# Vector Calculus

*Fourth Edition*

Susan Jane Colley

**INSTRUCTOR SOLUTIONS MANUAL**

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# VECTOR CALCULUS

## FOURTH EDITION

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*Oberlin College*

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Publishing as Pearson, 75 Arlington Street, Boston, MA 02116.



ISBN-13: 978-0-321-78066-9
ISBN-10: 0-321-78066-3

www.pearsonhighered.com

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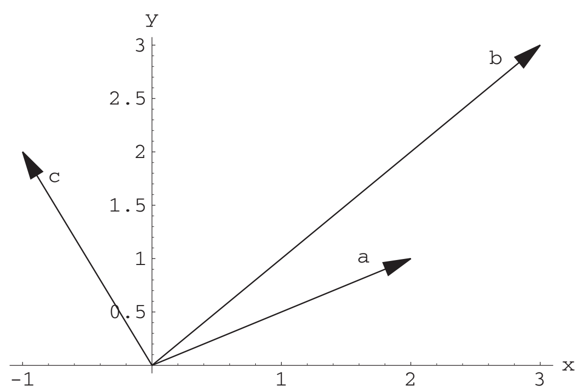
## Chapter 1

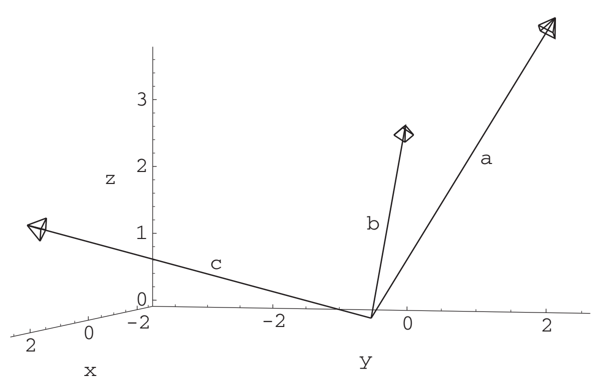
# Vectors

### 1.1 Vectors in Two and Three Dimensions

**1.** *Here we just connect the point (0, 0) to the points indicated*:




**2.** *Although more difficult for students to represent this on paper, the figures should look something like the following. Note that the origin is not at a corner of the frame box but is at the tails of the three vectors*.

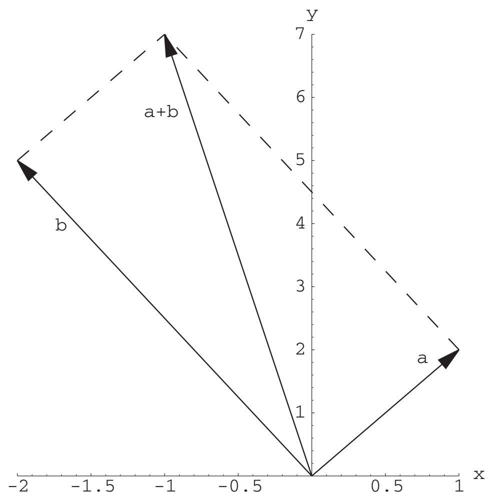


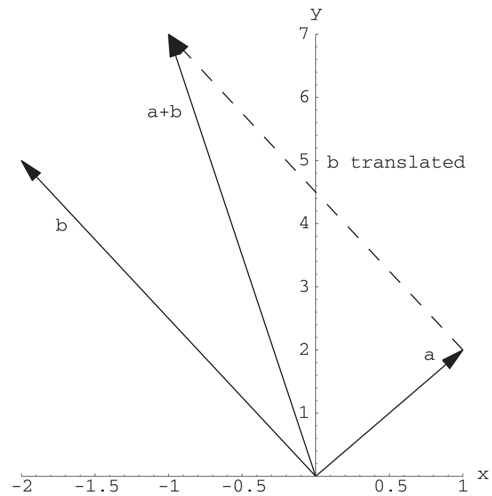

*In problems 3 and 4, we supply more detail than is necessary to stress to students what properties are being used*:

**3. (a)** $(3,1)+(-1,7)=(3+[-1],1+7)=(2,8)$.
**(b)** $-2(8,12)=(-2\cdot 8,-2\cdot 12)=(-16,-24)$.
**(c)** $(8,9)+3(-1,2)=(8+3(-1),9+3(2))=(5,15)$.
**(d)** $(1,1)+5(2,6)-3(10,2)=(1+5\cdot 2-3\cdot 10,1+5\cdot 6-3\cdot 2)=(-19,25)$.
**(e)** $(8,10)+3((8,-2)-2(4,5))=(8+3(8-2\cdot 4),10+3(-2-2\cdot 5))=(8,-26)$.

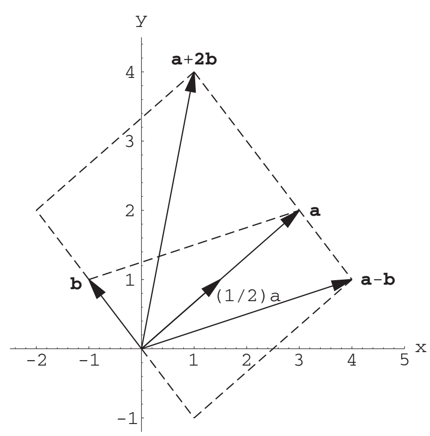
**4. (a)** $(2,1,2)+(-3,9,7)=(2-3,1+9,2+7)=(-1,10,9)$.
**(b)** $\frac{1}{2}(8,4,1)+2\left(5,-7,\frac{1}{4}\right)=\left(4,2,\frac{1}{2}\right)+\left(10,-14,\frac{1}{2}\right)=(14,-12,1)$.
**(c)** $-2\left((2,0,1)-6\left(\frac{1}{2},-4,1\right)\right)=-2((2,0,1)-(3,-24,6))=-2(-1,24,-5)=(2,-48,10)$.

**5.** We start with the two vectors **a** and **b**. We can complete the parallelogram as in the figure on the left. The vector from the origin to this new vertex is the vector $\mathbf{a}+\mathbf{b}$. In the figure on the right we have translated vector **b** so that its tail is the head of vector **a**. The sum $\mathbf{a}+\mathbf{b}$ is the directed third side of this triangle.



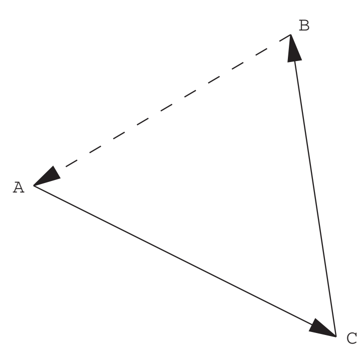

**6.** $\mathbf{a} = (3, 2) \quad \mathbf{b} = (-1, 1)$
$\mathbf{a} - \mathbf{b} = (3 - (-1), 2 - 1) = (4, 1) \qquad \frac{1}{2}\mathbf{a} = \left(\frac{3}{2}, 1\right) \quad \mathbf{a} + 2\mathbf{b} = (1, 4)$




**7.** **(a)** $\overrightarrow{AB} = (-3 - 1, 3 - 0, 1 - 2) = (-4, 3, -1) \quad \overrightarrow{BA} = -\overrightarrow{AB} = (4, -3, 1)$

**(b)** $\overrightarrow{AC} = (2 - 1, 1 - 0, 5 - 2) = (1, 1, 3)$
$\overrightarrow{BC} = (2 - (-3), 1 - 3, 5 - 1) = (5, -2, 4)$
$\overrightarrow{AC} + \overrightarrow{CB} = (1, 1, 3) - (5, -2, 4) = (-4, 3, -1)$

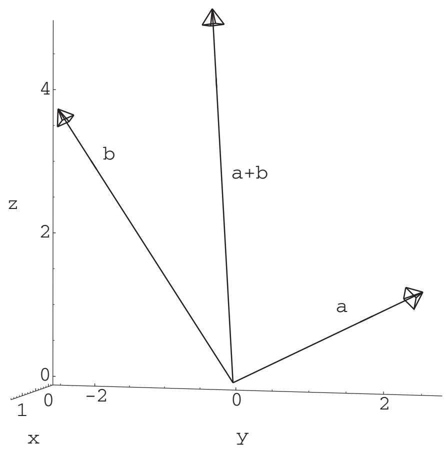
**(c)** This result is true in general:




Head-to-tail addition demonstrates this.

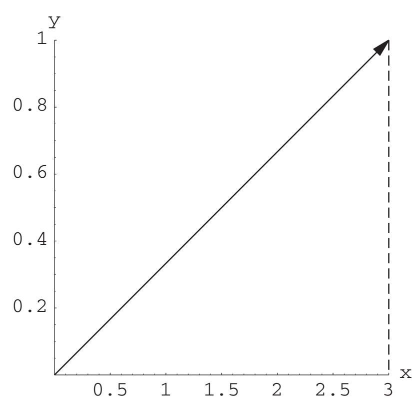
**8.** The vectors $\mathbf{a} = (1, 2, 1)$, $\mathbf{b} = (0, -2, 3)$ and $\mathbf{a} + \mathbf{b} = (1, 2, 1) + (0, -2, 3) = (1, 0, 4)$ are graphed below. *Again note that the origin is at the tails of the vectors in the figure*.

Also, $-1(1, 2, 1) = (-1, -2, -1)$. This would be pictured by drawing the vector (1, 2, 1) in the opposite direction. Finally, $4(1, 2, 1) = (4, 8, 4)$ which is four times vector **a** and so is vector **a** stretched four times as long in the same direction.

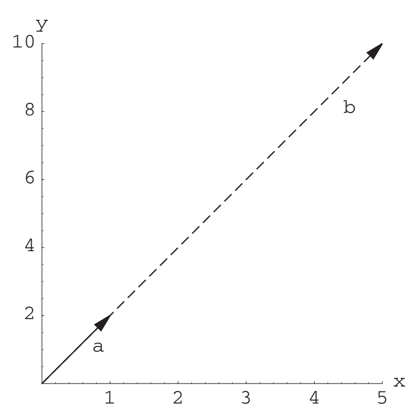



**9.** *Since the sum on the left must equal the vector on the right componentwise*:
$-12 + x = 2$, $9 + 7 = y$, and $z + -3 = 5$. Therefore, $x = 14$, $y = 16$, and $z = 8$.

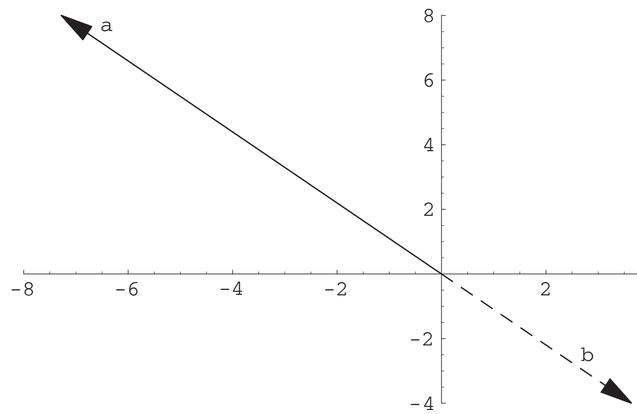
**10.** If we drop a perpendicular from (3, 1) to the $x$-axis we see that by the Pythagorean Theorem the length of the vector $(3, 1) = \sqrt{3^2 + 1^2} = \sqrt{10}$.




**11.** Notice that $\mathbf{b}$ (represented by the dotted line) $= 5\mathbf{a}$ (represented by the solid line).



**12.** Here the picture has been projected into two dimensions so that you can more clearly see that **a** (represented by the solid line) $= -2\mathbf{b}$ (represented by the dotted line).




**13.** The natural extension to higher dimensions is that we still add componentwise and that multiplying a scalar by a vector means that we multiply each component of the vector by the scalar. In symbols this means that:

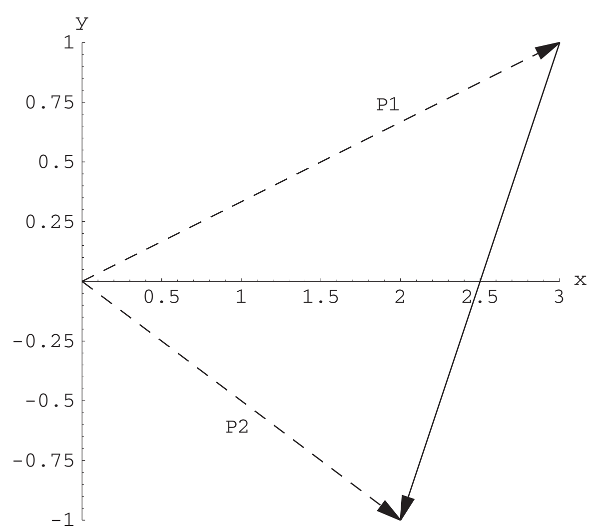
$\mathbf{a} + \mathbf{b} = (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$ and $k\mathbf{a} = (ka_1, ka_2, \ldots, ka_n)$.

In our particular examples, $(1, 2, 3, 4) + (5, -1, 2, 0) = (6, 1, 5, 4)$, and $2(7, 6, -3, 1) = (14, 12, -6, 2)$.

**14.** The diagrams for parts (a), (b) and (c) are similar to Figure 1.12 from the text. The displacement vectors are:

**(a)** $(1, 1, 5)$
**(b)** $(-1, -2, 3)$
**(c)** $(1, 2, -3)$
**(d)** $(-1, -2)$

*Note: The displacement vectors for (b) and (c) are the same but in opposite directions (i.e., one is the negative of the other)*. The displacement vector in the diagram for (d) is represented by the solid line in the figure below:

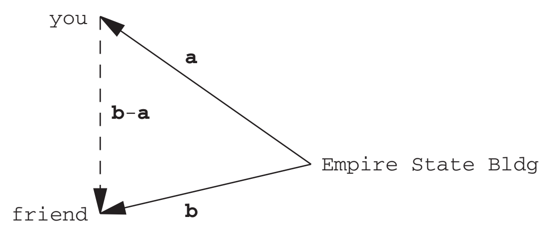



**15.** In general, we would define the displacement vector from $(a_1, a_2, \ldots, a_n)$ to $(b_1, b_2, \ldots, b_n)$ to be $(b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n)$.

In this specific problem the displacement vector from $P_1$ to $P_2$ is $(1, -4, -1, 1)$.

**16.** Let $B$ have coordinates $(x, y, z)$. Then $\overrightarrow{AB} = (x - 2, y - 5, z + 6) = (12, -3, 7)$ so $x = 14$, $y = 2$, $z = 1$ so $B$ has coordinates (14, 2, 1).

**17.** If $\mathbf{a}$ is your displacement vector from the Empire State Building and $\mathbf{b}$ your friend's, then the displacement vector from you to your friend is $\mathbf{b} - \mathbf{a}$.




**18.** Property 2 follows immediately from the associative property of the reals:

$$\begin{aligned}(\mathbf{a}+\mathbf{b})+\mathbf{c} &= ((a_1,a_2,a_3)+(b_1,b_2,b_3))+(c_1,c_2,c_3)\\ &= ((a_1+b_1,a_2+b_2,a_3+b_3)+(c_1,c_2,c_3)\\ &= ((a_1+b_1)+c_1,(a_2+b_2)+c_2,(a_3+b_3)+c_3)\\ &= (a_1+(b_1+c_1),a_2+(b_2+c_2),a_3+(b_3+c_3))\\ &= (a_1,a_2,a_3)+((b_1+c_1),(b_2+c_2),(b_3+c_3))\\ &= \mathbf{a}+(\mathbf{b}+\mathbf{c}).\end{aligned}$$

Property 3 also follows from the corresponding componentwise observation:

$$\mathbf{a}+\mathbf{0}=(a_1+0,a_2+0,a_3+0)=(a_1,a_2,a_3)=\mathbf{a}.$$

**19.** *We provide the proofs for* $\mathbf{R}^3$:

$$\begin{aligned}(1)\ (k+l)\mathbf{a} &= (k+l)(a_1,a_2,a_3)=((k+l)a_1,(k+l)a_2,(k+l)a_3)\\ &= (ka_1+la_1,ka_2+la_2,ka_3+la_3)=k\mathbf{a}+l\mathbf{a}.\\ (2)\ k(\mathbf{a}+\mathbf{b}) &= k((a_1,a_2,a_3)+(b_1,b_2,b_3))=k(a_1+b_1,a_2+b_2,a_3+b_3)\\ &= (k(a_1+b_1),k(a_2+b_2),k(a_3+b_3))=(ka_1+kb_1,ka_2+kb_2,ka_3+kb_3)\\ &= (ka_1,ka_2,ka_3)+(kb_1,kb_2,kb_3)=k\mathbf{a}+k\mathbf{b}.\\ (3)\ k(l\mathbf{a}) &= k(l(a_1,a_2,a_3))=k(la_1,la_2,la_3)\\ &= (kla_1,kla_2,kla_3)=(lka_1,lka_2,lka_3)\\ &= l(ka_1,ka_2,ka_3)=l(k\mathbf{a}).\end{aligned}$$

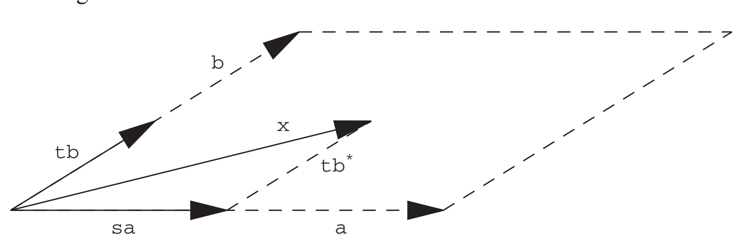
**20.** **(a)** $0\mathbf{a}$ is the zero vector. For example, in $\mathbf{R}^3$:

$$0\mathbf{a}=0(a_1,a_2,a_3)=(0\cdot a_1,0\cdot a_2,0\cdot a_3)=(0,0,0).$$

**(b)** $1\mathbf{a}=\mathbf{a}$. Again in $\mathbf{R}^3$:

$$1\mathbf{a}=1(a_1,a_2,a_3)=(1\cdot a_1,1\cdot a_2,1\cdot a_3)=(a_1,a_2,a_3)=\mathbf{a}.$$

**21.** **(a)** The head of the vector $s\mathbf{a}$ is on the $x$-axis between 0 and 2. Similarly the head of the vector $t\mathbf{b}$ lies somewhere on the vector **b**. Using the head-to-tail method, $s\mathbf{a}+t\mathbf{b}$ is the result of translating the vector $t\mathbf{b}$, in this case, to the right by $2s$ (represented in the figure by $t$**b***). The result is clearly inside the parallelogram determined by **a** and **b** (and is only on the boundary of the parallelogram if either $t$ or $s$ is 0 or 1.



**(b)** Again the vectors **a** and **b** will determine a parallelogram (with vertices at the origin, and at the heads of **a**, **b**, and $\mathbf{a}+\mathbf{b}$. The vectors $s\mathbf{a}+t\mathbf{b}$ will be the position vectors for all points in that parallelogram determined by (2, 2, 1) and (0, 3, 2).

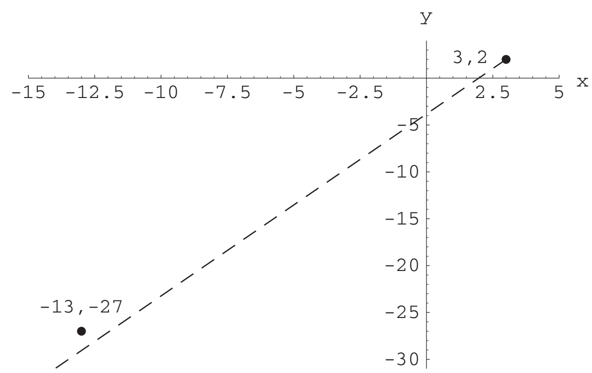
**22.** Here we are translating the situation in Exercise 21 by the vector $\overrightarrow{OP_0}$. The vectors will all be of the form $\overrightarrow{OP_0}+s\mathbf{a}+t\mathbf{b}$ for $0 \le s, t \le 1$.

**23. (a)** The speed of the flea is the length of the velocity vector $= \sqrt{(-1)^2+(-2)^2} = \sqrt{5}$ units per minute.

**(b)** After 3 minutes the flea is at $(3,2)+3(-1,-2)=(0,-4)$.

**(c)** We solve $(3,2)+t(-1,-2)=(-4,-12)$ for $t$ and get that $t=7$ minutes. Note that *both* $3-7=-4$ and $2-14=-12$.

**(d)** We can see this algebraically or geometrically: Solving the $x$ part of $(3,2)+t(-1,-2)=(-13,-27)$ we get that $t=16$. But when $t=16$, $y=-30$ not $-27$. Also in the figure below we see the path taken by the flea will miss the point $(-13,-27)$.




**24. (a)** The plane is climbing at a rate of 4 miles per hour.

**(b)** To make sure that the axes are oriented so that the plane passes over the building, the positive $x$ direction is east and the positive $y$ direction is north. Then we are heading east at a rate of 50 miles per hour at the same time we're heading north at a rate of 100 miles per hour. We are directly over the skyscraper in 1/10 of an hour or 6 minutes.

**(c)** Using our answer in (b), we have traveled for 1/10 of an hour and so we've climbed 4/10 of a mile or 2112 feet. The plane is $2112-1250$ or 862 feet about the skyscraper.

**25. (a)** Adding we get: $\mathbf{F}_1+\mathbf{F}_2=(2,7,-1)+(3,-2,5)=(5,5,4)$.

**(b)** You need a force of the same magnitude in the opposite direction, so $\mathbf{F}_3=-(5,5,4)=(-5,-5,-4)$.

**26. (a)** Measuring the force in pounds we get $(0,0,-50)$.

**(b)** The $z$ components of the two vectors along the ropes must be equal and their sum must be opposite of the $z$ component in part (a). Their $y$ components must also be opposite each other. Since the vector points in the direction (0, ±2, 1), the $y$ component will be twice the $z$ component. Together this means that the vector in the direction of $(0,-2,1)$ is $(0,-50,25)$ and the vector in the direction (0, 2, 1) is (0, 50, 25).

**27.** The force **F** due to gravity on the weight is given by $\mathbf{F}=(0,0,-10)$. The forces along the ropes are each parallel to the displacement vectors from the weight to the respective anchor points. That is, the tension vectors along the ropes are

$$\mathbf{F}_1 = k((3,0,4)-(1,2,3)) = k(2,-2,1)$$
$$\mathbf{F}_2 = l((0,3,5)-(1,2,3)) = l(-1,1,2),$$

where $k$ and $l$ are appropriate scalars. For the weight to remain in equilibrium, we must have $\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}=\mathbf{0}$, or, equivalently, that

$$k(2,-2,1)+l(-1,1,2)+(0,0,-10)=(0,0,0).$$

Taking components, we obtain a system of three equations:

$$\begin{cases} 2k-l=0 \\ -2k+l=0 \\ k+2l=10. \end{cases}$$

Solving, we find that $k = 2$ and $l = 4$, so that

$$\mathbf{F}_1=(4,-4,2) \text{ and } \mathbf{F}_2=(-4,4,8).$$

## 1.2 More about Vectors

*It may be useful to point out that the answers to Exercises 1 and 5 are the "same", but that in Exercise 1,* $\mathbf{i} = (1, 0)$ *and in Exercise 5,* $\mathbf{i} = (1, 0, 0)$. *This comes up when going the other direction in Exercises 9 and 10. In other words, it's not always clear whether the exercise "lives" in* $\mathbf{R}^2$ *or* $\mathbf{R}^3$.

**1.** $(2, 4) = 2(1, 0) + 4(0, 1) = 2\mathbf{i} + 4\mathbf{j}$.

**2.** $(9, -6) = 9(1, 0) - 6(0, 1) = 9\mathbf{i} - 6\mathbf{j}$.

**3.** $(3, \pi, -7) = 3(1, 0, 0) + \pi(0, 1, 0) - 7(0, 0, 1) = 3\mathbf{i} + \pi\mathbf{j} - 7\mathbf{k}$.

**4.** $(-1, 2, 5) = -1(1, 0, 0) + 2(0, 1, 0) + 5(0, 0, 1) = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$.

**5.** $(2, 4, 0) = 2(1, 0, 0) + 4(0, 1, 0) = 2\mathbf{i} + 4\mathbf{j}$.

**6.** $\mathbf{i} + \mathbf{j} - 3\mathbf{k} = (1, 0, 0) + (0, 1, 0) - 3(0, 0, 1) = (1, 1, -3)$.

**7.** $9\mathbf{i} - 2\mathbf{j} + \sqrt{2}\mathbf{k} = 9(1, 0, 0) - 2(0, 1, 0) + \sqrt{2}(0, 0, 1) = (9, -2, \sqrt{2})$.

**8.** $-3(2\mathbf{i} - 7\mathbf{k}) = -6\mathbf{i} + 21\mathbf{k} = -6(1, 0, 0) + 21(0, 0, 1) = (-6, 0, 21)$.

**9.** $\pi\mathbf{i} - \mathbf{j} = \pi(1, 0) - (0, 1) = (\pi, -1)$.

**10.** $\pi\mathbf{i} - \mathbf{j} = \pi(1, 0, 0) - (0, 1, 0) = (\pi, -1, 0)$.

*Note: You may want to assign both Exercises 11 and 12 together so that the students may see the difference. You should stress that the reason the results are different has nothing to do with the fact that Exercise 11 is a question about* $\mathbf{R}^2$ *while Exercise 12 is a question about* $\mathbf{R}^3$.

**11. (a)** $(3, 1) = c_1(1, 1) + c_2(1, -1) = (c_1 + c_2, c_1 - c_2)$, so $\begin{cases} c_1 + c_2 = 3, \text{ and} \\ c_1 - c_2 = 1. \end{cases}$

Solving simultaneously (for instance by adding the two equations), we find that $2c_1 = 4$, so $c_1 = 2$ and $c_2 = 1$. So $\mathbf{b} = 2\mathbf{a}_1 + \mathbf{a}_2$.

**(b)** Here $c_1 + c_2 = 3$ and $c_1 - c_2 = -5$, so $c_1 = -1$ and $c_2 = 4$. So $\mathbf{b} = -\mathbf{a}_1 + 4\mathbf{a}_2$.

**(c)** More generally, $(b_1, b_2) = (c_1 + c_2, c_1 - c_2)$, so $\begin{cases} c_1 + c_2 = b_1, \text{ and} \\ c_1 - c_2 = b_2. \end{cases}$

Again solving simultaneously, $c_1 = \dfrac{b_1 + b_2}{2}$ and $c_2 = \dfrac{b_1 - b_2}{2}$. So

$$\mathbf{b} = \left(\frac{b_1 + b_2}{2}\right)\mathbf{a}_1 + \left(\frac{b_1 - b_2}{2}\right)\mathbf{a}_2.$$

**12.** Note that $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$, so really we are only working with two (linearly independent) vectors.

**(a)** $(5, 6, -5) = c_1(1, 0, -1) + c_2(0, 1, 0) + c_3(1, 1, -1)$; this gives us the equations:

$$\begin{cases} 5 = c_1 + c_3 \\ 6 = c_2 + c_3 \\ -5 = -c_1 - c_3. \end{cases}$$

The first and last equations contain the same information and so we have infinitely many solutions. You will quickly see one by letting $c_3 = 0$. Then $c_1 = 5$ and $c_2 = 6$. So we could write $\mathbf{b} = 5\mathbf{a}_1 + 6\mathbf{a}_2$. More generally, you can choose any value for $c_1$ and then let $c_2 = c_1 + 1$ and $c_3 = 5 - c_1$.

**(b)** We cannot write (2, 3, 4) as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. Here we get the equations:

$$\begin{cases} c_1 + c_3 = 2 \\ c_2 + c_3 = 3 \\ -c_1 - c_3 = 4. \end{cases}$$

The first and last equations are inconsistent and so the system cannot be solved.

**(c)** As we saw in part (b), not all vectors in $\mathbf{R}^3$ can be written in terms of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. In fact, only vectors of the form $(a, b, -a)$ can be written in terms of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. For your students who have had linear algebra, this is because the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are not linearly independent.

*Note: As pointed out in the text, the answers for 13–21 are not unique.*

**13.** $\mathbf{r}(t) = (2, -1, 5) + t(1, 3, -6)$ so $\begin{cases} x = 2 + t \\ y = -1 + 3t \\ z = 5 - 6t. \end{cases}$

**14.** $\mathbf{r}(t) = (12, -2, 0) + t(5, -12, 1)$ so $\begin{cases} x = 12 + 5t \\ y = -2 - 12t \\ z = t. \end{cases}$

**15.** $\mathbf{r}(t) = (2, -1) + t(1, -7)$ so $\begin{cases} x = 2 + t \\ y = -1 - 7t. \end{cases}$

**16.** $\mathbf{r}(t) = (2, 1, 2) + t(3 - 2, -1 - 1, 5 - 2)$ so $\begin{cases} x = 2 + t \\ y = 1 - 2t \\ z = 2 + 3t. \end{cases}$

**17.** $\mathbf{r}(t) = (1, 4, 5) + t(2 - 1, 4 - 4, -1 - 5)$ so $\begin{cases} x = 1 + t \\ y = 4 \\ z = 5 - 6t. \end{cases}$

**18.** $\mathbf{r}(t) = (8, 5) + t(1 - 8, 7 - 5)$ so $\begin{cases} x = 8 - 7t \\ y = 5 + 2t. \end{cases}$

*Note: In higher dimensions, we switch our notation to $x_i$.*

**19.** $\mathbf{r}(t) = (1, 2, 0, 4) + t(-2, 5, 3, 7)$ so $\begin{cases} x_1 = 1 - 2t \\ x_2 = 2 + 5t \\ x_3 = 3t \\ x_4 = 4 + 7t. \end{cases}$

**20.** $\mathbf{r}(t) = (9, \pi, -1, 5, 2) + t(-1 - 9, 1 - \pi, \sqrt{2} + 1, 7 - 5, 1 - 2)$ so $\begin{cases} x_1 = 9 - 10t \\ x_2 = \pi + (1 - \pi)t \\ x_3 = -1 + (\sqrt{2} + 1)t \\ x_4 = 5 + 2t \\ x_5 = 2 - t. \end{cases}$

**21. (a)** $\mathbf{r}(t) = (-1, 7, 3) + t(2, -1, 5)$ so $\begin{cases} x = -1 + 2t \\ y = 7 - t \\ z = 3 + 5t. \end{cases}$

**(b)** $\mathbf{r}(t) = (5, -3, 4) + t(0 - 5, 1 + 3, 9 - 4)$ so $\begin{cases} x = 5 - 5t \\ y = -3 + 4t \\ z = 4 + 5t. \end{cases}$

**(c)** Of course, there are infinitely many solutions. For our variation on the answer to (a) we note that a line parallel to the vector $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ is also parallel to the vector $-(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$ so another set of equations for part (a) is:

$$\begin{cases} x = -1 - 2t \\ y = 7 + t \\ z = 3 - 5t. \end{cases}$$

For our variation on the answer to (b) we note that the line passes through both points so we can set up the equation with respect to the other point:

$$\begin{cases} x = -5t \\ y = 1 + 4t \\ z = 9 + 5t. \end{cases}$$

**(d)** The symmetric forms are:

$$\frac{x+1}{2} = 7 - y = \frac{z-3}{5} \qquad \text{(for (a))}$$

$$\frac{5-x}{5} = \frac{y+3}{4} = \frac{z-4}{5} \qquad \text{(for (b))}$$

$$\frac{x+1}{-2} = y - 7 = \frac{z-3}{-5} \qquad \text{(for the variation of (a))}$$

$$\frac{x}{-5} = \frac{y-1}{4} = \frac{z-9}{5} \qquad \text{(for the variation of (b))}$$

**22.** Solve for $t$ in each of the parametric equations. Thus

$$t = \frac{x-5}{-2}, t = \frac{y-1}{3}, t = \frac{z+4}{6}$$

and the symmetric form is

$$\frac{x-5}{-2} = \frac{y-1}{3} = \frac{z+4}{6}.$$

**23.** Solving for $t$ in each of the parametric equations gives $t = x-7$, $t = (y+9)/3$, and $t = (z-6)/(-8)$, so that the symmetric form is

$$\frac{x-7}{1} = \frac{y+9}{3} = \frac{z-6}{-8}.$$

**24.** Set each piece of the equation equal to $t$ and solve:

$$\frac{x-2}{5} = t \Rightarrow x-2 = 5t \Rightarrow x = 2+5t$$

$$\frac{y-3}{-2} = t \Rightarrow y-3 = -2t \Rightarrow y = 3-2t$$

$$\frac{z+1}{4} = t \Rightarrow z+1 = 4t \Rightarrow z = -1+4t.$$

**25.** Let $t = (x+5)/3$. Then $x = 3t-5$. In view of the symmetric form, we also have that $t = (y-1)/7$ and $t = (z+10)/(-2)$. Hence a set of parametric equations is $x = 3t-5$, $y = 7t+1$, and $z = -2t-10$.

*Note: In Exercises 26–29, we could say for certain that two lines are not the same if the vectors were not multiples of each other. In other words, it takes two pieces of information to specify a line. You either need two points, or a point and a direction (or in the case of* $\mathbf{R}^2$*, equivalently, a slope).*

**26.** The first line is parallel to the vector $\mathbf{a}_1 = (5, -3, 4)$, while the second is parallel to $\mathbf{a}_2 = (10, -5, 8)$. Since $\mathbf{a}_1$ and $\mathbf{a}_2$ are not parallel, the lines cannot be the same.

**27.** If we multiply each of the pieces in the second symmetric form by $-2$, we are effectively just traversing the same path at a different speed and with the opposite orientation. So the second set of equations becomes:

$$\frac{x+1}{3} = \frac{y+6}{7} = \frac{z+5}{5}.$$

This looks a lot more like the first set of equations. If we now subtract one from each piece of the second set of equations (as suggested in the text), we are effectively just changing our initial point but we are still on the same line:

$$\frac{x+1}{3} - \frac{3}{3} = \frac{y+6}{7} - \frac{7}{7} = \frac{z+5}{5} - \frac{5}{5}.$$

We have transformed the second set of equations into the first and therefore see that they both represent the same line in $\mathbf{R}^3$.

**28.** If you first write the equation of the two lines in vector form, we can see immediately that their direction vectors are the same so either they are parallel or they are the same line:

$$\mathbf{r_1}(t) = (-5, 2, 1) + t(2, 3, -6)$$

$$\mathbf{r_2}(t) = (1, 11, -17) - t(2, 3, -6).$$

The first line contains the point $(-5, 2, 1)$. If the second line contains $(-5, 2, 1)$, then the equations represent the same line. Solve just the $x$ component to get that $-5 = 1-2t \Rightarrow t = 3$. Checking we see that $\mathbf{r_2}(3) = (1, 11, -17) - 3(2, 3, -6) = (-5, 2, 1)$ so the lines are the same.

**29.** Here again the vector forms of the two lines can be written so that we see their headings are the same:

$$\mathbf{r_1}(t) = (2, -7, 1) + t(3, 1, 5)$$

$$\mathbf{r_2}(t) = (-1, -8, -3) + 2t(3, 1, 5).$$

The point $(2, -7, 1)$ is on line one, so we will check to see if it is also on line two. As in Exercise 28 we check the equation for the $x$ component and see that $-1+6t = 2 \Rightarrow t = 1/2$. Checking we see that $\mathbf{r_2}(1/2) = (-1, -8, -3) + (1/2)(2)(3, 1, 5) = (2, -7, 2) \neq (2, -7, 1)$ so the equations do not represent the same lines.

*Note: It is a good idea to assign both Exercises 30 and 31 together. Although they look similar, there is a difference that students might miss.*

**30.** If you make the substitution $u = t^3$, the equations become: $\begin{cases} x = 3u + 7, \\ y = -u + 2, \text{ and} \\ z = 5u + 1. \end{cases}$

The map $u = t^3$ is a bijection. The important fact is that $u$ takes on exactly the same values that $t$ does, just at different times. Since $u$ takes on all reals, the parametric equations do determine a line (it's just that the speed along the line is not constant).

**31.** This time if you make the substitution $u = t^2$, the equations become: $\begin{cases} x = 5u - 1, \\ y = 2u + 3, \text{ and} \\ z = -u + 1. \end{cases}$

The problem is that $u$ cannot take on negative values so these parametric equations are for a ray with endpoint $(-1, 3, 1)$ and heading $(5, 2, -1)$.

**32.** **(a)** The vector form of the equations is: $\mathbf{r}(t) = (7, -2, 1) + t(2, 1, -3)$. The initial point is then $\mathbf{r}(0) = (7, -2, 1)$, and after 3 minutes the bird is at $\mathbf{r}(3) = (7, -2, 1) + 3(2, 1, -3) = (13, 1, -8)$.

**(b)** $(2, 1, -3)$

**(c)** We only need to check one component (say the $x$): $7 + 2t = 34/3 \Rightarrow t = 13/6$. Checking we see that $\mathbf{r}\left(\frac{13}{6}\right) = (7, -2, 1) + \left(\frac{13}{6}\right)(2, 1, -3) = \left(\frac{34}{3}, \frac{1}{6}, -\frac{11}{2}\right)$.

**(d)** As in part (c), we'll check the $x$ component and see that $7 + 2t = 17$ when $t = 5$. We then check to see that $\mathbf{r}(5) = (7, -2, 1) + 5(2, 1, -3) = (17, 3, -14) \neq (17, 4, -14)$ so, no, the bird doesn't reach $(17, 4, -14)$.

**33.** We can substitute the parametric forms of $x$, $y$, and $z$ into the equation for the plane and solve for $t$. So $(3t - 5) + 3(2 - t) - (6t) = 19$ which gives us $t = -3$. Substituting back in the parametric equations, we find that the point of intersection is $(-14, 5, -18)$.

**34.** Using the same technique as in Exercise 33, $5(1 - 4t) - 2(t - 3/2) + (2t + 1) = 1$ which simplifies to $t = 2/5$. This means the point of intersection is $(-3/5, -11/10, 9/5)$.

**35.** We will set each of the coordinate equations equal to zero in turn and substitute that value of $t$ into the other two equations.

$$x = 2t - 3 = 0 \Rightarrow t = 3/2. \text{ When } t = 3/2, y = 13/2 \text{ and } z = 7/2.$$

$$y = 3t + 2 = 0 \Rightarrow t = -2/3, \text{ so } x = -13/3 \text{ and } z = 17/3.$$

$$z = 5 - t = 0 \Rightarrow t = 5, \text{ so } x = 7 \text{ and } y = 17.$$

The points are (0, 13/2, 7/2), $(-13/3, 0, 17/3)$, and (7, 17, 0).

**36.** We could show that two points on the line are also in the plane or that for points on the line:
$2x - y + 4z = 2(5 - t) - (2t - 7) + 4(t - 3) = 5$, so they are in the plane.

**37.** For points on the line we see that $x - 3y + z = (5 - t) - 3(2t - 3) + (7t + 1) = 15$, so the line does not intersect the plane.

**38.** First we parametrize the line by setting $t = (x - 3)/6$, which gives us $x = 6t + 3$, $y = 3t - 2$, $z = 5t$. Plugging these parametric values into the equation for the plane gives

$$2(6t + 3) - 5(3t - 2) + 3(5t) + 8 = 0 \iff 12t + 24 = 0 \iff t = -2.$$

The parameter value $t = -2$ yields the point $(6(-2) + 3, 3(-2) - 2, 5(-2)) = (-9, -8, -10)$.

**39.** We find parametric equations for the line by setting $t = (x - 3)/(-2)$, so that $x = 3 - 2t$, $y = t + 5$, $z = 3t - 2$. Plugging these parametric values into the equation for the plane, we find that

$$3(3 - 2t) + 3(t + 5) + (3t - 2) = 9 - 6t + 3t + 15 + 3t - 2 = 22$$

for *all* values of $t$. Hence the line is contained in the plane.

**40.** Again we find parametric equations for the line. Set $t = (x + 4)/3$, so that $x = 3t - 4$, $y = 2 - t$, $z = 1 - 9t$. Plugging these parametric values into the equation for the plane, we find that

$$2(3t - 4) - 3(2 - t) + (1 - 9t) = 7 \iff 6t - 8 - 6 + 3t + 1 - 9t = 7 \iff -13 = 7.$$

Hence we have a contradiction; that is, *no* value of $t$ will yield a point on the line that is also on the plane. Thus the line and the plane do *not* intersect.

**41.** We just plug the parametric expressions for $x$, $y$, $z$ into the equation for the surface:

$$\frac{(at+a)^2}{a^2}+\frac{b^2}{b^2}-\frac{(ct+c)^2}{c^2}=\frac{c^2(t+1)^2}{a^2}+1-\frac{c^2(t+1)^2}{c^2}=1$$

for all values of $t \in \mathbf{R}$. Hence all points on the line satisfy the equation for the surface.

**42.** As explained in the text, we can't just set the two sets of equations equal to each other and solve. If the two lines intersect at a point, we may get to that point at two different times. Let's call these times $t_1$ and $t_2$ and solve the equations

$$\begin{cases} 2t_1+3=15-7t_2, \\ 3t_1+3=t_2-2, \text{ and} \\ 2t_1+1=3t_2-7. \end{cases}$$

Eliminate $t_1$ by subtracting the third equation from the first to get $t_2 = 2$. Substitute back into any of the equations to get $t_1 = -1$. Using either set of equations, you'll find that the point of intersection is $(1, 0, -1)$.

**43.** The way the problem is phrased tips us off that something is going on. Let's handle this the same way we did in Exercise 42.

$$\begin{cases} 2t_1+1=3t_2+1, \\ -3t_1=t_2+5, \text{ and} \\ t_1-1=7-t_2. \end{cases}$$

Adding the last two equations eliminates $t_2$ and gives us $t_1 = 13/2$. This corresponds to the point (14, $-39/2$, 11/2). Substituting this value of $t_1$ into the third equation gives us $t_2 = 3/2$, while substituting this into the first equation gives us $t_2 = 13/3$. This inconsistency tells us that the second line doesn't pass through the point $(14, -39/2, 11/2)$.

**44.** **(a)** The distance is $\sqrt{(3t-5+2)^2+(1-t-1)^2+(4t+7-5)^2}=\sqrt{26t^2-2t+13}$.

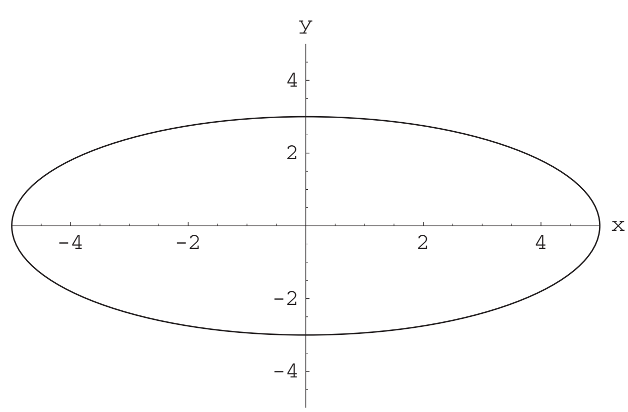
**(b)** Using a standard first year calculus trick, the distance is minimized when the square of the distance is minimized. So we find $D = 26t^2 - 2t + 13$ is minimized (at the vertex of the parabola) when $t = 1/26$. Substitute back into our answer for (a) to find that the minimal distance is $\sqrt{337/26}$.

**45.** **(a)** As in Example 2, this is the equation of a circle of radius 2 centered at the origin. The difference is that you are traveling around it three times as fast. This means that if $t$ varied between 0 and $2\pi$ that the circle would be traced three times.

**(b)** This is just like part (a) except the radius of the circle is 5.

**(c)** This is just like part (b) except the $x$ and $y$ coordinates have been switched. This is the same as reflecting the circle about the line $y = x$ and so this is also a circle of radius 5. If you care, the circle in (b) was drawn starting at the point (5, 0) counterclockwise while this circle is drawn starting at (0, 5) clockwise.

**(d)** This is an ellipse with major axis along the $x$-axis intersecting it at $(\pm 5, 0)$ and minor axis along the $y$-axis intersecting it at $(0, \pm 3)$ : $\frac{x^2}{25}+\frac{y^2}{9}=1$.



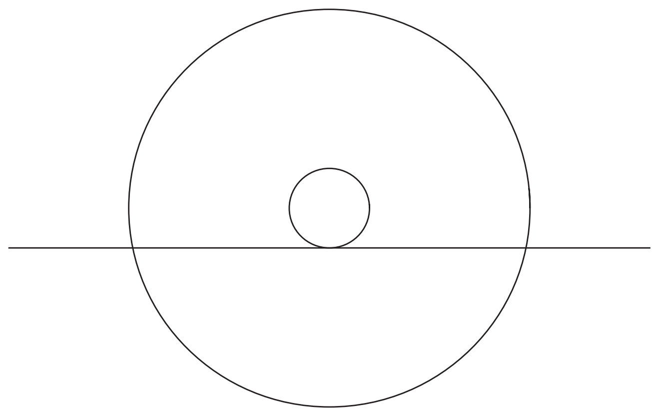

**46.** The discussion in the text of the cycloid looked at the path traced by a point on the circumference of a circle of radius $a$ as it is rolled without slipping on the $x$-axis. The vector from the origin to our point $P$ was split into two pieces: $\overrightarrow{OA}$ (the vector from the origin to the center of the circle) and $\overrightarrow{AP}$ (the vector from the center of the circle to $P$). This split remains the same in our problem.

The center of the circle is always $a$ above the $x$-axis, and after the wheel has rolled through a central angle of $t$ radians the $x$ coordinate is just *at*. So $\overrightarrow{OA} = (at, a)$. This does not change in our problem.

The vector $\overrightarrow{AP}$ was calculated to be $(-a\sin t, -a\cos t)$. The direction of the vector is still correct but the length is not. If we are $b$ units from the center then $\overrightarrow{AP} = -b(\sin t, \cos t)$.

We conclude then that the parametric equations are $x = at - b\sin t, y = a - b\cos t$. When $a = b$ this is the case of the cycloid described in the text; when $a > b$ we have the curtate cycloid; and when $a < b$ we have the prolate cycloid.

For a picture of how to generate one consider the diagram:



Here the inner circle is rolling along the ground and the prolate cycloid is the path traced by a point on the outer circle. There is a classic toy with a plastic wheel that runs along a handheld track, but your students are too young for that. You could pretend that the big circle is the end of a round roast and the little circle is the end of a skewer. In a regular rotisserie the roast would just rotate on the skewer, but we could imagine rolling the skewer along the edges of the grill. The motion of a point on the outside of the roast would be a prolate cycloid.

**47.** You are to picture that the circular dispenser stays still so Egbert has to unwind around the dispenser. The direction is $(\cos\theta, \sin\theta)$. The length is the radius of the circle $a$, plus the amount of tape that's been unwound. The tape that's been unwound is the distance around the circumference of the circle. This is $a\theta$ where $\theta$ is again in radians. The equation is therefore $(x, y) = a(1 + \theta)(\cos\theta, \sin\theta)$.

