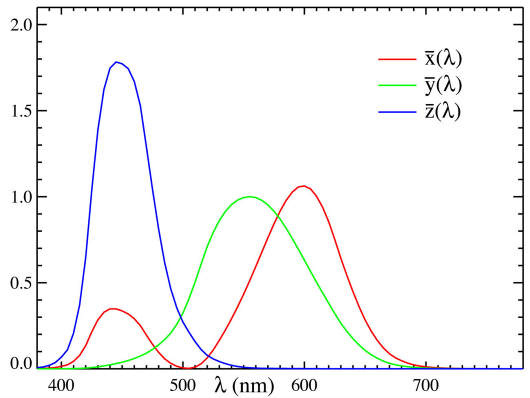
$$c = S^T f, \qquad c \in \mathbb{R}^3, \quad S \in \mathbb{R}^{n \times 3}, \quad f \in \mathbb{R}^n.$$

For standardized S, c is the **tristimulus** vector.

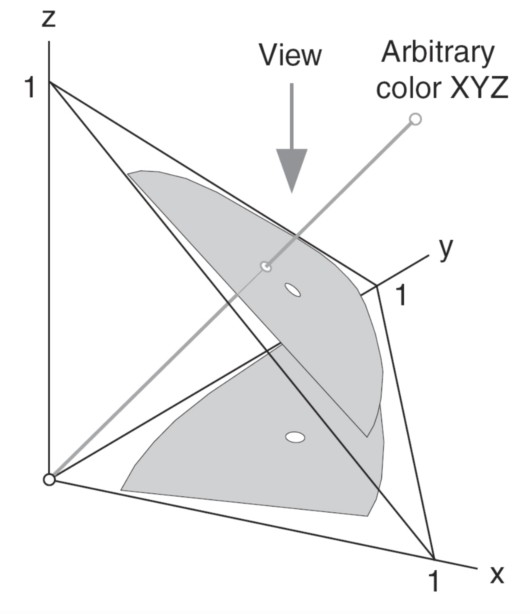


Standardization

- Commission Internationale de l'Éclairage (CIE) defined standard **colour matching functions** $s_i(\lambda)$ (1931, 1964).
- CIE RGB space.
- CIE XYZ space: nonnegative $s_i(\lambda)$, Y corresponds to perceived brightness.