## 1.3 The Dot Product

*Exercises 1–16 are just straightforward calculations. For 1–6 use Definition 3.1 and formula (1). For 7–11 use formula (4). For 12–16 use formula (5).*

**1.** $(1, 5)\cdot(-2, 3) = 1(-2) + 5(3) = 13, \quad \|(1, 5)\| = \sqrt{1^2 + 5^2} = \sqrt{26},$
$\|(-2, 3)\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$

**2.** $(4, -1)\cdot(1/2, 2) = 4(1/2) - 1(2) = 0, \quad \|(4, -1)\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$
$\|(1/2, 2)\| = \sqrt{(1/2)^2 + 2^2} = \sqrt{17}/2.$

**3.** $(-1, 0, 7)\cdot(2, 4, -6) = -1(2) + 0(4) + 7(-6) = -44, \quad \|(-1, 0, 7)\| = \sqrt{(-1)^2 + 0^2 + 7^2} = \sqrt{50} = 5\sqrt{2}$, and $\|(2, 4, -6)\| = \sqrt{2^2 + 4^2 + (-6)^2} = \sqrt{56} = 2\sqrt{14}.$

**4.** $(2, 1, 0)\cdot(1, -2, 3) = 2(1)+1(-2)+0(3) = 0, \quad \|(2, 1, 0)\| = \sqrt{2^2 + 1} = \sqrt{5}$, and $\|(1, -2, 3)\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}.$

**5.** $(4\mathbf{i} - 3\mathbf{j} + \mathbf{k})\cdot(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 4(1) + -3(1) + 1(1) = 2, \quad \|4\mathbf{i} - 3\mathbf{j} + \mathbf{k}\| = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{26}$, and $\|\mathbf{i} + \mathbf{j} + \mathbf{k}\| = \sqrt{1 + 1 + 1} = \sqrt{3}.$

**6.** $(\mathbf{i} + 2\mathbf{j} - \mathbf{k})\cdot(-3\mathbf{j} + 2\mathbf{k}) = 2(-3) - 1(2) = -8, \quad \|\mathbf{i} + 2\mathbf{j} - \mathbf{k}\| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$, and $\| - 3\mathbf{j} + 2\mathbf{k}\| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}.$

**7.** $\theta = \cos^{-1}\left(\dfrac{(\sqrt{3}\mathbf{i} + \mathbf{j})\cdot(-\sqrt{3}\mathbf{i} + \mathbf{j})}{\|(\sqrt{3}\mathbf{i} + \mathbf{j})\|\ \| - \sqrt{3}\mathbf{i} + \mathbf{j}\|}\right) = \cos^{-1}\left(\dfrac{-3 + 1}{(2)(2)}\right) = \cos^{-1}\left(\dfrac{-1}{2}\right) = \dfrac{2\pi}{3}.$

**8.** $\theta = \cos^{-1}\left(\frac{(-1,2)\cdot(3,1)}{\|(-1,2)\|\,\|(3,1)\|}\right) = \cos^{-1}\left(\frac{-3+2}{\sqrt{5}\,\sqrt{10}}\right) = \cos^{-1}\left(-\frac{1}{5\sqrt{2}}\right)$.

**9.** $\theta = \cos^{-1}\left(\frac{(\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})}{\|\mathbf{i}+\mathbf{j}\|\,\|\mathbf{i}+\mathbf{j}+\mathbf{k}\|}\right) = \cos^{-1}\left(\frac{1+1}{\sqrt{2}\,\sqrt{3}}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$.

**10.** $\theta = \cos^{-1}\left(\frac{(\mathbf{i}+\mathbf{j}-\mathbf{k})\cdot(-\mathbf{i}+2\mathbf{j}+2\mathbf{k})}{\|\mathbf{i}+\mathbf{j}-\mathbf{k}\|\,\|-\mathbf{i}+2\mathbf{j}+2\mathbf{k}\|}\right) = \cos^{-1}\left(\frac{-1+2-2}{(\sqrt{3})(\sqrt{3})}\right) = \cos^{-1}\left(\frac{-1}{3\sqrt{3}}\right)$.

**11.** $\theta = \cos^{-1}\left(\frac{(1,-2,3)\cdot(3,-6,-5)}{\|(1,-2,3)\|\,\|(3,-6,-5)\|}\right) = \cos^{-1}\left(\frac{3+12-15}{\sqrt{14}\sqrt{70}}\right) = \cos^{-1}(0) = \frac{\pi}{2}$.

*Note: The answers to 12 and 13 are the same. You may want to assign both exercises and ask your students why this should be true. You might then want to ask what would happen if vector* **a** *was the same but vector* **b** *was divided by* $\sqrt{2}$.

**12.** $\text{proj}_{\mathbf{i}+\mathbf{j}}(2\mathbf{i}+3\mathbf{j}-\mathbf{k}) = \left(\frac{(\mathbf{i}+\mathbf{j})\cdot(2\mathbf{i}+3\mathbf{j}-\mathbf{k})}{(\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}+\mathbf{j})}\right)(\mathbf{i}+\mathbf{j}) = \frac{2+3}{1+1}(1,1,0) = \left(\frac{5}{2},\frac{5}{2},0\right)$.

**13.** $\text{proj}_{\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}}(2\mathbf{i}+3\mathbf{j}-\mathbf{k}) = \left(\frac{\left(\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\right)\cdot(2\mathbf{i}+3\mathbf{j}-\mathbf{k})}{\left(\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\right)\cdot\left(\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\right)}\right)\left(\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\right) = \frac{\frac{1}{\sqrt{2}}(2+3)}{\frac{1+1}{2}}\frac{(1,1,0)}{\sqrt{2}} = \left(\frac{5}{2},\frac{5}{2},0\right)$.

**14.** $\text{proj}_{5\mathbf{k}}(\mathbf{i}-\mathbf{j}+2\mathbf{k}) = \left(\frac{(5\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}+2\mathbf{k})}{(5\mathbf{k})\cdot(5\mathbf{k})}\right)(5\mathbf{k}) = \frac{10}{25}(5\mathbf{k}) = 2\mathbf{k}$.

**15.** $\text{proj}_{-3\mathbf{k}}(\mathbf{i}-\mathbf{j}+2\mathbf{k}) = \left(\frac{(-3\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}+2\mathbf{k})}{(-3\mathbf{k})\cdot(-3\mathbf{k})}\right)(-3\mathbf{k}) = \frac{-6}{9}(-3\mathbf{k}) = 2\mathbf{k}$.

**16.** $\text{proj}_{\mathbf{i}+\mathbf{j}+2\mathbf{k}}(2\mathbf{i}-4\mathbf{j}+\mathbf{k}) = \left(\frac{(\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(2\mathbf{i}-4\mathbf{j}+\mathbf{k})}{(\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+2\mathbf{k})}\right)(\mathbf{i}+\mathbf{j}+2\mathbf{k}) = \frac{2-4+2}{1+1+4}(1,1,2) = 0$.

**17.** We just divide the vector by its length: $\frac{2\mathbf{i}-\mathbf{j}+\mathbf{k}}{||2\mathbf{i}-\mathbf{j}+\mathbf{k}||} = \frac{1}{\sqrt{6}}(2,-1,1)$.

**18.** Here we take the negative of the vector divided by its length: $\frac{\mathbf{i}-2\mathbf{k}}{\|\mathbf{i}-2\mathbf{k}\|} = \frac{1}{\sqrt{5}}(1,0,-2)$.

**19.** Same idea as Exercise 17, but multiply by 3: $\frac{3(\mathbf{i}+\mathbf{j}-\mathbf{k})}{\|\mathbf{i}+\mathbf{j}-\mathbf{k}\|} = \frac{3}{\sqrt{3}}(1,1,-1) = \sqrt{3}(1,1,-1)$.

**20.** There are a whole plane full of perpendicular vectors. The easiest three to find are when we set the coefficients of the coordinate vectors equal to zero in turn: $\mathbf{i}+\mathbf{j}$, $\mathbf{j}+\mathbf{k}$, and $-\mathbf{i}+\mathbf{k}$.

**21.** We have two cases to consider.
If either of the projections is zero: $\text{proj}_{\mathbf{a}}\mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a}\cdot\mathbf{b} = 0 \Leftrightarrow \text{proj}_{\mathbf{b}}\mathbf{a} = \mathbf{0}$.
If neither of the projections is zero, then the directions must be the same. This means that **a** must be a multiple of **b**. Let $\mathbf{a} = c\mathbf{b}$, then on the one hand

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \text{proj}_{c\mathbf{b}}\mathbf{b} = \frac{c\mathbf{b}\cdot\mathbf{b}}{c\mathbf{b}\cdot c\mathbf{b}}c\mathbf{b} = \mathbf{b}.$$

On the other hand

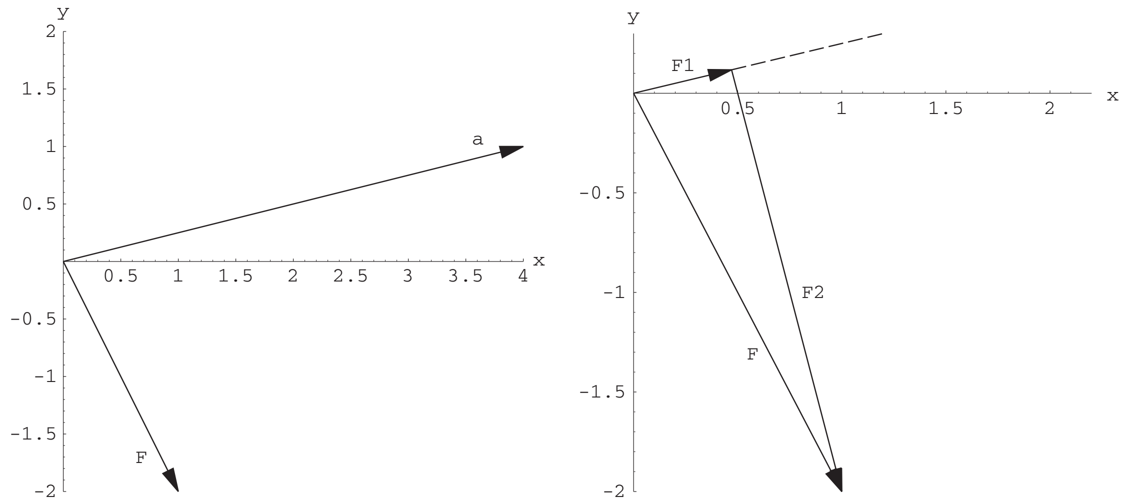
$$\text{proj}_{\mathbf{b}}\mathbf{a} = \text{proj}_{\mathbf{b}}c\mathbf{b} = \frac{\mathbf{b}\cdot c\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} = c\mathbf{b}.$$

These are equal only when $c = 1$.
In other words, $\text{proj}_{\mathbf{a}}\mathbf{b} = \text{proj}_{\mathbf{b}}\mathbf{a}$ when $\mathbf{a}\cdot\mathbf{b} = 0$ or when $\mathbf{a} = \mathbf{b}$.

**22.** Property 2: $\mathbf{a}\cdot\mathbf{b} = (a_1,a_2,a_3)\cdot(b_1,b_2,b_3) = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3 = \mathbf{b}\cdot\mathbf{a}$.
Property 3: $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = (a_1,a_2,a_3)\cdot((b_1,b_2,b_3)+(c_1,c_2,c_3)) = (a_1,a_2,a_3)\cdot(b_1+c_1,b_2+c_2,b_3+c_3) = a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3) = (a_1b_1+a_2b_2+a_3b_3) + (a_1c_1+a_2c_2+a_3c_3) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$.
Property 4: $(k\mathbf{a})\cdot b = (k(a_1,a_2,a_3))\cdot(b_1,b_2,b_3) = (ka_1,ka_2,ka_3)\cdot(b_1,b_2,b_3) = ka_1b_1 + ka_2b_2 + ka_3b_3$(*for the* $1^{st}$ *equality*) $= k(a_1b_1+a_2b_2+a_3b_3) = k(\mathbf{a}\cdot\mathbf{b})$. (*for the* $2^{nd}$ *equality*) $= a_1\,kb_1 + a_2\,kb_2 + a_3\,kb_3 = (a_1,a_2,a_3)\cdot(kb_1,kb_2,kb_3) = \mathbf{a}\cdot(k\mathbf{b})$.

**23.** We have $\|k\mathbf{a}\| = \sqrt{k\mathbf{a}\cdot k\mathbf{a}} = \sqrt{k^2(\mathbf{a}\cdot\mathbf{a})} = \sqrt{k^2}\sqrt{\mathbf{a}\cdot\mathbf{a}} = |k|\,\|\mathbf{a}\|$.

**24.** The following diagrams might be helpful:




To find $\mathbf{F}_1$, the component of $\mathbf{F}$ in the direction of $\mathbf{a}$, we project $\mathbf{F}$ onto $\mathbf{a}$:

$$\mathbf{F}_1 = \text{proj}_{\mathbf{a}}\mathbf{F} = \left(\frac{(\mathbf{i} - 2\mathbf{j}) \cdot (4\mathbf{i} + \mathbf{j})}{(4\mathbf{i} + \mathbf{j}) \cdot (4\mathbf{i} + \mathbf{j})}\right)(4\mathbf{i} + \mathbf{j}) = \frac{2}{17}(4, 1).$$

To find $\mathbf{F}_2$, the component of $\mathbf{F}$ in the direction perpendicular to $\mathbf{a}$, we can just subtract $\mathbf{F}_1$ from $\mathbf{F}$:

$$\mathbf{F}_2 = (1, -2) - \frac{2}{17}(4, 1) = \left(\frac{9}{17}, \frac{-36}{17}\right) = \frac{9}{17}(1, -4).$$

Note that $\mathbf{F}_1$ is a multiple of $\mathbf{a}$ so that $\mathbf{F}_1$ does point in the direction of $\mathbf{a}$ and that $\mathbf{F}_2 \cdot \mathbf{a} = 0$ so $\mathbf{F}_2$ is perpendicular to $\mathbf{a}$.

**25. (a)** The work done by the force is given to be the product of the length of the displacement ($\|\overrightarrow{PQ}\|$) and the component of force in the direction of the displacement ($\pm\|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|$ or in the case pictured in the text, $\|\mathbf{F}\| \cos\theta$). That is,

$$\text{Work} = \|\overrightarrow{PQ}\|\,\|\mathbf{F}\| \cos\theta = \mathbf{F} \cdot \overrightarrow{PQ}$$

using Theorem 3.3.

**(b)** The displacement vector is $\overrightarrow{PQ} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and so, using part (a), we have

$$\text{Work} = \mathbf{F} \cdot \overrightarrow{PQ} = (\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 1 + 5 - 4 = 2.$$

**26.** The amount of work is

$$\|\mathbf{F}\|\,\|\overrightarrow{PQ}\| \cos 20^\circ = 60 \cdot 12 \cdot \cos 20^\circ \approx 676.6 \text{ ft-lb}.$$

**27.** To move the bananas, one must exert an *upward* force of 500 lb. Such a force makes an angle of $60^\circ$ with the ramp, and it is the ramp that gives the direction of displacement. Thus the amount of work done is

$$\|\mathbf{F}\|\,\|\overrightarrow{PQ}\| \cos 60^\circ = 500 \cdot 40 \cdot \tfrac{1}{2} = 10{,}000 \text{ ft-lb}.$$

**28.** Note that $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ each point along the positive $x$-, $y$-, and $z$-axes. Therefore, we may use Theorem 3.3 to calculate that

$$\cos\alpha = \frac{(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot \mathbf{i}}{\|\mathbf{i} + 2\mathbf{j} - \mathbf{k}\|(1)} = \frac{1}{\sqrt{6}};$$

$$\cos\beta = \frac{(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot \mathbf{j}}{\|\mathbf{i} + 2\mathbf{j} - \mathbf{k}\|(1)} = \frac{2}{\sqrt{6}};$$

$$\cos\gamma = \frac{(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot \mathbf{k}}{\|\mathbf{i} + 2\mathbf{j} - \mathbf{k}\|(1)} = -\frac{1}{\sqrt{6}}.$$

**29.** As in the previous problem, we use $\mathbf{a} = 3\mathbf{i} + 4\mathbf{k}$ to find that

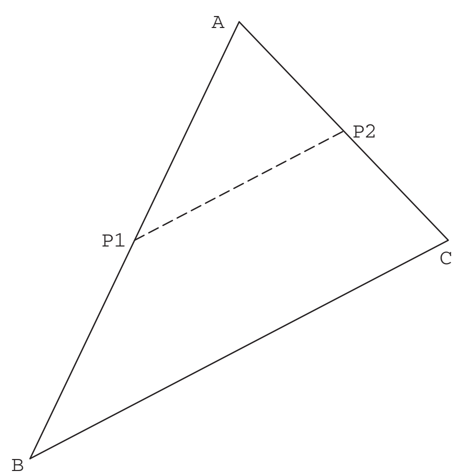
$$\cos\alpha = \frac{(3\mathbf{i}+4\mathbf{k})\cdot\mathbf{i}}{\|3\mathbf{i}+4\mathbf{k}\|(1)} = \frac{3}{5};$$

$$\cos\beta = \frac{(3\mathbf{i}+4\mathbf{k})\cdot\mathbf{j}}{\|3\mathbf{i}+4\mathbf{k}\|(1)} = 0;$$

$$\cos\gamma = \frac{(3\mathbf{i}+4\mathbf{k})\cdot\mathbf{k}}{\|3\mathbf{i}+4\mathbf{k}\|(1)} = \frac{4}{5}.$$

**30.** You could either use the three right triangles determined by the vector **a** and the three coordinate axes, or you could use Theorem 3.3. By that theorem, $\cos\alpha = \dfrac{\mathbf{a}\cdot\mathbf{i}}{\|\mathbf{a}\|\,\|\mathbf{i}\|} = \dfrac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}}$. Similarly, $\cos\beta = \dfrac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}}$ and $\cos\gamma = \dfrac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}$.

**31.** Consider the figure:

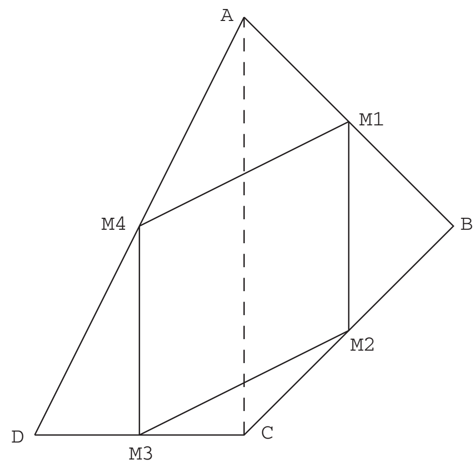



If $P_1$ is the point on $\overline{AB}$ located $r$ times the distance from $A$ to $B$, then the vector $\overrightarrow{AP_1} = r\overrightarrow{AB}$. Similarly, since $P_2$ is the point on $\overline{AC}$ located $r$ times the distance from $A$ to $C$, then the vector $\overrightarrow{AP_2} = r\overrightarrow{AC}$. So now we can look at the line segment $\overline{P_1P_2}$ using vectors.

$$\overrightarrow{P_1P_2} = \overrightarrow{AP_2} - \overrightarrow{AP_1} = r\overrightarrow{AC} - r\overrightarrow{AB} = r(\overrightarrow{AC} - \overrightarrow{AB}) = r\overrightarrow{BC}.$$

The two conclusions now follow. Because $\overrightarrow{P_1P_2}$ is a scalar multiple of $\overrightarrow{BC}$, they are parallel. Also the positive scalar $r$ pulls out of the norm so $\|\overrightarrow{P_1P_2}\| = \|r\overrightarrow{BC}\| = r\|\overrightarrow{BC}\|$.

**32.** This now follows immediately from Exercise 31 or Example 6 from the text. Consider first the triangle *ABC*.




If $M_1$ is the midpoint of $\overline{AB}$ and $M_2$ is the midpoint of $\overline{BC}$, we've just shown that $\overline{M_1M_2}$ is parallel to $\overline{AC}$ and has half its length. Similarly, consider triangle *DAC* where $M_3$ is the midpoint of $\overline{CD}$ and $M_4$ is the midpoint of $\overline{DA}$. We see that $\overline{M_3M_4}$ is parallel to *AC* and has half its length. The first conclusion is that $\overline{M_1M_2}$ and $\overline{M_3M_4}$ have the same length and are parallel. Repeat this process for triangles *ABD* and *CBD* to conclude that $\overline{M_1M_4}$ and $\overline{M_2M_3}$ have the same length and are parallel. We conclude that $M_1M_2M_3M_4$ is a parallelogram. *For kicks—have your students draw the figure for ABCD a non-convex quadrilateral. The argument and the conclusion still hold even though one of the "diagonals" is not inside of the quadrilateral*.

**33.** In the diagram in the text, the diagonal running from the bottom left to the top right is $\mathbf{a}+\mathbf{b}$ and the diagonal running from the bottom right to the top left is $\mathbf{b}-\mathbf{a}$.

$$\begin{aligned}
\|\mathbf{a}+\mathbf{b}\| &= \|-\mathbf{a}+\mathbf{b}\| && \Leftrightarrow \\
\sqrt{(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})} &= \sqrt{(-\mathbf{a}+\mathbf{b})\cdot(-\mathbf{a}+\mathbf{b})} && \Leftrightarrow \\
\sqrt{\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}+2\mathbf{a}\cdot\mathbf{b}} &= \sqrt{(-1)^2\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{b}} && \Leftrightarrow \\
\mathbf{a}\cdot\mathbf{b} &= 0
\end{aligned}$$

Since neither **a** nor **b** is zero, they must be orthogonal.

**34.** Using the same set up as that in Exercise 33, we note first that

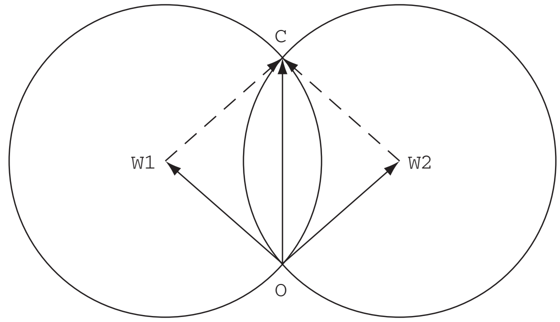
$$(\mathbf{a}+\mathbf{b})\cdot(-\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot(-\mathbf{a})+\mathbf{b}\cdot(-\mathbf{a})+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b} = -\|\mathbf{a}\|^2+\|\mathbf{b}\|^2.$$

It follows immediately that

$$(\mathbf{a}+\mathbf{b})\cdot(-\mathbf{a}+\mathbf{b}) = 0 \Leftrightarrow \|\mathbf{a}\| = \|\mathbf{b}\|.$$

In other words that the diagonals of the parallelogram are perpendicular if and only if the parallelogram is a rhombus.

**35. (a)** Let's start with the two circles with centers at $W_1$ and $W_2$. Assume that in addition to their intersection at point $O$ that they also intersect at point $C$ as shown below.




The polygon $OW_1CW_2$ is a parallelogram. In fact, because all sides are equal, it is a rhombus. We can, therefore, write the vector $\mathbf{c} = \overrightarrow{OC} = \overrightarrow{OW}_1 + \overrightarrow{OW}_2 = \mathbf{w}_1 + \mathbf{w}_2$. Similarly, we can write $\mathbf{b} = \mathbf{w}_1 + \mathbf{w}_3$ and $\mathbf{a} = \mathbf{w}_2 + \mathbf{w}_3$.

**(b)** Let's use the results of part (a) together with the hint. We need to show that the distance from each of the points $A$, $B$, and $C$ to $P$ is $r$. Let's show, for example, that $\|\overrightarrow{CP}\|$ is $r$:

$$\|\overrightarrow{CP}\| = \|\overrightarrow{OP} - \overrightarrow{OC}\| = \|(\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3) - (\mathbf{w}_1 + \mathbf{w}_2)\| = \|\mathbf{w}_3\| = r.$$

The arguments for the other two points are analogous.

**(c)** What we really need to show is that each of the lines passing through $O$ and one of the points $A$, $B$, or $C$ is perpendicular to the line containing the two other points. Using vectors we will show that $\overrightarrow{OA} \perp \overrightarrow{BC}$, $\overrightarrow{OB} \perp \overrightarrow{AC}$, and $\overrightarrow{OC} \perp \overrightarrow{AB}$ by showing their dot products are 0. It's enough to show this for one of them: $\overrightarrow{OA} \cdot \overrightarrow{BC} = (\mathbf{w}_2 + \mathbf{w}_3) \cdot ((\mathbf{w}_1 + \mathbf{w}_2) - (\mathbf{w}_1 + \mathbf{w}_3)) = (\mathbf{w}_2 + \mathbf{w}_3) \cdot (\mathbf{w}_2 - \mathbf{w}_3) = \mathbf{w}_2 \cdot \mathbf{w}_2 + \mathbf{w}_3 \cdot \mathbf{w}_2 - \mathbf{w}_2 \cdot \mathbf{w}_3 - \mathbf{w}_3 \cdot \mathbf{w}_3 = r^2 - r^2 = 0$.

**36. (a)** This follows immediately from Exercise 34 if you notice that the vectors are the diagonals of the rhombus with two sides $\|\mathbf{b}\|\mathbf{a}$ and $\|\mathbf{a}\|\mathbf{b}$.

Or we can proceed with the calculation: $(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|) \cdot (\|\mathbf{b}\|\mathbf{a} - \|\mathbf{a}\|\mathbf{b})$. The only bit of good news here is that the cross terms clearly cancel each other out and we're left with: $\|\mathbf{b}\|^2(\mathbf{a} \cdot \mathbf{a}) - \|\mathbf{a}\|^2(\mathbf{b} \cdot \mathbf{b}) = \|\mathbf{b}\|^2\|\mathbf{a}\|^2 - \|\mathbf{a}\|^2\|\mathbf{b}\|^2 = 0$.

**(b)** As in (a), the slicker way is to recall (or reprove geometrically) that the diagonals of a rhombus bisect the vertex angles. Then note that $(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})$ is the diagonal of the rhombus with sides $\|\mathbf{b}\|\mathbf{a}$ and $\|\mathbf{a}\|\mathbf{b}$ and so bisects the angle between them which is the same as the angle between **a** and **b**.

Another way is to let $\theta_1$ be the angle between **a** and $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$, and let $\theta_2$ be the angle between **b** and $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$. Then

$$\cos^{-1}\theta_1 = \frac{\mathbf{a} \cdot (\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})}{(\|\mathbf{a}\|)\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|} = \frac{\|\mathbf{a}\|^2\|\mathbf{b}\| + \|\mathbf{a}\|\mathbf{a} \cdot \mathbf{b}}{(\|\mathbf{a}\|)\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|} = \frac{\|\mathbf{a}\|\,\|\mathbf{b}\| + \mathbf{a} \cdot \mathbf{b}}{\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|}.$$

Also

$$\cos^{-1}\theta_2 = \frac{\mathbf{b} \cdot (\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})}{(\|\mathbf{b}\|)\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|} = \frac{\|\mathbf{b}\|\mathbf{b} \cdot \mathbf{a} + \|\mathbf{b}\|^2\|\mathbf{a}\|}{(\|\mathbf{b}\|)\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|} = \frac{\mathbf{b} \cdot \mathbf{a} + \|\mathbf{a}\|\,\|\mathbf{b}\|}{\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|}.$$

So $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$ bisects the angle between the vectors **a** and **b**.

## 1.4 The Cross Product

*For Exercises 1–4 use Definition 4.2.*

**1.** $(2)(3) - (4)(1) = 2$.

**2.** $(0)(6) - (5)(-1) = 5$.

**3.** $(1)(2)(3) + (3)(7)(-1) + (5)(0)(0) - (5)(2)(-1) - (1)(7)(0) - (3)(0)(3) = -5$.

**4.** $(-2)(6)(2) + (0)(-1)(4) + (1/2)(3)(-8) - (1/2)(6)(4) - (-2)(-1)(-8) - (0)(3)(2) = -32$.

*Note: In Exercises 5–7, the difference between using (2) and (3) really amounts to changing the coefficient of* **j** *from* $(a_3b_1 - a_1b_3)$ *in formula (2) to* $-(a_1b_3 - a_3b_1)$ *in formula (3). The details are only provided in Exercise 5.*

**5.** First we'll use formula (2):

$$\begin{aligned}(1,3,-2) \times (-1,5,7) &= [(3)(7) - (-2)(5)]\mathbf{i} + [(-2)(-1) - (1)(7)]\mathbf{j} + [(1)(5) - (3)(-1)]\mathbf{k} \\ &= 31\mathbf{i} - 5\mathbf{j} + 8\mathbf{k} = (31, -5, 8).\end{aligned}$$

If instead we use formula (3), we get:

$$\begin{aligned}(1,3,-2) \times (-1,5,7) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ -1 & 5 & 7 \end{vmatrix} \\ &= \begin{vmatrix} 3 & -2 \\ 5 & 7 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ -1 & 7 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix}\mathbf{k} \\ &= 31\mathbf{i} - 5\mathbf{j} + 8\mathbf{k} = (31, -5, 8).\end{aligned}$$

**6.** Just using formula (3):

$$(3\mathbf{i}-2\mathbf{j}+\mathbf{k})\times(\mathbf{i}+\mathbf{j}+\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&-2&1\\1&1&1\end{vmatrix}=\begin{vmatrix}-2&1\\1&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}3&1\\1&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}3&-2\\1&1\end{vmatrix}\mathbf{k}$$
$$=-3\mathbf{i}-2\mathbf{j}+5\mathbf{k}=(-3,-2,5).$$

**7.** Note that these two vectors form a basis for the $xy$-plane so the cross product will be a vector parallel to (0, 0, 1). Again, just using formula (3):

$$(\mathbf{i}+\mathbf{j})\times(-3\mathbf{i}+2\mathbf{j})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\-3&2&0\end{vmatrix}=\begin{vmatrix}1&0\\2&0\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&0\\-3&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&1\\-3&2\end{vmatrix}\mathbf{k}=5\mathbf{k}=(0,0,5).$$

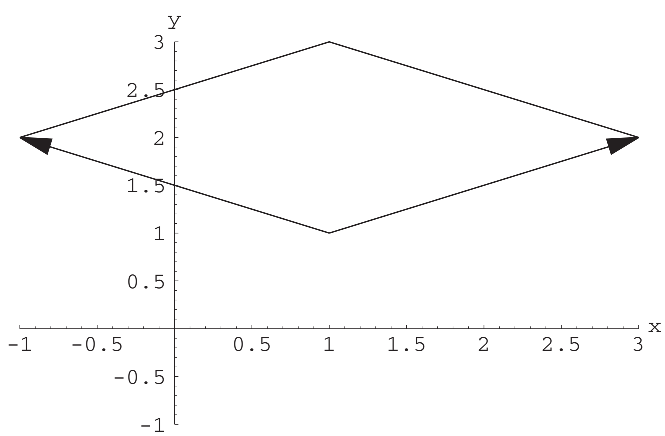
**8.** By (1) $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=-\mathbf{c}\times(\mathbf{a}+\mathbf{b})$.
By (2), this $=-\mathbf{c}\times\mathbf{a}+-\mathbf{c}\times\mathbf{b}$.
By (1), this $=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$.

**9.** $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=(\mathbf{a}\times\mathbf{a})+(\mathbf{b}\times\mathbf{a})-(\mathbf{a}\times\mathbf{b})-(\mathbf{b}\times\mathbf{b})$. The cross product of a vector with itself is $\mathbf{0}$ and also $(\mathbf{b}\times\mathbf{a})=-(\mathbf{a}\times\mathbf{b})$, so

$$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=-2(\mathbf{a}\times\mathbf{b}).$$

*You may wish to have your students consider what this means about the relationship between the cross product of the sides of a parallelogram and the cross product of its diagonals*. In any case, we are given that $\mathbf{a}\times\mathbf{b}=(3,-7,-2)$, so $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=(-6,14,4)$.

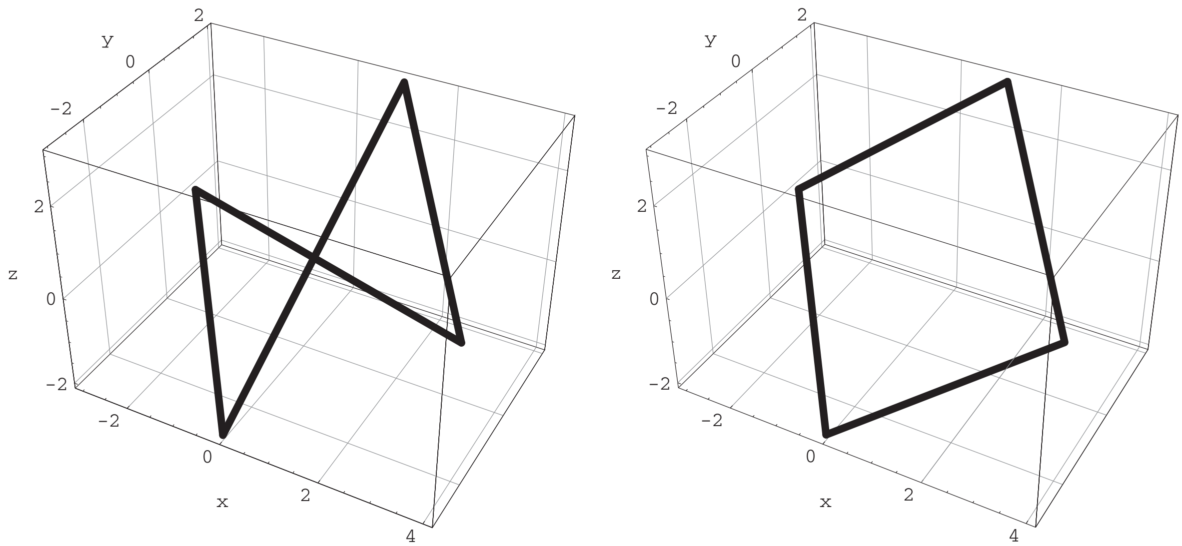
**10.** If you plot the points you'll see that they are given in a counterclockwise order of the vertices of a parallelogram. To find the area we will view the sides from (1, 1) to (3, 2) and from (1, 1) to $(-1, 2)$ as vectors by calculating the displacement vectors: $(3,2)-(1,1)$ and $(-1,2)-(1,1)$. We then embed the problem in $\mathbf{R}^3$ and take a cross product. The length of this cross product is the area of the parallelogram.




$$(3-1,2-1,0)\times(-1-1,2-1,0)=(2,1,0)\times(-2,1,0)=4\mathbf{k}=(0,0,4).$$

So the area is $\|(0,0,4)\|=4$.

**11.** This is tricky, as the points are not given in order. The figure on the left shows the sides connected in the order that the points are given.




As the figure on the right shows, if you take the first side to be the side that joins the points $(1, 2, 3)$ and $(4, -2, 1)$ then the next side is the side that joins $(4, -2, 1)$ and $(0, -3, -2)$. We will again calculate the length of the cross product of the displacement vectors. So the area of the parallelogram will be the length of

$$(0-4, -3-(-2), -2-1) \times (1-4, 2-(-2), 3-1) = (-4, -1, -3) \times (-3, 4, 2) = (10, 17, -19).$$

The length of $(10, 17, -19)$ is $\sqrt{10^2 + 17^2 + (-19)^2} = \sqrt{750} = 5\sqrt{30}$.

**12.** The cross product will give us the right direction; if we then divide this result by its length we will get a unit vector:

$$\frac{(2, 1, -3) \times (1, 0, 1)}{\|(2, 1, -3) \times (1, 0, 1)\|} = \frac{(1, -5, -1)}{\|(1, -5, -1)\|} = \frac{1}{\sqrt{27}}(1, -5, -1).$$

**13.** For $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ to be zero either

- One or more of the three vectors is $\mathbf{0}$,
- $(\mathbf{a} \times \mathbf{b}) = \mathbf{0}$ which would happen if $\mathbf{a} = k\mathbf{b}$ for some real $k$, or
- $\mathbf{c}$ is in the plane determined by $\mathbf{a}$ and $\mathbf{b}$.

*For Exercises 14–17 we'll just take half of the length of the cross product. Unlike Exercises 10 and 11, in Exercises 16 and 17 we don't have to worry about the ordering of the points. In a triangle, whichever order we choose we are traveling either clockwise or counterclockwise. Just choose any of the vertices as the base for the cross product. Our choices may differ, but the solution won't.*

**14.** $(1/2)\|(1, 1, 0) \times (2, -1, 0)\| = (1/2)\|(0, 0, -3)\| = 3/2$.

**15.** $(1/2)\|(1, -2, 6) \times (4, 3, -1)\| = (1/2)\|(-16, 25, 11)\| = \sqrt{1002}/2$.

**16.** $(1/2)\|(-1-1, 2-1, 0) \times (-2-1, -1-1, 0)\| = (1/2)\|(-2, 1, 0) \times (-3, -2, 0)\| =$
$(1/2)\|(0, 0, 7)\| = 7/2$.

**17.** $(1/2)\|(0-1, 2, 3-1) \times (-1-1, 5, -2-1)\| = (1/2)\|(-1, 2, 2) \times (-2, 5, -3)\| =$
$(1/2)\|(-16, -7, -1)\| = \sqrt{306}/2 = 3\sqrt{34}/2$.

*The triple scalar product is used in Exercises 18 and 19 and the equivalent determinant form mentioned in the text is proved in Exercise 20.*

*Some people write this product as* $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ *instead of* $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. *Exercise 28 shows that these are equivalent.*

**18.** Here we are given the vectors so we can just use the triple scalar product:

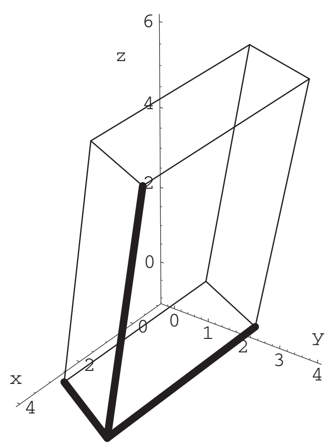
$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= ((3\mathbf{i}-\mathbf{j})\times(-2\mathbf{i}+\mathbf{k}))\cdot(\mathbf{i}-2\mathbf{j}+4\mathbf{k}) = \begin{vmatrix} 3 & -1 & 0 \\ -2 & 0 & 1 \\ 1 & -2 & 4 \end{vmatrix} \\ &= 3\begin{vmatrix} 0 & 1 \\ -2 & 4 \end{vmatrix} - (-1)\begin{vmatrix} -2 & 1 \\ 1 & 4 \end{vmatrix} + 0\begin{vmatrix} -2 & 0 \\ 1 & -2 \end{vmatrix} = 3(2) + (-9) = -3.\end{aligned}$$

Volume $= |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| = 3$.

**19.** You need to figure out a useful ordering of the vertices. You can either plot them by hand or use a computer package to help or you can make some observations about them. First look at the $z$ coordinates. Two points have $z=-1$ and two have $z=0$. These form your bottom face. Of the remaining points two have $z=5$—these will match up with the bottom points with $z=-1$, and two have $z=6$—these will match up with the bottom points with $z=0$. The parallelepiped is shown below.

We'll use the highlighted edges as our three vectors **a**, **b**, and **c**. You could have based the calculation at any vertex. I have chosen $(4,2,-1)$. The three vectors are:

$$\begin{aligned}\mathbf{a} &= (0,3,0)-(4,2,-1) = (-4,1,1)\\ \mathbf{b} &= (4,3,5)-(4,2,-1) = (0,1,6)\\ \mathbf{c} &= (3,0,-1)-(4,2,-1) = (-1,-2,0)\end{aligned}$$




We can now calculate

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= ((-4,1,1)\times(0,1,6))\cdot(-1,-2,0) = \begin{vmatrix} -4 & 1 & 1 \\ 0 & 1 & 6 \\ -1 & -2 & 0 \end{vmatrix} \\ &= -4\begin{vmatrix} 1 & 6 \\ -2 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 6 \\ -1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} = -4(12) - (6) + (1) = -53.\end{aligned}$$

Finally, Volume $= |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| = 53$.

*Note: The proofs of Exercises 20 and 28 are easier if you remember that if matrix $A$ is just matrix $B$ with any two rows interchanged then the determinant of $A$ is the negative of the determinant of $B$. If you don't use this fact (which is explored in exercises later in this chapter), you can prove this with a long computation. That is why the author of the text suggests that a computer algebra system could be helpful—and this would be a great place to use it in a class demonstration.*

**20.** This is not as bad as it might first appear.

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\left(\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}\right)\cdot(c_1,c_2,c_3)$$
$$=\left(\mathbf{i}\begin{vmatrix}a_2&a_3\\b_2&b_3\end{vmatrix}-\mathbf{j}\begin{vmatrix}a_1&a_3\\b_1&b_3\end{vmatrix}+\mathbf{k}\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}\right)\cdot(c_1,c_2,c_3)$$
$$=c_1\begin{vmatrix}a_2&a_3\\b_2&b_3\end{vmatrix}-c_2\begin{vmatrix}a_1&a_3\\b_1&b_3\end{vmatrix}+c_3\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}$$
$$=\begin{vmatrix}c_1&c_2&c_3\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=-\begin{vmatrix}a_1&a_2&a_3\\c_1&c_2&c_3\\b_1&b_2&b_3\end{vmatrix}=\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}$$

**21.** $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}$ by Exercise 20. Similarly, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}=\begin{vmatrix}b_1&b_2&b_3\\c_1&c_2&c_3\\a_1&a_2&a_3\end{vmatrix}$ by Exercise 20. Expand these determinants to see that they are equal.

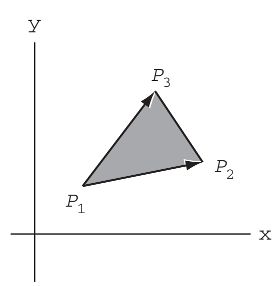
$$\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}=a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)$$
$$\begin{vmatrix}b_1&b_2&b_3\\c_1&c_2&c_3\\a_1&a_2&a_3\end{vmatrix}=b_1(a_3c_2-a_2c_3)-b_2(a_3c_1-a_1c_3)+b_3(a_2c_1-a_1c_2)$$

**22.** The value of $|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$ is the volume of the parallelepiped determined by the vectors $\mathbf{a},\mathbf{b},\mathbf{c}$. But so is $|\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})|$, so the quantities must be equal.

**23. (a)** We have

$$\text{Area}=\tfrac{1}{2}\|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}\|$$
$$=\tfrac{1}{2}\|(x_2-x_1,y_2-y_1,0)\times(x_3-x_1,y_3-y_1,0)\|$$

$$\text{Now }\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\x_2-x_1&y_2-y_1&0\\x_3-x_1&y_3-y_1&0\end{vmatrix}$$
$$=[(x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1)]\mathbf{k}$$




Hence the area is $\frac{1}{2}|(x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1)|$. On the other hand

$$\frac{1}{2}\begin{vmatrix}1&1&1\\x_1&x_2&x_3\\y_1&y_2&y_3\end{vmatrix}=\frac{1}{2}\left(\begin{vmatrix}x_2&x_3\\y_2&y_3\end{vmatrix}-\begin{vmatrix}x_1&x_3\\y_1&y_3\end{vmatrix}+\begin{vmatrix}x_1&x_2\\y_1&y_2\end{vmatrix}\right).$$

Expanding and taking absolute value, we obtain

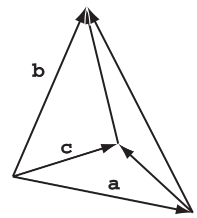
$$\tfrac{1}{2}|x_2y_3-x_3y_2+x_3y_1-x_1y_3+x_1y_2-x_2y_1|.$$

From here, its easy to see that this agrees with the formula above.

**(b)** We compute the absolute value of $\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -4 \\ 2 & 3 & -4 \end{vmatrix} = \frac{1}{2}(-8-8+3-4+12+4) = \frac{1}{2}(-1) = -\frac{1}{2}$.

Thus the area is $\left|-\frac{1}{2}\right| = \frac{1}{2}$.

**24.** Surface area $= \frac{1}{2}(\|\mathbf{a}\times\mathbf{b}\| + \|\mathbf{b}\times\mathbf{c}\| + \|\mathbf{a}\times\mathbf{c}\| + \|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\|)$




**25.** We assume that **a**, **b**, and **c** are non-zero vectors in $\mathbf{R}^3$.

**(a)** The cross product $\mathbf{a}\times\mathbf{b}$ is orthogonal to both **a** and **b**.

**(b)** Scale the cross product to a unit vector by dividing by the length and then multiply by 2 to get $2\left(\dfrac{\mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\times\mathbf{b}\|}\right)$.

**(c)** $\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$.

**(d)** Here we divide vector **a** by its length and multiply it by the length of **b** to get $\left(\dfrac{\|\mathbf{b}\|}{\|\mathbf{a}\|}\right)\mathbf{a}$.

**(e)** The cross product of two vectors is orthogonal to each: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

**(f)** A vector perpendicular to $\mathbf{a}\times\mathbf{b}$ will be back in the plane determined by **a** and **b**, so our answer is $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

**26.** *I love this problem—students tend to go ahead and calculate without thinking through what they're doing first. This would make a great quiz at the beginning of class.*

**(a)** Vector: The cross product of the vectors **a** and **b** is a vector so you can take its cross product with vector **c**.

**(b)** Nonsense: The dot product of the vectors **a** and **b** is a scalar so you can't dot it with a vector.

**(c)** Nonsense: The dot products result in scalars and you can't find the cross product of two scalars.

**(d)** Scalar: The cross product of the vectors **a** and **b** is a vector so you can take its dot product with vector **c**.

**(e)** Nonsense: The cross product of the vectors **a** and **b** is a vector so you can take its cross product with vector that is the result of the cross product of **c** and **d**.

**(f)** Vector: The dot product results in a scalar that is then multiplied by vector **d**. We can evaluate the cross product of vector **a** with this result.

**(g)** Scalar: We are taking the dot product of two vectors.

**(h)** Vector: You are subtracting two vectors.

*Note: You can have your students use a computer algebra system for these as suggested in the text. I've included worked out solutions for those as old fashioned as I am.*

**27.** Exercise 25(f) shows us that $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ is in the plane determined by **a** and **b** and so we expect the solution to be of the form $k_1\mathbf{a}+k_2\mathbf{b}$ for scalars $k_1$ and $k_2$.

Using formula (3) from the text for $\mathbf{a}\times\mathbf{b}$:

$$
\begin{aligned}
(\mathbf{a}\times\mathbf{b})\times\mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= \left(-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}c_3 - \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}c_2\right)\mathbf{i} - \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}c_3 - \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}c_1\right)\mathbf{j} \\
&\quad + \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}c_2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}c_1\right)\mathbf{k}
\end{aligned}
$$

Look first at the coefficient of **i**: $-a_1b_3c_3 + a_3b_1c_3 - a_1b_2c_2 + a_2b_1c_2$. If we add and subtract $a_1b_1c_1$ and regroup we have: $b_1(a_1c_1 + a_2c_2 + a_3c_3) - a_1(b_1c_1 + b_2c_2 + b_3c_3) = b_1(\mathbf{a}\cdot\mathbf{c}) - a_1(\mathbf{b}\cdot\mathbf{c})$. Similarly for the coefficient of **j**. Expand then add and subtract $a_2b_2b_3$ and regroup to get $b_2(\mathbf{a}\cdot\mathbf{c}) - a_2(\mathbf{b}\cdot\mathbf{c})$. Finally for the coefficient of **k**, expand then add and subtract $a_3b_3c_3$ and regroup to obtain $b_3(\mathbf{a}\cdot\mathbf{c}) - a_3(\mathbf{b}\cdot\mathbf{c})$. This shows that $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$.

Now here's a version of Exercise 27 worked on *Mathematica*. First you enter the following to define the vectors **a**, **b**, and **c**.

$$a = \{a1, a2, a3\}$$
$$b = \{b1, b2, b3\}$$
$$c = \{c1, c2, c3\}$$

The reply from *Mathematica* is an echo of your input for **c**. Let's begin by calculating the cross product. You can either select the cross product operator from the typesetting palette or you can type the escape key followed by "cross" followed by the escape key. *Mathematica* should reformat this key sequence as $\times$ and you should be able to enter

$$(a \times b) \times c.$$

*Mathematica* will respond with the calculated cross product

$$\begin{aligned}&\{a2b1c2 - a1b2c2 + a3b1c3 - a1b3c3,\\ &-a2b1c1 + a1b2c1 + a3b2c3 - a2b3c3,\\ &-a3b1c1 + a1b3c1 - a3b2c2 + a2b3c2\}.\end{aligned}$$

Now you can check the other expression. Use a period for the dot in the dot product.

$$(a.c)b - (b.c)\,a$$

*Mathematica* will immediately respond

$$\begin{aligned}&\{b1(a1c1 + a2c2 + a3c3) - a1(b1c1 + b2c2 + b3c3),\\ &b2(a1c1 + a2c2 + a3c3) - a2(b1c1 + b2c2 + b3c3),\\ &b3(a1c1 + a2c2 + a3c3) - a3(b1c1 + b2c2 + b3c3)\}\end{aligned}$$

This certainly looks different from the previous expression. Before giving up hope, note that this one has been factored and the earlier one has not. You can expand this by using the command

$$\text{Expand}[(a.c)b - (b.c)a]$$

or use *Mathematica*'s command % to refer to the previous entry and just type

$$\text{Expand}[\%].$$

This still might not look familiar. So take a look at

$$(a \times b) \times c - [(a.c)b - (b.c)a].$$

If this *still* isn't what you are looking for, simplify it with the command

$$\text{Simplify}[\%]$$

and *Mathematica* will respond

$$\{0, 0, 0\}.$$

**28.** The exercise asks us to show that six quantities are equal.

The most important pair is $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$. Because of the commutative property of the dot product $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ and so we are showing that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.

$$\begin{aligned}\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\\ &= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}\\ &= (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}).\end{aligned}$$

The determinants of the 3 by 3 matrices above are equal because we had to interchange two rows twice to get from one to the other. This fact has not yet been presented in the text. This would be an excellent time to use a computer algebra system to show the two determinants are equal. Of course, you could use *Mathematica* or some other such system to do the entire problem.

To show that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$ we use a similar approach:

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} &= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b} = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}).\end{aligned}$$

So we've established that the first three triple scalars are equal.
We get the rest almost for free by noticing that three pairs of equations are trivial:

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= -\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}),\\ \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) &= -\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}),\ \text{and}\\ \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) &= -\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a}).\end{aligned}$$

Each of the above pairs are equal by the anticommutativity property of the cross product. If you prefer the matrix approach, this also follows from the fact that interchanging two rows changes the sign of the determinant.

**29.** By Exercise 28, $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{c}\cdot(\mathbf{d}\times(\mathbf{a}\times\mathbf{b}))$.
By anticommutativity, $\mathbf{c}\cdot(\mathbf{d}\times(\mathbf{a}\times\mathbf{b})) = -\mathbf{c}\cdot((\mathbf{a}\times\mathbf{b})\times\mathbf{d})$.
By Exercise 27, $-\mathbf{c}\cdot((\mathbf{a}\times\mathbf{b})\times\mathbf{d}) = -\mathbf{c}\cdot((\mathbf{a}\cdot\mathbf{d})\mathbf{b} - (\mathbf{b}\cdot\mathbf{d})\mathbf{a}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}$.

**30.** Apply the results of Exercise 27 to each of the three components:

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times\mathbf{c} + (\mathbf{b}\times\mathbf{c})\times\mathbf{a} + (\mathbf{c}\times\mathbf{a})\times\mathbf{b} &= [(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}] + [(\mathbf{b}\cdot\mathbf{a})\mathbf{c} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}]\\ &\quad + [(\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}] = \mathbf{0}.\end{aligned}$$

(For example, the $(\mathbf{a}\cdot\mathbf{c})\mathbf{b}$ cancels with the $(\mathbf{c}\cdot\mathbf{a})\mathbf{b}$ because of the commutative property for the dot product.)

**31.** If your students are using a computer algebra system, they may not notice that this is *exactly* the same problem as Exercise 27. Just replace **c** with $(\mathbf{c}\times\mathbf{d})$ on both sides of the equation in Exercise 27 to obtain the result here.

**32.** First apply Exercise 29 to the dot product to get

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a}) = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})][\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})] - [\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})][\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c})].$$

You can either observe that two of these quantities must be 0, or you can apply Exercise 28 to see $\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{a}) = 0$. Exercise 28 also shows that $\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. The result follows.

**33.** We did this above in Exercise 29.

**34.** The amount of torque is the product of the length of the "wrench" and the component of the force perpendicular to the "wrench". In this case, the wrench is the door—so the length is four feet. The 20 lb force is applied perpendicular to the plane of the doorway and the door is open $45^\circ$. So from the text, the amount of torque $= \|\mathbf{a}\|\|\mathbf{F}\|\sin\theta = (4)(20)(\sqrt{2}/2) = 40\sqrt{2}$ ft-lb.

**35.** **(a)** Here the length of **a** is 1 foot, the force $\mathbf{F} = 40$ pounds and angle $\theta = 120$ degrees. So

$$\textbf{Torque} = (1)(40)\sin 120^\circ = 40\left(\frac{\sqrt{3}}{2}\right) = 20\sqrt{3}\ \text{foot-pounds}.$$

**(b)** Here all that has changed is that $\|\mathbf{a}\|$ is 1.5 feet, so

$$\textbf{Torque} = (3/2)(40)\sin 120^\circ = 60\left(\frac{\sqrt{3}}{2}\right) = 30\sqrt{3}\ \text{foot-pounds}.$$

**36.** $\mathbf{a} = 2$ in but torque is measured in foot-pounds so $\|\mathbf{a}\| = (1/6)$ ft.

$$\textbf{Torque} = \mathbf{a} \times \mathbf{F} = \left(\frac{1}{6}, 0, 0\right) \times (0, 15, 0) = \left(0, 0, \frac{5}{2}\right).$$

So Egbert is using 5/2 foot-pounds straight up.

**37.** From the figure

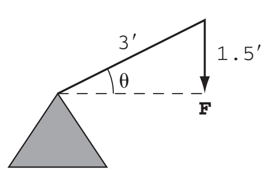
$$\begin{aligned} \sin\theta &= \frac{1.5}{3} = \frac{1}{2} \\ \Rightarrow \theta &= \pi/6. \end{aligned}$$

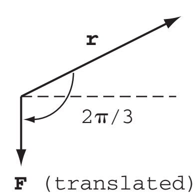
This is the angle the seesaw makes with horizontal. The angle we want is

$$\pi/6 + \pi/2 = 2\pi/3.$$

Since $\|\mathbf{r}\| = 3$ and $\|\mathbf{F}\| = 50$, the amount of torque is

$$\begin{aligned} \|\mathbf{T}\| &= \|\mathbf{r} \times \mathbf{F}\| = \|\mathbf{r}\|\,\|\mathbf{F}\| \sin\frac{2\pi}{3} \\ &= 3 \cdot 50 \cdot \frac{\sqrt{3}}{2} = 75\sqrt{3} \text{ ft-lb} \end{aligned}$$




**38.** **(a)** The linear velocity is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ so that

$$\|\mathbf{v}\| = \|\boldsymbol{\omega}\|\,\|\mathbf{r}\| \sin\theta.$$

We have that the angular speed is $\frac{2\pi \text{ radians}}{24 \text{ hrs}} = \frac{\pi}{12}$ radians/hr (this is $\|\boldsymbol{\omega}\|$.) Also $\|\mathbf{r}\| = 3960$, so at $45°$ North latitude, $\|\mathbf{v}\| = \frac{\pi}{12} \cdot 3960 \cdot \sin 45° = \frac{330\pi}{\sqrt{2}} \approx 733.08$ mph.

**(b)** Here the only change is that $\theta = 90°$. Thus $\|\mathbf{v}\| = \frac{\pi}{2} \cdot 3960 \cdot \sin 90° = 330\pi \approx 1036.73$ mph.

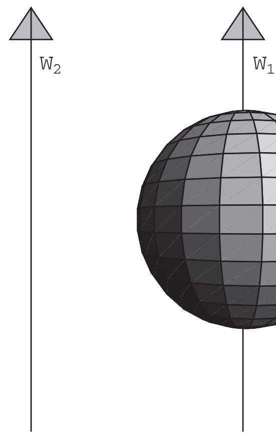
**39.** Archie's actual experience isn't important in solving this problem; he could have ridden closer to the center. Since we are only interested in comparing Archie's experience with Annie's, it turns out that their difference would be the same so long as the difference in their distance from the center remained at 2 inches. The difference in speed is $(331/3)(2\pi)(6) - (331/3)(2\pi)(4) = (331/3)(2\pi)(2) = 4\pi(331/3) = 1331/3\pi = 400\pi/3$ in/ min.

**40.** **(a)** $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = (0, 0, 12) \times (2, -1, 3) = (12, 24, 0) = 12\mathbf{i} + 24\mathbf{j}$.

**(b)** The height of the point doesn't change so we can view this as if it were a problem in $\mathbf{R}^2$. When $x = 2$ and $y = -1$, we can find the central angle by taking $\tan^{-1}(-1/2)$. In one second the angle has moved 12 radians so the new point is

$$(\sqrt{5}\cos(\tan^{-1}(-1/2) + 12), \sqrt{5}\sin(\tan^{-1}(-1/2) + 12), 3) \approx (1.15, -1.92, 3).$$

**41.** Consider the rotations of a sphere about each of the two parallel axes pictured below.




Assume the two corresponding angular velocity vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ (denoted $w_1$ and $w_2$ in the diagram) are "parallel" and even have the same magnitude. Let them both point straight up (parallel to (0, 0, 1)) with magnitude $2\pi$ radians per second. The idea is that as "free vectors" $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are both equal to $(0, 0, 2\pi)$. but that the corresponding rotational motions are very different.

In the case of $\boldsymbol{\omega}_1$, each second every point on the sphere has made a complete orbit around the axis. The corresponding motion is that the sphere is rotating about this axis. (More concretely, take your *Vector Calculus* book and stand it up on its end. Imagine an axis anywhere and spin it around that axis at a constant speed.)

In the case of $\boldsymbol{\omega}_2$, each second every point on the sphere has made a complete orbit around the axis. In this case that means that the corresponding motion is that the sphere is orbiting about this axis. (Hold your *Vector Calculus* book at arms length and you spin around your axis.)

## 1.5 Equations for Planes; Distance Problems

**1.** This is a straightforward application of formulas (1) and (2):

$$1(x-3) - (y+1) + 2(z-2) = 0 \quad \Longleftrightarrow \quad x - y + 2z = 8.$$

**2.** Again we apply formula (2):

$$(x-9) - 2(z+1) = 0 \quad \Longleftrightarrow \quad x - 2z = 11.$$

So what happened to the $y$ term? The equation is independent of $y$. In the $x - z$ plane draw the line $x - 2z = 11$ and then the plane is generated by "dragging" the line either way in the $y$ direction.

**3.** We first need to find a vector normal to the plane, so we take the cross product of two displacement vectors:

$$(3-2, -1-0, 2-5) \times (1-2, -2-0, 4-5) = (1, -1, -3) \times (-1, -2, -1) = (-5, 4, -3).$$

Now we can apply formula (2) using any of the three points:

$$-5(x-3) + 4(y+1) - 3(z-2) = 0 \quad \Longleftrightarrow \quad -5x + 4y - 3z = -25.$$

**4.** We'll again find the cross product of two displacement vectors:

$$(A, -B, 0) \times (0, -B, C) = (-BC, -AC, -AB).$$

Now we apply formula (2):

$$-BC(x-A) - AC(y) - AB(z) = 0 \quad \Longleftrightarrow \quad BCx + ACy + ABz = ABC.$$

**5.** If the planes are parallel, then a vector normal to one is normal to the other. In this case the normal vector is $\mathbf{n} = (5, -4, 1)$. So using formula (2) we get:

$$5(x-2) - 4(y+1) + (z+2) = 0 \quad \Longleftrightarrow \quad 5x - 4y + z = 12.$$

**6.** The plane must have a normal vector parallel to the normal $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ of the given plane; therefore, the vector $\mathbf{n}$ may also be taken to be the normal to the desired plane. Hence an equation is

$$2(x+1) - 3(y-1) + 1(z-2) = 0 \quad \Longleftrightarrow \quad 2x - 3y + z = -3.$$

**7.** We may take the normal to the plane to be the same as a normal to the given plane; thus we may let $\mathbf{n} = \mathbf{i} - \mathbf{j} + 7\mathbf{k}$. Hence an equation for the desired plane is

$$1(x+2) - 1(y-0) + 7(z-1) = 0 \quad \Longleftrightarrow \quad x - y + 7z = 5.$$

**8.** We may take the normal to the desired plane to be $\mathbf{n} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Therefore, the equation of the plane must be of the form $2x + 2y + z = D$ for some constant $D$. For the plane to contain the given line, *every* point on the line must satisfy the equation for the plane. Thus for *all* $t \in \mathbf{R}$ we must have

$$2(2-t) + 2(2t+1) + (3-2t) = D$$

$$\Longleftrightarrow \quad 4 - 2t + 4t + 2 + 3 - 2t = D$$
$$\Longleftrightarrow \quad 9 = D.$$

Hence the desired equation is $2x + 2y + z = 9$.

**9.** Any plane parallel to $5x - 3y + 2z = 10$ can be written in the form $5x - 3y + 2z = D$ for some constant $D$. For this plane to contain the given line, it must be the case that for *all* $t \in \mathbf{R}$ we have

$$5(t+4) - 3(3t-2) + 2(5-2t) = D$$

$$\Longleftrightarrow \quad 5t + 20 - 9t + 6 + 10 - 4t = D$$
$$\Longleftrightarrow \quad 36 - 8t = D \quad \Longleftrightarrow \quad 8t = 36 - D.$$

However, there is no *constant* value for $D$ for which $8t = 36 - D$ for *all* $t \in \mathbf{R}$. Hence the given line will *intersect* each plane parallel to $5x - 3y + 2z = 10$, but it will never be completely contained in any of them.

**10.** The plane contains the line $\mathbf{r}(t) = (-1, 4, 7) + (2, 3, -1)t$ and the point (2, 5, 0). Choose two points on the line, for example $(-1, 4, 7)$ and (13, 25, 0) and proceed as in Exercises 3 and 4.

$$\begin{aligned}(-1-2, 4-5, 7-0) \times (13-2, 25-5, 0) = (-3, -1, 7) \times (11, 20, 0) &= (-140, 77, -49) \\ &= 7(-20, 11, -7).\end{aligned}$$

We are just looking for the plane perpendicular to this vector so we can ignore the scalar 7.

$$-20(x-2) + 11(y-5) - 7(z) = 0 \quad \Longleftrightarrow \quad -20x + 11y - 7z = 15.$$

**11.** The only relevant information contained in the equation of the line $\mathbf{r}(t) = (-5, 4, 7) + (3, -2, -1)t$ is the vector coefficient of $t$. This is the normal vector $\mathbf{n} = (3, -2, -1)$.

$$3(x-1) - 2(y+1) - (z-2) = 0 \quad \Longleftrightarrow \quad 3x - 2y - z = 3.$$

**12.** We have two lines given by the vector equations:

$$\mathbf{r}_1(t) = (2, -5, 1) + (1, 3, 5)t$$
$$\mathbf{r}_2(t) = (5, -10, 9) + (-1, 3, -2)t$$

The vector $(1, 3, 5) \times (-1, 3, -2) = (-21, -3, 6) = -3(7, 1, -2)$ is orthogonal to both lines. So the equation of the plane containing both lines is:

$$7(x-2) + y + 5 - 2(z-1) = 0 \quad \Longleftrightarrow \quad 7x + y - 2z = 7.$$

**13.** The line shared by two planes will be orthogonal each of their normal vectors. First, calculate: $(1, 2, -3) \times (5, 5, -1) = (13, -14, -5)$. Now find a point on the line by setting $z = 0$ and solving the two equations

$$\begin{cases} x + 2y = 5 \\ 5x + 5y = 1 \end{cases}$$

to get $x = -23/5$ and $y = 24/5$. The equation of the line is $\mathbf{r}(t) = (-23/5, 24/5, 0) + (13, -14, -5)t$, or in parametric form:

$$\begin{cases} x &= 13t - \frac{23}{5} \\ y &= -14t + \frac{24}{5} \\ z &= -5t. \end{cases}$$

**14.** The normal to the plane is $\mathbf{n} = (2, -3, 5)$ and the line passes through the point $P = (5, 0, 6)$. The equation of the line

$$\mathbf{r}(t) = P + \mathbf{n}t = (5, 0, 6) + (2, -3, 5)\,t.$$

In parametric form this is:

$$\begin{cases} x = 2t + 5 \\ y = -3t \\ z = 5t + 6. \end{cases}$$

**15.** The easiest way to solve this is to check that the vector from the coefficients of the first equation $(8, -6, 9A)$ is a multiple of the coefficients of the second equation $(A, 1, 2)$. In this case the first is $-6$ times the second. This means that $8 = -6A$ or $A = -4/3$. Checking we see this is confirmed by $9A = -6(2)$.

**16.** For perpendicular planes we check that $0 = (A, -1, 1) \cdot (3A, A, -2)$. This yields the quadratic $0 = 3A^2 - A - 2 = (3A + 2)(A - 1)$. The two solutions are $A = -2/3$ and $A = 1$.

**17.** This is a direct application of formula (10):

$$\mathbf{x}(s, t) = s\mathbf{a} + t\mathbf{b} + \mathbf{c} = s(2, -3, 1) + t(1, 0, -5) + (-1, 2, 7).$$

In parametric form this is:

$$\begin{cases} x = 2s + t - 1 \\ y = -3s + 2 \\ z = s - 5t + 7 \end{cases}$$

**18.** Again this follows from formula (10):

$$\mathbf{x}(s, t) = s(-8, 2, 5) + t(3, -4, -2) + (2, 9, -4) \quad \text{or} \quad \begin{cases} x = -8s + 3t + 2 \\ y = 2s - 4t + 9 \\ z = 5s - 2t - 4 \end{cases}$$

**19.** The plane contains the lines given by the equations:

$$\mathbf{r}_1(t) = (5, -6, 10) + t(2, -3, 4), \text{ and}$$
$$\mathbf{r}_2(t) = (-1, 3, -2) + t(5, 10, 7).$$

So we use formula (10) with the vectors $(2, -3, 4)$ and (5, 10, 7) and either of the two points to get:

$$\mathbf{x}(s, t) = t(2, -3, 4) + s(5, 10, 7) + (-1, 3, -2) \quad \text{or} \quad \begin{cases} x = 2t + 5s - 1 \\ y = -3t + 10s + 3 \\ z = 4t + 7s - 2. \end{cases}$$

**20.** We need to find two out of the three displacement vectors and use any of the three points:

$$\mathbf{a} = (0, 2, 1) - (7, -1, 5) = (-7, 3, -4) \quad \text{and} \quad \mathbf{b} = (0, 2, 1) - (-1, 3, 0) = (1, -1, 1) \quad \text{so}$$

$$\mathbf{x}(s, t) = s(-7, 3, -4) + t(1, -1, 1) + (0, 2, 1) \quad \text{or} \quad \begin{cases} x = -7s + t \\ y = 3s - t + 2 \\ z = -4s + t + 1. \end{cases}$$

**21.** The equation of the line $\mathbf{r}(t) = (-5, 10, 9) + t(3, -3, 2)$ immediately gives us one of the two vectors $\mathbf{a} = (3, -3, 2)$. The displacement vector from a point on the line to our given point gives us the vector $\mathbf{b} = (-5, 10, 9) - (-2, 4, 7) = (-3, 6, 2)$. So our equations are:

$$\mathbf{x}(s,t) = s(3,-3,2) + t(-3,6,2) + (-5,10,9) \quad \text{or} \quad \begin{cases} x = 3s - 3t - 5 \\ y = -3s + 6t + 10 \\ z = 2s + 2t + 9. \end{cases}$$

**22.** To convert to the parametric form we will need two vectors orthogonal to the normal direction $\mathbf{n} = (2, -3, 5)$ and a point on the plane. The easiest way to find an orthogonal vector is to let one coordinate be zero and find the other two. For example if the $x$ component is zero then $(2, -3, 5) \cdot (0, y, z) = -3y + 5z$ is solved when $y = 5k$ and $z = 3k$ for any scalar $k$. In other words, the vectors $\mathbf{a} = (0, 5, 3)$ and $\mathbf{b} = (3, 2, 0)$ are orthogonal to $\mathbf{n}$. For a point in the plane $2x - 3y + 5z = 30$, set any two of $x$, $y$, and $z$ to zero. For example (0, 0, 6) is in the plane. Our parametric equations are:

$$\mathbf{x}(s,t) = s(0,5,3) + t(3,2,0) + (0,0,6) \quad \text{or} \quad \begin{cases} x = 3t \\ y = 5s + 2t \\ z = 3s + 6. \end{cases}$$

**23.** We combine the parametric equations into the single equation:

$$\mathbf{x}(s,t) = s(3,4,1) + t(-1,1,5) + (2,0,3).$$

Use the cross product to find the normal vector to the plane:

$$\mathbf{n} = (3,4,1) \times (-1,1,5) = (19,-16,7).$$

So the equation of the plane is:

$$19(x-2) - 16y + 7(z-3) = 0 \quad \text{or} \quad 19x - 16y + 7z = 59.$$

**24.** Using method 1 of Example 7, choose a point $B$ on the line, say $B = (-5, 3, 4)$. Then $\overrightarrow{BP_0} = (-5, 3, 4) - (1, -2, 3) = (-6, 5, 1)$, and $\mathbf{a} = (2, -1, 0)$. So

$$\text{proj}_{\mathbf{a}}\overrightarrow{BP_0} = \left(\frac{\mathbf{a} \cdot \overrightarrow{BP_0}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} = \left(\frac{(2,-1,0)\cdot(-6,5,1)}{(2,-1,0)\cdot(2,-1,0)}\right)(2,-1,0) = \frac{-17}{5}(2,-1,0).$$

The distance is

$$\|\overrightarrow{BP_0} - \text{proj}_{\mathbf{a}}\overrightarrow{BP_0}\| = \left\|(-6,5,1) - \frac{-17}{5}(2,-1,0)\right\| = (1/5)\|(4,8,5)\| = \sqrt{105}/5.$$

**25.** This time we'll use method 2 of Example 7. Again choose a point $B$ on the line and a vector **a** parallel to the line. The distance is then

$$D = \frac{\|\mathbf{a} \times \overrightarrow{BP_0}\|}{\|\mathbf{a}\|} = \frac{\|(3,5,0) \times (7-2, -3+1, 0)\|}{\|(3,5,0)\|} = \frac{31}{\sqrt{34}}.$$

For a method 3, you could have solved for an arbitrary point on the line $B$ such that $\overrightarrow{BP_0} \cdot \mathbf{a} = 0$ and then found the length of $\overrightarrow{BP_0}$. In $\mathbf{R}^2$, the calculation is not too bad.

**26.** Using method 1 of Example 7, choose a point $B$ on the line, say $B = (5, 3, 8)$. Then $\overrightarrow{BP_0} = (-11, 10, 20) - (5, 3, 8) = (-16, 7, 12)$, and $\mathbf{a} = (-1, 0, 7)$. So

$$\text{proj}_{\mathbf{a}}\overrightarrow{BP_0} = \left(\frac{\mathbf{a} \cdot \overrightarrow{BP_0}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} = \left(\frac{(-1,0,7)\cdot(-16,7,12)}{(-1,0,7)\cdot(-1,0,7)}\right)(-1,0,7) = (-2,0,14).$$

The distance is

$$\|\overrightarrow{BP_0} - \text{proj}_{\mathbf{a}}\overrightarrow{BP_0}\| = \|(-16,7,12) - (-2,0,14)\| = \|(-14,7,-2)\| = \sqrt{249}.$$

**27.** Use Example 9 and for two points $B_1 = (-1, 3, 5)$ on $l_1$ and $B_2 = (0, 3, 4)$ on $l_2$ calculate $\overrightarrow{B_1B_2} = (1, 0, -1)$. To find the vector **n**, calculate the cross product $\mathbf{n} = (8, -1, 0) \times (0, 3, 1) = (-1, -8, 24)$.

$$\begin{aligned}\text{proj}_{\mathbf{n}}\overrightarrow{B_1B_2} &= \left(\frac{\mathbf{n}\cdot\overrightarrow{B_1B_2}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \left(\frac{(-1,-8,24)\cdot(1,0,-1)}{(-1,-8,24)\cdot(-1,-8,24)}\right)(-1,-8,24)\\ &= -\frac{25}{641}(-1,-8,24).\end{aligned}$$

Finally $\left\|-\frac{25}{641}(-1,-8,24)\right\| = \frac{25}{\sqrt{641}}$.

**28.** Again, use Example 9 and for two points $B_1 = (-7, 1, 3)$ on $l_1$ and $B_2 = (0, 2, 1)$ on $l_2$ calculate $\overrightarrow{B_1B_2} = (7, 1, -2)$. To find the vector **n**, calculate the cross product $\mathbf{n} = (1, 5, -2) \times (4, -1, 8) = (38, -16, -21)$.

$$\begin{aligned}\text{proj}_{\mathbf{n}}\overrightarrow{B_1B_2} &= \left(\frac{\mathbf{n}\cdot\overrightarrow{B_1B_2}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \left(\frac{(38,-16,-21)\cdot(7,1,-2)}{(38,-16,-21)\cdot(38,-16,-21)}\right)(38,-16,-21)\\ &= \frac{292}{2141}(38,-16,-21).\end{aligned}$$

Finally $\left\|\frac{292}{2141}(38,-16,-21)\right\| = \frac{292}{\sqrt{2141}}$.

**29.** **(a)** Again, use Example 9 with the two points $B_1 = (4, 0, 2)$, and $B_2 = (2, 1, 3)$ and normal vector $\mathbf{n} = (3, 1, 2)\times(1, 2, 3) = (-1, -7, 5)$. The displacement vector is $\overrightarrow{B_1B_2} = (-2, 1, 1)$. Note that $\overrightarrow{B_1B_2}$ is orthogonal to **n** and so the projection $\text{proj}_{\mathbf{n}}\overrightarrow{B_1B_2} = \mathbf{0}$ (if you'd like, you can go ahead and calculate this) and so the lines are distance 0 apart.

**(b)** This means that the lines must have a point in common (that they intersect at least once). The lines are not parallel so they have exactly one point in common (i.e., they aren't the same line).

**30.** **(a)** The shortest distance between a point $P_0$ and a line $l$ is a straight line that meets $P_0$ orthogonally. If we have two nonparallel lines then we can use the cross product to find the one direction **n** that is orthogonal to each. The shortest segment between two lines will meet each orthogonally, for two skew lines $l_1$ and $l_2$ the line that joins them at these closest points will be parallel to **n**.

If instead $l_1$ is parallel to $l_2$ we get a whole plane's worth of orthogonal directions. We have no way of choosing a unique vector **n** that is used in the calculation.

O.K., that's why we can't use the method of Example 9. What can we do instead?

**(b)** Fix a point on $l_1$, say $P_1 = (2, 0, -4)$. Then as we saw in an earlier exercise, the distance from $P_1$ to an arbitrary point $P_2 = (1 + t, 3 - t, -5 + 5t)$ on $l_2$ is

$$\|\overrightarrow{P_1P_2}\| = \sqrt{(t-1)^2 + (3-t)^2 + (-1+5t)^2} = \sqrt{27t^2 - 18t + 11}.$$

$\|\overrightarrow{P_1P_2}\|$ is minimized when $\|\overrightarrow{P_1P_2}\|^2$ is minimized. This is at the vertex of the parabola, when $54t - 18 = 0$ or $t = 1/3$. At this point the distance is

$$\sqrt{27(1/3)^2 - 18(1/3) + 11} = \sqrt{3 - 6 + 11} = \sqrt{8} = 2\sqrt{2}.$$

*Note: In Exercises 31–33 we could just cut to the end of Example 8 and realize that the length of* $\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2} = \dfrac{|\mathbf{n}\cdot\overrightarrow{P_1P_2}|}{\|\mathbf{n}\|}$. *Instead we will stay true to the spirit of the examples and follow the argument through.*

**31.** These planes are parallel so we can use Example 8. The point $P_1 = (1, 0, 0)$ is on plane one and the point $P_2 = (8, 0, 0)$ is on plane two. We project the displacement vector $\overrightarrow{P_1P_2} = (7, 0, 0)$ onto the normal direction $\mathbf{n} = (1, -3, 2)$:

$$\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2} = \left(\frac{(7,0,0)\cdot(1,-3,2)}{(1,-3,2)\cdot(1,-3,2)}\right)(1,-3,2) = \frac{7}{14}(1,-3,2) = \frac{1}{2}(1,-3,2).$$

So the distance is $\|\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2}\| = \sqrt{14}/2$.

**32.** These planes are also parallel. We choose point $P_1 = (0,0,6)$ on plane one and $P_2 = (0,0,-2)$ on plane two. The displacement vector is therefore $\overrightarrow{P_1P_2} = (0,0,-8)$, and the normal vector is $\mathbf{n} = (5,-2,2)$. So

$$\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2} = \left(\frac{(0,0,-8)\cdot(5,-2,2)}{(5,-2,2)\cdot(5,-2,2)}\right)(5,-2,2) = \frac{-16}{33}(5,-2,2).$$

The distance is $\|\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2}\| = \frac{16}{\sqrt{33}}$.

**33.** As in Exercises 27 and 28, we'll choose a point $P_1 = (D_1/A, 0, 0)$ on plane one and $P_2 = (D_2/A, 0, 0)$ on plane two. The displacement vector is

$$\overrightarrow{P_1P_2} = \left(\frac{D_2 - D_1}{A}, 0, 0\right).$$

A vector normal to the plane is $\mathbf{n} = (A, B, C)$.

$$\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2} = \left(\frac{\left(\frac{D_2-D_1}{A}, 0, 0\right)\cdot(A,B,C)}{(A,B,C)\cdot(A,B,C)}\right)(A,B,C) = \frac{D_2 - D_1}{A^2+B^2+C^2}(A,B,C).$$

The distance between the two planes is:

$$\|\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2}\| = \frac{|D_2 - D_1|}{A^2+B^2+C^2}\|(A,B,C)\| = \frac{|D_2 - D_1|}{\sqrt{A^2+B^2+C^2}}.$$

**34. (a)** Plane one is normal to $\mathbf{n_1} = (9,-5,9)\times(3,-2,3) = (3,0,-3)$ while plane two is normal to $\mathbf{n_2} = (-9,2,-9)\times(-4,7,-4) = (55,0,-55)$. So $\mathbf{n_1} = (3/55)\mathbf{n_2}$, i.e. they normal vectors are parallel so the planes are parallel.

**(b)** We'll use the two points in the given equations to get the displacement vector $\overrightarrow{P_1P_2} = (8,-4,12)$, and the normal vector $\mathbf{n} = (3,0,-3)$. So

$$\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2} = \left(\frac{(8,-4,12)\cdot(3,0,-3)}{(3,0,-3)\cdot(3,0,-3)}\right)(3,0,-3) = \frac{-12}{18}(3,0,-3).$$

The distance is $\|\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2}\| = \frac{12}{\sqrt{18}} = \frac{12}{3\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$.

**35.** This exercise follows immediately from Exercise 33 (and can be very difficult without it). Here $A = 1, B = 3, C = -5$ and $D_1 = 2$. The equation in Exercise 33 becomes:

$$3 = \frac{|2 - D_2|}{\sqrt{1^2 + 3^2 + (-5)^2}} = \frac{|2 - D_2|}{\sqrt{35}}.$$

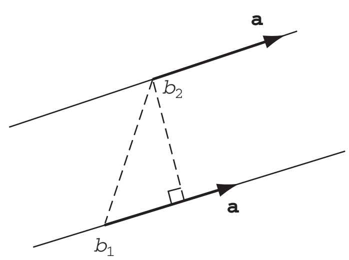
So

$$3\sqrt{35} = |2 - D_2| \quad \text{or} \quad 2 - D_2 = \pm 3\sqrt{35}.$$

So $D_2 = 2 \pm 3\sqrt{35}$ and the equations of the two planes are:

$$x + 3y - 5z = 2 \pm 3\sqrt{35}.$$

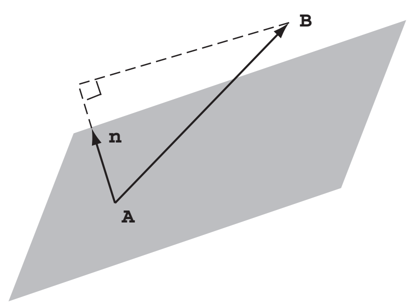
**36.** The lines are parallel, so the distance between them is the same as the distance between any point on one of the lines and the other line. Thus take $\mathbf{b}_2$—the position vector of a point on the second line—and use Example 7. Then $D = \dfrac{\|\mathbf{a} \times (\mathbf{b}_2 - \mathbf{b}_1)\|}{\|\mathbf{a}\|}$.




**37.** We have $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

$$D = \|\text{proj}_{\mathbf{n}}(\mathbf{b} - \mathbf{a})\| = \frac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}\|^2}\|\mathbf{n}\|$$

$$= \frac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}\|}.$$




(As for the motivation, consider Example 8 with $A$ as $P_1$, $B$ as $P_2$.)

**38.** The parallel planes have equations $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_1) = 0$ and $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_2) = 0$. The desired distance is given by $\|\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2}\|$ where $P_i$ is the point whose position vector is $\mathbf{x}_i$. Thus $\overrightarrow{P_1P_2} = \mathbf{x}_2 - \mathbf{x}_1$ so

$$\|\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2}\| = \frac{|\mathbf{n} \cdot (\mathbf{x}_2 - \mathbf{x}_1)|}{\|\mathbf{n}\|^2}\|\mathbf{n}\|$$

$$= \frac{|\mathbf{n} \cdot (\mathbf{x}_2 - \mathbf{x}_1)|}{\|\mathbf{n}\|}.$$

**39.** By letting $t = 0$ in each vector parametric equation, we obtain $\mathbf{b}_1, \mathbf{b}_2$ as position vectors of points $B_1, B_2$ on the respective lines. Hence $\overrightarrow{B_1B_2} = \mathbf{b}_2 - \mathbf{b}_1$. A vector $\mathbf{n}$ perpendicular to both lines is given by $\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2$. Thus

$$D = \|\text{proj}_{\mathbf{n}}\overrightarrow{B_1B_2}\| = \frac{|\mathbf{n} \cdot \overrightarrow{B_1B_2}|}{\|\mathbf{n}\|^2}\|\mathbf{n}\| = \frac{|\mathbf{n} \cdot \overrightarrow{B_1B_2}|}{\|\mathbf{n}\|}$$

$$= \frac{|(\mathbf{a}_1 \times \mathbf{a}_2) \cdot (\mathbf{b}_2 - \mathbf{b}_1)|}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}.$$

## 1.6 Some $n$-dimensional Geometry

**1. (a)** $(1,2,3,\ldots,n)=(1,0,0,\ldots,0)+2(0,1,0,0,\ldots,0)+\cdots+n(0,0,0,\ldots,0,1)=\mathbf{e}_1+2\mathbf{e}_2+3\mathbf{e}_3+\cdots+n\mathbf{e}_n$.
**(b)** $(1,0,-1,1,0,-1,\ldots,1,0,-1)=\mathbf{e}_1-\mathbf{e}_3+\mathbf{e}_4-\mathbf{e}_6+\mathbf{e}_7-\mathbf{e}_9+\cdots+\mathbf{e}_{n-2}-\mathbf{e}_n$.

**2.** $\mathbf{e}_1+\mathbf{e}_2+\cdots+\mathbf{e}_n=(1,1,1,\ldots,1)$.

**3.** $\mathbf{e}_1-2\mathbf{e}_2+3\mathbf{e}_3-4\mathbf{e}_4+\cdots+(-1)^{n+1}n\mathbf{e}_n=(1,-2,3,-4,\ldots,(-1)^{n+1}n)$.

**4.** $\mathbf{e}_1+\mathbf{e}_n=(1,0,0,\ldots,0,1)$.

**5. (a)** $\mathbf{a}+\mathbf{b}=(1+2,3-4,5+6,7-8,\ldots,2n-1+(-1)^{n+1}2n)=(3,-1,11,-1,19,-1,\ldots,2n-1+(-1)^{n+1}2n)$.
The $n^{\text{th}}$ term is $\begin{cases}4n-1 & \text{if } n \text{ is odd, and}\\ -1 & \text{if } n \text{ is even.}\end{cases}$
**(b)** $\mathbf{a}-\mathbf{b}=(1-2,3+4,5-6,7+8,\ldots,2n-1-(-1)^{n+1}2n)=(-1,7,-1,15,-1,\ldots,2n-1-(-1)^{n+1}2n)$. The $n^{\text{th}}$ term is $\begin{cases}4n-1 & \text{if } n \text{ is even, and}\\ -1 & \text{if } n \text{ is odd.}\end{cases}$
**(c)** $-3(1,3,5,7,\ldots,2n-1)=(-3,-9,-15,-21,\ldots,-6n+3)$.
**(d)** $\|\mathbf{a}\|=\sqrt{\mathbf{a}\cdot\mathbf{a}}=\sqrt{1^2+3^2+5^2+\cdots+(2n-1)^2}$.
**(e)** $\mathbf{a}\cdot\mathbf{b}=1(2)+3(-4)+5(6)+\cdots+(2n-1)(-1)^{n+1}2n=2-12+30-56+\cdots+(-1)^{n+1}2n(2n-1)$.

**6.** We want to show that $\|\mathbf{a}+\mathbf{b}\|\le\|\mathbf{a}\|+\|\mathbf{b}\|$. Here $n$ is even and **a** and **b** are vectors in $\mathbf{R}^n$,

$$\mathbf{a}=(1,0,1,0,\ldots,0)$$
$$\mathbf{b}=(0,1,0,1,\ldots,1),\ \text{and}$$
$$\mathbf{a}+\mathbf{b}=(1,1,1,1,\ldots,1).$$

$$\|\mathbf{a}+\mathbf{b}\|=\underbrace{\sqrt{1^2+1^2+\cdots+1^2}}_{n\text{ times}}=\sqrt{n}=2\sqrt{n/4}\le 2\sqrt{n/2}=2\underbrace{\sqrt{1^2+1^2+\cdots+1^2}}_{n/2\text{ times}}=\|\mathbf{a}\|+\|\mathbf{b}\|.$$

**7.** First we calculate

$$\|\mathbf{a}\|=\sqrt{1^2+2^2+3^2+\cdots+n^2}=\sqrt{\frac{n(n+1)(2n+1)}{6}}$$
$$\|\mathbf{b}\|=\underbrace{\sqrt{1^2+1^2+\cdots+1^2}}_{n\text{ times}}=\sqrt{n},\ \text{and}$$
$$|\mathbf{a}\cdot\mathbf{b}|=1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

So

$$\|\mathbf{a}\|\,\|\mathbf{b}\|=\left(\sqrt{\frac{n(n+1)(2n+1)}{6}}\right)(\sqrt{n})=\left(\frac{n}{2}\right)\left(\sqrt{\frac{2(n+1)(2n+1)}{3}}\right).$$

For $n=1$, $\sqrt{\frac{2(n+1)(2n+1)}{3}}=2=n+1$.
For $n=2$, $\sqrt{\frac{2(n+1)(2n+1)}{3}}=\sqrt{10}\ge 3=n+1$.
For $n\ge 3$,

$$\left(\frac{n}{2}\right)\left(\sqrt{\frac{2(n+1)(2n+1)}{3}}\right)\ge\left(\frac{n}{2}\right)\frac{2n+1}{\sqrt{3}}\ge\left(\frac{n}{2}\right)(n+1)=|\mathbf{a}\cdot\mathbf{b}|.$$

**8.** As always,

$$\text{proj}_{\mathbf{a}}\mathbf{b}=\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}=\frac{2-5+27-2}{1+1+49+9+4}\mathbf{a}=\frac{22}{64}\mathbf{a}$$
$$=\frac{11}{32}(1,-1,7,3,2)=\left(\frac{11}{32},\frac{-11}{32},\frac{77}{32},\frac{11}{16}\right).$$

**9.** This is just the triangle inequality:

$$\|\mathbf{a}-\mathbf{b}\|=\|(\mathbf{a}-\mathbf{c})+(\mathbf{c}-\mathbf{b})\|\le\|\mathbf{a}-\mathbf{c}\|+\|\mathbf{c}-\mathbf{b}\|.$$

**10.** We are given that $\mathbf{c} = \mathbf{a} + \mathbf{b}$ so

$$\|\mathbf{c}\|^2 = \|\mathbf{a} + \mathbf{b}\|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b}.$$

But $\mathbf{a} \cdot \mathbf{b} = 0 = \mathbf{b} \cdot \mathbf{a}$, so

$$\mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2.$$

This is analogous to the Pythagorean Theorem. Here **a** and **b** are playing the role of the legs. They are orthogonal vectors. The third side of the triangle is $\mathbf{a} + \mathbf{b} = \mathbf{c}$. The theorem in this case says that the sum of the squares of the lengths of the "legs" is the square of the length of the "hypotenuse".

**11.** We have

$$\|\mathbf{a} + \mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\| \Rightarrow \|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a} - \mathbf{b}\|^2$$
$$\Rightarrow (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}).$$

Expand to find

$$\mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}$$
$$\Rightarrow 4\mathbf{a} \cdot \mathbf{b} = 0,$$

so **a** and **b** are orthogonal.

**12.** As above, if $\|\mathbf{a} - \mathbf{b}\| > \|\mathbf{a} + \mathbf{b}\|$, then $-2\mathbf{a} \cdot \mathbf{b} > 2\mathbf{a} \cdot \mathbf{b}$ so $-4\mathbf{a} \cdot \mathbf{b} > 0 \Leftrightarrow \mathbf{a} \cdot \mathbf{b} < 0$. Thus $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|} < 0$. Hence $\frac{\pi}{2} < 0 \leq \pi$.

**13.** The equation could also be written in the more suggestive form:

$$(2, 3, -7, 1, -5) \cdot [(x_1, x_2, x_3, x_4, x_5) - (1, -2, 0, 4, -1)] = 0.$$

These are the points in $\mathbf{R}^5$ so that $(x_1, x_2, x_3, x_4, x_5) - (1, -2, 0, 4, -1)$ is orthogonal to the vector $(2, 3, -7, 1, -5)$. This is the four dimensional hyperplane in $\mathbf{R}^5$ orthogonal to $(2, 3, -7, 1, -5)$ containing the point $(1, -2, 0, 4, -1)$.

**14.** Half of each type of your inventory gives T-shirts in quantities of 10, 15, 12, 10 (in order of lowest to highest selling price). Half of each type of your friend's inventory gives 15, 8, 10, 14 baseball caps. The value of your half of the inventory is

$$(8, 10, 12, 15) \cdot (10, 15, 12, 10) = \$524.$$

The value of your friend's inventory is

$$(8, 10, 12, 15) \cdot (15, 8, 10, 14) = \$530.$$

Thus your friend might be reluctant to accept your offer, unless he's quite a good friend.

**15.** **(a)** We have

$$\mathbf{p} = (200, 250, 300, 375, 450, 500)$$
$$\text{Total cost} = \mathbf{p} \cdot \mathbf{x} = 200x_1 + 250x_2 + 300x_3 + 375x_4 + 450x_5 + 500x_6$$

**(b)** With **p** as in part (a), the customer can afford commodity bundles **x** in the set

$$\{\mathbf{x} \in \mathbf{R}^6 | \mathbf{p} \cdot \mathbf{x} \leq 100{,}000\}.$$

The budget hyperplane is $\mathbf{p} \cdot \mathbf{x} = 100{,}000$ or $200x_1 + 250x_2 + 300x_3 + 375x_4 + 450x_5 + 500x_6 = 100{,}000$.

**16.**

$$3A - 2B = 3\begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} -4 & 9 & 5 \\ 0 & 3 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 6 & 9 \\ -6 & 0 & 3 \end{bmatrix} - \begin{bmatrix} -8 & 18 & 10 \\ 0 & 6 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 11 & -12 & -1 \\ -6 & -6 & 3 \end{bmatrix}$$

**17.**

$$AC = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 7 \\ 0 & 3 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 1(1)+2(2)+3(0) & 1(-1)+2(0)+3(3) & 1(0)+2(7)+3(-2) \\ -2(1)+0(2)+1(0) & -2(-1)+0(0)+1(3) & -2(0)+0(7)+1(-2) \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 8 & 8 \\ -2 & 5 & -2 \end{bmatrix}.$$

**18.**

$$DB = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} -4 & 9 & 5 \\ 0 & 3 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1(-4)+0(0) & 1(9)+0(3) & 1(5)+0(0) \\ 2(-4)-3(0) & 2(9)-3(3) & 2(5)-3(0) \end{bmatrix}$$
$$= \begin{bmatrix} -4 & 9 & 5 \\ -8 & 9 & 10 \end{bmatrix}.$$

**19.**

$$B^T D = \begin{bmatrix} -4 & 0 \\ 9 & 3 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} -4(1)+0(2) & -4(0)+0(-3) \\ 9(1)+3(2) & 9(0)+3(-3) \\ 5(1)+0(2) & 5(0)+0(-3) \end{bmatrix}$$
$$= \begin{bmatrix} -4 & 0 \\ 15 & -9 \\ 5 & 0 \end{bmatrix}$$

**20. (a)**

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

**(b)** The *ij*th entry of the product of matrices $A$ and $B$ is the product of the $i$th row of $A$ and the $j$th column of $B$. So in case i. we have:

$$(AI_n)_{ij} = [a_{i1} \quad a_{i2} \quad a_{i3} \quad \ldots \quad a_{in}](\mathbf{e}_j)^T = (a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in}) \cdot \mathbf{e}_j = a_{ij}.$$

In case ii. we have:

$$(I_n A)_{ij} = (\mathbf{e}_i) \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{nj} \end{pmatrix} = \mathbf{e}_i \cdot (a_{1j}, a_{2j}, a_{3j}, \ldots, a_{nj}) = a_{ij}.$$

In both cases we've shown that the *ij*th component of the product is the *ij*th component of matrix $A$, so $AI_n = A = I_n A$.

**21.** We'll expand on the first row:

$$\begin{vmatrix} 7 & 0 & -1 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & -3 & 0 & 2 \\ 0 & 5 & 1 & -2 \end{vmatrix} = 7 \begin{vmatrix} 0 & 1 & 3 \\ -3 & 0 & 2 \\ 5 & 1 & -2 \end{vmatrix} - \begin{vmatrix} 2 & 0 & 3 \\ 1 & -3 & 2 \\ 0 & 5 & -2 \end{vmatrix}$$
$$= 7\left(-1\begin{vmatrix} -3 & 2 \\ 5 & -2 \end{vmatrix} + 3\begin{vmatrix} -3 & 0 \\ 5 & 1 \end{vmatrix}\right) - \left(2\begin{vmatrix} -3 & 2 \\ 5 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & -3 \\ 0 & 5 \end{vmatrix}\right)$$
$$= 7(-1(-4)+3(-3)) - (2(-4)+3(5)) = -42.$$

*Note: Exercises 22 and 23 are good exploration problems for students before they've done Exercise 25.*

**22.** Note that if we expand along the first row, only one term survives. If at each step we expand along the first row, the pattern continues. What we are left with is the product of the elements along the diagonal.

$$\begin{vmatrix} 8 & 0 & 0 & 0 \\ 15 & 1 & 0 & 0 \\ -7 & 6 & -1 & 0 \\ 8 & 1 & 9 & 7 \end{vmatrix} = 8 \begin{vmatrix} 1 & 0 & 0 \\ 6 & -1 & 0 \\ 1 & 9 & 7 \end{vmatrix}$$

$$= (8)(1) \begin{vmatrix} -1 & 0 \\ 9 & 7 \end{vmatrix}$$

$$= (8)(1)(-1)(7) = -56.$$

**23.** This is similar to Exercise 22. Either we could expand along the last row of each matrix at each step or we could expand along the first column at each step. It is easier to keep track of signs if we choose this second approach. We again find that the determinant is the product of the diagonal elements.

$$\begin{vmatrix} 5 & -1 & 0 & 8 & 11 \\ 0 & 2 & 1 & 9 & 7 \\ 0 & 0 & 4 & -3 & 5 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{vmatrix} = 5 \begin{vmatrix} 2 & 1 & 9 & 7 \\ 0 & 4 & -3 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix}$$

$$= (5)(2) \begin{vmatrix} 4 & -3 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{vmatrix}$$

$$= (5)(2)(4) \begin{vmatrix} 2 & 1 \\ 0 & -3 \end{vmatrix}$$

$$= (5)(2)(4)(2)(-3) = -240.$$

**24.** There really isn't anything to show. Using the convenient fact provided after Example 8:

- If row $i$ consists of all zeros (i.e., $a_{ij} = 0$ for $1 \le j \le n$) then expand along row $i$. Using the cofactor notation:

$$\begin{aligned} |A| &= (-1)^{i+1} a_{i1}|A_{i1}| + (-1)^{i+2} a_{i2}|A_{i2}| + \cdots + (-1)^{i+n} a_{in}|A_{in}| \\ &= (-1)^{i+1}(0)|A_{i1}| + (-1)^{i+2}(0)|A_{i2}| + \cdots + (-1)^{i+n}(0)|A_{in}| = 0. \end{aligned}$$

- If column $j$ consists of all zeros (i.e., $a_{ij} = 0$ for all $1 \le i \le n$) then expand along column $j$. As above we get

$$\begin{aligned} |A| &= (-1)^{1+j} a_{1j}|A_{1j}| + (-1)^{2+j} a_{2j}|A_{2j}| + \cdots + (-1)^{n+j} a_{nj}|A_{nj}| \\ &= (-1)^{1+j}(0)|A_{1j}| + (-1)^{2+j}(0)|A_{2j}| + \cdots + (-1)^{n+j}(0)|A_{nj}| = 0. \end{aligned}$$

**25. (a)** **A lower triangular** matrix is an $n \times n$ matrix whose entries above the main diagonal are all zero. For example the matrix in Exercise 22 is lower triangular.