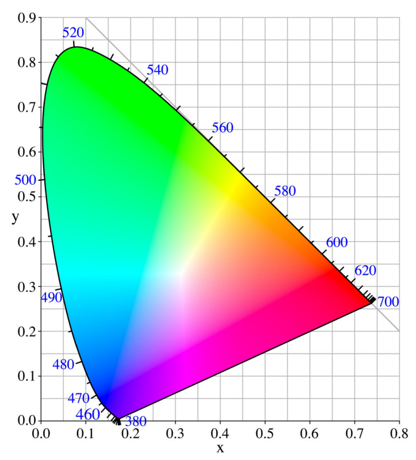
Projective Transformation

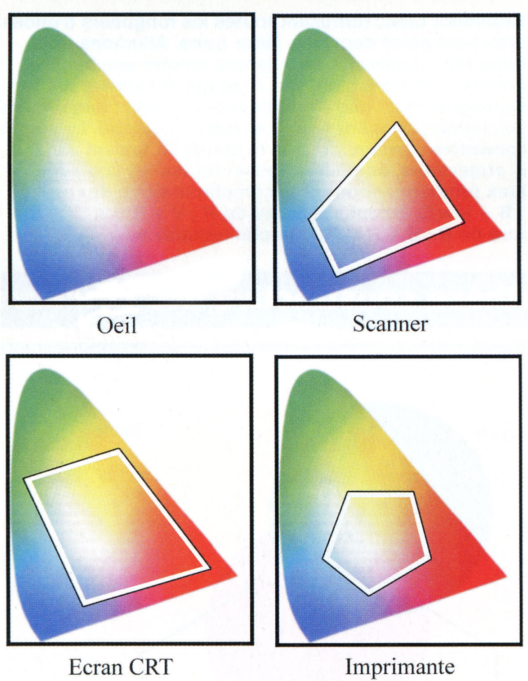


CIE Chromacity Coordinates

$$x = \frac{X}{X+Y+Z}, \qquad y = \frac{Y}{X+Y+Z} \qquad (z = 1 - x - y).$$

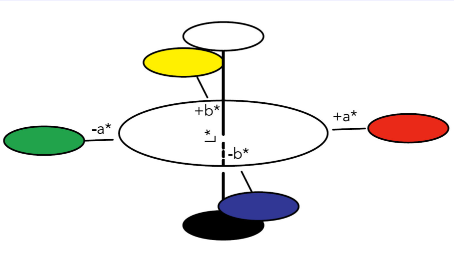
(x, y) chromacity diagram:



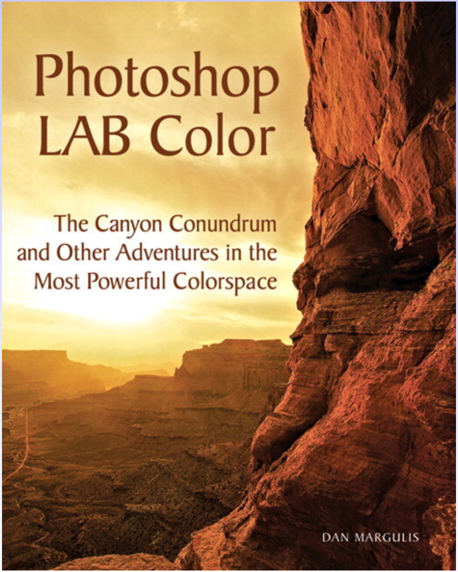

Oeil
Scanner
Ecran CRT
Imprimante

Perceptual Uniformity: LAB Space

- XYZ and RGB far from perceptually uniform.
- Search for (non)linear transformations that give more uniform colour spaces.
- **CIE L*a*b*** (**LAB**, 1976) is more uniform:
  L = lightness, A = green–magenta, B = blue–yellow.
- LAB supported by Adobe Photoshop, MATLAB Image Processing Toolbox.



Dan Margulis on LAB (2006)



CMYK

Printers use **subtractive colour model**: dyes absorb power from spectrum. To produce wide range of colours need **cyan**, **yellow**, **magenta** primaries.

But $C + M + Y = K = \text{black}$: why do we need K?

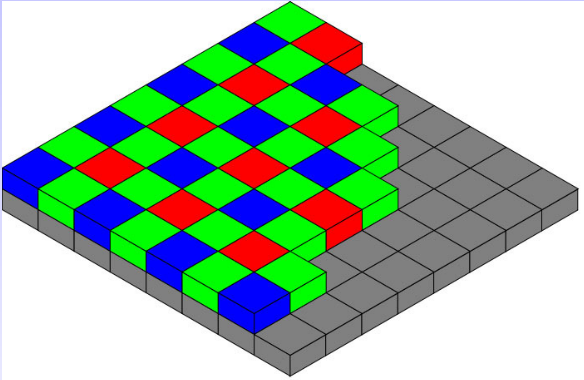
CMYK

Printers use **subtractive colour model**: dyes absorb power from spectrum. To produce wide range of colours need **cyan**, **yellow**, **magenta** primaries.

But $C + M + Y = K =$ black: why do we need K?

- Printing 3 layers makes the paper very wet.
- Black as 3 layers requires accurate registration.
- $C + M + Y$ will not give a true, deep black due to imperfections.
- Coloured ink is more expensive.