**(b)** If we expand the determinant of an upper triangular matrix along its first column we get:

$$\begin{aligned} |A| &= (-1)^{1+1} a_{11}|A_{11}| + (-1)^{2+1} a_{21}|A_{21}| + \cdots + (-1)^{n+1} a_{n1}|A_{n1}| \\ &= (-1)^{1+1}(a_{11})|A_{11}| + (-1)^{2+1}(0)|A_{2j}| + \cdots + (-1)^{n+1}(0)|A_{nj}| = (a_{11})|A_{11}|. \end{aligned}$$

Looking back on what we have found: The determinant of an upper triangular matrix is equal to the term in the upper left position multiplied by the determinant of the matrix that's left when the top most row and left most column are removed. Each time we remove the top row and left column we are left with an upper triangular matrix of one dimension lower. Repeat the process $n$ times and it is clear that

$$|A| = a_{11}|A_{11}| = a_{11}(a_{22}|(A_{11})_{11}|) = \cdots = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

**26. (a) Type I Rule**: If matrix $B$ results from matrix $A$ by exchanging rows $i$ and $j$ then $|A| = -|B|$.

As one example,

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1, \quad \text{while} \quad \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1.$$

A more important example is

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = -(b_1a_2 - b_2a_1) = -\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}.$$

The reason this second example is more important is that you can always expand the determinants of $A$ and $B$ so that you are left with a sum of scalars times the determinants of 2 by 2 matrices involving only the two rows being switched. Since the scalars will be the same in both cases, this second example shows that the effect of switching rows $i$ and $j$ is to switch the sign of every component in the sum and so $|A| = -|B|$.

**(b) Type III Rule**: If matrix $B$ results from matrix $A$ by adding a multiple of row $i$ to row $j$ and leaving row $i$ unchanged then $|A| = |B|$.

As one example,

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1, \quad \text{and also} \quad \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

To see what's going on, let's look at the example

$$\begin{aligned} \begin{vmatrix} a_1 + nb_1 & a_2 + nb_2 \\ b_1 & b_2 \end{vmatrix} &= (a_1 + nb_1)b_2 - (a_2 + nb_2)b_1 = a_1b_2 - a_2b_1 + n(b_1b_2 - b_2b_1) \\ &= a_1b_2 - a_2b_1. \end{aligned}$$

Another way to look at the example above is to see that the determinant splits into two pieces:

$$a_1b_2 - a_2b_1 + n(b_1b_2 - b_2b_1) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + n\begin{vmatrix} b_1 & b_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

*Note: A more general case of this rule will be proved in Exercise 28.*

**(c) Type II Rule**: If matrix $B$ results from matrix $A$ by multiplying the entries in the $i$th row of $A$ by the scalar $c$ then $|B| = c|A|$.

We will prove this by expanding the determinant for $B$ along the $i$th row. Because row $i$ is the only one changed, the cofactors $B_{ij}$ are the same as the cofactors $A_{ij}$.

$$\begin{aligned} |B| &= (-1)^{i+1}b_{i1}|B_{i1}| + (-1)^{i+2}b_{i2}|B_{i2}| + \cdots + (-1)^{i+n}b_{in}|B_{in}| \\ &= (-1)^{i+1}ca_{i1}|A_{i1}| + (-1)^{i+2}ca_{i2}|A_{i2}| + \cdots + (-1)^{i+n}ca_{in}|A_{in}| \\ &= c((-1)^{i+1}a_{i1}|A_{i1}| + (-1)^{i+2}a_{i2}|A_{i2}| + \cdots + (-1)^{i+n}a_{in}|A_{in}|) = c|A|. \end{aligned}$$

**27.** Here we go: at each step we'll specify what we've done.

$$\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 1 & 0 & 1 & -2 & 4 \\ -1 & 1 & 2 & 3 & -5 \\ 0 & 2 & 3 & 1 & 7 \\ -3 & 2 & -1 & 0 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ -1 & 1 & 2 & 3 & -5 \\ 1 & 0 & 1 & -2 & 4 \\ 0 & 2 & 3 & 1 & 7 \\ -3 & 2 & -1 & 0 & 1 \end{vmatrix} \quad \text{switched rows 2 and 3}$$

$$= (-1)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 1 & 0 & 1 & -2 & 4 \\ 0 & 2 & 3 & 1 & 7 \\ 0 & 2 & 2 & -6 & 13 \end{vmatrix} \quad \begin{matrix} \\ \leftarrow \text{row } 2 + \text{row } 3 \\ \\ \\ \leftarrow \text{row } 5 + 3(\text{row } 3) \end{matrix}$$

$$= \left(\frac{-1}{2}\right)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 2 & 0 & 2 & -4 & 8 \\ 0 & 2 & 3 & 1 & 7 \\ 0 & 2 & 2 & -6 & 13 \end{vmatrix} \quad \leftarrow 2(\text{row } 3)$$

$$= \left(\frac{-1}{2}\right)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & -1 & 4 & -11 & 0 \\ 0 & 2 & 3 & 1 & 7 \\ 0 & 2 & 2 & -6 & 13 \end{vmatrix} \quad \leftarrow \text{row } 3 - \text{row } 1$$

$$= \left(\frac{-1}{2}\right)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 7 & -10 & -1 \\ 0 & 0 & -3 & -1 & 9 \\ 0 & 0 & -4 & -8 & 15 \end{vmatrix} \quad \begin{matrix} \\ \\ \leftarrow \text{row } 3 + \text{row } 2 \\ \leftarrow \text{row } 4 - 2(\text{row } 2) \\ \leftarrow \text{row } 5 - 2(\text{row } 2) \end{matrix}$$

$$= \left(\frac{-1}{2(7)(7)}\right)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 7 & -10 & -1 \\ 0 & 0 & -21 & -7 & 63 \\ 0 & 0 & -28 & -56 & 105 \end{vmatrix} \quad \begin{matrix} \\ \\ \\ \leftarrow 7(\text{row } 4) \\ \leftarrow 7(\text{row } 5) \end{matrix}$$

$$= \left(\frac{-1}{2(7)(7)}\right)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 7 & -10 & -1 \\ 0 & 0 & 0 & -37 & 60 \\ 0 & 0 & 0 & -96 & 101 \end{vmatrix} \quad \begin{matrix} \\ \\ \\ \leftarrow \text{row } 4 + 3(\text{row } 4) \\ \leftarrow \text{row } 5 + 4(\text{row } 3) \end{matrix}$$

$$= \left(\frac{-1}{2(7)(7)(-37)}\right)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 7 & -10 & -1 \\ 0 & 0 & 0 & -37 & 60 \\ 0 & 0 & 0 & 3552 & -3737 \end{vmatrix} \quad \begin{matrix} \\ \\ \\ \\ \leftarrow -37(\text{row } 5) \end{matrix}$$

$$= \left(\frac{-1}{2(7)(7)(-37)}\right)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 7 & -10 & -1 \\ 0 & 0 & 0 & -37 & 60 \\ 0 & 0 & 0 & 0 & 2023 \end{vmatrix} \quad \begin{matrix} \\ \\ \\ \\ \leftarrow \text{row } 5 + 96(\text{row } 4) \end{matrix}$$

$$= \left(\frac{-1}{2(7)(7)(-37)}\right)(2)(1)(7)(-37)(2023) = \frac{2023}{7} = -289.$$

**28. (a)** If you let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ then $1 = \det(A+B)$ but $\det(A) = \det(B) = 0$. So in general $\det(A+B) \neq \det(A) + \det(B)$.

**(b)** $\begin{vmatrix} 1 & 2 & 7 \\ 3+2 & 1-1 & 5+1 \\ 0 & -2 & 0 \end{vmatrix} = -58$, while $\begin{vmatrix} 1 & 2 & 7 \\ 3 & 1 & 5 \\ 0 & -2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 7 \\ 2 & -1 & 1 \\ 0 & -2 & 0 \end{vmatrix} = -32 - 26 = -58$. It makes sense that these should be equal; if you imagine expanding on the second row we see that

$$\begin{vmatrix} 1 & 2 & 7 \\ 3+2 & 1-1 & 5+1 \\ 0 & -2 & 0 \end{vmatrix} = (3+2)\begin{vmatrix} 2 & 7 \\ -2 & 0 \end{vmatrix} + (1-1)\begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix} + (5+1)\begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix}$$

$$= \left(3\begin{vmatrix} 2 & 7 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix} + 5\begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix}\right) + \left(2\begin{vmatrix} 2 & 7 \\ -2 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix}\right)$$

$$= \begin{vmatrix} 1 & 2 & 7 \\ 3 & 1 & 5 \\ 0 & -2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 7 \\ 2 & -1 & 1 \\ 0 & -2 & 0 \end{vmatrix}.$$

**(c)** $\begin{vmatrix} 1 & 3 & 2+3 \\ 0 & 4 & -1+5 \\ -1 & 0 & 0-2 \end{vmatrix} = 0$, while $\begin{vmatrix} 1 & 3 & 2 \\ 0 & 4 & -1 \\ -1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 3 \\ 0 & 4 & 5 \\ -1 & 0 & -2 \end{vmatrix} = -11 + 11 = 0$.

**(d)** We might characterize the rules for rows as follows:
Let $A$, $B$ and $C$ be three matrices whose elements are the same except for those in row $i$ where $c_{ij} = a_{ij} + b_{ij}$ for $1 \leq j \leq n$. Then $\det(C) = \det(A) + \det(B)$. We prove this by expanding the determinant along row $i$ noting that in that case the cofactors for all three matrices are equal (i.e., $A_{ij} = B_{ij} = C_{ij}$ for $1 \leq j \leq n$):

$$\begin{aligned}
|C| &= (-1)^{i+1}c_{i1}|C_{i1}| + (-1)^{i+2}c_{i2}|C_{i2}| + \cdots (-1)^{i+n}c_{in}|C_{in}| \\
&= (-1)^{i+1}(a_{i1}+b_{i1})|C_{i1}| + (-1)^{i+2}(a_{i2}+b_{i2})|C_{i2}| + \cdots (-1)^{i+n}(a_{in}+b_{in})|C_{in}| \\
&= (-1)^{i+1}(a_{i1})|C_{i1}| + (-1)^{i+2}(a_{i2})|C_{i2}| + \cdots (-1)^{i+n}(a_{in})|C_{in}| \\
&\quad + (-1)^{i+1}(b_{i1})|C_{i1}| + (-1)^{i+2}(b_{i2})|C_{i2}| + \cdots (-1)^{i+n}(b_{in})|C_{in}| \\
&= (-1)^{i+1}(a_{i1})|A_{i1}| + (-1)^{i+2}(a_{i2})|A_{i2}| + \cdots (-1)^{i+n}(a_{in})|A_{in}| \\
&\quad + (-1)^{i+1}(b_{i1})|B_{i1}| + (-1)^{i+2}(b_{i2})|B_{i2}| + \cdots (-1)^{i+n}(b_{in})|B_{in}| \\
&= |A| + |B|.
\end{aligned}$$

The rule for columns is exactly the same:
Let $A$, $B$ and $C$ be three matrices whose elements are the same except for those in column $j$ where $c_{ij} = a_{ij} + b_{ij}$ for $1 \leq i \leq n$. Then $\det(C) = \det(A) + \det(B)$. We could prove this by expanding the determinant along column $j$ just as above. Instead note that $A^T$, $B^T$, and $C^T$ satisfy the above rule for rows and that the determinant of a matrix is equal to the determinant of its transpose. Our proof is then:

$$|C| = |C^T| = |A^T| + |B^T| = |A| + |B|.$$

**29.** This is a pretty cool fact. If *AB* and *BA* both exist, these two matrices may not be equal. It doesn't matter. They still have the same determinant. The proof is straightforward: $\det(AB) = (\det\ A)(\det B) = (\det\ B)(\det\ A) = \det\,(BA)$.

**30. (a)** Check the products in both directions . . .

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} (1+0) & (0+0) \\ (1-1) & (0+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1+0) & (0+0) \\ (-1+1) & (0+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

**(b)** Again, the products in both directions yield the identity matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} = \begin{bmatrix} (-40+26+15) & (16-10-6) & (9-6-3) \\ (-80+65+15) & (32-25-6) & (18-15-3) \\ (-40+0+40) & (16+0-16) & (9+0-8) \end{bmatrix}$$

$$= \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} (-40+32+9) & (-80+80+0) & (-120+48+72) \\ (13-10-3) & (26-25+0) & (39-15-24) \\ (5-4-1) & (10-10+0) & (15-6-8) \end{bmatrix}.$$

**31.** Say the given matrix is $A$. Then the top left entry in the inverse must be 1/2 because 1 is the top left entry of the product of $A^{-1}A$ and it is twice the top left entry in the inverse matrix.

Looking at the second row of $A$, in the product $AA^{-1}$ it "picks out" the element in the second row. This means that the second row of $A^{-1}$ is (0, 1, 0). Similarly, the third row of $A$ picks out the opposite of the element in the third row in the product $AA^{-1}$ so the third row of $A^{-1}$ is $(0, 0, -1)$.

The third column of $A$ tells us that the first and third elements of the top row of $A^{-1}$ must be the same. The final element to solve for is the middle element of the top row of $A^{-1}$. It must be the opposite of the middle element of the third row of $A^{-1}$. Putting this information together, we have that

$$A^{-1} = \begin{bmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

**32.** Since the first column is 0, the determinant is 0. This means that the matrix could not have an inverse. We'll actually show this in Exercise 35 below. Say, for a minute that you don't accept the results of Exercise 35 and you think you have found an inverse matrix $A^{-1}$ for the given matrix $A$. Then look at the product $A^{-1}A$. It should be the identity matrix $I_3$ but the first column of the product will be all 0's. For this reason, no inverse for $A$ could exist.

**33.** Using the hint, assume that $A$ has two inverses $B$ and $C$. Then

$$B = BI = B(AC) = (BA)C = IC = C.$$

**34.** We just verify that $B^{-1}A^{-1}$ behaves as an inverse:

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$

**35. (a)** If $A$ is invertible, consider the product $AA^{-1} = I$. By the formula in Exercise 29, $(\det\ A)(\det\ A^{-1}) = \det(AA^{-1}) = \det\ I = 1$. From this we see that $\det\ A \neq 0$. In fact, we see more – the results of part (b) follow immediately.

**(b)** See part (a).

**36. (a)**

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = I_2$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = I_2$$

**(b)**

$$\begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix}^{-1} = \frac{1}{(2\cdot 2-(4)(-1))}\begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1/4 & -1/2 \\ 1/8 & 1/4 \end{bmatrix}$$

**37.** If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 1 & 0 & 3 \end{bmatrix}$, then det $A = 12 + 4 - 2 = 14$, so the formula gives

$$A^{-1} = \frac{1}{14}\begin{bmatrix} \begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} \\ -\begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 6 & -3 & 2 \\ 4 & 5 & -8 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & -\frac{3}{14} & \frac{1}{7} \\ \frac{2}{7} & \frac{5}{14} & -\frac{4}{7} \\ -\frac{1}{7} & \frac{1}{14} & \frac{2}{7} \end{bmatrix}$$

**38.** If $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix}$, then det $A = 4 + 6 - 18 + 1 = -7$, so the formula gives

$$A^{-1} = -\frac{1}{7}\begin{bmatrix} \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \end{bmatrix}$$

$$= -\frac{1}{7}\begin{bmatrix} 2 & 1 & -4 \\ -7 & -7 & 7 \\ -6 & -3 & 5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{7} & -\frac{1}{7} & \frac{4}{7} \\ 1 & 1 & -1 \\ \frac{6}{7} & \frac{3}{7} & -\frac{5}{7} \end{bmatrix}$$

**39.** We'll transform the cross product into a determinant. To make the determinant easier to calculate we'll replace the fourth row with the sum of the fourth row and five times the second row. Finally we'll expand along the first column.

$$(1, 2, -1, 3) \times (0, 2, -3, 1) \times (-5, 1, 6, 0) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ 1 & 2 & -1 & 3 \\ 0 & 2 & -3 & 1 \\ -5 & 1 & 6 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ 1 & 2 & -1 & 3 \\ 0 & 2 & -3 & 1 \\ 0 & 11 & 1 & 15 \end{vmatrix} \quad \leftarrow \text{row } 4 + 5(\text{row } 2) \qquad = \mathbf{e}_1 \begin{vmatrix} 2 & -1 & 3 \\ 2 & -3 & 1 \\ 11 & 1 & 15 \end{vmatrix} - \begin{vmatrix} \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ 2 & -3 & 1 \\ 11 & 1 & 15 \end{vmatrix}$$

$$= 32\mathbf{e}_1 + 46\mathbf{e}_2 + 19\mathbf{e}_3 - 35\mathbf{e}_4 = (32, 46, 19, -35).$$

**40. (a)** We use the matrix form to write the cross product as a determinant. We then switch row $i+1$ (the row consisting of $a_{i1}, a_{i2}, \ldots, a_{in}$) with row $j+1$ (the row consisting of $a_{j1}, a_{j2}, \ldots, a_{jn}$) which multiplies the determinant by $-1$:

$$\mathbf{a}_1 \times \cdots \times \mathbf{a}_i \times \cdots \times \mathbf{a}_j \times \cdots \times \mathbf{a}_{n-1} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ a_{11} & a_{12} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix}$$

$$= (-1) \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{Switch this row } (i+1) \\ \\ \leftarrow \text{with this row } (j+1) \\ \\ \\ \\ \end{matrix}$$

$$= -(\mathbf{a}_1 \times \cdots \times \mathbf{a}_j \times \cdots \times \mathbf{a}_i \times \cdots \times \mathbf{a}_{n-1})$$

**(b)** Again we will change to the matrix form and then use the rule for the row operation of type II to pull the scalar $k$ out and then rewrite as a cross product.

$$\mathbf{a}_1 \times \cdots \times k\mathbf{a}_i \times \cdots \times \mathbf{a}_{n-1} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \ldots & \mathbf{e}_n \\ a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{i1} & ka_{i2} & \ldots & ka_{in} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \ldots & a_{(n-1)n} \end{vmatrix}$$

$$= (k) \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \ldots & \mathbf{e}_n \\ a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \ldots & a_{(n-1)n} \end{vmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{row } (i+1) \text{ divided by } k \\ \\ \\ \\ \end{matrix}$$

$$= k(\mathbf{a}_1 \times \cdots \times \mathbf{a}_i \times \cdots \times \mathbf{a}_{n-1})$$

**(c)** Once again, we will change to the matrix form. This time we will use the rule we developed in Exercise 28 to write this as two determinants. Finally we will convert each back to the cross product form.

$$\mathbf{a}_1 \times \cdots \times (\mathbf{a}_i + \mathbf{b}) \times \cdots \times \mathbf{a}_{n-1} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \ldots & \mathbf{e}_n \\ a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ (a_{i1}+b_1) & (a_{i2}+b_2) & \ldots & (a_{in}+b_n) \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \ldots & a_{(n-1)n} \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \ldots & \mathbf{e}_n \\ a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \ldots & a_{(n-1)n} \end{vmatrix} + \begin{vmatrix} l\mathbf{e}_1 & \mathbf{e}_2 & \ldots & \mathbf{e}_n \\ a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ b_1 & b_2 & \ldots & b_n \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \ldots & a_{(n-1)n} \end{vmatrix}$$

$$= (\mathbf{a}_1 \times \cdots \times \mathbf{a}_i \times \cdots \times \mathbf{a}_{n-1}) + (\mathbf{a}_1 \times \cdots \times \mathbf{b} \times \cdots \times \mathbf{a}_{n-1})$$

**(d)** Expand the determinant along the first row; we'll refer to the cross product matrix as $C$:

$$\mathbf{b} \cdot |C| = \mathbf{b} \cdot ((\mathbf{a}_1 \times \cdots \times \mathbf{a}_i \times \cdots \times \mathbf{a}_{n-1}) = \mathbf{b} \cdot \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \ldots & \mathbf{e}_n \\ a_{11} & a_{12} & \ldots & a_{1n} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \ldots & a_{(n-1)n} \end{vmatrix}$$

$$= \mathbf{b} \cdot (\mathbf{e}_1|C_{11}| - \mathbf{e}_2|C_{12}| + \cdots + (-1)^{1+n}\mathbf{e}_n|C_{1n}|)$$

$$= b_1|C_{11}| - b_2|C_{12}| + \cdots + (-1)^{1+n}b_n|C_{1n}| = \begin{vmatrix} b_1 & b_2 & \cdots & b_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix}$$

**41.** This follows immediately from part (d) of Exercise 28. For $1 \leq i \leq n-1$,

$$\mathbf{a}_i \cdot (\mathbf{a}_1 \times \cdots \times \mathbf{a}_i \times \cdots \times \mathbf{a}_{n-1}) = \begin{vmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix}.$$

Replace the first row with the difference between row 1 and row $i+1$ and you will get (by Exercise 26) a matrix with the same determinant, namely:

$$\begin{vmatrix} 0 & 0 & \cdots & 0 \\ a_{11} & a_{12} & \cdots & a_{1n} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix} = 0.$$

Therefore **b** is orthogonal to $\mathbf{a}_i$ for $1 \leq i \leq n-1$.

**42.** To find the normal direction **n** we'll take the cross product of the displacement vectors:

$$\overrightarrow{P_0P_1} = (2,-1,0,0,5) - (1,0,3,0,4) = (1,-1,-3,0,1)$$
$$\overrightarrow{P_0P_2} = (7,0,0,2,0) - (1,0,3,0,4) = (6,0,-3,2,-4)$$
$$\overrightarrow{P_0P_3} = (2,0,3,0,4) - (1,0,3,0,4) = (1,0,0,0,0)$$
$$\overrightarrow{P_0P_4} = (1,-1,3,0,4) - (1,0,3,0,4) = (0,-1,0,0,0)$$

We take the cross product which is the determinant (expand along the fourth row, and then along the last row):

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 \\ 1 & -1 & -3 & 0 & 1 \\ 6 & 0 & -3 & 2 & -4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 \\ -1 & -3 & 0 & 1 \\ 0 & -3 & 2 & -4 \\ -1 & 0 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 \\ -3 & 0 & 1 \\ -3 & 2 & -4 \end{vmatrix}$$
$$= 2\mathbf{e}_3 + 15\mathbf{e}_4 + 6\mathbf{e}_5 = (0,0,2,15,6).$$

We can choose any of the points, say $P_0$ to find the equation of the hyperplane:

$$2(x_3-3) + 15(x_4) + 6(x_5-4) = 0 \quad \text{or} \quad 2x_3 + 15x_4 + 6x_5 = 30.$$

## 1.7 New Coordinate Systems

*In Exercises 1–3 use equations (1) $x = r\cos\theta$ and $y = r\sin\theta$.*

**1.** $x = \sqrt{2}\cos\pi/4 = (\sqrt{2})(\sqrt{2}/2) = 1$, and $y = \sqrt{2}\sin\pi/4 = 1$. The rectangular coordinates are (1, 1).

**2.** $x = \sqrt{3}\cos 5\pi/6 = (\sqrt{3})(-\sqrt{3}/2) = -3/2$, and $y = \sqrt{3}\sin 5\pi/6 = (\sqrt{3})(1/2) = \sqrt{3}/2$. The rectangular coordinates are $(-3/2, \sqrt{3}/2)$.

**3.** $x = 3\cos 0 = 3(1) = 3$, and $y = 3\sin 0 = 0$. The rectangular coordinates are (3, 0).

*In Exercises 4–6 use equations (2) $r^2 = x^2 + y^2$, and $\tan\theta = y/x$.*

**4.** $r^2 = (2\sqrt{3})^2 + 2^2 = 16$, so $r = 4$. Also, $\tan\theta = 2/2\sqrt{3} = (1/2)/(\sqrt{3}/2)$. Since we are in the first quadrant the polar coordinates are $(4, \pi/6)$.

**5.** $r^2 = (-2)^2 + 2^2 = 8$, so $r = 2\sqrt{2}$. Also, $\tan\theta = 2/(-2) = -1$. Since we are in the second quadrant the polar coordinates are $(2\sqrt{2}, 3\pi/4)$.

**6.** $r^2 = (-1)^2 + (-2)^2 = 5$, so $r = \sqrt{5}$. Also, $\tan\theta = -2/(-1) = 2$. If the point were in the first quadrant, then the angle would be $\tan^{-1} 2$. Since we are in the third quadrant the polar coordinates are $(\sqrt{5}, \pi + \tan^{-1} 2)$.

*Exercises 7–9 involve exactly the same idea as Exercises 1–3. Since the z coordinates are the same again we use equations (1) or (3).*

**7.** Here there's nothing to do; the rectangular coordinates are (2 cos 2, 2 sin 2, 2).

**8.** $x = \pi\cos\pi/2 = (\pi)(0)$, $y = \pi\sin\pi/2 = (\pi)(1)$, and $z = 1$. The rectangular coordinates are $(0, \pi, 1)$.

**9.** $x = 1\cos 2\pi/3 = -1/2$, $y = 1\sin 2\pi/3 = \sqrt{3}/2$, and $z = -2$. The rectangular coordinates are $(-1/2, \sqrt{3}/2, -2)$.

*In Exercises 10–13 use equations (7) $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$.*

**10.** $x = 4(\sin\pi/2)(\cos\pi/3) = 4(1)(1/2) = 2$, $y = 4(\sin\pi/2)(\sin\pi/3) = 4(1)(\sqrt{3}/2) = 2\sqrt{3}$, and $z = 4\cos\pi/2 = 4(0) = 0$. So the rectangular coordinates are $(2, 2\sqrt{3}, 0)$.

**11.** $x = 3(\sin\pi/3)(\cos\pi/2) = 3(\sqrt{3}/2)(0) = 0$, $y = 3(\sin\pi/3)(\sin\pi/2) = 3(\sqrt{3}/2)(1) = 3\sqrt{3}/2$, and $z = 3\cos\pi/3 = 3(1/2) = 3/2$. So the rectangular coordinates are $(0, 3\sqrt{3}/2, 3/2)$.

**12.** $x = (\sin 3\pi/4)(\cos 2\pi/3) = (\sqrt{2}/2)(-1/2) = -\sqrt{2}/4$, $y = (\sin 3\pi/4)(\sin 2\pi/3) = (\sqrt{2}/2)(\sqrt{3}/2) = \sqrt{6}/4$, and $z = \cos 3\pi/4 = -\sqrt{2}/2$. So the rectangular coordinates are $(-\sqrt{2}/4, \sqrt{6}/4, -\sqrt{2}/2)$. I gave the answer in this form because most students have been told throughout high school that you can never leave a square root in the denominator. They should, of course, feel comfortable leaving the answer as $(-1/\sqrt{8}, \sqrt{3}/\sqrt{8}, -1/\sqrt{2})$, but most won't.

**13.** $x = 2(\sin\pi)(\cos\pi/4) = 2(0)(\sqrt{2}/2) = 0$, $y = 2(\sin\pi)(\sin\pi/4) = 2(0)(\sqrt{2}/2) = 0$, and $z = 2\cos\pi = 2(-1) = -2$. So the rectangular coordinates are $(0, 0, -2)$.

*Exercises 14–16 are basically the same as Exercises 4–6 since the $z$ coordinates are the same in both coordinate systems. Use equations (2) or (4).*

**14.** $r^2 = (-1)^2 + 0^2 = 1$, so $r = 1$. Also, $\tan\theta = 0/(-1) = 0$, so $\theta = \pi$. The cylindrical coordinates are $(1, \pi, 2)$.

**15.** $r^2 = (-1)^2 + (\sqrt{3})^2$, so $r = 2$. Also, $\tan\theta = \sqrt{3}/(-1) = (\sqrt{3}/2)/(-1/2)$, so $\theta = 2\pi/3$. The cylindrical coordinates are $(2, 2\pi/3, 13)$.

**16.** $r^2 = 5^2 + 6^2$, so $r = \sqrt{61}$. Also $\tan\theta = 6/5$, so $\theta = \tan^{-1} 6/5$. The cylindrical coordinates are $(\sqrt{61}, \tan^{-1} 6/5, 3)$.

*In Exercises 17 and 18 use equations (7) $\rho^2 = x^2 + y^2 + z^2$, $\tan\varphi = \sqrt{x^2 + y^2}/z$, and $\tan\theta = y/x$.*

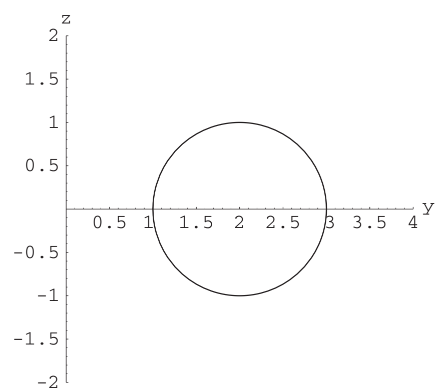
**17.** $\rho^2 = (1)^2 + (-1)^2 + (\sqrt{6})^2 = 8$, so $\rho = \sqrt{8} = 2\sqrt{2}$. Also, $\tan\varphi = \sqrt{1^2 + (-1)^2}/\sqrt{6} = \sqrt{2}/\sqrt{6} = (1/2)/(\sqrt{3}/2)$, so $\varphi = \pi/6$. Finally, $\tan\theta = -1/1 = -1$, so $\theta = 7\pi/4$ (since the point, when projected onto the $xy$-plane is in the fourth quadrant). In spherical coordinates the point is $(2\sqrt{2}, \pi/6, 7\pi/4)$.

**18.** $\rho^2 = 0^2 + (\sqrt{3})^2 + 1^2 = 4$, so $\rho = 2$. Also $\tan\varphi = \sqrt{0^2 + (\sqrt{3})^2}/1 = \sqrt{3}$, so $\varphi = \pi/3$. Finally, when we project the point onto the $xy$-plane we see that the point is on the positive $y$-axis so $\theta = \pi/2$. Or, just using the equation $\tan\theta = \sqrt{3}/0$, so $\theta = \pi/2$. In spherical coordinates the point is $(2, \pi/3, \pi/2)$.

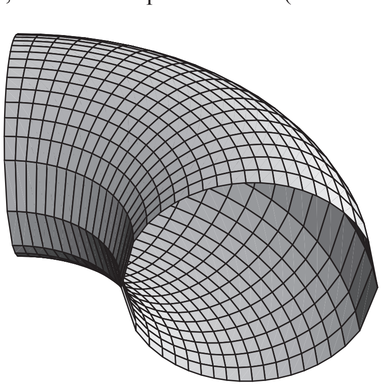
*The figures in Exercises 19–21 form a progression. To complete it, the next in line following Exercise 21 would be a sphere.*

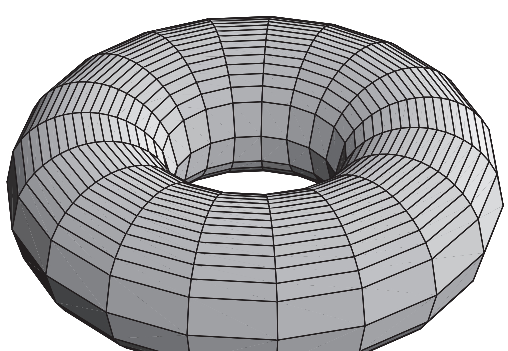
**19.** As in Example 5, $\theta$ does not appear so the surface will be circularly symmetric about the $z$-axis. Once we have our answer to part (a), we can just rotate it about the $z$-axis to generate the answer to part (b).

**(a)** We are slicing in the direction $\pi/2$ which puts us in the $yz$-plane for positive $y$. This means that $(r-2)^2 + z^2 = 1$ becomes $(y-2)^2 + z^2 = 1$. This is a circle of radius 1 centered at (0, 2, 0).

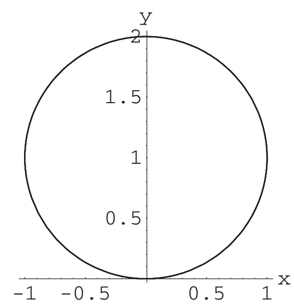



**(b)** As we start to rotate this about the $z$-axis, we get a feel for the shape being generated (see below left). In the figure above we see the result of the condition that $r \geq 0$. Without that restriction we would see two circles, each sweeping out a trail like that above. We would end up tracing our surface twice. Rotating this circle (with the restriction on $r$) about the $z$-axis, we will end up with a torus (see below right).

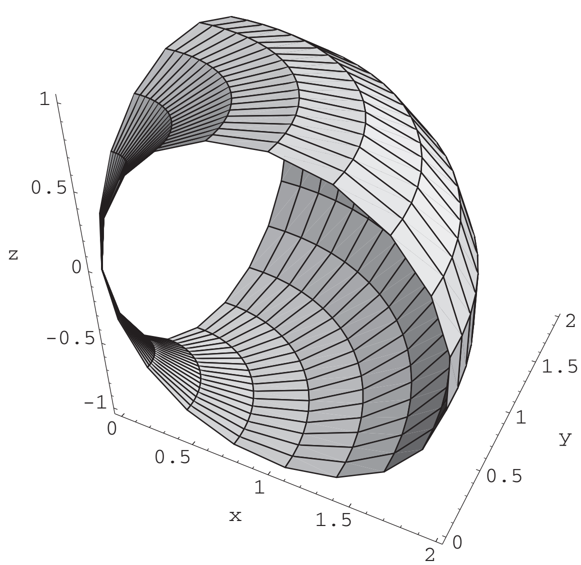


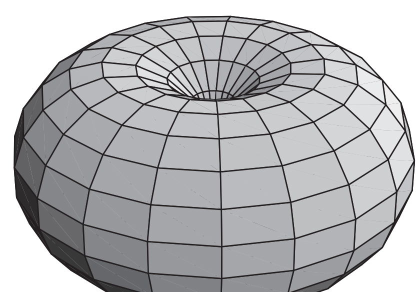


**20. (a)** As in Example 2, we could reason that our result is a circle that is traced twice (in the figures $a$ is taken to be 1):




**(b)** When we move to spherical coordinates $\varphi$ takes on the role of $\theta$ from part (a). Note $\theta$ does not explicitly appear in this spherical equation. As in the case for cylindrical equations, this means that the surface will be circularly symmetric about the $z$-axis. As we start to revolve about the $z$-axis we get the figure on the left.

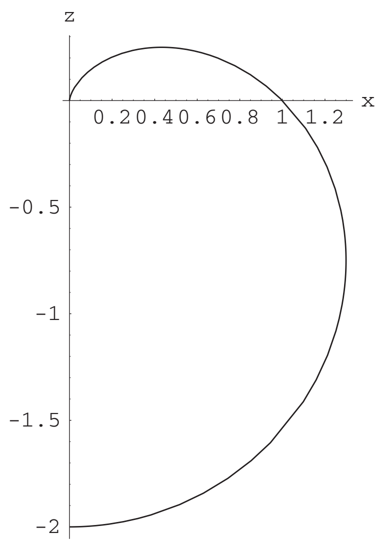


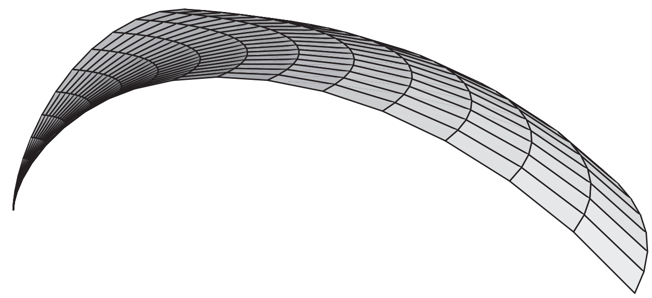



Again, the completed figure is a torus (see above right), but this time the "hole" closes off at the origin.

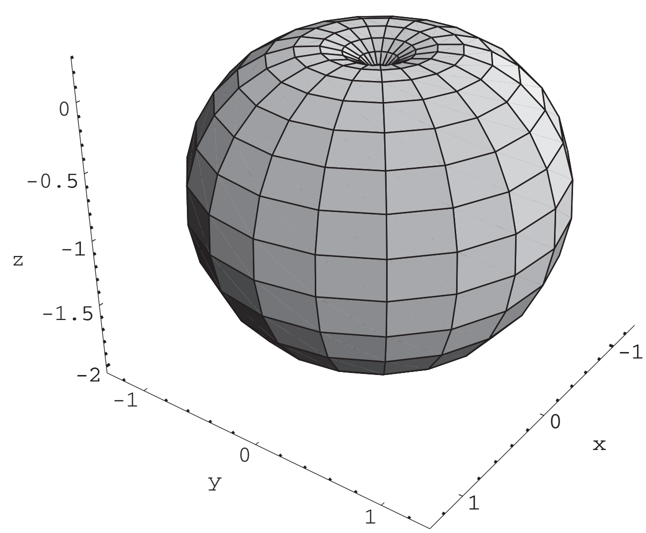
*Note: You might want to assign both Exercises 21 and 22. They look so similar and yet the results are very different.*

**21.** As noted above, surface will be circularly symmetric about the $z$-axis (the equation does not involve $\theta$). In this case we are rotating a piece of the cardioid $1 - \cos\varphi$ shown below left:

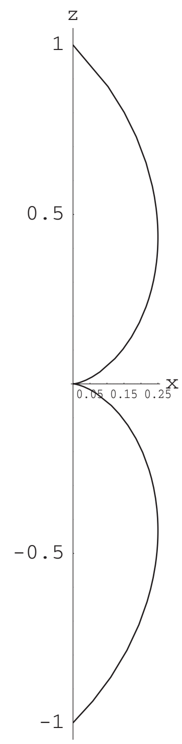


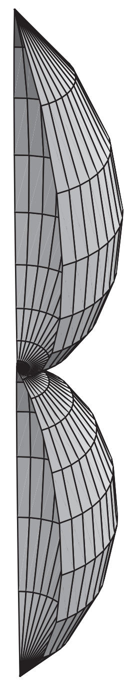



As we start to rotate it we see a "flattened" circle sweeping out the figure pictured above right. The completed figure is like a "dimpled" sphere:

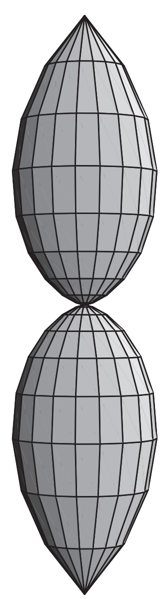


**22.** Once again, the surface will be circularly symmetric about the $z$-axis (the equation does not involve $\theta$). In this case we are rotating a piece of the cardioid $1 - \sin\varphi$ shown below left:



As we start to rotate it we see a "double hump" sweeping out the figure pictured above right. The completed figure is shown below:



**23.** The equation: $\rho \sin\varphi \sin\theta = 2$ is clearly a spherical equation (it involves all three of the spherical coordinates).

- Use equation (7) to convert it to cartesian coordinates: $y = \rho \sin\varphi \sin\theta$ so the cartesian form is simply

$$y = 2.$$

This is a vertical plane parallel to the *xz*-plane.

- Use equation (6) to convert to cylindrical coordinates. $\sin\theta$ stays $\sin\theta$ and $\rho\sin\varphi = r$. So the cylindrical form is

$$r\sin\theta = 2.$$

**24.** The equation

$$z^2 = 2x^2 + 2y^2$$

is clearly a cartesian equation (it involves all three of the cartesian coordinates).

- Use equation (4) to convert it to cylindrical coordinates: $z^2 = 2(x^2+y^2) = 2r^2$ so the cylindrical form is simply

$$z^2 = 2r^2.$$

This is a cone which is symmetric about the $z$-axis, whose vertex is at the origin, one nappe above and one below the $xy$-plane.

- Use equation (7) to convert to spherical coordinates. $z^2 = 2x^2 + 2y^2$, so $0 = 2(x^2+y^2+z^2) - 3z^2$. So the cylindrical form is

$$0 = 2\rho^2 - 3(\rho\cos\varphi)^2 \quad \text{or} \quad \cos\varphi = \pm\sqrt{2/3}.$$

In this final form it is again clear that the surface is a cone.

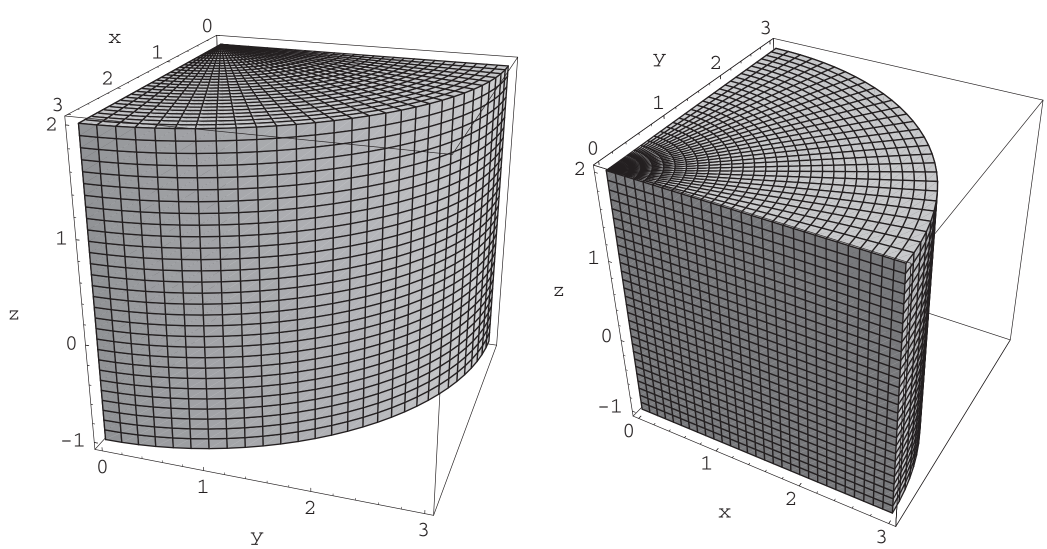
**25.** $r = 0$ is an equation in cylindrical coordinates. If $r = 0$ then it doesn't matter what $\theta$ is and $z$ is free to take on any value. This is the $z$-axis. In cartesian coordinates this is

$$x = y = 0,$$

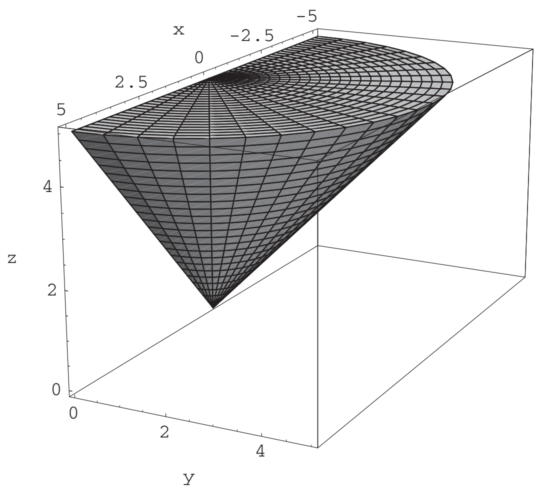
and in spherical coordinates $\rho$ and $\theta$ are not constrained but

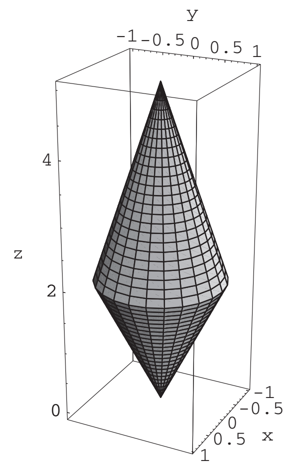
$$\varphi = 0 \quad \text{or} \quad \varphi = \pi.$$

**26.** You are slicing a wedge out of a cylinder. The result looks like a quarter of a wheel of cheese.



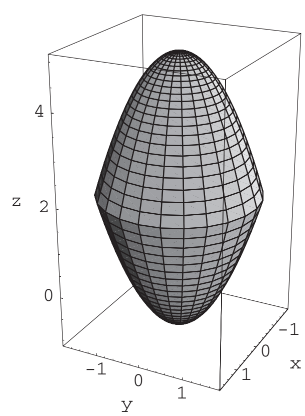
**27.** Here you are taking the triangular region above the ray $z = r$ and below the ray $z = 5$ in a plane for which $\theta$ is fixed (say $\theta = 0$) and rotating it through half a rotation to get half of a cone. The figure is below left.




**28.** Again we are rotating a triangular region—but this time it is above the line $z = 2r$ and below the line $z = 5 - 3r$. This gives us an image that looks like a diamond spun on a diagonal. The figure is above right.

**29.** This solid is bound by two paraboloids.



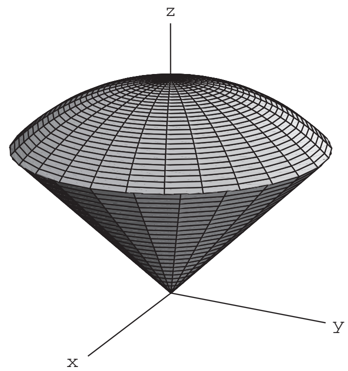

*Note: For Exercises 30–32 no sketch is included. I've just roughly described the figure.*

**30.** This is a hollow sphere. The sphere of radius 2 is missing a spherical hole of radius 1.

**31.** This is the top half of the unit sphere.

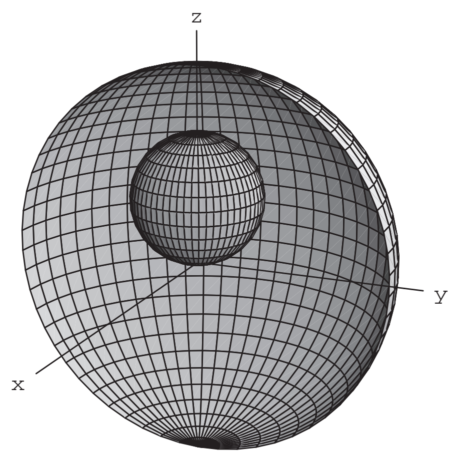
**32.** This is a quarter of the unit sphere sitting over (and under) the first quadrant.

**33.** This looks like an ice cream cone:




**34.** This may look complicated, but it is the cone without the ice cream from the previous problem. The equation $\rho = 2/\cos\varphi$ looks worse than it is. Remember that $z = \rho\cos\varphi$ so this is equivalent to $z = 2$. So we get a flat topped cone with height 2 and tip on the origin.

**35.** This is a sphere of radius 3 centered at the origin from which we've removed a sphere of radius 1 centered at (0, 0, 1).




**36.** **(a)** Look for where $(x, y) = (r, \theta)$. We know also that $x = r\cos\theta$, so $r\cos\theta = r$. This implies that $\cos\theta = 1$ so $\theta = 0$. Also $y = r\sin\theta$, but $\sin\theta = 0$ so $y = 0$. So points of the form $(a, 0)$ are the same in both cartesian and polar coordinates.

**(b)** The only difference between this and part (a) is that a $z$ coordinate has been added to each. So points of the form $(a, 0, b)$ are the same in both rectangular and cylindrical coordinates.

**(c)** Here $(x, y, z)$ must equal $(\rho, \varphi, \theta)$, where $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$.
By the first equation $\rho\sin\varphi\cos\theta = \rho$. This implies that $\sin\varphi = 1$ which in turn implies that $\cos\varphi = 0$, $\cos\theta = 1$, and $\sin\theta = 0$. But then $z = \rho\cos\varphi = 0$, and $y = \rho\sin\varphi\sin\theta = \rho(1)(0) = 0$. It looks as if we're headed to solutions on the $x$-axis again. But wait a minute, if $y = 0$, then $\varphi = 0$, but if $\sin\varphi = 1$ then $\varphi$ can't be zero. The only point satisfying all of the conditions is the origin (0, 0, 0).

**37.** **(a)** Picture drawing the graph of the polar equation $r = f(\theta)$ by standing at the origin and turning to angle $\theta$ and then walking radially out to $f(\theta)$. You can see that if instead you walked radially out to $-f(\theta)$ you would be heading the same distance in the opposite direction. This tells you that the graph $r = -f(\theta)$ is just the graph $r = f(\theta)$ reflected through the origin.

**(b)** Although we now have an additional degree of freedom the idea is the same. For each direction specified by $\varphi$ and $\theta$ we would be heading the same distance in the opposite direction. Again this tells you that the graph $\rho = -f(\varphi, \theta)$ is just the graph $\rho = f(\varphi, \theta)$ reflected through the origin.

**(c)** We're back to the situation in part (a). This time you head in the same direction, you just walk three times as far. So $r = 3f(\theta)$ is as if we expanded the graph $r = f(\theta)$ to three times its original size without changing its shape or orientation.