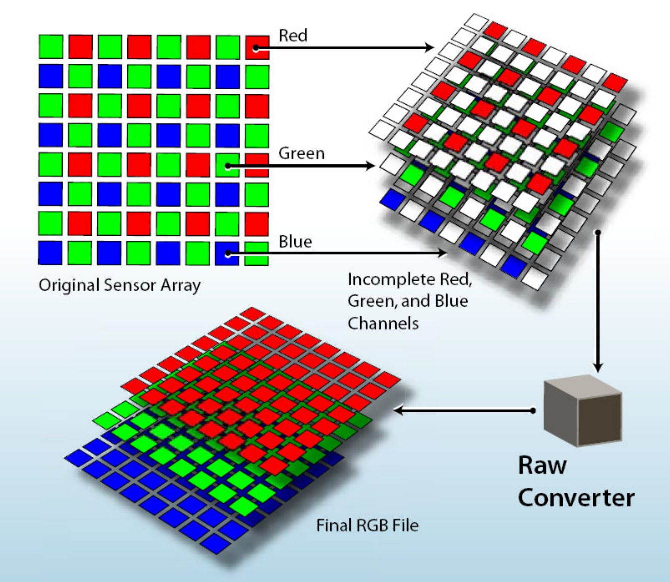
Bayer Filter



Sensor has 2 green filters for each red and blue.

- **Raw files** are the unprocessed data off the sensor.

Demosaicing



Converts to RGB colour image by **interpolation**.

Compression

Original number string

1	2	0	0	0	0	8	7	7	7	7	7	6

13

Lossless compression

1	2	0	*4	8	7	*5	6

8

Lossy compression

0	*6	7	*7

4

Jpeg is a lossy compression scheme.

Jpeg

- Compressed RGB file.
- Filesizes reduced by orders of magnitude.
- Used by all digital cameras and imaging software.

tif (LZW)	12111 k
jpg 12	1892 k
jpg 8	917 k
jpg 0	221 k

Jpeg 200 $\times$ 200 px

Quality 8

Quality 0

Colour Space

Jpeg compression first converts from RGB to YC_bC_r colour space where $Y =$ luminance, $C_b, C_r =$ blue, red chrominances, by

$$\begin{bmatrix} Y \\ C_b \\ C_r \end{bmatrix} = \begin{bmatrix} 0.299 & 0.587 & 0.114 \\ -0.1687 & -0.3313 & 0.5 \\ 0.5 & -0.4187 & -0.0813 \end{bmatrix} \begin{bmatrix} R \\ G \\ B \end{bmatrix}.$$

- Vision has poor response to spatial detail in coloured areas of same luminance $\Rightarrow$ C_b, C_r can take greater compression.
- Note: $\sum \text{row}_1 = 1$, $\sum \text{row}_2 = \sum \text{row}_3 = 0$.

Discrete Cosine Transform

Algorithm breaks image into 8×8 blocks.
For each block luminance values expressed as linear combination of cosine functions of increasing frequency

$$\ell_{x,y} = \sum_{i=0}^{7} \sum_{j=0}^{7} f_{ij} \cos\left(\frac{(2x+1)i\pi}{16}\right) \cos\left(\frac{(2y+1)i\pi}{16}\right),$$

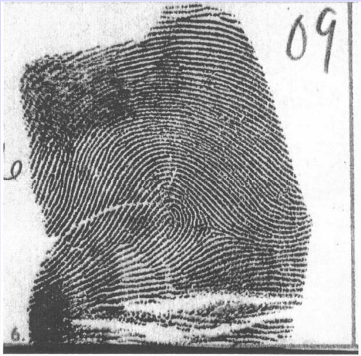
where f_{ij} computed by a discrete cosine transform:

$$f_{ij} = \sum_{x=0}^{7} \sum_{y=0}^{7} \ell_{x,y} \cos\left(\frac{(2x+1)i\pi}{16}\right) \cos\left(\frac{(2y+1)i\pi}{16}\right).$$

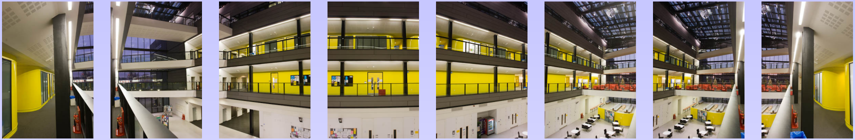
- Coefficients rounded, higher freqs de-emphasized.
- Same for C_b, C_r but more aggressive compression done.