**(d)** Analogously, $\rho = 3f(\varphi, \theta)$ is as if we expanded the graph $\rho = f(\varphi, \theta)$ to three times its original size without changing its shape or orientation.

**38.** Because there is no dependence on $\theta$ it means that for each $r$ and the corresponding $z = f(r)$ you have a solution set that corresponds to rotating the point $(r, f(r))$ about the $z$-axis.

**39.** **(a)** We need to take six dot products. Each vector dotted with itself must be 1 and each vector dotted with any other must be 0.

$$\mathbf{e}_r \cdot \mathbf{e}_r = (\cos\theta, \sin\theta, 0) \cdot (\cos\theta, \sin\theta, 0) = \cos^2\theta + \sin^2\theta = 1.$$
$$\mathbf{e}_\theta \cdot \mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0) \cdot (-\sin\theta, \cos\theta, 0) = \sin^2\theta + \cos^2\theta = 1.$$
$$\mathbf{e}_z \cdot \mathbf{e}_z = (0, 0, 1) \cdot (0, 0, 1) = 1.$$
$$\mathbf{e}_r \cdot \mathbf{e}_\theta = (\cos\theta, \sin\theta, 0) \cdot (-\sin\theta, \cos\theta, 0) = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0.$$
$$\mathbf{e}_r \cdot \mathbf{e}_z = (\cos\theta, \sin\theta, 0) \cdot (0, 0, 1) = 0.$$
$$\mathbf{e}_\theta \cdot \mathbf{e}_z = (-\sin\theta, \cos\theta, 0) \cdot (0, 0, 1) = 0.$$

**(b)** We now do the same for the spherical basis vectors.

$$\begin{aligned}\mathbf{e}_\rho \cdot \mathbf{e}_\rho &= (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \cdot (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) = \sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi \\ &= \sin^2\varphi + \cos^2\varphi = 1.\end{aligned}$$
$$\begin{aligned}\mathbf{e}_\varphi \cdot \mathbf{e}_\varphi &= (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi) \cdot (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi) = \cos^2\varphi\cos^2\theta \\ &\quad + \cos^2\varphi\sin^2\theta + \sin^2\varphi = \cos^2\varphi + \sin^2\varphi = 1.\end{aligned}$$
$$\mathbf{e}_\theta \cdot \mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0) \cdot (-\sin\theta, \cos\theta, 0) = \sin^2\theta + \cos^2\theta = 1.$$
$$\begin{aligned}\mathbf{e}_\rho \cdot \mathbf{e}_\varphi &= (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \cdot (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi) = \sin\varphi\cos\varphi\cos^2\theta + \sin\varphi\cos\varphi\cos^2\theta \\ &\quad - \sin\varphi\cos\varphi = \sin\varphi\cos\varphi - \sin\varphi\cos\varphi = 0.\end{aligned}$$
$$\mathbf{e}_\rho \cdot \mathbf{e}_\theta = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \cdot (-\sin\theta, \cos\theta, 0) = -\sin\varphi\cos\theta\sin\theta + \sin\varphi\sin\theta\cos\theta = 0.$$
$$\mathbf{e}_\varphi \cdot \mathbf{e}_\theta = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi) \cdot (-\sin\theta, \cos\theta, 0) = -\cos\varphi\cos\theta\sin\theta + \cos\varphi\sin\theta\cos\theta = 0.$$

**40.** Begin with

$$\begin{aligned}\mathbf{e}_r &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \\ \mathbf{e}_\theta &= -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.\end{aligned}$$

Then

$$\begin{aligned}\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta &= (\sin\theta\ \cos\theta\,\mathbf{i} + \sin^2\theta\,\mathbf{j}) + (-\cos\theta\ \sin\theta\,\mathbf{i} + \cos^2\theta\,\mathbf{j}) \\ &= \mathbf{j}.\end{aligned}$$

Similarly, $\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta = \mathbf{i}$. Thus, all together

$$\begin{aligned}\mathbf{i} &= \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta \\ \mathbf{j} &= \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta \\ \mathbf{k} &= \mathbf{e}_z.\end{aligned}$$

**41.** First note that, from (9),

$$\begin{aligned}\sin\varphi\,\mathbf{e}_\rho + \cos\varphi\,\mathbf{e}_\varphi &= (\sin^2\varphi\cos\theta\,\mathbf{i} + \sin^2\varphi\sin\theta\,\mathbf{j}) + (\cos^2\varphi\cos\theta\,\mathbf{i} + \cos^2\varphi\sin\theta\,\mathbf{j}) \\ &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}.\end{aligned}$$

Hence

$$\begin{aligned}\cos\theta\,(\sin\varphi\,\mathbf{e}_\rho + \cos\varphi\,\mathbf{e}_\varphi) - \sin\theta\,\mathbf{e}_\theta &= (\cos^2\theta\,\mathbf{i} + \cos\theta\ \sin\theta\,\mathbf{j}) + (\sin^2\theta\,\mathbf{i} - \sin\theta\ \cos\theta\,\mathbf{j}) \\ &= \mathbf{i}.\end{aligned}$$

and, similarly, $\sin\theta\,(\sin\varphi\,\mathbf{e}_\rho + \cos\varphi\,\mathbf{e}_\varphi) + \cos\theta\,\mathbf{e}_\theta = \mathbf{j}$.

Finally, verify that $\cos\varphi\,\mathbf{e}_\rho - \sin\varphi\,\mathbf{e}_\varphi = \mathbf{k}$.

So our results are

$$\begin{aligned}\mathbf{i} &= \sin\varphi\cos\theta\,\mathbf{e}_\rho + \cos\varphi\cos\theta\,\mathbf{e}_\varphi - \sin\theta\,\mathbf{e}_\theta\\ \mathbf{j} &= \sin\varphi\sin\theta\,\mathbf{e}_\rho + \cos\varphi\sin\theta\,\mathbf{e}_\varphi + \cos\theta\,\mathbf{e}_\theta\\ \mathbf{k} &= \cos\varphi\,\mathbf{e}_\rho - \sin\varphi\,\mathbf{e}_\varphi.\end{aligned}$$

**42.** The exercise is more naturally set up for spherical coordinates.

**(a)** Here we are inside the portion of the sphere $\rho = 3$ for $|\tan\varphi| \le 1/\sqrt{8}$.

$$\text{Ice cream cone} = \{(\rho,\varphi,\theta)|0 \le \rho \le 3, 0 \le \varphi \le \tan^{-1}(1/\sqrt{8}),\ \text{and } 0 \le \theta < 2\pi\}.$$

**(b)** Here, $z$'s lower limit is the cone portion so $z \ge \sqrt{8}r$. The upper limit is the portion of the sphere so $z \le \sqrt{3^2 - r^2}$. The variable $r$ is free to be anything between 0 and 1 and $\theta$ is free to take on values between 0 and $2\pi$. The cylindrical description is:

$$\{(r,\theta,z)|\sqrt{8}r \le z \le \sqrt{9-r^2}, 0 \le r \le 1,\ \text{and } 0 \le \theta \le 2\pi\}.$$

**43.** From the formulas in (10) in §1.7, we have that

$$x_1 = \rho\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\cos\varphi_{n-1}$$

and

$$x_2 = \rho\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\sin\varphi_{n-1}.$$

Thus when we take the ratio $x_2/x_1$, everything cancels to leave us with

$$\frac{x_2}{x_1} = \frac{\sin\varphi_{n-1}}{\cos\varphi_{n-1}} = \tan\varphi_{n-1}.$$

**44. (a)** Using the formulas in (10), we have that

$$\begin{aligned}x_1^2 + x_2^2 &= \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-2}\cos^2\varphi_{n-1} + \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-2}\sin^2\varphi_{n-1}\\ &= \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-2}\left(\cos^2\varphi_{n-1} + \sin^2\varphi_{n-1}\right)\\ &= \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-2}.\end{aligned}$$

**(b)** If we assume the restrictions given by the inequalities in (11), then the result in part (a) implies that

$$\begin{aligned}\frac{\sqrt{x_1^2+x_2^2}}{x_3} &= \frac{\rho\sin\varphi_1\cdots\sin\varphi_{n-3}\sin\varphi_{n-2}}{\rho\sin\varphi_1\cdots\sin\varphi_{n-3}\cos\varphi_{n-2}}\\ &= \frac{\sin\varphi_{n-2}}{\cos\varphi_{n-2}} = \tan\varphi_{n-2}.\end{aligned}$$

**45. (a)** From part (a) of the previous exercise, we know that $x_1^2 + x_2^2 = \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-2}$. Thus

$$\begin{aligned}\left(x_1^2 + x_2^2\right) + x_3^2 &= \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-3}\sin^2\varphi_{n-2}\\ &\quad + \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-3}\cos^2\varphi_{n-2}\\ &= \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-3}\left(\sin^2\varphi_{n-2} + \cos^2\varphi_{n-2}\right)\\ &= \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-3}.\end{aligned}$$

**(b)** Assuming the restrictions given by the inequalities in (11), we obtain

$$\begin{aligned}\frac{\sqrt{x_1^2+x_2^2+x_3^2}}{x_4} &= \frac{\rho\sin\varphi_1\cdots\sin\varphi_{n-4}\sin\varphi_{n-3}}{\rho\sin\varphi_1\cdots\sin\varphi_{n-4}\cos\varphi_{n-3}}\\ &= \frac{\sin\varphi_{n-3}}{\cos\varphi_{n-3}} = \tan\varphi_{n-3}.\end{aligned}$$

**46. (a)** By the work in the previous two exercises, the result holds when $k = 2$ and $k = 3$. To establish the result in general by mathematical induction, we suppose that

$$x_1^2 + \cdots + x_{k-1}^2 = \rho^2 \sin^2 \varphi_1 \cdots \sin^2 \varphi_{n-(k-1)}.$$

Then

$$\begin{aligned}\left(x_1^2 + \cdots + x_{k-1}^2\right) + x_k^2 &= \rho^2 \sin^2 \varphi_1 \cdots \sin^2 \varphi_{n-k} \sin^2 \varphi_{n-k+1} \\ &\quad + \rho^2 \sin^2 \varphi_1 \cdots \sin^2 \varphi_{n-k} \cos^2 \varphi_{n-k+1} \\ &= \rho^2 \sin^2 \varphi_1 \cdots \sin^2 \varphi_{n-k} \left(\sin^2 \varphi_{n-k+1} + \cos^2 \varphi_{n-k+1}\right) \\ &= \rho^2 \sin^2 \varphi_1 \cdots \sin^2 \varphi_{n-k}.\end{aligned}$$

**(b)** Assuming the restrictions given by the inequalities in (11), then the result in part (a) implies that

$$\begin{aligned}\frac{\sqrt{x_1^2 + \cdots + x_k^2}}{x_{k+1}} &= \frac{\rho \sin \varphi_1 \cdots \sin \varphi_{n-k-1} \sin \varphi_{n-k}}{\rho \sin \varphi_1 \cdots \sin \varphi_{n-k-1} \cos \varphi_{n-k}} \\ &= \frac{\sin \varphi_{n-k}}{\cos \varphi_{n-k}} = \tan \varphi_{n-k}.\end{aligned}$$

**47.** By part (a) of the previous exercise with $k = n - 1$, we have

$$x_1^2 + \cdots + x_{n-1}^2 = \rho^2 \sin^2 \varphi_1 \cdots \sin^2 \varphi_{n-(n-1)} = \rho^2 \sin^2 \varphi_1.$$

Hence

$$\left(x_1^2 + \cdots + x_{n-1}^2\right) + x_n^2 = \rho^2 \sin^2 \varphi_1 + \rho^2 \cos^2 \varphi_1 = \rho^2.$$

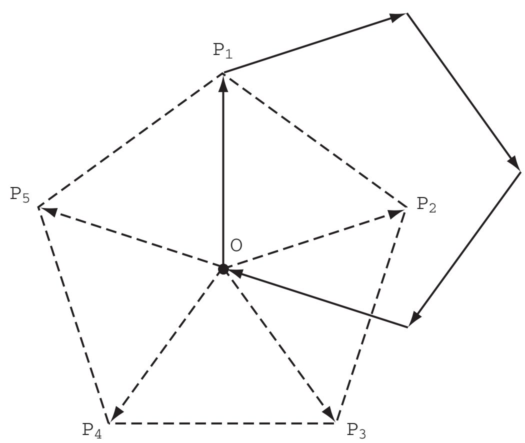
### True/False Exercises for Chapter 1

**1.** False. (The corresponding components must be equal.)
**2.** True. (Apply two kinds of distributive laws.)
**3.** False. ($(-4, -3, -3)$ is the displacement vector from $P_2$ to $P_1$.)
**4.** True.
**5.** False. (Velocity is a vector, but speed is a scalar.)
**6.** False. (Distance is a scalar, but displacement is a vector.)
**7.** False. (The particle will be at $(2, -1) + 2(1, 3) = (4, 5)$.)
**8.** True.
**9.** False. (From the parametric equations, we may read a vector parallel to the line to be $(-2, 4, 0)$. This vector is not parallel to $(-2, 4, 7)$.)
**10.** True. (Note that a vector parallel to the line is $(1, 2, 3) - (4, 3, 2) = (-3, -1, 1)$.)
**11.** False. (The line has symmetric form $\frac{x-2}{-3} = y - 1 = \frac{z+3}{2}$.)
**12.** True. (Check that the points $(-1, 2, 5)$ and $(2, 1, 7)$ lie on both lines.)
**13.** False. (The parametric equations describe a *semicircle* because of the restriction on $t$.)
**14.** False. (The dot product is the cosine of the angle between the vectors.)
**15.** False. ($\|k\mathbf{a}\| = |k|\,\|\mathbf{a}\|$.)
**16.** True.
**17.** False. (Let $\mathbf{a} = \mathbf{b} = \mathbf{i}$, and $\mathbf{c} = \mathbf{j}$.)
**18.** True.
**19.** True.
**20.** True.
**21.** True. (Check that each point satisfies the equation.)
**22.** False. (No values of $s$ and $t$ give the point $(1, 2, 1)$.)
**23.** False. (The product *BA* is not defined.)
**24.** False. (The expression gives the opposite of the determinant.)

**25.** False. $(\det(2A) = 2^n \det\ A.)$
**26.** True.
**27.** False. (The surface with equation $\rho = 4\cos\varphi$ is a sphere.)
**28.** True. (It's the plane $x = 3$.)
**29.** True.
**30.** False. (The spherical equation should be $\varphi = \tan^{-1}\frac{1}{2}$.)

## Miscellaneous Exercises for Chapter 1

**1.** *Solution 1.* We add the vectors head-to-tail by parallel translating $\overrightarrow{OP}_2$ so its tail is at the vertex $P_1$, translating $\overrightarrow{OP}_3$ so that its tail is at the head of the translated $\overrightarrow{OP}_2$, etc. Since each vector $\overrightarrow{OP}_i$ has the same length and, for $i = 2, \ldots, n$, the vector $\overrightarrow{OP}_i$ is rotated $2\pi/n$ from $\overrightarrow{OP}_{i-1}$, the translated vectors will form a closed (regular) $n$-gon, as the figure below in the case $n = 5$ demonstrates.




Thus, using head-to-tail addition with the closed $n$-gon, we see that $\sum_{i=1}^{n} \overrightarrow{OP}_i = \mathbf{0}$.

*Solution 2.* Suppose that $\sum_{i=1}^{n} \overrightarrow{OP}_i = \mathbf{a} \neq \mathbf{0}$. Imagine rotating the entire configuration through an angle of $2\pi/n$ about the center $O$ of the polygon. The vector **a** will have rotated to a different nonzero vector **b**. However, the original polygon will have rotated to an identical polygon (except for the vertex labels), so the new vector sum $\sum_{i=1}^{n} \overrightarrow{OP}_i$ must be unchanged. Hence $\mathbf{a} = \mathbf{b}$, which is a contradiction. Thus $\mathbf{a} = \mathbf{0}$.

**2.** The line will be $\mathbf{r}(t) = (1, 0, -2) + t(3, -7, 1)$, or $\begin{cases} x = 1 + 3t \\ y = -7t \\ z = -2 + t. \end{cases}$

**3.** The displacement vector $(3t_0 + 1, 5 - 7t_0, t_0 + 12) - (1, 0, -2) = (3t_0, 5 - 7t_0, t_0 + 14)$ is orthogonal to $(3, -7, 1)$. This means that

$$0 = (3t_0, 5 - 7t_0, t_0 + 14) \cdot (3, -7, 1) = 9t_0 - 35 + 49t_0 + t_0 + 14 = 59t_0 - 21.$$

So $t_0 = 21/59$. The displacement vector gives us the direction of the line:

$$(3t_0, 5 - 7t_0, t_0 + 14) = (1/59)(63, 148, 847).$$

So the equation of the line is

$$\mathbf{r}(t) = (1, 0, -2) + t(63, 148, 847), \quad \text{or} \quad \begin{cases} x = 1 + 63t \\ y = 148t \\ z = -2 + 847t. \end{cases}$$

**4. (a)** If $\mathbf{r}(t) = \overrightarrow{OP}_0 + t\overrightarrow{P_0P}_1$, then $\mathbf{r}(0) = \overrightarrow{OP}_0$ and $\mathbf{r}(1) = \overrightarrow{OP}_0 + \overrightarrow{P_0P}_1 = \overrightarrow{OP}_1$.

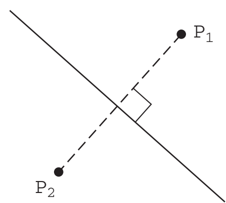
**(b)** Part (a) set us up for part (b). We know that $\mathbf{r}(0)$ and $\mathbf{r}(1)$ give us the end points of the line segment so $\mathbf{r}(t) = \overrightarrow{OP}_0 + t\overrightarrow{P_0P}_1$, for $0 \leq t \leq 1$ is the equation of the line segment.

**(c)** We can just plug into our equation in part (b) to get $\mathbf{r}(t) = (0, 1, 3) + t(2, 4, -10)$ for $0 \le t \le 1$. In parametric form this is

$$\begin{cases} x = 2t \\ y = 1 + 4t \qquad \text{for } 0 \le t \le 1. \\ z = 3 - 10t \end{cases}$$

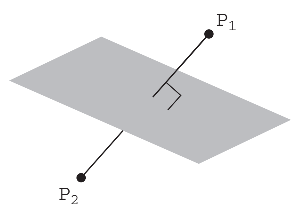
**5. (a)** The desired line must pass through the midpoint of $\overline{P_1P_2}$, which has coordinates $\left(\frac{-1+5}{2}, \frac{3-7}{2}\right) = (2, -2)$. The line must also be perpendicular to $\overrightarrow{P_1P_2}$. The vector $\overrightarrow{P_1P_2}$ is $(5+1, -7-3) = (6, -10)$. A vector perpendicular to this must satisfy $(6, -10) \cdot (a_1, a_2) = 0$ so $3a_1 - 5a_2 = 0$ Hence $\mathbf{a} = (5, 3)$ will serve. A vector parametric equation for the line is $\mathbf{l}(t) = (2, -2) + t(5, 3)$, yielding

$$\begin{cases} x = 5t + 2 \\ y = 3t - 2. \end{cases}$$




**(b)** We generalize part (a). Midpoint of $\overline{P_1P_2}$ is $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right)$. Vector $\overrightarrow{P_1P_2}$ is $(b_1 - a_1, b_2 - a_2)$. A vector $\mathbf{v}$ perpendicular to $\overrightarrow{P_1P_2}$ satisfies $(b_1 - a_1, b_2 - a_2) \cdot \mathbf{v} = 0$ We may therefore take $\mathbf{v}$ to be $\mathbf{v} = (b_2 - a_2, a_1 - b_1)$ so $\mathbf{l}(t) = \left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right) + t(b_2 - a_2, a_1 - b_1)$ yielding

$$x = (b_2 - a_2)t + \frac{a_1 + b_1}{2}$$
$$y = (a_1 - b_1)t + \frac{a_2 + b_2}{2}.$$




**6. (a)** Desired plane passes through midpoint $M(1, 2, -1)$ and has $\overrightarrow{P_1P_2} = (-10, -2, 2)$ as normal vector. So the equation is

$$-10(x-1) - 2(y-2) + 2(z+1) = 0 \iff 5x + y - z = 8.$$

**(b)** $M$ is $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}\right)$; $\overrightarrow{P_1P_2} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$.
Equation for plane is

$$(b_1 - a_1)\left(x - \frac{a_1 + b_1}{2}\right) + (b_2 - a_2)\left(y - \frac{a_2 + b_2}{2}\right) + (b_3 - a_3)\left(z - \frac{a_3 + b_3}{2}\right) = 0$$

or

$$(b_1 - a_1)x + (b_2 - a_2)y + (b_3 - a_3)z = \frac{1}{2}(b_1^2 + b_2^2 + b_3^2 - a_1^2 - a_2^2 - a_3^2).$$

**7. (a)** Midpoint of segment is $\left(\frac{1-3}{2}, \frac{6-2}{2}, \frac{0+4}{2}, \frac{3+1}{2}\ \frac{-2+0}{2}\right) = (-1, 2, 2, 2, -1)$. Normal to hyperplane is $\overrightarrow{P_1P_2} = (-4, -8, 4, -2, 2)$ so the equation of the hyperplane is $-4(x_1 + 1) - 8(x_2 - 2) + 4(x_3 - 2) - 2(x_4 - 2) + 2(x_5 + 1) = 0$ or $2x_1 + 4x_2 - 2x_3 + x_4 - x_5 = 5$.

**(b)** Very similar to 6(b). Equation for plane is

$$(b_1 - a_1)\left(x_1 - \frac{a_1 + b_1}{2}\right) + \cdots + (b_n - a_n)\left(x_n - \frac{a_n + b_n}{2}\right) = 0$$

or

$$(b_1 - a_1)x_1 + \cdots + (b_n - a_n)x_n = \frac{1}{2}(b_1^2 + \cdots + b_n^2 - a_1^2 - \cdots - a_n^2).$$

**8.** We have

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\, \|\mathbf{b}\| \sin\theta = \sin\theta, \quad \mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\, \|\mathbf{b}\| \cos\theta = \cos\theta,$$

since **a**, **b** are unit vectors. Thus $\|\mathbf{a} \times \mathbf{b}\|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = \sin^2\theta + \cos^2\theta = 1$.

**9. (a)** No. $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ just means that the angle between vectors **a** and **b** and the angle between vectors **a** and **b** have the same cosine. If you would prefer, rewrite the equation as $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$ and you can see that what this says is that one of the following is true: vector **a** is orthogonal to the vector $\mathbf{b} - \mathbf{c}$ or $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} - \mathbf{c} = \mathbf{0}$.

**(b)** No. Use the distributive property of cross products to rewrite the equation as $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$. This could be true if **a** is parallel to $\mathbf{b} - \mathbf{c}$ or if $\mathbf{a} = \mathbf{0}$ or if $\mathbf{b} - \mathbf{c} = \mathbf{0}$.

**10.** The lines are $\mathbf{r}_1(t) = (-3, 1, 5) + t(1, -2, 2)$ and $\mathbf{r}_2(t) = (4, 3, 6) + t(-2, 4, -4)$. The direction vector for line 2 is $-2$ times the direction vector for line 1 so either they are parallel or they are the same line. Look at the displacement vector from a point on line 1 to a point on line 2, for example $(4, 3, 6) - (-3, 1, 5) = (7, 2, 1)$. This is not a multiple of the direction vector so they are not the same line. Now, to find the normal direction we'll take

$$(7, 2, 1) \times (1, -2, 2) = (6, -13, -16).$$

The equation of the plane is therefore

$$6(x + 3) - 13(y - 1) - 16(z - 5) = 0 \quad \text{or} \quad 6x - 13y - 16z = -111.$$

**11. (a)** The angle between the two planes will be the same as the angle between the normal vectors. The normal to $x + y = 1$ is $\mathbf{n}_1 = (1, 1, 0)$, and the normal to $y + z = 1$ is $\mathbf{n}_2 = (0, 1, 1)$.

The angle is then

$$\cos^{-1}\left(\frac{(1, 1, 0) \cdot (0, 1, 1)}{\|(1, 1, 0)\|\, \|(0, 1, 1)\|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

**(b)** The line common to both planes must be orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$. We use the cross product to find:

$$\mathbf{n}_1 \times \mathbf{n}_2 = (1, 1, 0) \times (0, 1, 1) = (1, -1, 1).$$

The line must also pass through the point (0, 1, 0). Of course this isn't the only point you could have come up with, but it is the easiest to see. So parametric equations for the line are:

$$\begin{cases} x = t \\ y = 1 - t \\ z = t. \end{cases}$$

**12.** We begin by computing vectors that are parallel to each of the given lines. In particular, we have

$$\begin{aligned} \mathbf{a} &= 4\mathbf{i} - 2\mathbf{j} + 8\mathbf{k} && \text{for line (a),} \\ \mathbf{b} &= -6\mathbf{i} + 3\mathbf{j} - 9\mathbf{k} && \text{for line (b),} \\ \mathbf{c} &= -2\mathbf{i} + \mathbf{j} - 4\mathbf{k} && \text{for line (c),} \\ \mathbf{d} &= 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} && \text{for line (d).} \end{aligned}$$

Note that $\mathbf{a} = -2\mathbf{c}$ and $\mathbf{b} = -3\mathbf{d}$, but $\mathbf{c}$ and $\mathbf{d}$ are not scalar multiples of one another. Hence lines (a) and (c) are at least parallel, as are lines (b) and (d), but line (a) is *not* parallel to (b). To see if any of the parallel pairs coincide, note that by letting $t = 0$ in the parametric equations for line (a) we obtain the point (6, 2, 1). This point also lies in line (c): let $t = -2$ in the parametric equations for (c) to obtain it. Hence since we already know that the lines are parallel, this shows that they must in fact be the same. However, if we let $t = 0$ in the parametric equations for (b), we obtain the point (3, 0, 4). This point does *not* lie on line (d) because the only point on (d) with a $y$-coordinate of 0 is (6, 0, 1). Hence lines (b) and (d) are only parallel.

**13.** First note that vectors normal to the respective planes are given by:

$$\begin{aligned} \mathbf{a} &= 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} && \text{for plane (a),} \\ \mathbf{b} &= -6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} && \text{for plane (b),} \\ \mathbf{c} &= \mathbf{i} + \mathbf{j} - \mathbf{k} && \text{for plane (c),} \\ \mathbf{d} &= 10\mathbf{i} + 15\mathbf{j} - 5\mathbf{k} && \text{for plane (d),} \\ \mathbf{e} &= 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} && \text{for plane (e).} \end{aligned}$$

It is easy to see that $\mathbf{d} = 5\mathbf{a}$ and $\mathbf{b} = -2\mathbf{e}$ and that $\mathbf{c}$ is not a scalar multiple of any of the other vectors (also that $\mathbf{b}$ and $\mathbf{d}$ are not multiples of one another). Hence planes (a) and (d) must be at least parallel; so must planes (b) and (e). In the case of (b) and (e) note that the equation for (b) may be written as

$$-2(3x - 2y + z) = -2(1).$$

That is, the equation for (b) may be transformed into that for (e) by dividing terms by $-2$. Hence (b) and (e) are equations for the same plane. In the case of (a) and (d), note that $(0, 0, -3)$ lies on plane (a), but not on (d). Hence (a) and (d) are parallel, but not identical. Finally, it is easy to check that $\mathbf{c} \cdot \mathbf{e} = 3 - 2 - 1 = 0$. Thus the normal vectors to planes (c) and (e) are perpendicular, so that the corresponding planes are perpendicular as well. ($\mathbf{c} \cdot \mathbf{a} = 2 + 3 + 1 = 6 \neq 0$, so plane (c) is perpendicular to neither plane (a) nor (d).)

**14.** Set up a cube so that one vertex is at the origin and the rest of the bottom face has vertices at (1, 0, 0), (0, 1, 0), and (1, 1, 0). Then the top face will have vertices at (0, 0, 1), (1, 0, 1), (0, 1, 1), and (1, 1, 1).

**(a)** The angle between the diagonal and one of the edges is

$$\cos^{-1}\left(\frac{(1,1,1)\cdot(1,0,0)}{\|(1,1,1)\|\;\|(1,0,0)\|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

**(b)** You might be tempted to think that the angle between the diagonal of the cube and the diagonal of one of its faces is (by inspection) half of a right angle. The triangle with the diagonal of the cube as its hypotenuse and the diagonal of one of the faces as one of the legs is a $1 : \sqrt{2} : \sqrt{3}$ right triangle. The cosine of the angle between the diagonal of the cube and the diagonal of a side is $\sqrt{2}/\sqrt{3}$. Using the formula above we also see

$$\cos^{-1}\left(\frac{(1,1,1)\cdot(1,1,0)}{\|(1,1,1)\|\;\|(1,1,0)\|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right) = \cos^{-1}\left(\frac{\sqrt{6}}{3}\right).$$

**15.** The dot product of your two vectors indicates how much you agree with your friend on these five questions. When you both agree or both disagree with an item, the contribution to your dot product is 1. When one of you agrees and the other disagrees the contribution is $-1$. Your dot product will be an odd number between $-5$ and 5.

**16.** **(a)** Following the instructions, we can write $\overrightarrow{BM_1} = \overrightarrow{AM_1} - \overrightarrow{AB} = \frac{1}{2}\overrightarrow{AC} - \overrightarrow{AB}$ because $M_1$ is the midpoint of $\overline{AC}$. Similarly, $\overrightarrow{CM_2} = \frac{1}{2}\overrightarrow{AB} - \overrightarrow{AC}$.

**(b)** $P$ is on $\overline{BM_1}$ so we can write $\overrightarrow{BP}$ as some multiple of $\overrightarrow{BM_1}$. For definiteness, let's say that $\overrightarrow{BP} = k\overrightarrow{BM_1}$ where $0 < k < 1$. Similarly, $\overrightarrow{CP} = l\overrightarrow{CM_2}$ where $0 < l < 1$. Putting this together with our results from part (a), $\overrightarrow{BP} = k(\frac{1}{2}\overrightarrow{AC} - \overrightarrow{AB})$ and $\overrightarrow{CP} = l(\frac{1}{2}\overrightarrow{AB} - \overrightarrow{AC})$.

**(c)** First, $\overrightarrow{CB} = \overrightarrow{CP} + \overrightarrow{PB} = \overrightarrow{CP} - \overrightarrow{BP}$. From part (b), this is $l(\frac{1}{2}\overrightarrow{AB} - \overrightarrow{AC}) - k(\frac{1}{2}\overrightarrow{AC} - \overrightarrow{AB}) = (\frac{l}{2} + k)\overrightarrow{AB} - (l + \frac{k}{2})\overrightarrow{AC}$. But, $\overrightarrow{CB}$ also equals $\overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{AB} - \overrightarrow{AC}$. Equating the coefficients gives us the simultaneous equations $(\frac{l}{2} + k) = 1$ and $(l + \frac{k}{2}) = 1$. This easily gives us $l = k = 2/3$.

**(d)** Repeat steps (a) through (c) with $\overrightarrow{AM_3}$ and either of the other median vectors. You will again get a point of intersection, say $Q$. You will show that $Q$ is 2/3 of the way down each median and so must be the same point as $P$.

**17.** We are assuming that the plane $\Pi$ contains the vectors **a, b, c**, and **d**. The vector $\mathbf{n}_1 = \mathbf{a} \times \mathbf{b}$ is orthogonal to $\Pi$, and the vector $\mathbf{n}_2 = \mathbf{c} \times \mathbf{d}$ is orthogonal to $\Pi$. So the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel. This means that $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{0}$.

**18.** The first two ways that may come to mind to your students each depends on prior knowledge:

*Method One:* Recall that the area of a triangle is $(1/2)\|\mathbf{a}\|\;\|\mathbf{b}\|\sin C$, where $C$ is the angle between **a** and **b**. So the area is

$$\begin{aligned}\left(\frac{1}{2}\right)\|\mathbf{a}\|\;\|\mathbf{b}\|\sin C &= \left(\frac{1}{2}\right)\sqrt{\|\mathbf{a}\|^2\;\|\mathbf{b}\|^2\sin^2 C}\\ &= \left(\frac{1}{2}\right)\sqrt{\|\mathbf{a}\|^2\;\|\mathbf{b}\|^2(1-\cos^2 C)}\\ &= \left(\frac{1}{2}\right)\sqrt{\|\mathbf{a}\|^2\;\|\mathbf{b}\|^2 - (\|\mathbf{a}\|^2\;\|\mathbf{b}\|^2)\cos^2 C}\\ &= \left(\frac{1}{2}\right)\sqrt{\|\mathbf{a}\|^2\;\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2}.\end{aligned}$$

*Method Two:* The area of a triangle is 1/2 the area of the parallelogram determined by the same two vectors. The area of the parallelogram is the length of the cross product. So the area is

$$\left(\frac{1}{2}\right)\|\mathbf{a}\times\mathbf{b}\| = \left(\frac{1}{2}\right)\sqrt{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})}$$

$$\text{(by Section 1.4, Exercise 29)} = \left(\frac{1}{2}\right)\sqrt{(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b})-(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{a})}$$

$$= \left(\frac{1}{2}\right)\sqrt{\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2-(\mathbf{a}\cdot\mathbf{b})^2}.$$

**19. (a)** The vertices are given so that if they are connected in order *ABDC* we will sketch a parallelogram. From Exercise 18 we could say that

$$\begin{aligned}\text{Area} &= \sqrt{\|\overrightarrow{AB}\|^2\,\|\overrightarrow{AC}\|^2-(\overrightarrow{AB}\cdot\overrightarrow{AC})^2}\\ &= [\|(4-1,-1-3,3+1)\|^2\,\|(2-1,5-3,2+1)\|^2\\ &\quad -((4-1,-1-3,3+1)\cdot(2-1,5-3,2+1))^2]^{1/2}\\ &= \sqrt{\|(3,-4,4)\|^2\,\|(1,2,3)\|^2-((3,-4,4)\cdot(1,2,3))^2}\\ &= \sqrt{(41)(14)-(7^2)}=\sqrt{525}=5\sqrt{21}.\end{aligned}$$

**(b)** When we project the parallelogram in the $xy$-plane we get the same points with the $z$ coordinate equal to 0. We do the same calculation as in part a with the new vectors:

$$\begin{aligned}\text{Area} &= \sqrt{\|\overrightarrow{AB}\|^2\,\|\overrightarrow{AC}\|^2-(\overrightarrow{AB}\cdot\overrightarrow{AC})^2}\\ &= \sqrt{\|(4-1,-1-3,0)\|^2\,\|(2-1,5-3,0)\|^2-((4-1,-1-3,0)\cdot(2-1,5-3,0))^2}\\ &= \sqrt{\|(3,-4,0)\|^2\,\|(1,2,0)\|^2-((3,-4,0)\cdot(1,2,0))^2}\\ &= \sqrt{(25)(5)-(5^2)}=\sqrt{100}=10.\end{aligned}$$

**20. (a)** Students raised on the slope-intercept form of a line may be more comfortable once you point out that the slope of the line $ax+by=d$ is $\frac{\Delta y}{\Delta x}=\frac{-a}{b}$. Now the direction that the vector points is clear: $\mathbf{v}=(b,-a)$.

**(b)** A vector **n** normal to the line $l$ must be orthogonal to the vector **v** you found in part (a). We are also told that the first component of **n** is $a$. This means that

$$0=\mathbf{n}\cdot\mathbf{v}=(a,?)\cdot(b,-a)=ab-?\,a.$$

So $\mathbf{n}=(a,b)$.

**(c)** Choose a point $P_1$ on the line $ax+by=d$. For example, if $P_1$ has $x$ component zero then $y=d/b$. In other words, choose $P_1=(0,d/b)$. It doesn't matter. We are going to project the displacement vector from the point $P_1$ to the point $P_0=(x_0,y_0)$ onto **n**.

$$\|\text{proj}_{\mathbf{n}}\overrightarrow{P_0P_1}\| = \left\|\left(\frac{\mathbf{n}\cdot\overrightarrow{P_0P_1}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}\right\| = \frac{|\mathbf{n}\cdot\overrightarrow{P_0P_1}|}{\|\mathbf{n}\|} = \frac{|(a,b)\cdot(x_0,y_0-d/b)|}{\|(a,b)\|} = \frac{|ax_0+by_0-d|}{\sqrt{a^2+b^2}}.$$

**(d)** We plug into our brand new formula:

$$\text{Distance from } (3,5) \text{ to } l:(3x-5y=2) \text{ is } \frac{|8(3)-5(5)-2|}{\sqrt{8^2+5^2}}=\frac{3}{\sqrt{89}}.$$

**21. (a)** As should be expected, this is similar to the calculation in Exercise 20. We choose any point $P_1$ in the plane $\Pi$ : $Ax+By+Cz=D$. For example, let $P_1=(0,0,D/C)$ and $P_0=(x_0,y_0,z_0)$. The normal vector $\mathbf{n}=(A,B,C)$. Again the distance from $P_0$ to $\Pi$ is

$$\begin{aligned}\|\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_0}\| &= \left\|\left(\frac{\mathbf{n}\cdot\overrightarrow{P_1P_0}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}\right\| \\ &= \frac{|\mathbf{n}\cdot\overrightarrow{P_1P_0}|}{\|\mathbf{n}\|} \\ &= \frac{|(A,B,C)\cdot(x_0,y_0,z_0-D/C)|}{\|(A,B,C)\|} \\ &= \frac{|Ax_0+By_0+Cz_0-D|}{\sqrt{A^2+B^2+C^2}}.\end{aligned}$$

**(b)** We plug into our formula from part (a):

$$\text{Distance from } (1,5,-3) \text{ to } \Pi\text{: } (x-2y+2z+12=0) \text{ is } \frac{|1(1)-2(5)+2(-3)+12|}{\sqrt{1^2+(-2)^2+2^2}} = \frac{3}{\sqrt{9}} = 1.$$

**22.** **(a)** A vector $\mathbf{n}$ normal to $\Pi$ may be obtained as $\mathbf{n}=\mathbf{b}\times\mathbf{c}$ as both $\mathbf{b}=\overrightarrow{AB}$ and $\mathbf{c}=\overrightarrow{AC}$ are parallel to $\Pi$. Thus the distance from $P$ to $\Pi$ may be found by taking $\|\text{proj}_{\mathbf{n}}\overrightarrow{AP}\| = \|\text{proj}_{\mathbf{n}}\mathbf{p}\|$. Now

$$\text{proj}_{\mathbf{n}}\mathbf{p} = \frac{\mathbf{n}\cdot\mathbf{p}}{\mathbf{n}\cdot\mathbf{n}}\,\mathbf{n} = \frac{\mathbf{n}\cdot\mathbf{p}}{\|\mathbf{n}\|^2}\,\mathbf{n}.$$

Thus

$$\|\text{proj}_{\mathbf{n}}\mathbf{p}\| = \frac{|\mathbf{n}\cdot\mathbf{p}|}{\|\mathbf{n}\|^2}\,\|\mathbf{n}\| = \frac{|\mathbf{n}\cdot\mathbf{p}|}{\|\mathbf{n}\|} = \frac{|(\mathbf{b}\times\mathbf{c})\cdot\mathbf{p}|}{\|\mathbf{n}\|}.$$

**(b)** We have $\mathbf{b}=(2,-3,1)-(1,2,3)=(1,-5,-2)$, $\mathbf{c}=(2,-1,0)-(1,2,3)=(1,-3,-3)$, and $\mathbf{p}=(1,0,-1)-(1,2,3)=(0,-2,-4)$. Thus

$$\mathbf{b}\times\mathbf{c} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -5 & -2\\ 1 & -3 & -3\end{vmatrix} = (9,1,2).$$

Hence the desired distance is

$$\frac{|(0,-2,-4)\cdot(9,1,2)|}{\|(9,1,2)\|} = \frac{|-10|}{\sqrt{86}} = \frac{10}{\sqrt{86}}.$$

**23.** **(a)** The vector $\overrightarrow{AB}\times\overrightarrow{AC}=\mathbf{0}$ if and only if $\overrightarrow{AB}$ is parallel to $\overrightarrow{AC}$. This happens if and only if $A$, $B$, and $C$ are collinear.

**(b)** We note that $\overrightarrow{CD}\neq\mathbf{0}$ since $C$ and $D$ are distinct points. Then $(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{CD}=0$ if and only if $\overrightarrow{AB}\times\overrightarrow{AC}=\mathbf{0}$ or $\overrightarrow{AB}\times\overrightarrow{AC}$ is perpendicular to $\overrightarrow{CD}$. The first case occurs exactly when $A$, $B$, and $C$ are collinear (so $A$, $B$, $C$ and $D$ are coplanar). In the second case, $\overrightarrow{AB}\times\overrightarrow{AC}$ is perpendicular to the plane containing $A$, $B$, and $C$ and so $\overrightarrow{CD}$ can only be perpendicular to it if and only if $D$ lies in this plane as well.

**24.** We have the equation that if $\alpha$ is the angle between vectors $\mathbf{x}$ and the vector $\mathbf{k}=(0,0,1)$, then

$$\cos\alpha = \frac{\mathbf{x}\cdot\mathbf{k}}{\|\mathbf{x}\|\,\|\mathbf{k}\|} = \frac{\mathbf{x}\cdot\mathbf{k}}{\|\mathbf{x}\|}.$$

Since we are given that this last quantity $=1/\sqrt{2}$, $\mathbf{x}$ makes an angle of 45 degrees with the positive $z$-axis. So the points $P$ satisfying the condition of this exercise sweep out the top nappe of the cone making an angle of 45 degrees with the positive $z$-axis minus the origin.

**25.** The equation $\mathbf{a}\times\mathbf{x}=\mathbf{b}$ tells us that $\mathbf{x}$ points in the direction of $\mathbf{b}\times\mathbf{a}$. Now we have to determine the length of $\mathbf{x}$. We can choose any vector in the direction of $\mathbf{x}$. For convenience, let $\mathbf{y}$ be the unit vector in direction of $\mathbf{x}$:

$$\mathbf{y} = \frac{\mathbf{b}\times\mathbf{a}}{\|\mathbf{b}\times\mathbf{a}\|}.$$

The angle between $\mathbf{a}$ and $\mathbf{x}$ is the same as that between $\mathbf{a}$ and $\mathbf{y}$ so

$$\frac{\mathbf{a}\cdot\mathbf{y}}{\|\mathbf{a}\|} = \frac{\mathbf{a}\cdot\mathbf{x}}{\|\mathbf{a}\|\,\|\mathbf{x}\|} = \frac{c}{\|\mathbf{a}\|\,\|\mathbf{x}\|}.$$

So if $c \neq 0$,

$$\|\mathbf{x}\| = \frac{c}{\mathbf{a}\cdot\mathbf{y}}, \quad \text{and,} \quad \mathbf{x} = \left(\frac{c}{\mathbf{a}\cdot\mathbf{y}}\right)\mathbf{y}.$$

If $c = 0$ then **a** is orthogonal to **x** (and **y**). Use the fact that

$$\|\mathbf{b}\| = \|\mathbf{a}\times\mathbf{x}\| = \|\mathbf{a}\|\,\|\mathbf{x}\|\sin\theta = \|\mathbf{a}\|\,\|\mathbf{x}\|\sin\pi/2 = \|\mathbf{a}\|\,\|\mathbf{x}\|.$$

So when $c = 0$,

$$\|\mathbf{x}\| = \frac{\|\mathbf{b}\|}{\|\mathbf{a}\|} \quad \text{and} \quad \mathbf{x} = \left(\frac{\|\mathbf{b}\|}{\|\mathbf{a}\|}\right)\mathbf{y}.$$

**26. (a)** Let $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{c} = \mathbf{j}$. Then

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{i}\times(\mathbf{j}\times\mathbf{j}) = \mathbf{i}\times\mathbf{0} = \mathbf{0},$$

but

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{i}\times\mathbf{j})\times\mathbf{j} = \mathbf{k}\times\mathbf{j} = -\mathbf{i}.$$

**(b)** The Jacobi identity states that

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} + (\mathbf{b}\times\mathbf{c})\times\mathbf{a} + (\mathbf{c}\times\mathbf{a})\times\mathbf{b} = \mathbf{0}.$$

This result is equivalent to

$$-(\mathbf{b}\times\mathbf{c})\times\mathbf{a} = (\mathbf{a}\times\mathbf{b})\times\mathbf{c} + (\mathbf{c}\times\mathbf{a})\times\mathbf{b}.$$

Since $-(\mathbf{b}\times\mathbf{c})\times\mathbf{a} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, we see that we *always* have

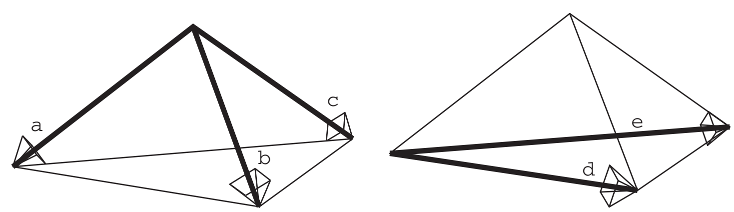
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c} + (\mathbf{c}\times\mathbf{a})\times\mathbf{b},$$

so that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$$

precisely when $(\mathbf{c}\times\mathbf{a})\times\mathbf{b} = \mathbf{0}$.

**27. (a)** In the figure below left, the cross product $\mathbf{a}\times\mathbf{b}$ is a vector outwardly normal to the face containing edges **a** and **b** with length equal to twice the area of the face. To keep the diagram uncluttered, it has been split into two:




So the sum of the four vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$ asked for in the exercise can be expressed as

$$(1/2)[(\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a}) + (\mathbf{e}\times\mathbf{d})].$$

But $\mathbf{d} = \mathbf{b} - \mathbf{a}$ and $\mathbf{e} = \mathbf{c} - \mathbf{a}$ so

$$\begin{aligned}\mathbf{e}\times\mathbf{d} &= (\mathbf{c}-\mathbf{a})\times(\mathbf{b}-\mathbf{a})\\ &= (\mathbf{c}\times\mathbf{b}) - (\mathbf{a}\times\mathbf{b}) - (\mathbf{c}\times\mathbf{a}) + (\mathbf{a}\times\mathbf{a})\\ &= -(\mathbf{a}\times\mathbf{b}) - (\mathbf{b}\times\mathbf{c}) - (\mathbf{c}\times\mathbf{a}).\end{aligned}$$

We put this together with the above to conclude:

$$\begin{aligned}\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_4 &= (1/2)[(\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a}) + (\mathbf{e}\times\mathbf{d})]\\ &= (1/2)[(\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a}) - (\mathbf{a}\times\mathbf{b}) - (\mathbf{b}\times\mathbf{c}) - (\mathbf{c}\times\mathbf{a})]\\ &= \mathbf{0}.\end{aligned}$$

**(b)** Denote the vectors associated with the first tetrahedron as $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$ and the vectors associated with the second tetrahedron as $\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3$, and $\mathbf{v}'_4$. Let the vectors associated with the sides being glued together be $\mathbf{v}_1$ and $\mathbf{v}'_1$.

By construction $\mathbf{v}_1$ and $\mathbf{v}'_1$ have equal lengths and point in opposite directions so $\mathbf{v}_1 + \mathbf{v}'_1 = \mathbf{0}$. From part (a) we know that

$$\mathbf{v}_1 = -(\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4) \quad \text{and} \quad \mathbf{v}'_1 = -(\mathbf{v}'_2 + \mathbf{v}'_3 + \mathbf{v}'_4).$$

This means that

$$(\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4) + (\mathbf{v}'_2 + \mathbf{v}'_3 + \mathbf{v}'_4) = \mathbf{0}.$$

**(c)** Just as we can break any polygon into triangles, we can break any polyhedron into tetrahedra. The key to part (b) was that when we glue two tetrahedra together, the vector of the face being hidden is equal to the sum of the three vectors being introduced. In symbols,

$$\mathbf{v}'_1 = \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4.$$

From part (a) we know that for any tetrahedron $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$. So as we build up our polyhedron by gluing tetrahedra together, at each stage (by parts (a) and (b)) the sum of the outward normals with length equal to the area of the face will be zero.

**28.** We may construct vectors $\mathbf{v}_1, \ldots, \mathbf{v}_4$ outwardly normal to each face of the tetrahedron and with length equal to the area of that face. Using the result of part (a) of Exercise 27, we have that $\mathbf{v}_1 + \cdots + \mathbf{v}_4 = \mathbf{0}$. Hence $\mathbf{v}_4 = -(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$. Let's assume that the vectors are indexed so that $\mathbf{v}_4$ is the vector normal to the face that is opposite to vertex $R$. Then $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are pairwise perpendicular and thus $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$.

Now we compute

$$\begin{aligned} d^2 = \|\mathbf{v}_4\|^2 &= \| - (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)\|^2 = \|\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\|^2 \\ &= (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \cdot (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \\ &= \mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_2 + \mathbf{v}_3 \cdot \mathbf{v}_3 + 2\mathbf{v}_1 \cdot \mathbf{v}_2 + 2\mathbf{v}_1 \cdot \mathbf{v}_3 + 2\mathbf{v}_2 \cdot \mathbf{v}_3 \\ &= \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + \|\mathbf{v}_3\|^2 + 0 + 0 + 0 \\ &= a^2 + b^2 + c^2. \end{aligned}$$

**29. (a)** Remember, if the adjacent sides of a parallelogram are $\mathbf{a}$ and $\mathbf{b}$, then the diagonals are $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$. So the sum of the squares of the lengths of the diagonals are

$$\begin{aligned} \|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) + (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= (\mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}) = 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2 \end{aligned}$$

which is the sum of the squares of the lengths of the four sides (opposite sides have equal lengths).

**(b)** $\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2)$.

**30.** The last line of the proof of the Cauchy–Schwarz inequality in Section 1.6 is

$$\|\mathbf{a}\|^2\|\mathbf{b}\|^2 \geq (\mathbf{a} \cdot \mathbf{b})^2.$$

Now we only need to notice that

$$(\mathbf{a} \cdot \mathbf{b})^2 = \left[\sum_{i=1}^{n} a_i b_i\right]^2$$

$$\|\mathbf{a}\|^2 = \sum_{i=1}^{n} a_i^2$$

$$\|\mathbf{b}\|^2 = \sum_{i=1}^{n} b_i^2$$

and the result follows immediately:

$$\left[\sum_{i=1}^{n} a_i^2\right]\left[\sum_{i=1}^{n} b_i^2\right] \geq \left[\sum_{i=1}^{n} a_i b_i\right]^2.$$

**31. (a)**

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, A^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

**(b)** It seems reasonable to guess that

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

**(c)** We need only show the inductive step:

$$A^{n+1} = AA^n = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}.$$

**32. (a)** There's nothing much to show. $A^2 = 0$.

**(b)** You shouldn't need a calculator or computer for this. The diagonal of 1's keeps moving to the left so that

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad A^5 = 0.$$

**33. (a)** The determinants are:

$$|H_2| = \frac{1}{12}, |H_3| = \frac{1}{2160}, |H_4| = \frac{1}{6048000},$$
$$|H_5| = \frac{1}{266716800000}, \text{ and } \quad |H_6| = \frac{1}{186313420339200000}$$

The determinants are going to 0 as $n$ gets larger. As for writing out the matrices, note that $H_2$ is the upper left two by two matrix in $H_{10}$ in part (b). Similarly, $H_3$ is the upper left three by three ... $H_6$ is the upper left six by six matrix in $H_{10}$. I would consider deducting points from any student who actually writes these out. They can use a computer algebra system to accomplish this. For *Mathematica* the command for generating $H_{10}$ would be

$$\text{Table}[1/(i+j-1), \{i, 10\}, \{j, 10\}]//\text{MatrixForm}.$$

The command for calculating the determinant would be

$$\text{Det}[\text{Table}[1/(i+j-1), \{i, 10\}, \{j, 10\}]].$$

**(b)** Using the *Mathematica* commands described in part (a), the determinant

$$|H_{10}| = 1/46206893947914691316295628839036278726983680000000000$$
$$\approx 2.16 \times 10^{-53}.$$

The matrix is

$$H_{10} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} \\ \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} \\ \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} \\ \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} & \frac{1}{19} \end{bmatrix}$$

**(c)** Again, the code examples will be from *Mathematica*. Let's first calculate a numerical approximation $A$ of $H_{10}$ with the command

$$A = \text{N[Table}[1/(i+j-1), \{i, 10\}, \{j, 10\}]].$$

We can then calculate the inverse $B$ and $A$ with the command

$$B = \text{Inverse}[A].$$

You can display these as matrices by appending "//MatrixForm" to the command. Now generate *AB* and *BA* with the commands

$$A.B//\text{MatrixForm} \quad \text{and} \quad B.A//\text{MatrixForm}.$$

You should note that these aren't equal and neither is the $10 \times 10$ identity matrix $I_{10}$.

**34.** The center of the moving circle is at $(a - b)(\cos t, \sin t)$. Notice that as the moving circle rolls so that its center moves counterclockwise it is turning clockwise relative to its center. When the small circle has traveled completely around the large circle it has rolled over a length of $2\pi(a)$. Its circumference is $2\pi b$ so if it were rolling along a straight line it would have revolved $a/b$ times. The problem is that it is rolling around in a circle and so it has lost a rotation each time the center has traveled completely around. In other words the smaller wheel is turning at a rate of

$$((a/b) - 1)t = (a - b)t/b.$$

The position of $P$ relative to the center of the moving circle is

$$b\left(\cos\left(-\frac{(a-b)t}{b}\right), \sin\left(-\frac{(a-b)t}{b}\right)\right) = b\left(\cos\frac{(a-b)t}{b}, -\sin\frac{(a-b)t}{b}\right).$$

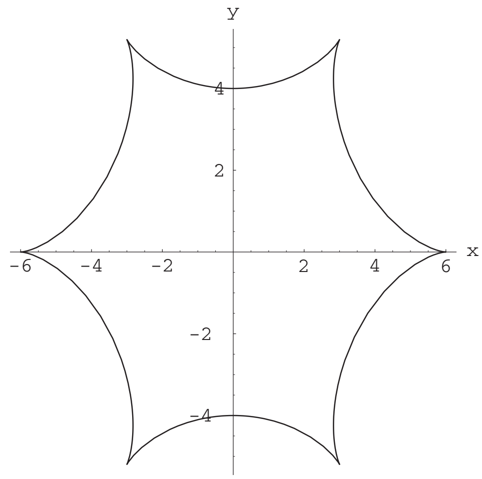
Putting this together, the position of $P$ is the sum of the vector from the origin to the center of the moving circle and the vector from the center of the moving circle to $P$. This is

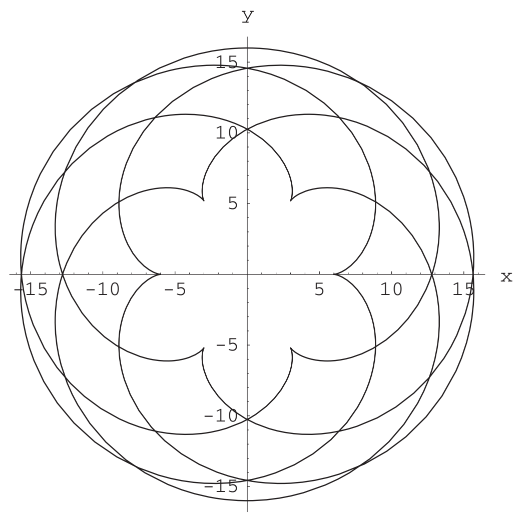
$$(a-b)(\cos t, \sin t) + b\left(\cos\left(\frac{(a-b)t}{b}\right), -\sin\left(\frac{(a-b)t}{b}\right)\right).$$

**35.** Not much changes here. The center of the moving circle is now at $(a + b)(\cos t, \sin t)$. Now the moving circle gains one revolution each time around the fixed circle and so turns at a rate of $((a/b) + 1)t = (a + b)t/b$. Since we are starting $P$ at $(a, 0)$, the initial angle from the center of the moving circle to $P$ is $\pi$ so the position of $P$ relative to the center of the moving circle is $b\left(\cos\left(\pi + \frac{(a+b)t}{b}\right), \sin\left(\frac{(a-b)t}{b}\right)\right) = -b\left(\cos\frac{(a+b)t}{b}, \sin\frac{(a+b)t}{b}\right)$. As in Exercise 34 we sum the same two vectors to get the expression:

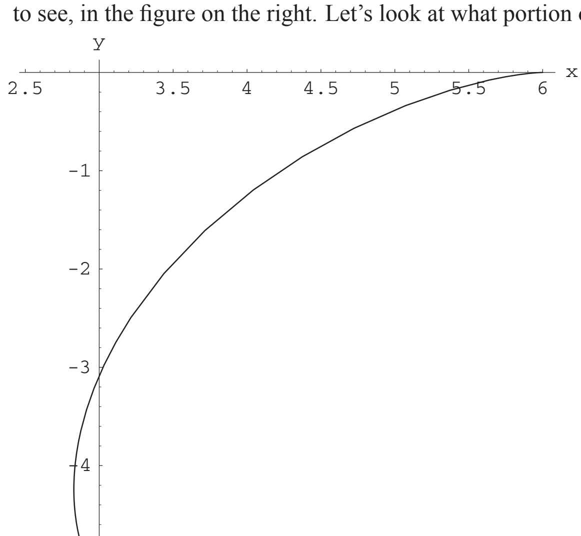
$$(a+b)(\cos t, \sin t) - b\left(\cos\left(\frac{(a+b)t}{b}\right), \sin\left(\frac{(a+b)t}{b}\right)\right).$$

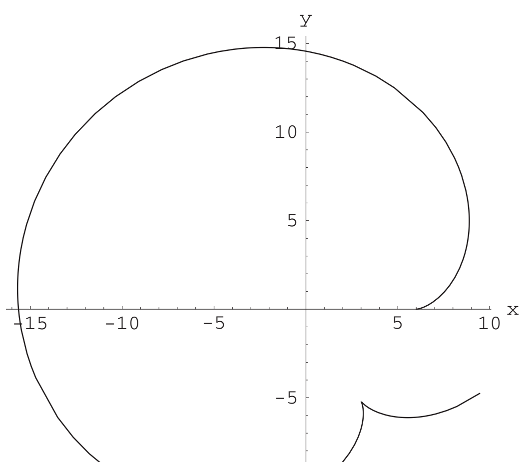
**36. (a)** Let's look at diagrams of hypocycloid (below on the left) and an epicycloid (below on the right) with $a = 6$ and $b = 5$:




What are the roles of $a$ and $b$? You can see in the figure on the left that there are 6 cusps. This is also true, but harder to see, in the figure on the right. Let's look at what portion of these curves correspond to $0 \le t \le 2\pi$.



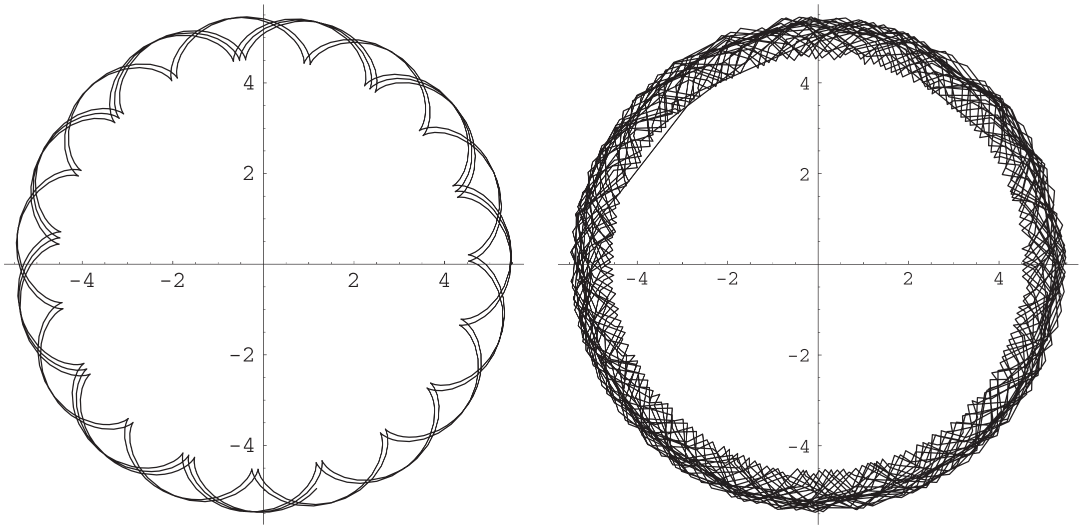




The figure on the left shows that 6/5 of the hypocycloid is covered for $0 \le t \le 2\pi$. The figure on the right is the corresponding portion of the epicycloid. Usually what we call the hypocycloid is what we draw until the ends close up. In this case, the hypocycloid is complete when $t = 5(2\pi)$. Again, although it is harder to see, this epicycloid will close up when $t = 5(2\pi)$.

If $a$ and $b$ have no common divisors and are both rational, then the hypocycloid or epicycloid will have $a$ cusps and will close up after $t = b(2\pi)$. If $a$ and $b$ have common divisors then write $a/b$ in lowest terms. The hypocycloid or epicycloid will have as many cusps as the numerator. The same answer holds for epicycloids.

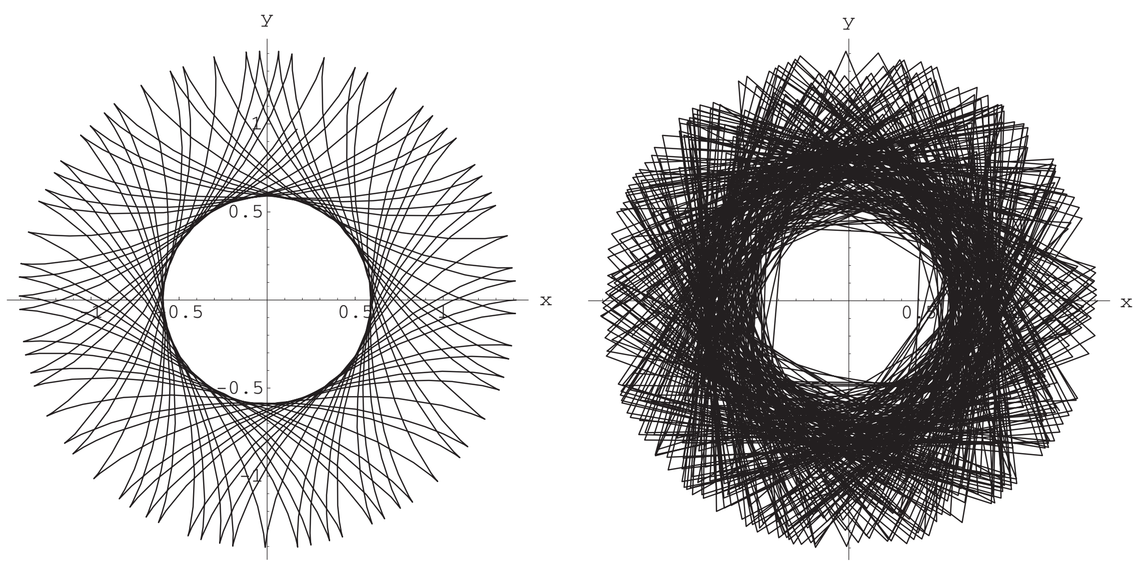
**(b)** We noted in part (a) that if $a/b$ is rational in lowest terms, the hypocycloid or epicycloid closes up when $t = b(2\pi)$. In the case of the hypocycloid, this is because then $(a-b)(\cos b(2\pi), \sin b(2\pi)) + b\left(\cos\left(\frac{a-b}{b}\right) b(2\pi), -\sin\left(\frac{a-b}{b}\right) b(2\pi)\right) = (a-b)(\cos 0, \sin 0) + b\left(\cos\left(\frac{a-b}{b}\right) 0, -\sin\left(\frac{a-b}{b}\right) 0\right)$. In words, its because the angle is a rational multiple of $2\pi$.

A picture of part of an epicycloid for which $a/b$ is irrational is:




If $a/b$ is irrational then the curve will never close up. It can't. At no time when the center of the moving circle comes back to its original position will $P$ be back in its original position.

A picture of part of a hypocycloid for which $a/b$ is irrational is:




In each case, the figure on the left shows several periods. For the figure on the right we let $t$ get larger. If we let $t$ get arbitrarily large the curve is dense.

**37.** Look at the second part of the answers in Exercises 34 and 35. The only difference is that we are changing the distance from the center of the moving wheel to $P$ from $b$ to $c$. The formula for a hypotrochoid is:

$$(a-b)(\cos t, \sin t) + c\left(\cos\left(\frac{(a-b)t}{b}\right), -\sin\left(\frac{(a-b)t}{b}\right)\right).$$

In parametric form, the formulas for a hypotrochoid are:

$$x = (a-b)\cos t + c\cos\left(\frac{(a-b)t}{b}\right),\ y = (a-b)\sin t - c\sin\left(\frac{(a-b)t}{b}\right).$$

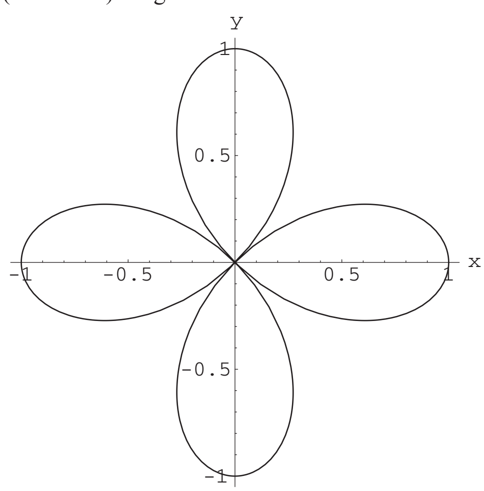
The formula for an epitrochoid is:

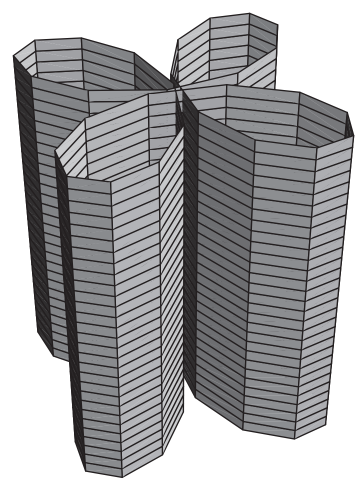
$$(a+b)(\cos t, \sin t) - c\left(\cos\left(\frac{(a+b)t}{b}\right), \sin\left(\frac{(a+b)t}{b}\right)\right).$$

In parametric form, the formulas for an epitrochoid are:

$$x = (a+b)\cos t - c\cos\left(\frac{(a+b)t}{b}\right),\ y = (a+b)\sin t - c\sin\left(\frac{(a+b)t}{b}\right).$$

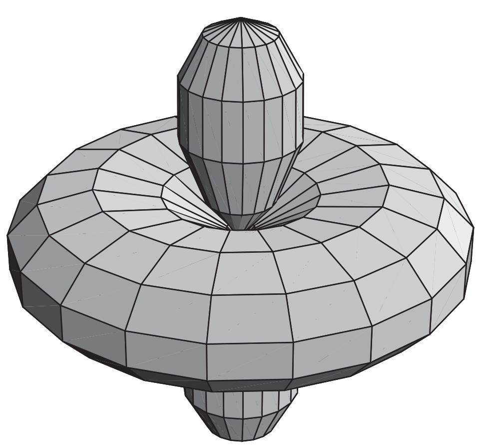
**38.** **(a)** Here (below left) we get the four leaf rose:

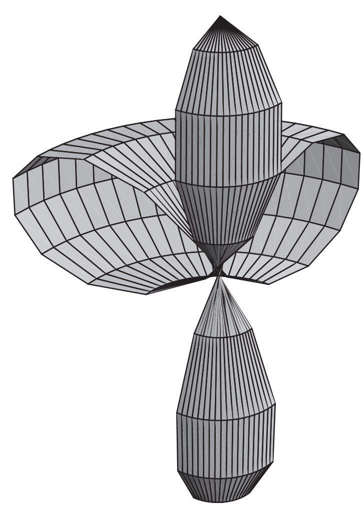


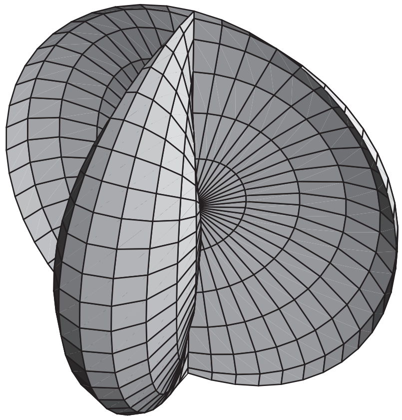
**(b)** We just erect a cylinder on that base and get the above right image.

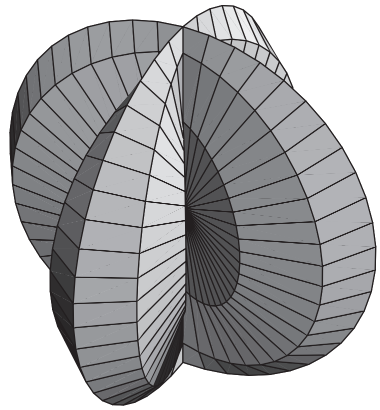
**(c)** There is no $\theta$ explicitly in the equation, so the rose is being rotated about the $z$-axis (we show both the completed figure and a partial to see how it is formed):



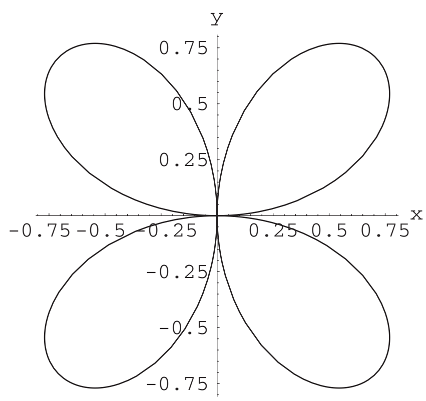


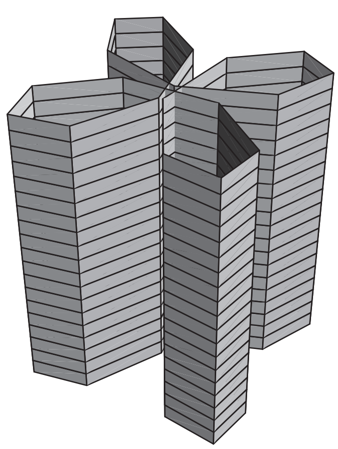
**(d)** Here we show half of the figure and then the completed figure. From the outside, the figure looks as if the rose has been first rotated about the $x$-axis and then about the $y$-axis.

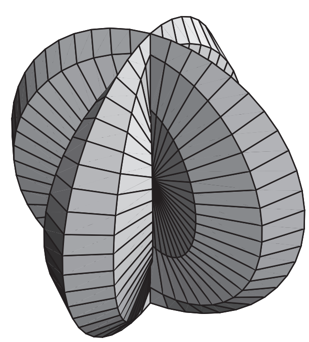




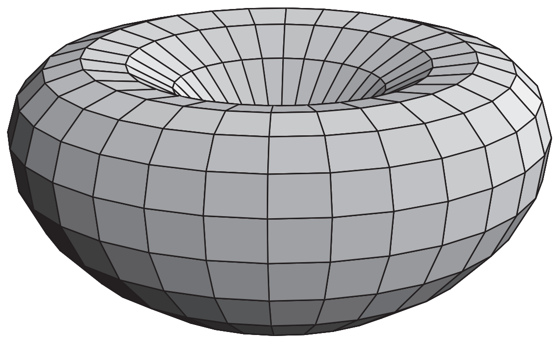
**39.** Parts (a), (b), and (d) are pictured below top, left, and right. They look very similar to the graphs from the previous exercise.



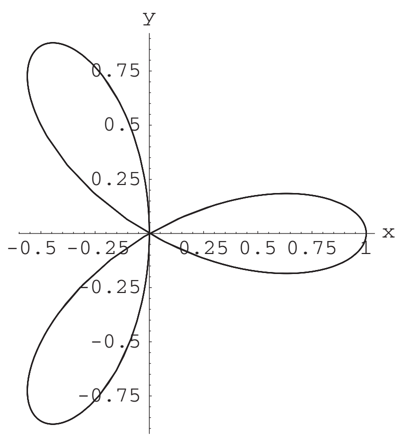



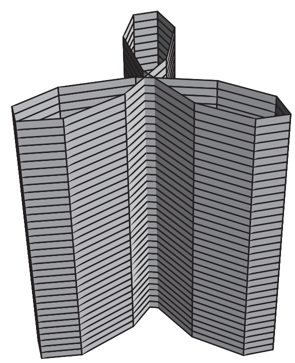


**(c)** This looks very different from its counterpart for Exercise 38. It looks like a dented sphere.



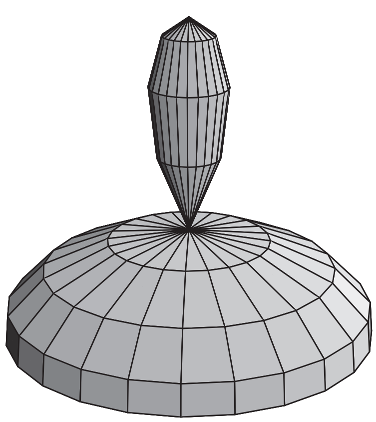
**40. (a)** We begin with a three leaf rose (the path is traced twice) shown below left.

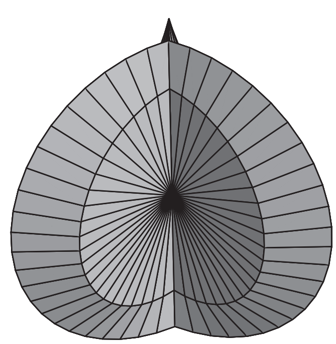


**(b)** The cylindrical equation again adds nothing. A cylinder is built over the rose. It is shown above right.

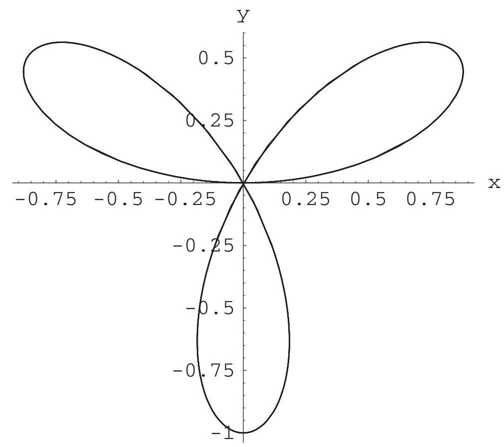
**(c)** This interesting and different image is shown below left.

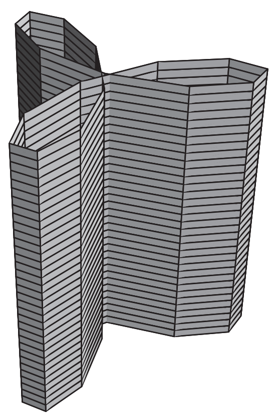


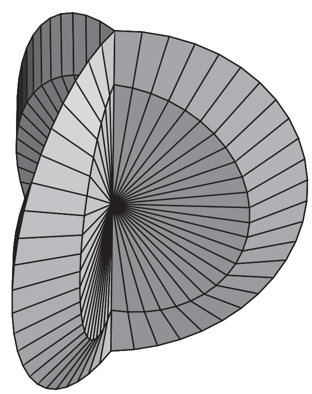


**(d)** This three leaf version of what we saw in Exercises 38 and 39 is shown above right.

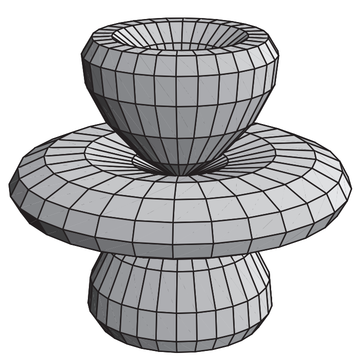
**41.** The polar plot, cylinder and part (d) are similar to the corresponding solutions for Exercise 40. They are shown below.



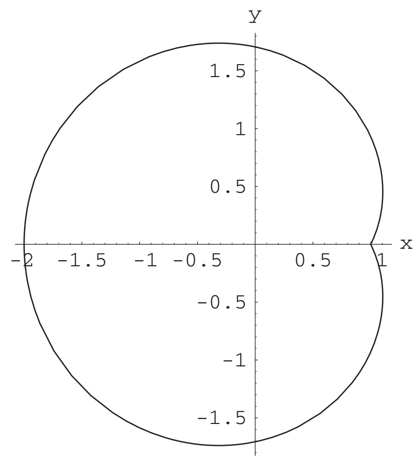



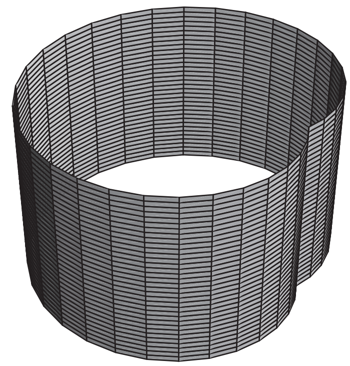


**(c)** Here in the figure shown below you see a difference in the solid generated by using sine instead of cosine.



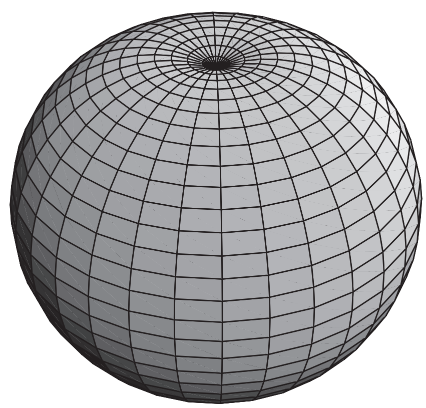
**42. (a)** The nephroid is shown below left.



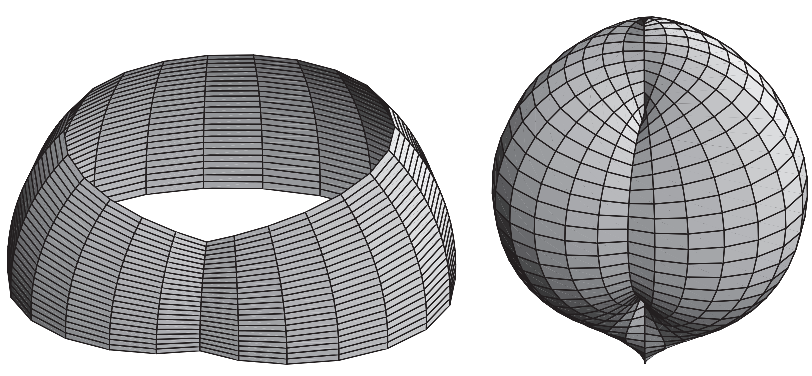



**(b)** The cylinder based on it is shown above right.

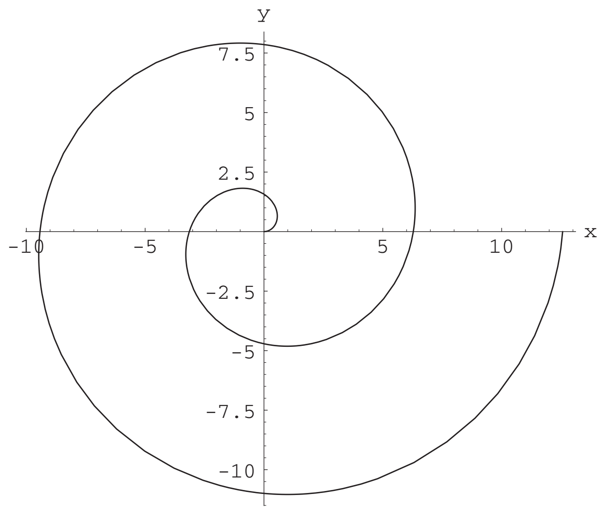
**(c)** The first spherical graph is a dimpled sphere.

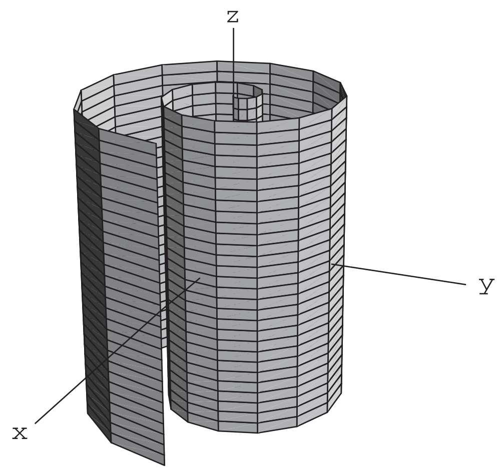


**(d)** The second spherical graph has a lot of complexity so I have included a partial graph and the completed graph.



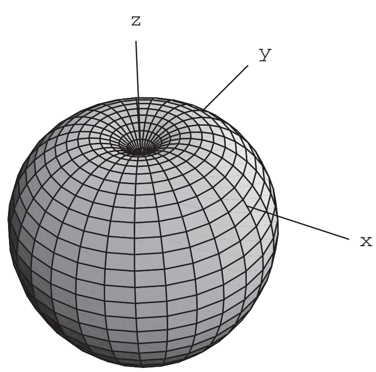
**43. (a)** The curve is a spiral and is pictured below left.



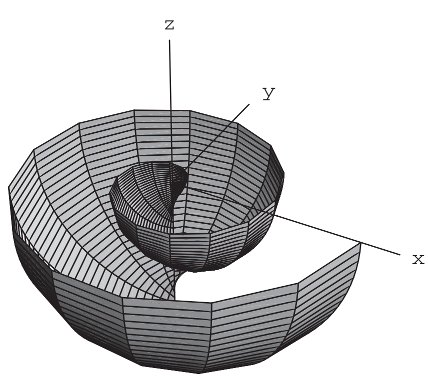




**(b)** The cylinder based on the spiral in part (a) is shown above right.

**(c)** Because only part of the spiral is used, the resulting surface is a dimpled ball.




**(d)** Finally, we see a lovely and intricate shell-like surface.




**44. (a)** In spherical coordinates the flat top of the hemisphere is the *xy*-plane with spherical equation $\varphi = \pi/2$. The hemispherical bottom has equation $\rho = 5$, but only with $\pi/2 \le \varphi \le \pi$. Thus we may describe the object as

$$\{(\rho, \varphi, \theta) | 0 \le \rho \le 5, \quad \pi/2 \le \varphi \le \pi, \quad 0 \le \theta < 2\pi\}.$$

**(b)** Now the flat top is described in cylindrical coordinates as $z = 0$ and the bottom hemisphere as $z^2 + r^2 = 25$ with $z \le 0$, that is, as $z = -\sqrt{25 - r^2}$. Bearing this in mind, the solid object is the set of points

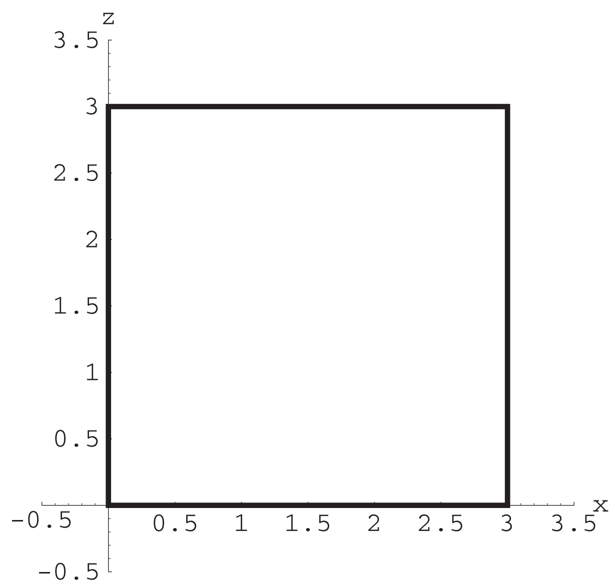
$$\{(r, \theta, z) | -\sqrt{25 - r^2} \le z \le 0, \quad 0 \le r \le 5, \quad 0 \le \theta < 2\pi\}.$$

**45.** Position the cylinder so that the center of the bottom disk is at the origin and the $z$-axis is the axis of the cylinder.

**(a)** In cylindrical coordinates $\theta$ is free to take on any values between $0$ and $2\pi$. The $z$-coordinate is bounded by $0$ and $3$, and $0 \le r \le 3$. To sum up:

$$\{(r, \theta, z) \mid 0 \le r \le 3,\ 0 \le z \le 3,\ 0 \le \theta \le 2\pi\}.$$

**(b)** Since the solid cylinder is rotationally symmetric about the $z$-axis, there is no restriction on the $\theta$ coordinate, and we may slice the cylinder with the half-plane $\theta =$ constant, in which case we see that the cross section is a filled-in square of side length $3$. Consider the cross section by the half-plane $\theta = 0$, pictured below:




The top of the square (which corresponds to the top of the cylinder) has equation $z = 3$, or $\rho \cos\varphi = 3$. Thus the top of the cylinder is the plane $\rho = 3\sec\varphi$. The bottom is, of course, the plane $z = 0$, which is given by $\rho\cos\varphi = 0$, which implies $\varphi = \pi/2$. The right side of the square, pictured as $x = 3$ in the figure above, corresponds to a cross section of the lateral surface of the cylinder given in cylindrical coordinates as $r = 3$, and thus in spherical coordinates by $\rho \sin\varphi = 3 \iff \rho = 3\csc\varphi$.

Now fix a value of $\varphi$. If this value of $\varphi$ is between $0$ and $\pi/4$, the spherical coordinate $\rho$ must be between $0$ and the top of the cylinder $\rho = 3\sec\varphi$. On the other hand, if this value of $\varphi$ is between $\pi/4$ and $\pi/2$, the spherical coordinate $\rho$ must be between $0$ and the lateral part of the cylinder $\rho = 3\csc\varphi$. If $\varphi$ is larger than $\pi/2$, no value of $\rho$ (other than zero) would give a point remaining inside the solid cylinder. To sum up:

$$\{(\rho, \varphi, \theta) \mid 0 \le \rho \le 3\sec\varphi,\ 0 \le \varphi \le \pi/4,\ 0 \le \theta \le 2\pi\}$$
$$\cup\{(\rho, \varphi, \theta) \mid 0 \le \rho \le 3\csc\varphi,\ \pi/4 \le \varphi \le \pi/2,\ 0 \le \theta \le 2\pi\}.$$

## Chapter 2

# Differentiation in Several Variables

### 2.1 Functions of Several Variables; Graphing Surfaces

**1.** $f\colon \mathbf{R} \to \mathbf{R}\colon x \mapsto 2x^2 + 1$
**(a)** Domain $f = \{x \in \mathbf{R}\}$, Range $f = \{y \in \mathbf{R} | y \geq 1\}$.
**(b)** No. For instance $f(1) = 3 = f(-1)$.
**(c)** No. For instance if $y = 0$, there is no $x$ such that $f(x) = 0$.

**2.** $f\colon \mathbf{R}^2 \to \mathbf{R}\colon (x, y) \mapsto 2x^2 + 3y^2 - 7$
**(a)** Domain $g = \{(x, y) \in \mathbf{R}^2\}$, Range $g = \{z \in \mathbf{R} | z \geq -7\}$.
**(b)** Let Domain $g = \{(x, x) \in \mathbf{R}^2 | x \geq 0\}$.
**(c)** Let Codomain $g$ = Range $g$.

**3.** Domain $f = \{(x, y) \in \mathbf{R}^2 | y \neq 0\}$, Range $f = \mathbf{R}$.

**4.** Domain $f = \{(x, y) \in \mathbf{R}^2 | x + y > 0\}$, Range $f = \mathbf{R}$.

**5.** Domain $g = \mathbf{R}^3$, Range $g = \{w \in \mathbf{R} | w \geq 0\}$.

**6.** Domain $g = \{\mathbf{x} \in \mathbf{R}^3 | \, \|\mathbf{x}\| < 2\}$, Range $g = \{y \in \mathbf{R} | y \geq 1/2\}$.

**7.** Domain $\mathbf{f} = \{(x, y) \in \mathbf{R}^2 | y \neq 1\}$, Range $\mathbf{f} = \{(x, y, z) \in \mathbf{R}^3 | y \neq 0, y^2 z = (xy - y - 1)^2 + (y + 1)^2\}$.

**8.** The component functions of $\mathbf{f}$ are $f_1(x, y) = x + y$, $f_2(x, y) = ye^x$, and $f_3(x, y) = x^2 y + 7$.

**9.** The component functions of $\mathbf{v}$ are obtained by extracting the $\mathbf{i}$-, $\mathbf{j}$- and $\mathbf{k}$-components of the expression for $\mathbf{v}(x, y, z, t)$. Thus we have

$$v_1(x, y, z, t) = xyzt, \; v_2(x, y, z, t) = x^2 - y^2, \; v_3(x, y, z, t) = 3z + t.$$

**10.** If $\mathbf{x} = (x_1, x_2, x_3) = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, then

$$\mathbf{f}(\mathbf{x}) = \mathbf{x} + 3\mathbf{j} = x_1\mathbf{i} + (x_2 + 3)\mathbf{j} + x_3\mathbf{k},$$

so that the component functions are

$$f_1(\mathbf{x}) = x_1, \quad f_2(\mathbf{x}) = x_2 + 3, \quad f_3(\mathbf{x}) = x_3.$$

**11.** **(a)** $\mathbf{f}(\mathbf{x}) = -2\mathbf{x}/\|\mathbf{x}\|$.
**(b)** The component functions are

$$f_1(x, y, z) = \frac{-2x}{\sqrt{x^2 + y^2 + z^2}}, \; f_2(x, y, z) = \frac{-2y}{\sqrt{x^2 + y^2 + z^2}}, \quad \text{and} \quad f_3(x, y, z) = \frac{-2z}{\sqrt{x^2 + y^2 + z^2}}.$$

**12.** **(a)** The component functions of $\mathbf{f}$ are just the components of the output vector $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$. Thus we calculate

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} 2 & -1 \\ 5 & 0 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ 5x_1 \\ -6x_1 + 3x_2 \end{bmatrix}.$$

Hence the component functions are:

$$f_1(x_1, x_2) = 2x_1 - x_2, \quad f_2(x_1, x_2) = 5x_1, \quad f_3(x_1, x_2) = -6x_1 + 3x_2.$$

**(b)** First note that, as $\mathbf{x}$ varies through all of $\mathbf{R}^2$, the expression $2x_1 - x_2$ can be any real number and $5x_1$ can be any real number. In addition, considering our answer in part (a), we see that

$$f_3(x_1, x_2) = -6x_1 + 3x_2 = -3(2x_1 - x_2) = -3f_1(x_1, x_2).$$

Thus the range of $\mathbf{f}$ consists of those vectors $\mathbf{y} = (y_1, y_2, y_3) \in \mathbf{R}^3$ with $y_3 = -3y_1$.

**13. (a)** The component functions of $\mathbf{f}$ are the components of the output vector $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$. Thus

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_3 + x_4 \\ 3x_2 \\ 2x_1 - x_3 + x_4 \end{bmatrix}.$$

The component functions are thus:

$$f_1(\mathbf{x}) = 2x_1 - x_3 + x_4, \quad f_2(\mathbf{x}) = 3x_2, \quad f_3(\mathbf{x}) = 2x_1 - x_3 + x_4.$$

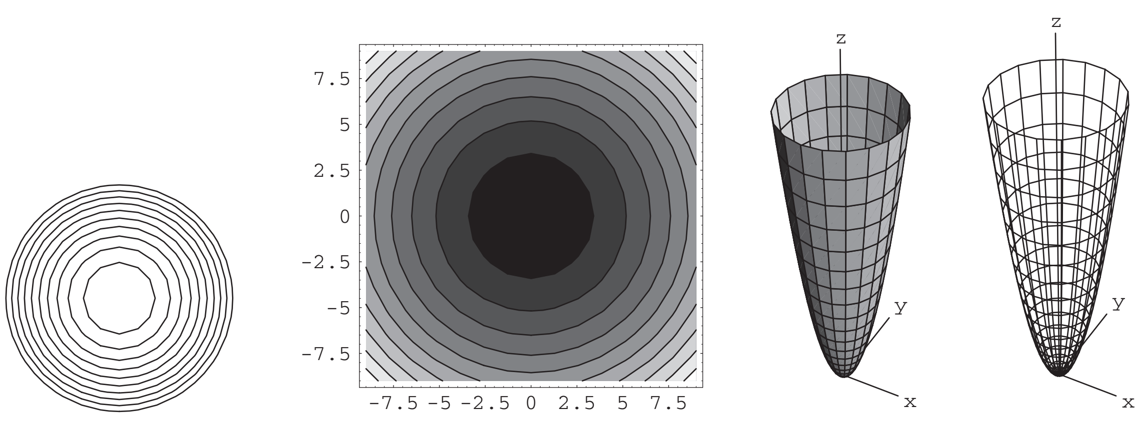
**(b)** Note that, as $\mathbf{x}$ varies through all of $\mathbf{R}^4$, the expression $2x_1 - x_3 + x_4$ can be any real number and $3x_2$ can be any real number. In addition, considering our answer in part (a), we see that

$$f_1(\mathbf{x}) = 2x_1 - x_3 + x_4 = f_3(\mathbf{x}).$$

Hence the range of $\mathbf{f}$ consists of those vectors $\mathbf{y} = (y_1, y_2, y_3) \in \mathbf{R}^3$ with $y_1 = y_3$.

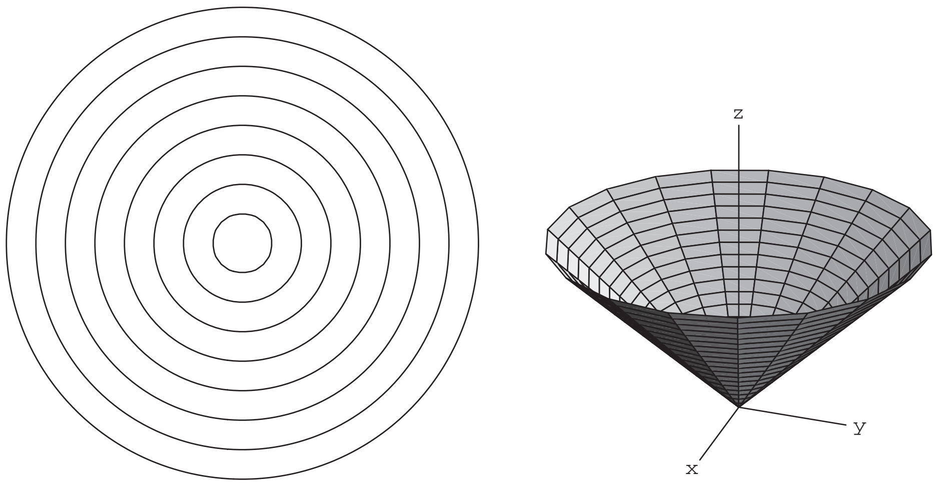
**14.** Here there is nothing to show. Everything is at level 3. This surface is a plane parallel to the *xy*-plane 3 units above it so the level set is the entire *xy*-plane if $c = 3$ and is the empty set if $c \neq 3$.

**15.** For $c > 0$ the level sets are circles centered at the origin of radius $\sqrt{c}$. For $c = 0$ the level set is just the origin. There are no values corresponding to $c < 0$. Note that the curves get closer together, indicating that we are climbing faster as we head out radially from the origin. The second figure below shows the plot of the level curves shaded to indicate the height of the level set (lighter is higher). The surface is therefore a paraboloid symmetric about the $z$-axis. We show it with and without the surface filled in.

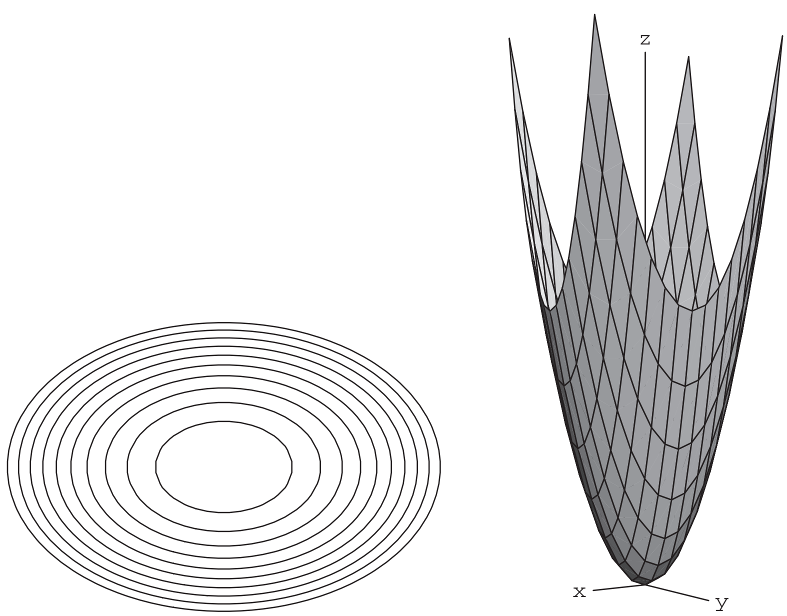



**16.** This is exactly the same as Exercise 15 except that the paraboloid has been shifted down 9 units so the level curves begin in the center at $c = -9$, not $c = 0$.