Fingerprints—FBI

- Digitized at 500dpi $\Rightarrow$ 10Mb. Compression $\gtrsim$ 10:1 req'd.
- Standardized on wavelet compression (1993).
- Jpeg: resonance of 8-pixel tiling w/ 500dpi scans, many edges.
- Wavelets: gradual blurring as compression increased.



Panoramic Stitching



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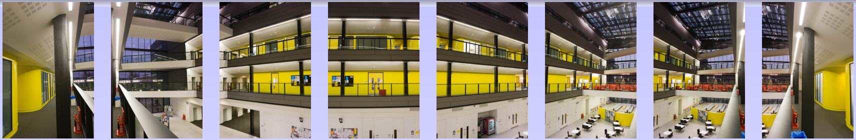

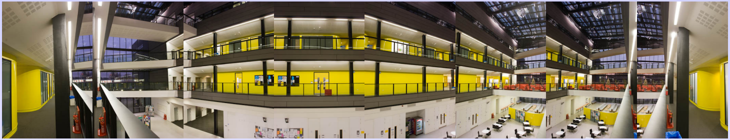

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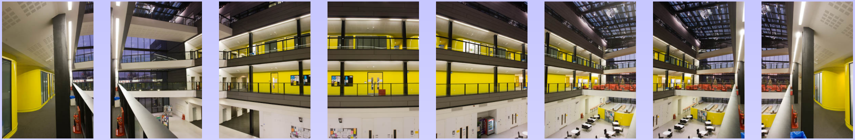

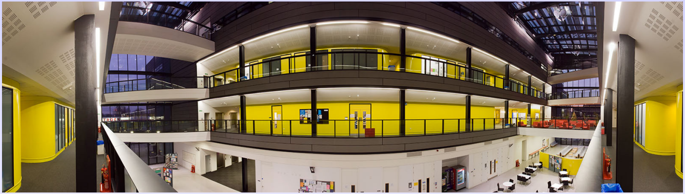

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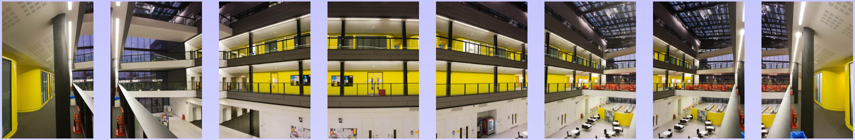
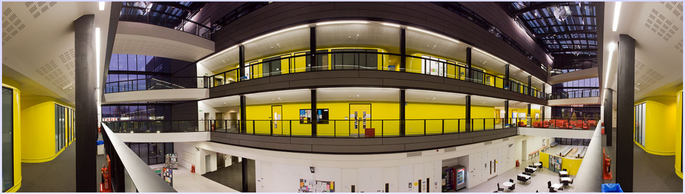
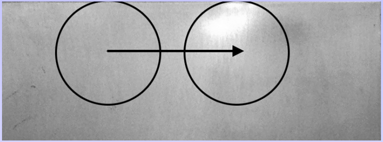


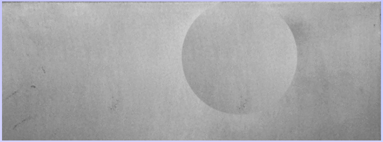


Nonlinear least squares, Levenberg–Marquardt:

$$(J^T J + \lambda D)d = J^T r, \quad J \in \mathbb{R}^{3200\times 32} \text{ for 8 images.}$$

Cloning/Healing





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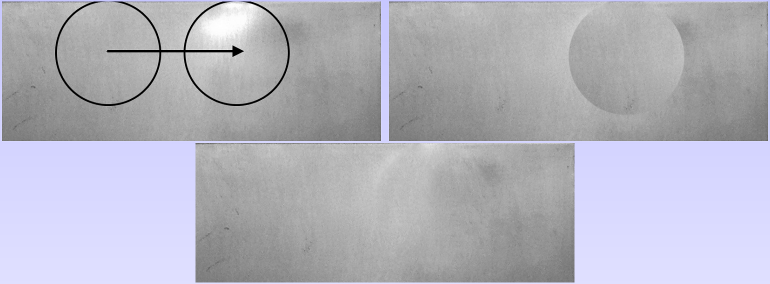

Photoshop blends the source into the target by solving the **biharmonic equation**

$$\frac{\partial^4 f}{\partial x^4} + 2\frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^4} = 0.$$

Originally used in mapping, contouring (1950s).

Transformations to improve images


Changing the nature
of investment.
Innovative, thoughtful,
Opportunities in


Changing the nature
of investment.
Opportunities in


The co-operative
We're diverse
We're ethical
We're commercial
We're ambitious
We're your future
The co-operative
We're food
We're funerals
We're pharmacies
We're travel
We're your future


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Kellogg's
CRUNCHY NUT
Kellogg's
FLAKES
Nutri-Grain
Soft Bake Bar
Nutri-Grain
Oat
Baked Bar
Kellogg's
RICE
CRACKLE
Kellogg's
COCO POPS
Kellogg's
CRUNCHY NUT
SQUARES
Special K
COCO POPS


Kellogg's
Kellogg's
N FLAKES
CRUNCHY NUT
CRUNCHY NUT
SQUARES
RICE
COCO POPS
Nutri-Grain
Soft Bake Bar
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Oat
Baked Bar
COCO POPS

Kellogg's
Crunchy Nut
Kellogg's
N FLAKES
Kellogg's
Rice
Krispies
Kellogg's
Coco
Pops
Kellogg's
Crunchy
Nut
Squares
Nutri-Grain
Soft Bake Bar
Strawberry
Nutri-Grain
Oat
Baked Bar
Special K
Coco Pops


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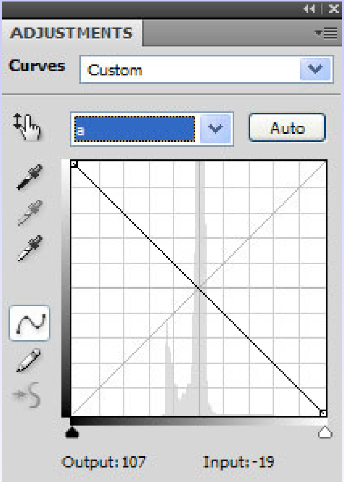
UoM turquoise is $(L, A, B) \approx (85, -12, -3)$.
Convert to LAB then $A \leftarrow -A$.



Now have $(L, A, B) \approx (85, 12, -3)$.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Mean

Median

Max

Min

Variance

Summary

- Mathematics is intrinsic to digital imaging: modelling the eye's response to colour, colour spaces, capturing images, storing and processing them.
- Modern developments in Photoshop, Lightroom, etc., rely on clever mathematical algorithms as well as exploiting faster processors—and, increasingly, GPUs.
- Most of the relevant mathematics is covered in our honours degree programme.

Talk, including references, available at
`http://www.maths.manchester.ac.uk/~higham/talks/digphot.pdf`

Acknowledgements for Graphics

- Wikipedia:
  http://en.wikipedia.org/wiki/Image:CIE1931_XYZCMF.png
  http://upload.wikimedia.org/wikipedia/commons/b/b0/CIExy1931.png
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