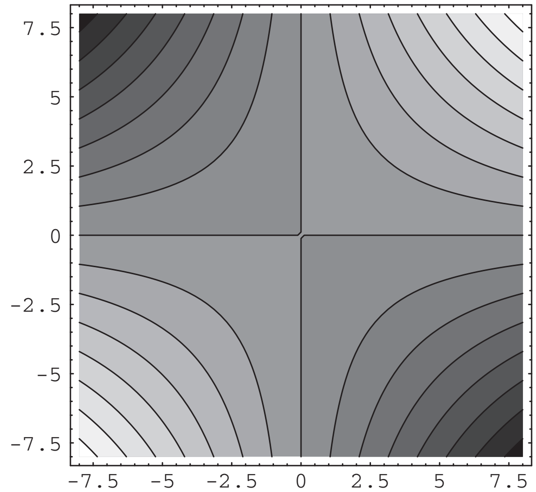
**17.** Again this time for $c > 0$ the level curves are circles. This time, however, the circles corresponding to the level sets at height $c$ are of radius $c$. In other words, they are evenly spaced. We are climbing at a constant rate as we head out radially, so the surface is a cone.

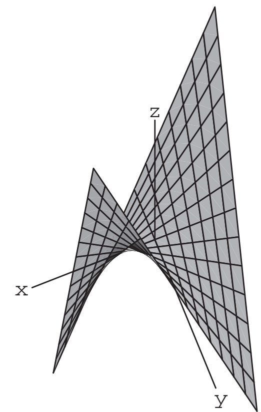



**18.** This time the level curves are ellipses. The sections as we cut in the direction $x$ is constant or $y$ is constant are still parabolas.



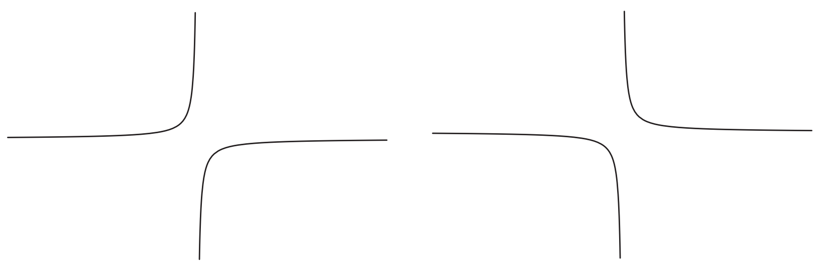

**19.** The graphs $xy = c$ are hyperbolas (unless $c = 0$ in which case it is the union of the two axes). When $x$ and $y$ are both positive the height of the level curves are positive and so the hyperboloid is increasing as we head away from the origin radially in either the first or third quadrant. When $x$ and $y$ are of different signs, the heights of the level curves are negative and so the hyperboloid is decreasing as we head out radially in either the second or fourth quadrant.



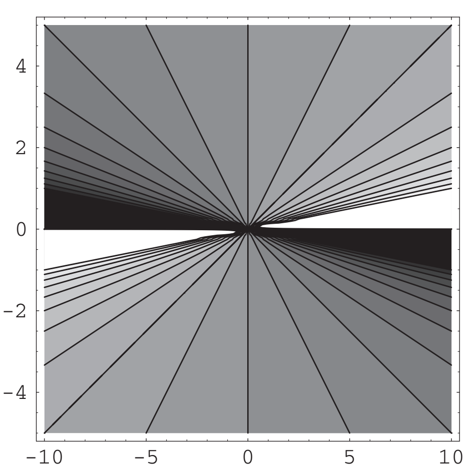




**20.** This is exactly the same as Exercise 21 except that the image has been reflected about the plane $y = x$.

**21.** We have a problem when $y = 0$. When $k < 0$, the section by $x = k$ looks like the hyperbola in the figure on the left, when $k > 0$, the section looks like the hyperbola in the figure on the right:

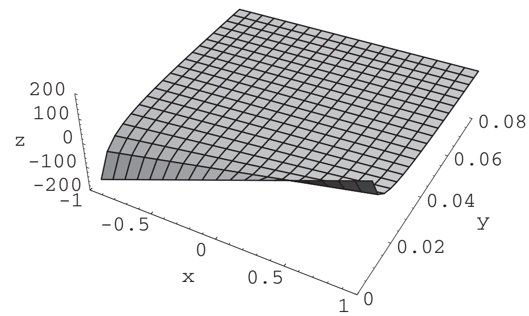


You can see that as $y \to 0$ from either side, along a line where $x$ is constant and not 0, the $z$ values won't match up. We are going to get a tear down the line $y = 0$. The level sets look like:

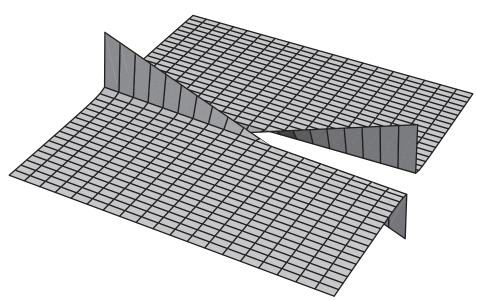



Notice that you can see that tear on the right center part of the above graph. The solid black and solid white areas which are on either side of the $x$-axis point to the behavior around the tear.

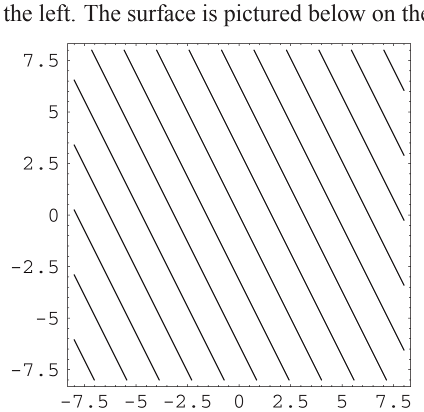
Graph each side of the $x$-axis and you will see the following piece of the surface:

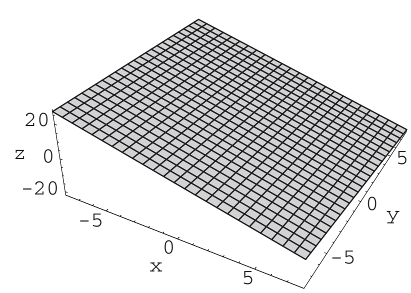



Our final surface is what you get when you try to glue two of those together:

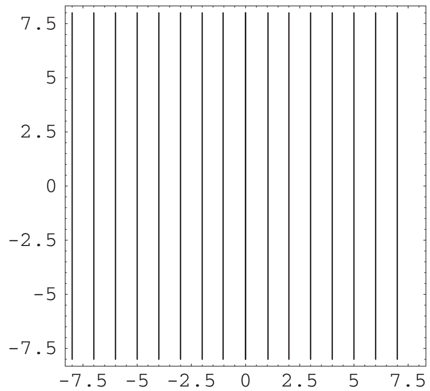


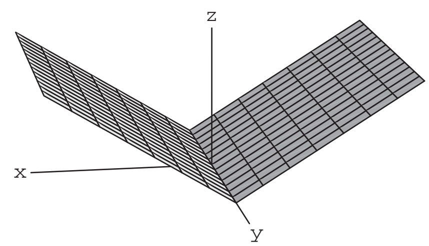
**22.** The surface is a plane. Level sets for which $f(x, y) = c$ are lines $c = 3 - 2x - y$ or $y = -2x + (3 - c)$. Level sets are pictured below on the left. The surface is pictured below on the right.



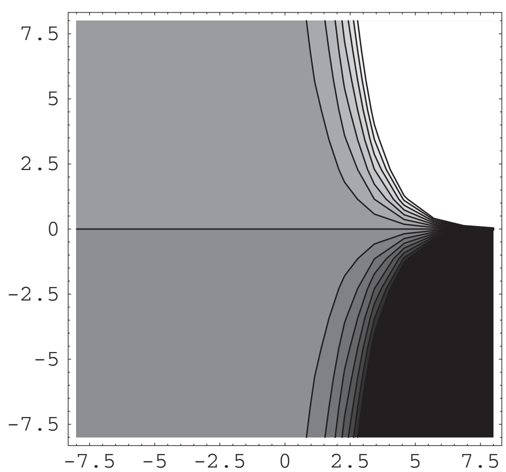

**23.** Here we are looking at the graph of $z = |x|$. For $c > 0$, level sets for $z = c$ will be the lines $x = \pm\, c$. For $c = 0$ the level set is the $y$-axis. The graph is like a folded plane.

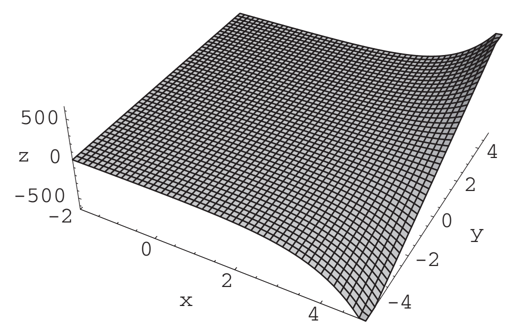


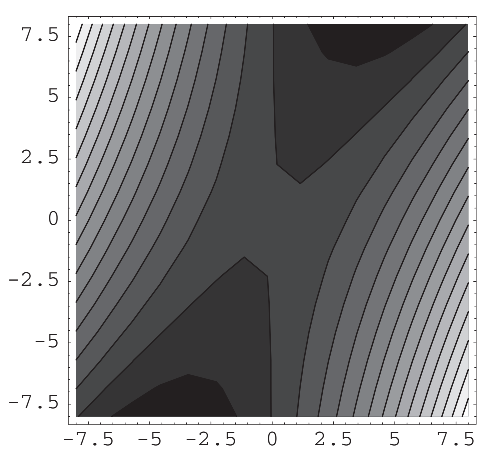

*Note: In Problems 24–27 the level curves are shown along with the contour shading so you get an idea at what height to hang the curves. You should be able to figure out the orientation of the surface from the contour plot.*

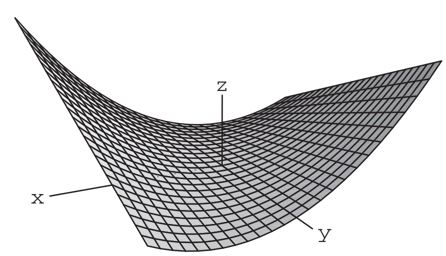
**24.** Figures below:



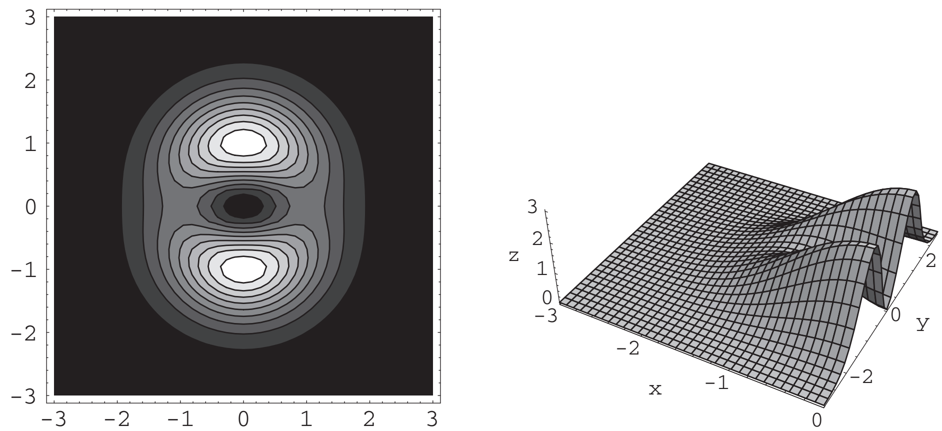




**25.** Figures below:

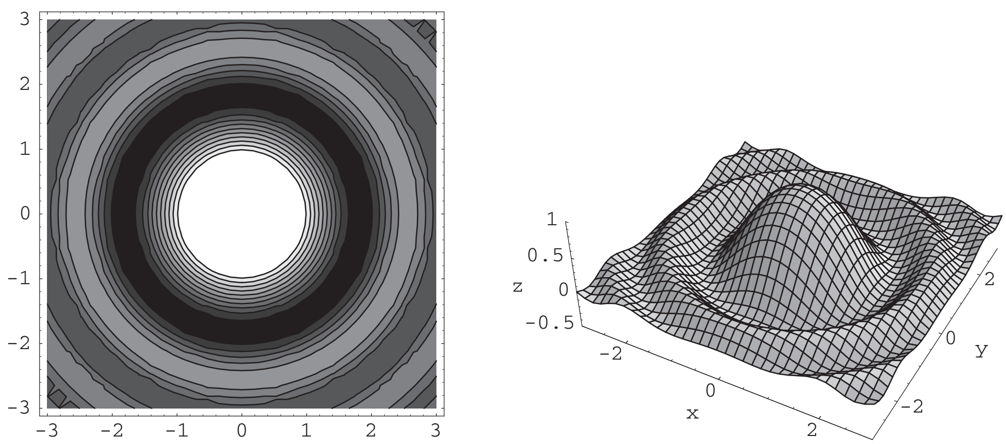


**26.** Figures below: (note only a portion of the surface has been sketched so that you get a better idea of what's going on)



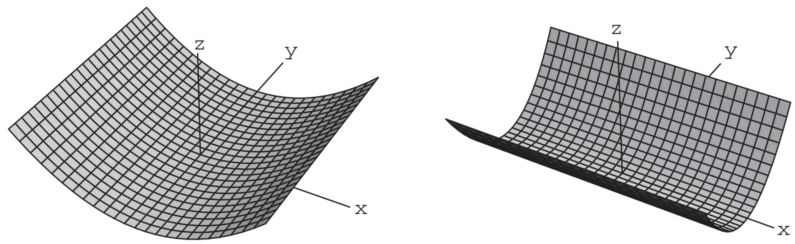
**27.** Figures below:



**28.** **(a)** We solve the equation $PV = kT$ for $T$, obtaining $T = f(P, V) = (1/k)PV$. This is the same as we considered in Exercise 15. See the figures for Exercise 19 for the general shape of the level curves.

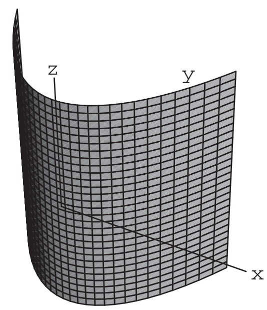
**(b)** Here $V = g(P, T) = \frac{kT}{P}$. This is the same as the cases we considered in Exercises 20 and 21. We will get a "torn" surface similar to the one shown in Exercise 21. The level curve $V = c$ is the line through the origin: $P = (k/c)T$.

**29.** **(a)** The surface $z = x^2$ is graphed below left and $z = y^2$ below right.



**(b)** Consider first the surface $z = f(x)$ by considering the curve in the $uv$-plane given by $v = f(u)$. The intersection of the surface with planes of the form $y = c$ will look the same as the curve in the $uv$-plane for any value of $y$. This helps us see that if we "drag" this curve in each direction along the $y$-axis, the trail will trace out the surface. Similarly, but along the $x$-axis for surfaces of the form $z = f(y)$. The lack of dependence on $x$ is our clue.

**(c)** The graph of the surface $y = x^2$ is shown below. It's what we would expect from parts (b) and (a).




**30.** See the solution to Exercise 21 and the note in Exercise 20.

**31.** They can't intersect—even though they may sometimes appear to. Say that two different level curves $f(x, y) = c_1$ and $f(x, y) = c_2$ where $c_1 \neq c_2$ intersect at some point (*a, b*). Then $f(a, b)$ would have assigned to it two non-equal values. This can't happen for a function (it's our vertical line test). On the other hand, if the limit as you approach (*a, b*) along different paths is different, those level curves may appear to intersect at (*a, b*) no matter how good the resolution on your contour plot.

**32.** The level surfaces are planes $x - 2y + 3z = c$.

**33.** The level surfaces at level $w = c$ are elliptic paraboloids.

**34.** The level surfaces at level $w = c$ are nested spheres of radius $\sqrt{c}$ centered at the origin.

**35.** The level surfaces at level $w = c$ are nested ellipsoids.

**36.** The level surfaces are of the form $y(x - z) = c$. If $c = 0$ we get the union of the *xz*-plane and the plane $x = z$. If $c \neq 0$ we get the hyperbola in the *xy*-plane $y = c/x$; this generates the solution surfaces when translated by $m(1, 0, -1)$.

**37. (a)** These are cylinders with the $z$-axis being the axis of the cylinder. For the surface at level $w = c$, the radius of the cylinder is $\sqrt{c}$.

**(b)** This is related to Exercise 29. A level surface at $w = c$ will be the surface generated by building a cylinder on the curve $h(x, y) = c$ in the $z = 0$ plane. You are dragging the curve both directions along the $z$-axis so that all cross sections for $z = c_1$ look identical.

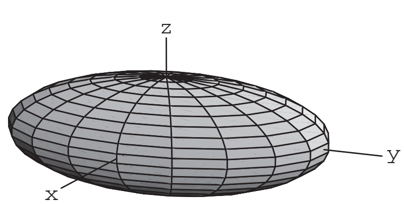
**(c)** Same thing in the $y$ direction.

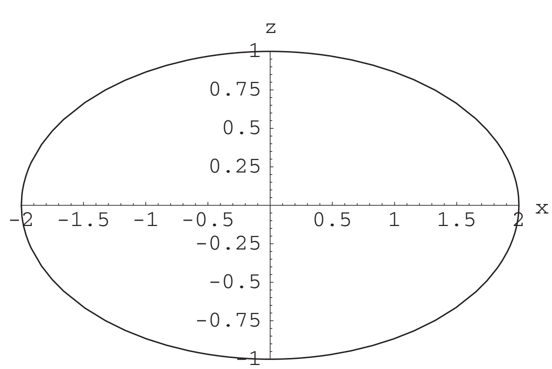
**(d)** If you said "same thing in the $x$ direction," read the problem again. You are solving equations that look like $h(x) = c$. For each $x_i$ that solves this equation, you have no dependency on $y$ or $z$ so the level set looks like a plane in $\mathbf{R}^3$ parallel to the *yz*-plane of the form $x = x_i$.

**38. (a)** $F$ is, of course, not uniquely determined. But if we let $F(x, y, z) = x^2 + xy - xz - 2$, then the surface is the level set $F(x, y, z) = 0$.

**(b)** $x^2 + xy - xz = 2$ is equivalent to $z = \dfrac{x^2 + xy - 2}{x} = f(x, y)$.

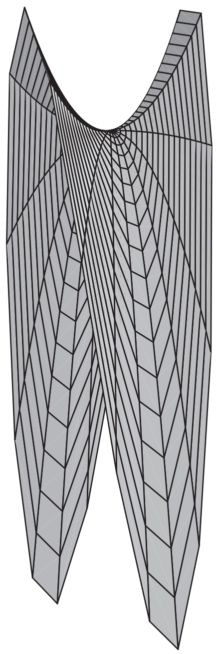
**39.** The ellipsoid is pictured below left. To see why you couldn't express the surface as one function $z = f(x, y)$, look for example at the intersection of the ellipsoid and the plane $y = 0$ pictured below on the right.

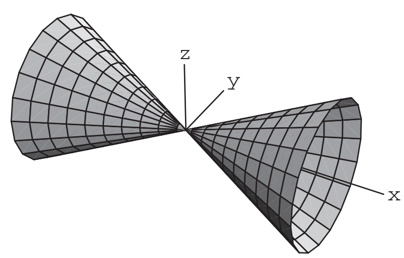


You can see that for $-2 < x < 2$ there correspond two values of $z$. We could express the top portion of the ellipsoid as $f(x, y) = \sqrt{1 - (x^2/4 + y^2/9)}$ and the bottom portion as $g(x, y) = -\sqrt{1 - (x^2/4 + y^2/9)}$.

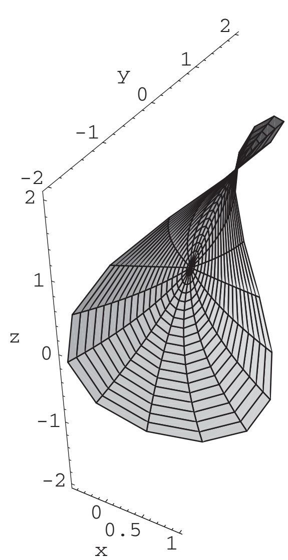
**40.** The figure is a hyperbolic paraboloid shown below left.

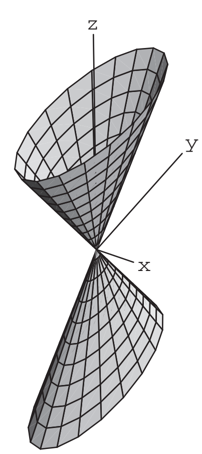





**41.** The only difference here is that $z$ is squared. Here we get a cone with axis of symmetry the $x$-axis. The figure is shown above right.

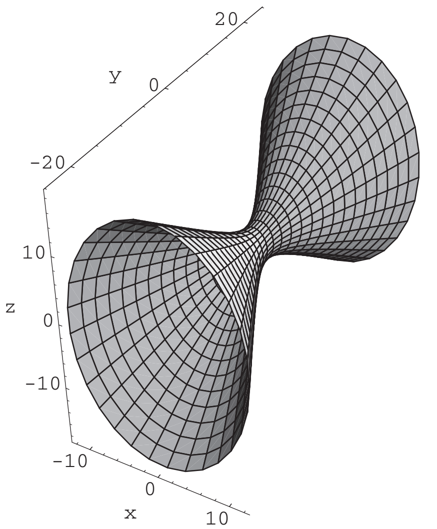
**42.** This is Exercise 40 with the roles of *x, y* and $z$ permuted and a change in the constants. The figure is shown below left.

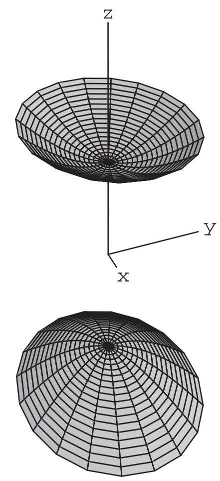



**43.** This is "cone" where the cross sections are ellipses, not circles. The figure is shown above right.

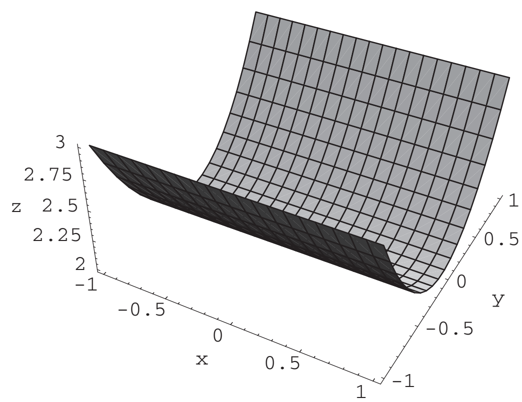
**44.** We see the figure is a hyperboloid. It is shown below left.



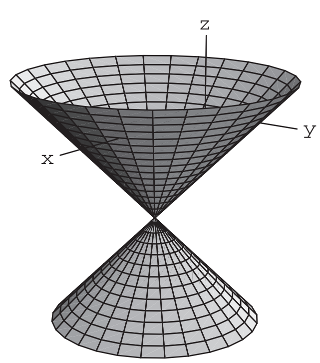

**45.** This is a hyperboloid of two sheets. It is shown above right.

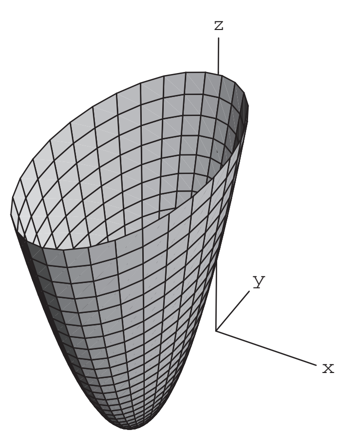
**46.** Here we have the parabola $z = y^2 + 2$ translated arbitrarily in the $x$ direction. This is what we call a cylinder over the parabola $z = y^2 + 2$.




*Note: Except that your students have to complete the square first, these are similar to Exercises 40–46 above. You may want them to be more explicit in reporting the translation as that's sometimes hard to pick up from a diagram.*

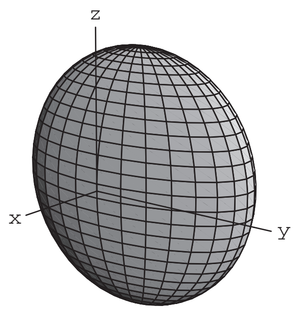
**47.** This is the equation of an elliptic cone with vertex at $(1, -1, -3)$. The graph is shown below left.

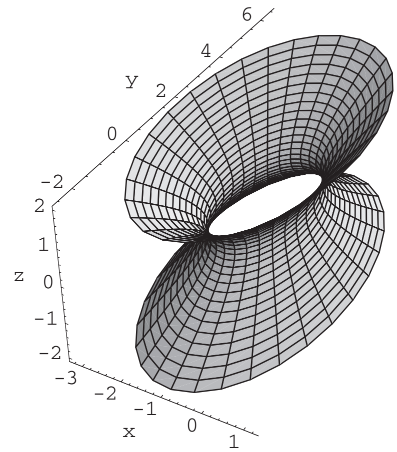


**48.** Here we have an elliptic paraboloid. The graph is shown above right.

**49.** This is the equation of an ellipsoid $4(x+1)^2 + y^2 + z^2 = 4$. The graph is shown below left.




**50.** This is the equation of a hyperboloid of one sheet $4(x+1)^2 + (y-2)^2 - 4z^2 = 4$. The graph is shown above right.

**51.** This is similar to Exercise 48. The equation is equivalent to $z - 1 = (x-3)^2 + 2y^2$.

**52.** Here we get $9x^2 + 4(y-1)^2 - 36(z+4)^2 = 684$ which is similar to Exercise 50.

## 2.2 Limits

*Note: In Exercises 1–6, the rule of thumb is that a set is closed if it contains all of its boundary points.*

**1.** This is an annulus which doesn't include its inner or outer boundary and so is **open**.

**2.** This is an annulus which includes all of its boundary points and so is **closed**.

**3.** This is an annulus which includes its inner boundary but not its outer boundary and so it is **neither open nor closed**.

**4.** This is a hollowed out sphere which includes its boundary points and so is **closed**.

**5.** This may be a bit harder to see. This is the union of an infinite open strip in the plane $(-1 < x < 1)$ and a closed line in the plane $(x = 2)$ and so is **neither open nor closed**.

**6.** This is the open infinite cylinder in $\mathbf{R}^3$ and so is **open**. *You could follow up on this by asking about* $\{(x,y,z) \in \mathbf{R}^3 | 1 \leq x^2 + y^2 \leq 4\}$.

*Note: As pointed out in the text, the most common and convincing way to prove that a limit of a function with domain in* $\mathbf{R}^2$ *doesn't exist is to show that you get two different answers when you follow two different paths. After doing Exercises 7–18 students may get in the habit of thinking that it is sufficient to check a few straight paths. Exercise 23 should make them think twice.*

**7.** There's no trick to taking this limit. Just let $(x,y,z) \to (0,0,0)$ and $x^2 + 2xy + yz + z^3 + 2 \to 2$.

**8.** We can see that $\lim_{(x,y)\to(0,0)} \frac{|y|}{\sqrt{x^2+y^2}}$ doesn't exist by looking at the limit along the paths $x = 0$ and $y = 0$. On the one hand

$$\lim_{(0,y)\to(0,0)} \frac{|y|}{\sqrt{x^2+y^2}} = \frac{|y|}{\sqrt{y^2}} = 1 \quad \text{while} \quad \lim_{(x,0)\to(0,0)} \frac{|y|}{\sqrt{x^2+y^2}} = \frac{0}{\sqrt{x^2}} = 0.$$

**9.** Again, the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x^2+y^2} = 1 + \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}.$$

When $x = y$,

$$1 + \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = 1 + \lim_{x\to 0} \frac{2x^2}{x^2+x^2} = 1 + 1 = 2.$$

When $x = 0$,

$$1 + \lim_{(0,y)\to(0,0)} \frac{2xy}{x^2+y^2} = 1 + \lim_{y\to 0} \frac{0}{y^2} = 1.$$

**10.** Here nothing goes wrong so we can evaluate the limit by substituting in the expression.

$$\lim_{(x,y)\to(0,0)} \frac{e^x e^y}{x+y+2} = \frac{e^0 e^0}{0+0+2} = \frac{1}{2}.$$

**11.** No limit exists.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2+y^2}{x^2+y^2} = 1 + \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}.$$

We reason, as above, that if $x = y$ then the limit is 3/2, but if $y = 0$ the limit is 2.

**12.** Here we can evaluate the function at the limit point and find that

$$\lim_{(x,y)\to(-1,2)} \frac{2x^2+y^2}{x^2+y^2} = \frac{6}{5}.$$

**13.** Just as with limits in first semester Calculus, this is begging to be simplified.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x+y} = \lim_{(x,y)\to(0,0)} (x+y) = 0.$$

**14.** This is the same as the limit in Exercise 9 (once we simplified it). The limit does not exist.

**15.** This, too, is begging to be simplified.

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)(x^2+y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} (x^2-y^2) = 0.$$

**16.** This is the same as the limit in Exercise 11 (once we simplified it). The limit does not exist.

**17.** This is another standard trick from first year Calculus.

$$\lim_{(x,y)\to(0,0),x\neq y} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0),x\neq y} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0),x\neq y} \frac{x(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}{\sqrt{x}-\sqrt{y}}$$
$$= \lim_{(x,y)\to(0,0),x\neq y} x(\sqrt{x}+\sqrt{y}) = 0.$$

**18.** You can see that you would get different values depending on the path you took to $(x, y) = (2, 0)$. If you followed the path $(2, y) \to (2, 0)$ the limit would be $-1$. If you followed the path $(x, 0) \to (2, 0)$ the limit would be 1. So the limit doesn't exist.

**19.** The function is continuous so the limit is $f(0, \sqrt{\pi}, 1) = e^0 \cos\pi - 0 = -1$.

**20.** As in Exercise 18, you get different values depending on the path you choose. Look, for example, at paths along the three axes. Along $(x, 0, 0) \to (0, 0, 0)$ the limit is 2, along $(0, y, 0) \to (0, 0, 0)$ the limit is 3 and along $(0, 0, z) \to (0, 0, 0)$ the limit is 1. We can see that no limit can exist.

**21.** Again the limit doesn't exist because the value would differ on different paths. If you followed a path $(t, t, t) \to (0, 0, 0)$ the limit would be 1/3. If you followed the path $(x, 0, 0) \to (0, 0, 0)$ the limit would be 0.

**22.** **(a)** We know from single-variable calculus (either using l'Hôpital's rule or the direct geometric argument) that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

**(b)** $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

**(c)** $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

*Note: Exercise 23 is a classic and cool problem. You may wish to set it up in class before assigning it. Write the function on the board and ask the students to evaluate the limit or explain why the limit fails to exist. For those who get it right, this is wonderful. For those who get it wrong, they are now in a position to appreciate the subtlety of the problem.*

**23.** Our goal is to evaluate $\lim_{(x,y)\to(0,0)} \dfrac{x^4y^4}{(x^2+y^4)^3}$ or explain why the limit fails to exist. We divide the answer into parts to make it easier to follow—there are no corresponding parts (a)–(d) in the text.

**(a)** If you evaluate the limit along the lines $x = 0$ and $y = 0$ the limit is 0. We might be tempted to guess that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$ but as we saw in Exercise 14, we could get a limit of 0 along the paths $x = 0$ and $y = 0$ but perhaps not along $x = y$.

**(b)** So now let's follow the line $y = mx$ into the origin and see where $f$ heads off to.

$$\begin{aligned}\lim_{(x,y)\to(0,0),y=mx} \frac{x^4y^4}{(x^2+y^4)^3} &= \lim_{x\to 0} \frac{x^4(mx)^4}{(x^2+(mx)^4)^3}\\ &= \lim_{x\to 0} \frac{m^4x^8}{(x^2(1+m^4x^2))^3}\\ &= m^4 \lim_{x\to 0} \frac{x^8}{(x^6)(1+m^4x^2)^3}\\ &= m^4 \lim_{x\to 0} \frac{x^2}{(1+m^4x^2)^3} = 0.\end{aligned}$$

This means then if we head into the origin along any straight line the limit of $f$ is 0. *Here is the point of this problem*: If we head into the origin in any constant direction, the limit of $f$ is 0 and yet $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist!

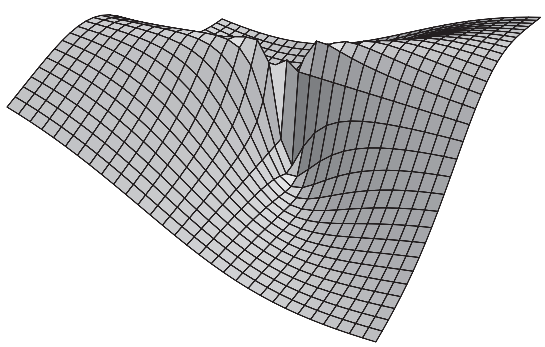
**(c)** For the limit to exist $f$ must approach the same number no matter what path we choose to take to the origin. So let's approach along the parabola $x = y^2$.

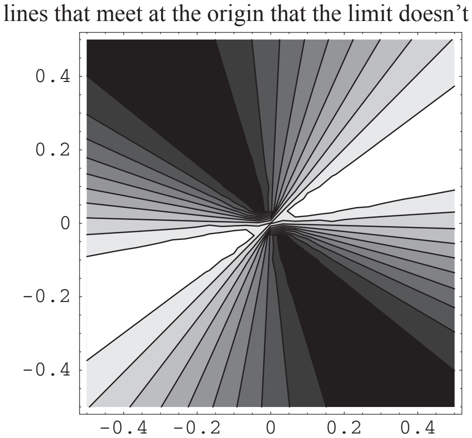
$$\begin{aligned}\lim_{(x,y)\to(0,0),x=y^2} \frac{x^4y^4}{(x^2+y^4)^3} &= \lim_{y\to 0} \frac{(y^2)^4y^4}{((y^2)^2+y^4)^3}\\ &= \lim_{y\to 0} \frac{y^{12}}{(2y^4)^3}\\ &= \lim_{y\to 0} \frac{y^{12}}{8y^{12}} = \frac{1}{8}.\end{aligned}$$

**(d)** So we get different answers for $\lim_{(x,y)\to(0,0)} \dfrac{x^4y^4}{(x^2+y^4)^3}$ depending on what path we follow into the origin. So the limit does not exist.

*Note—In Exercises 24–27 your students may find better visual information by using a contour plot than a three-dimensional plot.*

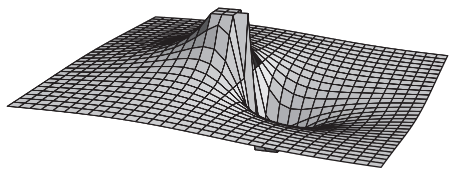
**24.** Below see two graphs of the function. The three-dimensional plot makes it seem as if there are mountains and valleys quite close to the origin. The contour plot helps you see from the diagonal lines that meet at the origin that the limit doesn't exist.

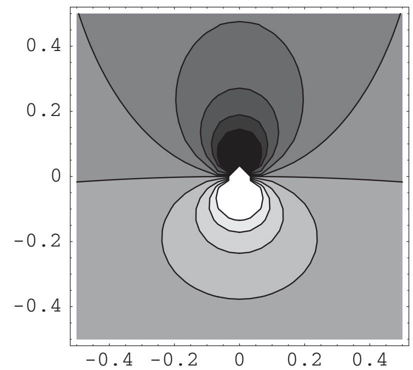





Analytically, $f$ is equivalent to $1 + (x^2 + 2xy)/(3x^2 + 5y^2)$. Head in toward the origin on a path where $x = y$ and the limit is 13/8. Head in toward the origin on a path where $x = 0$ and the limit is 1. Head in toward the origin on a path where $y = 0$ and the limit is 11/3. So the limit doesn't exist.

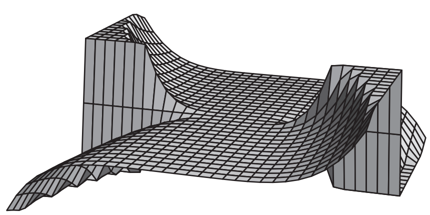
**25.** Below see two graphs of the function. You actually get most of the picture from the three-dimensional graph—except that it looks as if things are joined smoothly. The contour plot shows the dramatic problems near the origin. Particularly if you look along the vertical line $x = 0$ you'll see that the limit does not exist at the origin.

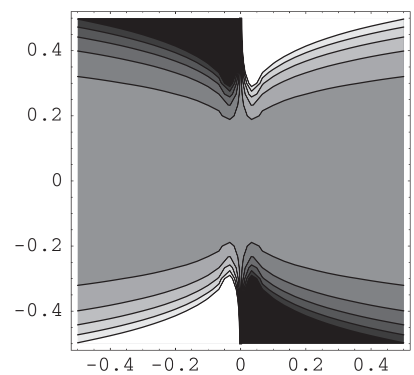




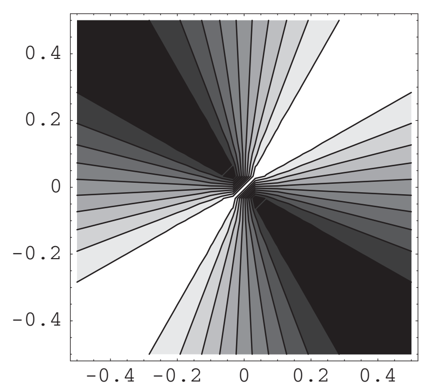

Analytically, look at the path $x = 0$. Here we're looking at the graph of $z = -1/y$. The limits as we approach from positive and negative $y$ values is $\pm\infty$ so no limit exists.

**26.** In the three-dimensional graph below you can see that the extreme behavior calms down near the origin. This is confirmed in the contour plot. From the graphs it appears that the limit exists at the origin.



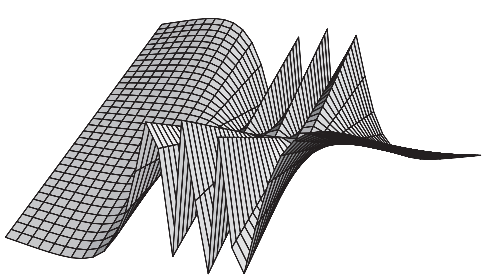


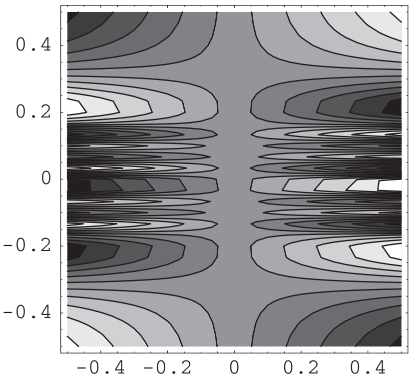

Before exploring this one analytically, consider the graph $g(x, t) = xt/(x^2 + t^2)$ Its contour plot is shown below.




So really the problem we are considering is the same with $t = y^5$. We're not looking along a path that shows us enough. Let's look at the limit for our original function $f$ as we approach the origin. Along a path where $x = 0$ or $y = 0$ the limit is 0. Along a path where $x = y^5$ the limit is 1/2. This is a good place to encourage your students to be careful drawing conclusions from even very good graphs.

**27.** You'd think we would have learned our lesson from Exercise 26. On the other hand, it sure looks as if things are calming down near the origin. Sure sin $1/y$ oscillates madly between $-1$ and 1 but $x$ seems to dampen it. We'll boldly assert that the limit exists at the origin.






Actually, the discussion above leads us to the truth. The product of a bounded function and one going to 0 goes to 0. The limit exists and is 0.

**28.** We rewrite $\dfrac{x^2y}{x^2+y^2}$ as

$$\frac{r^2\cos^2\theta\cdot r\sin\theta}{r^2\cos^2\theta+r^2\sin^2\theta}=r\cos^2\theta\sin\theta.$$

Because $0\le\cos^2\le 1$ and $-1\le\sin\theta\le 1$, we have

$$0\le r\cos^2\theta\sin\theta\le r.$$

Thus

$$\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^2+y^2}=\lim_{r\to 0} r\cos^2\theta\sin\theta=0$$

because the expression $r\cos^2\theta\sin\theta$ is squeezed between two others that have the same limits as $r\to 0$.

**29.**

$$\lim_{(x,y)\to(0,0)}\frac{x^2}{x^2+y^2}=\lim_{r\to 0}\frac{r^2\cos^2\theta}{r^2}=\lim_{r\to 0}\cos^2\theta=\cos^2\theta$$

Limit does not exist as the result depends on $\theta$.

**30.**

$$\lim_{(x,y)\to(0,0)}\frac{x^2+xy+y^2}{x^2+y^2}=\lim_{r\to 0}\frac{r^2+(r\cos\theta)r\sin\theta}{r^2}=\lim_{r\to 0}(1+\cos\theta\sin\theta)=1+\cos\theta\sin\theta.$$

Thus the limit does not exist.

**31.** We have

$$\begin{aligned}&\lim_{(x,y)\to(0,0)}\frac{x^5+y^4-3x^3y+2x^2+2y^2}{x^2+y^2}\\&\quad=\lim_{r\to 0}\frac{r^5\cos^5\theta+r^4\sin^4\theta-3r^4\cos^3\theta\sin\theta+2r^2\cos^2\theta+2r^2\sin^2\theta}{r^2\cos^2\theta+r^2\sin^2\theta}\\&\quad=\lim_{r\to 0}\frac{r^2(r^3\cos^5\theta+r^2\sin^4\theta-3r^2\cos^3\theta\sin\theta+2)}{r^2}\\&\quad=\lim_{r\to 0}\left[r^2(r\cos^5\theta+\sin^4\theta-3\cos^3\theta\sin\theta)+2\right]\end{aligned}$$

Note that $-1\le\cos^n\theta\le 1$ when $n$ is odd, $-1\le\sin\theta\le 1$, and $0\le\sin^m\theta\le 1$ when $m$ is even. Thus we have that

$$r^2(-r+0-3)+2\le r^2(r\cos^5\theta+\sin^4\theta-3\cos^3\theta\sin\theta)+2\le r^2(r+1+3)+2.$$

Now

$$\lim_{r\to 0}\left[-r^2(r+3)+2\right]=\lim_{r\to 0}\left[r^2(r+4)+2\right]=2;$$

thus $\lim_{r\to 0}\left[r^2(r\cos^5\theta+\sin^4\theta-3\cos^3\theta\sin\theta)+2\right]=2$ by squeezing.

**32.**

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$$
$$= \lim_{r\to 0^+} \frac{r^2\cos 2\theta}{r} = \lim_{r\to 0^+} r\cos 2\theta,$$

from the double-angle formula for cosine. (Note that since $\sqrt{r^2} = |r|$, we used a one-sided limit.) Since $-r \le r\cos 2\theta \le r$, we conclude that $\lim_{r\to 0^+} r\cos 2\theta = 0$ by squeezing.

**33.**

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{r\cos\theta + r\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$$
$$= \lim_{r\to 0^+} \frac{r\cos\theta + r\sin\theta}{r}$$
$$= \lim_{r\to 0^+} (\cos\theta + \sin\theta) = \cos\theta + \sin\theta.$$

Since this result depends on $\theta$, the limit does not exist.

**34.**

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y}{x^2+y^2+z^2} = \lim_{\rho\to 0} \frac{\left(\rho^2\sin^2\varphi\cos^2\theta\right)\left(\rho\sin\varphi\sin\theta\right)}{\rho^2}$$
$$= \lim_{\rho\to 0} \rho\sin^3\varphi\cos^2\theta\sin\theta$$

Since $0 \le \cos^2\theta \le 1$, we have $0 \le \rho\sin^3\varphi\cos^2\theta\sin\theta \le \rho$. Thus we conclude that $\lim_{\rho\to 0}\rho\sin^3\varphi\cos^2\theta\sin\theta = 0$ by squeezing.

**35.**

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim_{\rho\to 0} \frac{(\rho\sin\varphi\cos\theta)(\rho\sin\varphi\sin\theta)(\rho\cos\varphi)}{\rho^2}$$
$$= \lim_{\rho\to 0} \rho\sin^2\varphi\cos\varphi\cos\theta\sin\theta = 0$$

**36.**

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}} = \lim_{\rho\to 0} \frac{\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta}{\rho} = \lim_{\rho\to 0} \rho\sin^2\varphi = 0$$

**37.**

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xz}{x^2+y^2+z^2} = \lim_{\rho\to 0} \frac{\rho^2\sin\varphi\cos\varphi\cos\theta}{\rho^2} = \lim_{\rho\to 0} \sin\varphi\cos\varphi\cos\theta = \sin\varphi\cos\varphi\cos\theta$$

The limit does not exist.

*In Exercises 38–45: as the rules on continuity show, if the components are continuous and we put the functions together by adding, subtracting, multiplying, or composing, then the result is continuous. It should be clear to the students what points need checking.*

**38.** This is a polynomial and is continuous everywhere.

**39.** This too is a polynomial and is continuous everywhere.

*To make the point about composition, you may want to assign Exercises 40 and 41 together.*

**40.** The only place we could get into trouble is where the denominator is 0, but $x^2+1 \ne 0$ so $g$ is always continuous.

**41.** Here we are composing a continuous function (cos) with the continuous function $g$ from Exercise 22, so the composition is continuous.

**42.** You can even rewrite the function as $(\cos x)^2 - 2(\sin xy)^2$ so that it is clear that this is just the composition of continuous functions.

**43.** The only place we need to check is the origin. We need to show that the limit of $f$ as we approach (0, 0) is 0. If we add and subtract $y^2$ to the numerator we find that:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = 1 - 2 \lim_{(x,y)\to(0,0)} \frac{y^2}{x^2 + y^2}.$$

In Exercise 16 we showed that this limit doesn't exist (in this case you get two different answers if you follow the paths $y = 0$ and $y = x$) and so $f$ is not continuous at (0, 0).

**44.** As in Exercise 43, the only point we need to check is the origin.

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + x^2 + xy^2 + y^2}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2 + y^2)(x + 1)}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} (x + 1) = 1.$$

The good news is that the limit exists, the bad news is that

$$\lim_{(x,y)\to(0,0)} g(x, y) = 1 \neq 2 = g(0, 0),$$

so $g$ is not continuous at the origin.

**45.** A vector-valued function is continuous if each of its component functions is continuous. Each clearly is, so **F** is continuous.

**46.** Notice that when $(x, y) \neq 0$,

$$\frac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2 + y^2} = \frac{(x^2 + y^2)(x + 2)}{x^2 + y^2} = x + 2.$$

So $c = 2$ and the function $g(x, y)$ is seen to be equivalent to $x + 2$.

**47.** Here you can view $f$ as being a function $\mathbf{R}^3 \to \mathbf{R}$; then $f(x_1, x_2, x_3) = 2x_1 - 3x_2 + x_3$ which is linear in $x_1, x_2,$ and $x_3$ and therefore continuous.

**48.** This is equivalent to $\mathbf{f}(x, y, z) = (-5y, 5x - 6z, 6y)$. Since each of the component functions from $\mathbf{R}^3 \to \mathbf{R}$ is continuous, so is **f**.

*We make students do at least a few of the following because "it's good for them." Exercise 49 is a review of how they looked at limits in first semester Calculus—it prepares them for Exercise 50. Exercise 51 is a generalization of Exercise 50.*

**49.** Here $f(x) = 2x - 3$.

**(a)** If $|x - 5| < \delta$, then $|f(x) - 7| = |(2x - 3) - 7| = |2x - 10| = 2|x - 5| < 2\delta$.

**(b)** For any $\epsilon > 0$, if $0 < |x - 5| < \epsilon/2$, then $|f(x) - 7| < \epsilon$. This means that $\lim_{x\to5} f(x) = 7$.

**50.** Now the function is $f(x, y) = 2x - 10y + 3$.

**(a)** Really we're just arguing that the hypotenuse of a right triangle is at least as long as either leg.

$$\delta > \|(x, y) - (5, 1)\| = \sqrt{(x - 5)^2 + (y - 1)^2} \geq \sqrt{(x - 5)^2} = |x - 5|.$$

And

$$\delta > \|(x, y) - (5, 1)\| = \sqrt{(x - 5)^2 + (y - 1)^2} \geq \sqrt{(y - 1)^2} = |y - 1|.$$

**(b)** First:

$$|f(x, y) - 3| = |2x - 10y + 3 - 3| = |2x - 10y| = |2(x - 5) - 10(y - 1)|.$$

**(c)** By the triangle inequality

$$|2(x - 5) - 10(y - 1)| \leq |2(x - 5)| + |10(y - 1)| = 2|x - 5| + 10|y - 1|.$$

But we are assuming that $\|(x, y) - (5, 1)\| < \delta$ and from part (a) we know that this implies that $|x - 5| < \delta$ and $|y - 1| < \delta$, so

$$2|x - 5| + 10|y - 1| < 2\delta + 10\delta = 12\delta.$$

**(d)** We put these together to obtain: For any $\varepsilon > 0$, if $0 < \|(x, y) - (5, 1)\| < \varepsilon/12$, then $|f(x, y) - 3| < \varepsilon$. In other words,

$$\lim_{(x,y)\to(5,1)} f(x, y) = 3.$$

**51.** This is just a generalization of Exercise 50. We can use the same steps outlined there:

**(a)**

$$\delta > \|(x, y) - (x_0, y_0)\| = \sqrt{(x - x_0)^2 + (y - y_0)^2} \geq \sqrt{(x - x_0)^2} = |x - x_0|.$$

And

$$\delta > \|(x, y) - (x_0, y_0)\| = \sqrt{(x - x_0)^2 + (y - y_0)^2} \geq \sqrt{(y - y_0)^2} = |y - y_0|.$$

**(b)** Assume that $\|(x,y)-(x_0,y_0)\|<\delta$, then follow the steps in part (b) of Exercise 50:

$$\begin{aligned}|f(x,y)-(Ax_0+By_0+C)| &= |Ax+By+C-(Ax_0+By_0+C)|\\ &= |A(x-x_0)+B(y-y_0)| \le |A(x-x_0)|+|B(y-y_0)|\\ &= |A||x-x_0|+|B||y-y_0| < |A|\delta+|B|\delta = (|A|+|B|)\delta.\end{aligned}$$

**(c)** Now we're ready to put this together: For any $\varepsilon>0$, if $0<\|(x,y)-(x_0,y_0)\|<\varepsilon/(|A|+|B|)$, then $|f(x,y)-(Ax_0+By_0+C)|<\varepsilon$. In other words,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=Ax_0+By_0+C.$$

**52. (a)** This is really what we just showed in Exercise 51 with $x_0=0$ and $y_0=0$.

$$\|(x,y)\|=\sqrt{x^2+y^2}\ge\sqrt{x^2}=|x|.$$

And

$$\|(x,y)\|=\sqrt{x^2+y^2}\ge\sqrt{y^2}=|y|.$$

**(b)** We follow the hint given in the text: $|x^3+y^3|\le|x^3|+|y^3|=|x|^3+|y|^3$. But by part (a), $|x|\le\|(x,y)\|=\sqrt{x^2+y^2}$, and $|y|\le\|(x,y)\|=\sqrt{x^2+y^2}$. Therefore,

$$|x^3+y^3|\le|x|^3+|y|^3\le 2(\sqrt{x^2+y^2})^3=2(x^2+y^2)^{3/2}.$$

**(c)** If $0<\|(x,y)\|<\delta$ then by part (b),

$$\left|\frac{x^3+y^3}{x^2+y^2}\right|\le\left|\frac{2(x^2+y^2)^{3/2}}{x^2+y^2}\right|=2\sqrt{x^2+y^2}=2\|(x,y)\|<2\delta.$$

**(d)** First we know by part (c) that $\dfrac{x^3+y^3}{x^2+y^2}$ can be made to be arbitrarily close to 0 by choosing $(x, y)$ close enough to the origin. This means that the limit is 0.

Assemble the pieces: For any $\varepsilon>0$, if $0<\|(x,y)\|<\varepsilon/2$, then $|f(x,y)|<\varepsilon$. This shows that

$$\lim_{(x,y)\to(0,0)}\frac{x^3+y^3}{x^2+y^2}=0.$$

**53. (a)** $0\le(a+b)^2=a^2+2ab+b^2$, so $-2ab\le a^2+b^2$. Also $0\le(a-b)^2=a^2-2ab+b^2$, so $2ab\le a^2+b^2$. We combine these two results to get: $2|ab|\le a^2+b^2$.

**(b)** If $\|(x,y)\|<\delta$, then we'll use part (a) to rewrite $|xy|$ in the following calculation:

$$\left|xy\left(\frac{x^2-y^2}{x^2+y^2}\right)\right|=\frac{|xy|(|x^2-y^2|)}{x^2+y^2}\le\frac{(1/2)(x^2+y^2)|x^2-y^2|}{x^2+y^2}=\left(\frac{1}{2}\right)|x^2-y^2|.$$

We can apply part (a) again with $a=x+y$ and $b=x-y$ so that

$$|(x+y)(x-y)|\le\frac{(x+y)^2+(x-y)^2}{2}=x^2+y^2.$$

Noting that $x^2+y^2=\|(x,y)\|^2=\delta^2$, we have:

$$\left|xy\left(\frac{x^2-y^2}{x^2+y^2}\right)\right|\le\left(\frac{1}{2}\right)|x^2-y^2|=\frac{\delta^2}{2}.$$

**(c)** As in Exercise 52, the limit has to be 0 because we can make $f$ as small as we want by choosing $(x, y)$ close enough to the origin.

We summarize the above as: For any $\varepsilon>0$, if $0<\|(x,y)\|<\sqrt{2\varepsilon}$, then $|f(x,y)|<\varepsilon$. This shows that

$$\lim_{(x,y)\to(0,0)}\left|xy\left(\frac{x^2-y^2}{x^2+y^2}\right)\right|=0.$$

## 2.3 The Derivative

*The general strategy for Exercises 1–15 is to treat all variables except for the one with respect to which we are differentiating as constants.*

**1.** $f(x,y) = xy^2 + x^2y$, so $\partial f/\partial x = y^2 + 2xy$, and $\partial f/\partial y = 2xy + x^2$.

**2.** $f(x,y) = e^{x^2+y^2}$, so $\partial f/\partial x = 2xe^{x^2+y^2}$, and $\partial f/\partial y = 2ye^{x^2+y^2}$.

**3.** $f(x,y) = \sin xy + \cos xy$, so $\partial f/\partial x = y\cos xy - y\sin xy$, and $\partial f/\partial y = x\cos xy - x\sin xy$.

**4.** $f(x,y) = \dfrac{x^3 - y^2}{1 + x^2 + 3y^4}$, so

$$\frac{\partial f}{\partial x} = \frac{(1 + x^2 + 3y^4)(3x^2) - (x^2 - y^2)(2x)}{(1 + x^2 + 3y^4)^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{(1 + x^2 + 3y^4)(-2y) - (x^2 - y^2)(12y^3)}{(1 + x^2 + 3y^4)^2}.$$

**5.** $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$, so $\dfrac{\partial f}{\partial x} = \dfrac{(x^2 + y^2)(2x) - (x^2 - y^2)(2x)}{(x^2 + y^2)^2} = \dfrac{4xy^2}{(x^2 + y^2)^2}$
and $\dfrac{\partial f}{\partial y} = \dfrac{(x^2 + y^2)(-2y) - (x^2 - y^2)(2y)}{(x^2 + y^2)^2} = \dfrac{-4x^2y}{(x^2 + y^2)^2}$.

**6.** $f(x,y) = \ln(x^2 + y^2)$, so $\dfrac{\partial f}{\partial x} = \dfrac{1}{x^2 + y^2}(2x) = \dfrac{2x}{x^2 + y^2}$ and $\dfrac{\partial f}{\partial y} = \dfrac{2y}{x^2 + y^2}$.

**7.** $f(x,y) = \cos x^3y$, so $\dfrac{\partial f}{\partial x} = (-\sin x^3y)(3yx^2) = -3x^2y\sin x^3y$ and $\dfrac{\partial f}{\partial y} = -x^3\sin x^3y$.

**8.** $f(x,y) = \ln(x/y)$, so $\dfrac{\partial f}{\partial x} = \dfrac{1}{x/y}\cdot\dfrac{1}{y} = \dfrac{1}{x}$ and $\dfrac{\partial f}{\partial y} = \dfrac{1}{x/y}\left(-\dfrac{x}{y^2}\right) = -\dfrac{1}{y}$.

**9.** $f(x,y) = xe^y + y\sin(x^2 + y)$, so $\partial f/\partial x = e^y + 2xy\cos(x^2 + y)$ and $\partial f/\partial y = xe^y + \sin(x^2 + y) + y\cos(x^2 + y)$.

**10.** $F(x,y,z) = x + 3y - 2z$, so $\partial F/\partial x = 1$, $\partial F/\partial y = 3$, and $\partial F/\partial z = -2$.

**11.** $F(x,y,z) = \dfrac{x - y}{y + z}$, so $\dfrac{\partial F}{\partial x} = \dfrac{1}{y + z}$,

$$\frac{\partial F}{\partial y} = \frac{(y + z)(-1) - (x - y)(1)}{(y + z)^2} = -\frac{x + z}{(y + z)^2},$$

and

$$\frac{\partial F}{\partial z} = \frac{(y + z)(0) - (x - y)(1)}{(y + z)^2} = \frac{y - x}{(y + z)^2}.$$

**12.** $F(x,y,z) = xyz$, so $\partial F/\partial x = yz$, $\partial F/\partial y = xz$, and $\partial F/\partial z = xy$.

**13.** $F(x,y,z) = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{1/2}$. The partial derivatives are:

$$\frac{\partial F}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}},$$

$$\frac{\partial F}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}} \quad \text{and,}$$

$$\frac{\partial F}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

**14.** $F(x,y,z) = e^{ax}\cos by + e^{az}\sin bx$ so

$$\frac{\partial F}{\partial x} = ae^{ax}\cos by + be^{az}\cos bx,$$

$$\frac{\partial F}{\partial y} = -be^{ax}\sin by, \quad \text{and}$$

$$\frac{\partial F}{\partial z} = ae^{az}\sin bx.$$

**15.** $F(x,y,z) = \dfrac{x+y+z}{(1+x^2+y^2+z^2)^{3/2}}$

$$F_x(x,y,z) = \frac{(1+x^2+y^2+z^2)^{3/2} - (x+y+z)(3/2)(1+x^2+y^2+z^2)^{1/2}(2x)}{(1+x^2+y^2+z^2)^3}$$
$$= \frac{1-2x^2+y^2+z^2-3xy-3xz}{(1+x^2+y^2+z^2)^{5/2}}$$
$$F_y(x,y,z) = \frac{1+x^2-2y^2+z^2-3xy-3yz}{(1+x^2+y^2+z^2)^{5/2}}, \text{ and}$$
$$F_z(x,y,z) = \frac{1+x^2+y^2-2z^2-3xz-3yz}{(1+x^2+y^2+z^2)^{5/2}}.$$

**16.** $F(x,y,z) = \sin x^2y^3z^4$ so this is similar to Exercise 7 above. $F_x(x,y,z) = 2xy^3z^4\cos x^2y^3z^4$, $F_y(x,y,z) = 3x^2y^2z^4\cos x^2y^3z^4$ and $F_z(x,y,z) = 4x^2y^3z^3\cos x^2y^3z^4$.

**17.** $F(x,y,z) = \dfrac{x^3+yz}{(x^2+z^2+1)}$ We've seen this form a couple of times by now.

$$F_x(x,y,z) = \frac{(x^2+z^2+1)(3x^2)-(x^3+yz)(2x)}{(x^2+z^2+1)^2} = \frac{x^4+3x^2z^2+3x^2-2xyz}{(x^2+z^2+1)^2}$$
$$F_y(x,y,z) = \frac{(x^2+z^2+1)(z)-(x^3+yz)(0)}{(x^2+z^2+1)^2} = \frac{z}{x^2+z^2+1}$$
$$F_z(x,y,z) = \frac{(x^2+z^2+1)(y)-(x^3+yz)(2z)}{(x^2+z^2+1)^2} = \frac{x^2y-yz^2+y-2x^3z}{(x^2+z^2+1)^2}$$

*The gradient of f is the function* $(f_x(x,y,z), f_y(x,y,z), f_z(x,y,z))$. *In Exercises 18–25 we are evaluating the gradient at a given point*.

**18.** $f(x,y) = x^2y + e^{y/x}$, so $\nabla f(x,y) = (2xy + (-y/x^2)e^{y/x}, x^2 + (1/x)e^{y/x})$. This means that $\nabla f(1,0) = (0,2)$.

**19.** $f(x,y) = \dfrac{x-y}{x^2+y^2+1}$, so

$$\nabla f(x,y) = \left(\frac{(x^2+y^2+1)(1)-(x-y)(2x)}{(x^2+y^2+1)^2}, \frac{(x^2+y^2+1)(-1)-(x-y)(2y)}{(x^2+y^2+1)^2}\right)$$
$$= \left(\frac{-x^2+y^2+1+2xy}{(x^2+y^2+1)^2}, \frac{-x^2+y^2-1-2xy}{(x^2+y^2+1)^2}\right).$$

So

$$\nabla f(2,-1) = \left(-\frac{6}{36}, \frac{0}{36}\right) = \left(-\frac{1}{6}, 0\right).$$

**20.** $f(x,y,z) = \sin xyz$, so $\nabla f(x,y,z) = (\cos xyz)(yz, xz, xy)$. This means that

$$\nabla f(\pi, 0, \pi/2) = \cos 0(0, \pi^2/2, 0) = (0, \pi^2/2, 0).$$

**21.** $f(x,y,z) = xy + y\cos z - x\sin yz$, so $\nabla f(x,y,z) = (y - \sin yz, x + \cos z - xz\cos yz, -y\sin z - xy\cos yz)$. So,

$$\nabla f(2,-1,\pi) = (-1-\sin(-\pi), 2+\cos(\pi) - 2(\pi)\cos(-\pi), \sin(\pi) + 2\cos(-\pi))$$
$$= (-1, 1+2\pi, -2).$$

**22.** $f(x,y) = e^{xy} + \ln(x-y)$, so $\nabla f(x,y) = (ye^{xy} + 1/(x-y), xe^{xy} - 1/(x-y))$. This means that $\nabla f(2,1) = (e^2+1, 2e^2-1)$.

**23.** $f(x,y,z) = (x+y)e^{-z}$, so $\nabla f(x,y,z) = (e^{-z}, e^{-z}, -(x+y)e^{-z})$. So, $\nabla f(3,-1,0) = (1,1,-2)$.

**24.** $f(x,y,z) = \cos z\ln(x+y^2)$, so $\nabla f(x,y,z) = (1/(x+y^2), 2y/(x+y^2), -\sin z\ln(x+y^2))$. Hence $\nabla f(e, 0, \pi/4) = (1/e, 0, -1/\sqrt{2})$.

**25.** $f(x,y,z) = \dfrac{xy^2 - x^2z}{y^2+z^2+1}$, so we have $\dfrac{\partial f}{\partial x} = \dfrac{y^2 - 2xz}{y^2+z^2+1}$ and the quotient rule applied appropriately gives

$$\frac{\partial f}{\partial x} = \frac{(y^2+z^2+1)(2xy) - (xy^2 - x^2z)(2y)}{(y^2+z^2+1)^2} = \frac{2xy(xz+z^2+1)}{(y^2+z^2+1)^2}$$

and

$$\frac{\partial f}{\partial x} = \frac{(y^2+z^2+1)(-x^2) - (xy^2 - x^2z)(2z)}{(y^2+z^2+1)^2} = \frac{x(xz^2 - xy^2 - 2y^2z - x)}{(y^2+z^2+1)^2}.$$

Therefore, $\nabla f(-1,2,1) = (1, -1/9, 1/9)$.

*The nth row of the derivative matrix is the gradient of the nth component function.*

**26.** $f(x,y) = \frac{x}{y}, Df(x,y) = \left[\frac{1}{y}, \frac{-x}{y^2}\right]$. So $Df(3,2) = [1/2, -3/4]$.

**27.** $f(x,y,z) = x^2 + x\ln(yz)$, so $Df(x,y,z) = \begin{bmatrix} 2x+\ln(yz) & x/y & x/z\end{bmatrix}$ and thus $Df(-3,e,e) = \begin{bmatrix} -4 & -3/e & -3/e\end{bmatrix}$.

**28.** $\mathbf{f}(x,y,z) = \big(2x - 3y + 5z, x^2 + y, \ln(yz)\big)$, so $D\mathbf{f}(x,y,z) = \begin{bmatrix} 2 & -3 & 5 \\ 2x & 1 & 0 \\ 0 & 1/y & 1/z \end{bmatrix}$. Hence

$$D\mathbf{f}(3,-1,-2) = \begin{bmatrix} 2 & -3 & 5 \\ -6 & 1 & 0 \\ 0 & -1 & -1/2 \end{bmatrix}.$$

**29.** $\mathbf{f}(x,y,z) = (xyz, \sqrt{x^2+y^2+z^2})$, so

$$D\mathbf{f}(x,y,z) = \begin{bmatrix} yz & xz & xy \\ x/\sqrt{x^2+y^2+z^2} & y/\sqrt{x^2+y^2+z^2} & z/\sqrt{x^2+y^2+z^2} \end{bmatrix}.$$

This means,

$$D\mathbf{f}(1,0,-2) = \begin{bmatrix} 0 & -2 & 0 \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}.$$

**30.** $\mathbf{f}(t) = (t, \cos 2t, \sin 5t)$, so

$$D\mathbf{f}(t) = \begin{bmatrix} 1 \\ -2\sin 2t \\ 5\cos 5t \end{bmatrix} \quad \text{and so} \quad D\mathbf{f}(0) = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}.$$

**31.** $\mathbf{f}(x,y,z,w) = (3x - 7y + z, 5x + 2z - 8w, y - 17z + 3w)$ so

$$D\mathbf{f}(x,y,z,w) = \begin{bmatrix} 3 & -7 & 1 & 0 \\ 5 & 0 & 2 & -8 \\ 0 & 1 & -17 & 3 \end{bmatrix}.$$

Since all of the entries are constant, the matrix doesn't depend on **a**.

**32.** $\mathbf{f}(x,y) = (x^2y, x+y^2, \cos\pi xy)$, so

$$D\mathbf{f}(x,y) = \begin{bmatrix} 2xy & x^2 \\ 1 & 2y \\ -\pi y\sin\pi xy & -\pi x\sin\pi xy \end{bmatrix}.$$

This means,

$$D\mathbf{f}(2,-1) = \begin{bmatrix} -4 & 4 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}.$$

**33.** $\mathbf{f}(s,t) = (s^2, st, t^2)$, so

$$D\mathbf{f}(s,t) = \begin{bmatrix} 2s & 0 \\ t & s \\ 0 & 2t \end{bmatrix}.$$

This means,

$$D\mathbf{f}(-1,1)=\begin{bmatrix}-2 & 0\\ 1 & -1\\ 0 & 2\end{bmatrix}.$$

*We will appeal to Theorem 3.5 for Exercises 34–36.*

**34.** $f(x,y)=xy-7x^8y^2+\cos x$ is differentiable because the two partials $f_x(x,y)=y-56x^7y^2-\sin x$ and $f_y(x,y)=x-14x^8y$ are continuous.

**35.** $f(x,y,z)=\dfrac{x+y+z}{x^2+y^2+z^2}$ is differentiable because the three partials

$$f_x(x,y,z)=\frac{-x^2+y^2+z^2-2xy-2xz}{(x^2+y^2+z^2)^2}$$

$$f_y(x,y,z)=\frac{x^2-y^2+z^2-2xy-2yz}{(x^2+y^2+z^2)^2}$$

$$f_z(x,y,z)=\frac{x^2+y^2-z^2-2xz-2yz}{(x^2+y^2+z^2)^2}$$

are all continuous.

**36.** $\mathbf{f}(x,y)=\left(\dfrac{xy^2}{x^2+y^4},\dfrac{x}{y}+\dfrac{y}{x}\right)$ is differentiable because the partials in the matrix

$$D\mathbf{f}(x,y)=\begin{bmatrix}\dfrac{y^6-x^2y^2}{(x^2+y^4)^2} & \dfrac{2x^3y-2xy^5}{(x^2+y^4)^2}\\ \dfrac{1}{y}-\dfrac{y}{x^2} & \dfrac{-x}{y^2}+\dfrac{1}{x}\end{bmatrix}$$

are continuous in the domain of **f**.

**37.** **(a)** The graph of $z=x^3-7xy+e^y$ has continuous partial derivatives at $(-1,0,0)$.

**(b)** By Theorem 3.3, the equation for the tangent plane is: $z=f(-1,0)+f_x(-1,0)(x-(-1))+f_y(-1,0)(y-0)$. In this case $f_x(x,y)=3x^2-7y$ so $f_x(-1,0)=3$. Also $f_y(x,y)=-7x+e^y$ and so $f_y(-1,0)=8$. The equation of the plane is $z=3(x+1)+8y$.

**38.** Again using Theorem 3.3, the equation for the tangent plane is: $z=f(\pi/3,1)+f_x(\pi/3,1)(x-\pi/3)+f_y(\pi/3,1)(y-1)$. Here $z=4\cos xy$, so $f_x(x,y)=-4y\sin xy$ and $f_y(x,y)=-4x\sin xy$. Plugging in we get $z=2-2\sqrt{3}(x-\pi/3)-(2\pi/\sqrt{3})(y-1)$.

**39.** Again using Theorem 3.3, the equation for the tangent plane is: $z=f(0,1)+f_x(0,1)(x)+f_y(0,1)(y-1)$. Here $z=e^{x+y}\cos xy$, so $f_x(x,y)=e^{x+y}(\cos xy-y\sin xy)$ and $f_y(x,y)=e^{x+y}(\cos xy-x\sin xy)$. Plugging in we get $z=e+ex+e(y-1)$ or $z=ex+ey$.

**40.** First find the two partials $f_x(x,y)=2x-6$ and $f_y(x,y)=3y^2$. Then putting the tangent plane equation into the same form as the plane $4x-12y+z=7$ gives us $z-(2a-6)(x-a)-(3b^2)(y-b)=a^2-6a+b^3$ or $z-(2a-6)x-3b^2y=-a^2-2b^3$. So $2a-6=-4$ so $a=1$ and $3b^2=12$ so $b=\pm 2$. This gives two tangent planes. The equation for one is $4x-12y+z=-17$ and the equation for the other is $4x-12y+z=15$.

**41.** For $f(x_1,\ldots,x_4)=10-(x_1^2+3x_2^2+2x_3^2+x_4^2)$, we have

$$\nabla f=(-2x_1,-6x_2,-4x_3,-2x_4)\quad\text{so}\quad \nabla f(2,-1,1,3)=(-4,6,-4,-6).$$

Formula (8) gives that the hyperplane has equation

$$\begin{aligned}x_5&=-8+(-4,6,-4,-6)(x_1-2,x_2+1,x_3-1,x_4-3)\\&=-8-4(x_1-2)+6(x_2+1)-4(x_3-1)-6(x_4-3)\end{aligned}$$

or

$$x_5=-4x_1+6x_2-4x_3-6x_4+28.$$

**42. (a)**

$$f_x(2,3) \approx \frac{f(1.98,3) - f(2,3)}{1.98 - 2} = \frac{12.1 - 12}{-.02} = \frac{.1}{-.02} = -5$$
$$f_y(2,3) \approx \frac{f(2,3.01) - f(2,3)}{3.01 - 3} = \frac{12.2 - 12}{0.01} = \frac{.2}{.01} = 20$$

Thus, formula (4) of §2.3 would give an approximate equation for the tangent plane as

$$z = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3) \approx 12 - 5(x-2) + 20(y-3)$$

or

$$z = -5x + 20y - 8.$$

**(b)**

$$\begin{aligned} f(1.98, 2.98) &\approx 12 - 5(1.98 - 2) + 20(2.98 - 3) = 12 - 5(-0.02) + 20(-0.02) \\ &= 11.7 \end{aligned}$$

*Exercises 43–45 have the student investigate the linear approximation $h$ of $f$ near a given point $a$. We use the formula in Definition 3.8:*

$$\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + D\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a}).$$

**43.** Here $f(x,y) = e^{x+y}$ so the partials are $f_x(x,y) = e^{x+y} = f_y(x,y)$.
**(a)** $h(.1, -.1) = f(0,0) + (e^0, e^0) \cdot (.1, -.1) = 1$.
**(b)** $f(.1, -.1) = e^0 = 1$. So the approximation is exact.

**44.** Here $f(x,y) = 3 + \cos \pi xy$ so the partials are $f_x(x,y) = -\pi y \sin \pi xy$ and $f_y(x,y) = -\pi x \sin \pi xy$.
**(a)** $h(.98, .51) = 3 + \cos \pi(1)(.5) - (\pi(.5)\sin[\pi(1)(.5)], \pi(1)\sin[\pi(1)(.5)]) \cdot (-.02, .01) = 3 - \pi(.5, 1) \cdot (-.02, .01) = 3$.
**(b)** $f(.98, .51) = 3 + \cos \pi(.98)(.51) \approx 3.00062832$.

**45.** $f(x,y,z) = x^2 + xyz + y^3 z$, so the partials are $f_x(x,y,z) = 2x + yz$, $f_y(x,y,z) = xz + 3y^2 z$, and $f_z(x,y,z) = xy + y^3$.
**(a)** $h(1.01, 1.95, 2.2) = f(1,2,2) + (f_x(1,2,2), f_y(1,2,2), f_z(1,2,2)) \cdot (.01, -.05, .2) = 21 + (6, 26, 10) \cdot (.01, -.05, .2) = 21.76$.
**(b)** $f(1.01, 1.95, 2.2) = 21.665725$.

**46.**

$$f(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}, \text{ so}$$
$$\begin{aligned} f_{x_i}(x_1, x_2, \ldots, x_n) &= \frac{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} - x_i(x_1 + x_2 + \cdots + x_n)(x_1^2 + x_2^2 + \cdots + x_n^2)^{-1/2}}{x_1^2 + x_2^2 + \cdots + x_n^2} \\ &= \frac{x_1^2 + x_2^2 + \cdots + x_n^2 - x_i(x_1 + x_2 + \cdots + x_n)}{(x_1^2 + x_2^2 + \cdots + x_n^2)^{3/2}}. \end{aligned}$$

**47. (a)** For $(x,y) \neq (0,0)$ we can find a neighborhood that misses the origin. In this neighborhood

$$f(x,y) = \frac{xy^2 - x^2 y + 3x^3 - y^3}{x^2 + y^2} = x - y + \frac{2x^3}{x^2 + y^2}.$$

We can then easily compute the partials as

$$f_x(x,y) = 1 + \frac{2x^4 + 6x^2 y^2}{(x^2 + y^2)^2} \quad \text{and} \quad f_y(x,y) = -1 - \frac{4x^3 y}{(x^2 + y^2)^2}.$$

**(b)** Using Definition 3.2 of the partial derivative, if

$$f(x,y) = \begin{cases} x - y + \dfrac{2x^3}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases},$$

then

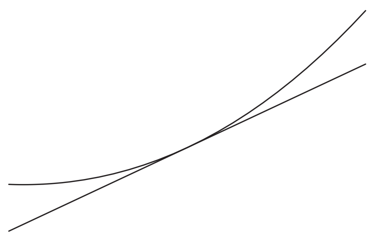
$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{3h}{h} = 3,$$

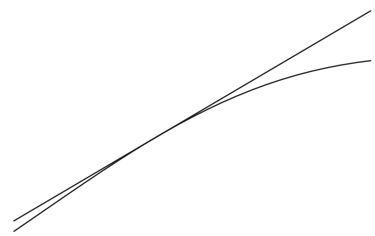
and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{-h}{h} = -1.$$

*Note: Exercises 48–51 are review exercises for single-variable calculus. The idea is to see that near a point, the tangent line approximates the curve. This idea will then be extended to a tangent plane and a surface in Exercises 53–57. For Exercises 48–51 use either the point-slope equation $y - f(a) = f'(a)(x-a)$ or solve for y to get $y = f'(a)x + f(a) - f'(a)a$.*

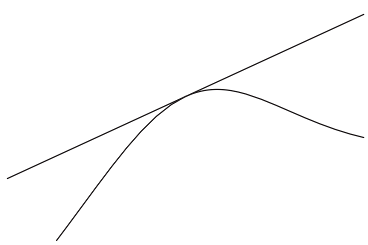
**48.** For the tangent line to $F(x) = x^3 - 2x + 3$ at $a = 1$ $F'(x) = 3x^2 - 2$ so $F'(1) = 1$. The tangent line is $y = x + 1$. The graph of $F$ and the tangent line near $x = 1$ (in this case for $.8 \le x \le 1.2$) is shown below left.

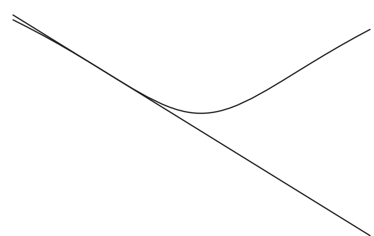




**49.** For the tangent line to $F(x) = x + \sin x$ at $a = \pi/4$ $F'(x) = 1 + \cos x$ so $F'(\pi/4) = 1 + \sqrt{2}/2$. The tangent line is $y = (1 + \sqrt{2}/2)x + (\pi/4 + \sqrt{2}/2 - (1 + \sqrt{2}/2)\pi/4)$. The graph of $F$ and the tangent line near $x = \pi/4$ is shown above right.

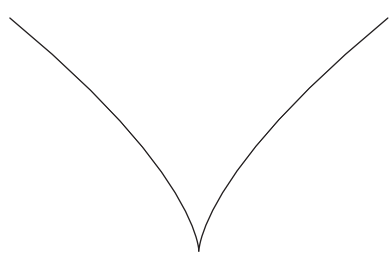
**50.** For the tangent line rewrite $F(x) = x - 3 + 3/(x^2+1)$. $F'(x) = 1 - 6x/(x^2+1)^2$ so $F'(0) = 1$ and $F(0) = 0$. The tangent line is $y = x$. We can see that by looking at our rewritten version of $F$. The graph of $F$ and the tangent line near $x = 0$ is shown below left.





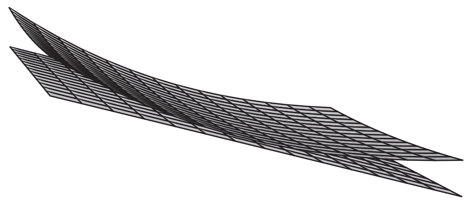
**51.** For the tangent line to $F(x) = \ln(x^2+1)$ at $a = -1$, $F'(x) = 2x/(x^2+1)$ so $F'(-1) = -1$. The tangent line is $y = -x + \ln\,2 - 1$. The graph of $F$ and the tangent line near $x = -1$ is shown above right.

**52.** Looking at the graph below, we can see that there is a cusp at $x = 2$ (trust me, that's where the cusp is). You can also see that the limit of the derivative using points to the left of 2 would not be the same as the derivative using points to the right of 2 as one set is negative and the other is positive. Finally, the tangent line looks to be a vertical line. This has no slope and so the derivative wouldn't exist.



**53. (a)** For the function $f(x,y) = x^3 - xy + y^2$, $f_x(x,y) = 3x^2 - y$ and $f_y(x,y) = -x + 2y$. So at the point (2, 1) these become $f(2,1) = 7$, $f_x(2,1) = 11$, and $f_y(2,1) = 0$. The equation of the tangent plane is $z = 7 + 11(x-2)$.

**(b)**



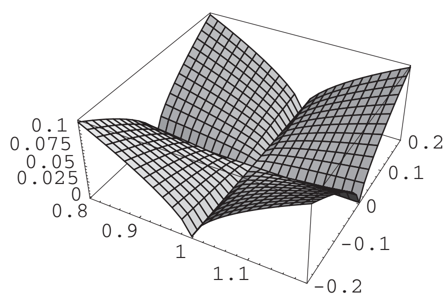
**(c)** The partials are continuous so by Theorem 3.5, $f$ is differentiable.

**54. (a)** To find the partial derivatives $f_x(1,0)$ and $f_y(1,0)$, we must look at appropriate partial functions of $f(x,y) = ((x-1)y)^{2/3}$:

$$f(x,0) \equiv 0 \Rightarrow f_x(1,0) = 0$$
$$f(1,y) \equiv 0 \Rightarrow f_y(1,0) = 0$$

Since $f(1,0) = 0$, the candidate tangent plane has equation $z = 0 + 0(x-1) + 0(y-0)$ or $z = 0$.

**(b)** A computer graph looks as follows.




Zooming in closer to the point (1, 0, 0) doesn't make things appear very different, so it's tempting to conclude that $f$ must not be differentiable at (1, 0).

**(c)** From our calculations in part (a), the linear function $h(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) = 0$. Thus, for $(x,y) \neq (1,0)$ we have

$$0 \leq \frac{|f(x,y) - h(x,y)|}{\|(x,y) - (1,0)\|} = \frac{|f(x,y)|}{\sqrt{(x-1)^2 + y^2}}.$$

Now

$$|f(x,y)| = |x-1|^{2/3}|y|^{2/3} \leq ((x-1)^2 + y^2)^{1/3}((x-1)^2 + y^2)^{1/3}$$
$$= ((x-1)^2 + y^2)^{2/3}.$$

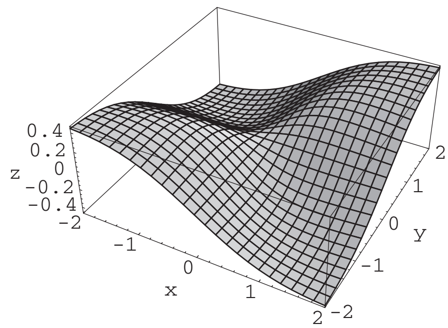
Thus

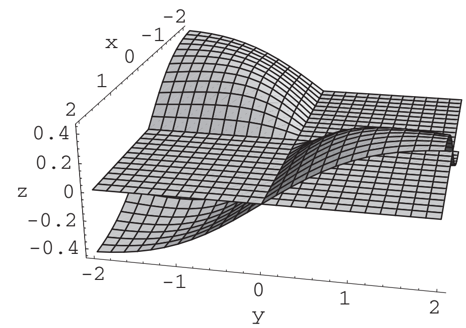
$$\frac{|f(x,y)|}{\sqrt{(x-1)^2 + y^2}} \leq \frac{((x-1)^2 + y^2)^{2/3}}{((x-1)^2 + y^2)^{1/2}} = ((x-1)^2 + y^2)^{1/6}.$$

Since this last expression approaches zero as $(x,y) \to (1,0)$, we see that $f$ must be differentiable at (1, 0) by Definition 3.4.

**55. (a)** For the function $f(x,y) = \dfrac{xy}{x^2 + y^2 + 1}$, $f_x(x,y) = \dfrac{-x^2y + y^3 + y}{(x^2 + y^2 + 1)^2}$ and $f_y(x,y) = \dfrac{x^3 - xy^2 + y}{(x^2 + y^2 + 1)^2}$. So at the point (0, 0) these become $f(0,0) = 0$, $f_x(0,0) = 0$, and $f_y(0,0) = 0$. The equation of the tangent plane is $z = 0$.

**(b)** The surface is shown below left. It is shown with the tangent plane below right.

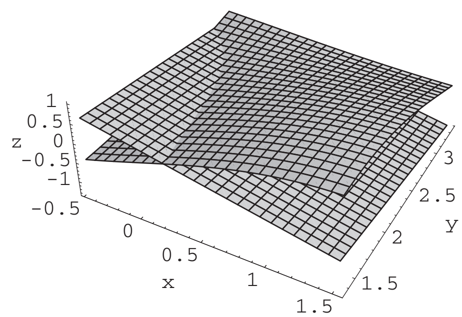



**(c)** This is the plane that best approximates the surface at that point. But we can see that it's not a very good approximation as you move way in any direction other than the two axes lines. Analytically, the reason is that the partials are continuous in a neighborhood of (0, 0).

**56.** **(a)** For the function $f(x, y) = \sin x \cos y$, $f_x(x, y) = \cos x \cos y$ and $f_y(x, y) = -\sin x \sin y$. So at the point $(\pi/6, 3\pi/4)$ these become $f(\pi/6, 3\pi/4) = -\sqrt{2}/4$, $f_x(\pi/6, 3\pi/4) = -\sqrt{6}/4$, and $f_y(\pi/6, 3\pi/4) = -\sqrt{2}/4$. The equation of the tangent plane is $z = -\sqrt{2}/4 - \sqrt{6}/4(x - \pi/6) - \sqrt{2}/4(y - 3\pi/4)$.

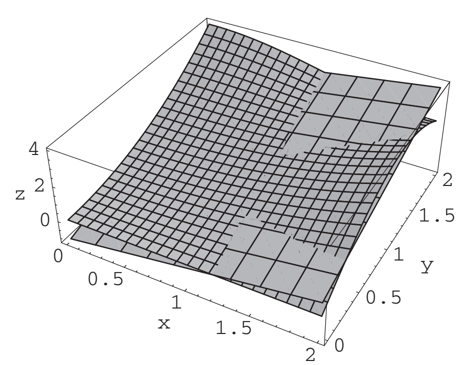
**(b)**




**(c)** Again the partials are continuous in a neighborhood of $(\pi/6, 3\pi/4)$ so by Theorem 3.5, $f$ is differentiable at the point.

**57.** **(a)** For the function $f(x, y) = x^2 \sin y + y^2 \cos x$, $f_x(x, y) = 2x \sin y - y^2 \sin x$ and $f_y(x, y) = x^2 \cos y + 2y \cos x$. So at the point $(\pi/3, \pi/4)$ these become $f(\pi/3, \pi/4) = \pi^2\sqrt{2}/18 + \pi^2/32$, $f_x(\pi/3, \pi/4) = \pi\sqrt{2}/3 - \pi^2\sqrt{3}/32$, and $f_y(\pi/3, \pi/4) = \pi^2\sqrt{2}/18 + \pi/4$. The equation of the tangent plane is $z = (\pi^2\sqrt{2}/18 + \pi^2/32) + (\pi\sqrt{2}/3 - \pi^2\sqrt{3}/32)(x - \pi/3) + (\pi^2\sqrt{2}/18 + \pi/4)(y - \pi/4)$.

**(b)**




**(c)** The partials are continuous near $(\pi/3, \pi/4)$ so by Theorem 3.5, $f$ is differentiable there.

**58.** **(a)** Yes $g(x, y) = (xy)^{1/3}$ is continuous at (0, 0).

**(b)** $\partial g/\partial x = (1/3)x^{-2/3}y^{1/3}$, and $\partial g/\partial y = (1/3)x^{1/3}y^{-2/3}$.

**(c)** Unfortunately we can't just substitute the point (0, 0) in our answers to (b), but using Definition 3.2 of partial derivatives, we see that the two partials must be 0. In other words we define $g_x(0, 0) = 0$, and $g_y(0, 0) = 0$.

**(d)** No (choose a path that crosses the $x$- and $y$-axes).

**(e)** You can see this answer if you look along the line $y = x$. There $g(x, x) = x^{2/3}$ which has a corner at (0, 0). So there can't be a tangent plane.

**(f)** No $g$ isn't differentiable at (0, 0).

**59.** If $\mathbf{f}(\mathbf{x}) = A\mathbf{x} = \left(\sum_{k=1}^{n} a_{1k}x_k, \sum_{k=1}^{n} a_{2k}x_k, \ldots, \sum_{k=1}^{n} a_{mk}x_k\right)$. Let's look at the entry in row $i$ column $j$ of $D\mathbf{f}(\mathbf{x})$. This will be

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\sum_{k=1}^{n} a_{ik}x_k\right) = a_{ij}.$$

So $D\mathbf{f}(\mathbf{x}) = A$.

**60.** By Theorem 2.6, if $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}) = \mathbf{0}$, then $\lim_{\mathbf{x}\to\mathbf{a}} F_i(\mathbf{x}) = 0$ for each component function $F_i$ of $\mathbf{F}$. Hence $\lim_{\mathbf{x}\to\mathbf{a}} \|\mathbf{F}(\mathbf{x})\| = 0$.

Conversely, assume that $\lim_{\mathbf{x}\to\mathbf{a}} \|\mathbf{F}(\mathbf{x})\| = 0$. This means that, given any $\epsilon > 0$, we can find an appropriate $\delta > 0$ such that if $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$, then $|\|\mathbf{F}(\mathbf{x})\| - 0| < 0$. But note that

$$|F_i(\mathbf{x})| \leq \sqrt{F_1(\mathbf{x})^2 + F_2(\mathbf{x})^2 + \cdots + F_m(\mathbf{x})^2} = |\|\mathbf{F}(\mathbf{x})\| - 0|.$$

Hence if $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$, then $|F_i(\mathbf{x}) - 0| < \epsilon$, so that $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}) = \mathbf{0}$.

**61. (a)** First, Exercise 60 shows that

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - [\mathbf{f}(\mathbf{a}) + A(\mathbf{x} - \mathbf{a})]}{\|\mathbf{x} - \mathbf{a}\|} = \lim_{\mathbf{x}\to\mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - [\mathbf{f}(\mathbf{a}) + B(\mathbf{x} - \mathbf{a})]}{\|\mathbf{x} - \mathbf{a}\|}.$$

Subtracting these limits we have

$$\begin{aligned}\mathbf{0} &= \lim_{\mathbf{x}\to\mathbf{a}} \left(\frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - A(\mathbf{x} - \mathbf{a})}{\|\mathbf{x} - \mathbf{a}\|} - \frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - B(\mathbf{x} - \mathbf{a})}{\|\mathbf{x} - \mathbf{a}\|}\right) \\ &= \lim_{\mathbf{x}\to\mathbf{a}} \frac{(B - A)(\mathbf{x} - \mathbf{a})}{\|\mathbf{x} - \mathbf{a}\|}.\end{aligned}$$

**(b)** When taking the limit, it's possible to have $\mathbf{x} \to \mathbf{a}$ in a completely arbitrary manner. But one way to have $\mathbf{x} \to \mathbf{a}$ is along a straight-line path, which may be described as $\mathbf{x} = \mathbf{a} + t\mathbf{h}$. For such paths, having $\mathbf{x} \to \mathbf{a}$ is achieved by letting $t \to 0$. Thus if we know that

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{(B - A)(\mathbf{x} - \mathbf{a})}{\|\mathbf{x} - \mathbf{a}\|} = \mathbf{0},$$

then it must follow that

$$\lim_{t\to 0} \frac{(B - A)(t\mathbf{h})}{\|t\mathbf{h}\|} = \mathbf{0}.$$

(Note: The converse need *not* be true.) It follows that we must have a consistent one-sided limit; hence

$$\lim_{t\to 0^+} \frac{(B - A)(t\mathbf{h})}{\|t\mathbf{h}\|} = \mathbf{0}.$$

Now, for $t > 0$, we have

$$\frac{(B - A)(t\mathbf{h})}{\|t\mathbf{h}\|} = \frac{(B - A)\mathbf{h}}{\|\mathbf{h}\|}.$$

Thus if

$$\lim_{t\to 0^+} \frac{(B - A)(t\mathbf{h})}{\|t\mathbf{h}\|} = \lim_{t\to 0^+} \frac{(B - A)\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0},$$

it must be the case that $(B - A)\mathbf{h} = \mathbf{0}$. Moreover, this must be true for *any* nonzero vector $\mathbf{h} \in \mathbf{R}^n$. By setting $\mathbf{h}$ in turn equal to the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$, we conclude that $B - A$ must be the zero matrix. Similarly, we must also have

$$\lim_{t\to 0^-} \frac{(B - A)(t\mathbf{h})}{\|t\mathbf{h}\|} = \mathbf{0}.$$

For $t < 0$, we have

$$\frac{(B - A)(t\mathbf{h})}{\|t\mathbf{h}\|} = \frac{(B - A)(t\mathbf{h})}{|t|\|\mathbf{h}\|} = -\frac{(B - A)\mathbf{h}}{\|\mathbf{h}\|}.$$

Thus if

$$\lim_{t\to 0^-} \frac{(B - A)(t\mathbf{h})}{\|t\mathbf{h}\|} = \lim_{t\to 0^-} -\frac{(B - A)\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0},$$

again it must be the case that $(B - A)\mathbf{h} = \mathbf{0}$. Hence $B - A$ must be the zero matrix.

**62. (a)** In the fraction that defines the function $\mathbf{F}$, the denominator $\|\mathbf{x}-\mathbf{a}\|$ is already a scalar-valued expression. Thus

$$F_i(\mathbf{x}) = \frac{f_i(\mathbf{x}) - f_i(\mathbf{a}) - (\text{row } i \text{ of } A)(\mathbf{x}-\mathbf{a})}{\|\mathbf{x}-\mathbf{a}\|}.$$

**(b)** By Theorem 2.6, since $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}) = \mathbf{0}$, we must have $\lim_{\mathbf{x}\to\mathbf{a}} F_i(\mathbf{x}) = 0$ for $i = 1, \ldots, m$ as well. Since the latter limit is known and is zero, it must be that the same limit is attained by letting $\mathbf{x}$ approach $\mathbf{a}$ along any straight-line path. We may describe staight-line paths that run parallel to the coordinate axes by $\mathbf{x} = \mathbf{a} + h\mathbf{e}_j$, where $\mathbf{e}_j$ is a standard basis vector for $\mathbf{R}^n$. Thus if $\lim_{\mathbf{x}\to\mathbf{a}} F_i(\mathbf{x}) = 0$, then $\lim_{h\to 0} F_i(\mathbf{a}+h\mathbf{e}_j) = 0$.

Now we determine $\lim_{h\to 0} F_i(\mathbf{a}+h\mathbf{e}_j)$. Using the description of the component function $F_i$ from part (a), we have

$$\begin{aligned}
0 &= \lim_{h\to 0} \frac{f_i(\mathbf{a}+h\mathbf{e}_j) - f_i(\mathbf{a}) - (\text{row } i \text{ of } A)(h\mathbf{e}_j)}{\|h\mathbf{e}_j\|} \\
&= \lim_{h\to 0} \left[\frac{f_i(\mathbf{a}+h\mathbf{e}_j) - f_i(\mathbf{a})}{\|h\mathbf{e}_j\|} - \frac{(\text{row } i \text{ of } A)(h\mathbf{e}_j)}{\|h\mathbf{e}_j\|}\right] \\
&= \lim_{h\to 0} \left[\frac{f_i(\mathbf{a}+h\mathbf{e}_j) - f_i(\mathbf{a})}{|h|\|\mathbf{e}_j\|} - \frac{(\text{row } i \text{ of } A)(h\mathbf{e}_j)}{|h|\|\mathbf{e}_j\|}\right] \\
&= \lim_{h\to 0} \left[\frac{f_i(\mathbf{a}+h\mathbf{e}_j) - f_i(\mathbf{a})}{|h|} - \frac{(\text{row } i \text{ of } A)(h\mathbf{e}_j)}{|h|}\right],
\end{aligned}$$

since the standard basis vectors are all unit vectors. Now consider one-sided limits. Suppose first that $h > 0$. Then

$$\begin{aligned}
0 &= \lim_{h\to 0^+} \left[\frac{f_i(\mathbf{a}+h\mathbf{e}_j) - f_i(\mathbf{a})}{|h|} - \frac{(\text{row } i \text{ of } A)(h\mathbf{e}_j)}{|h|}\right] \\
&= \lim_{h\to 0^+} \left[\frac{f_i(\mathbf{a}+h\mathbf{e}_j) - f_i(\mathbf{a})}{h} - (\text{row } i \text{ of } A)\mathbf{e}_j\right] \\
&= \lim_{h\to 0^+} \frac{f_i(a_1, \ldots, a_j + h, \ldots, a_n) - f_i(a_1, \ldots, a_n)}{h} - a_{ij}.
\end{aligned}$$

Similarly, if $h < 0$,

$$\begin{aligned}
0 &= \lim_{h\to 0^-} \left[\frac{f_i(\mathbf{a}+h\mathbf{e}_j) - f_i(\mathbf{a})}{|h|} - \frac{(\text{row } i \text{ of } A)(h\mathbf{e}_j)}{|h|}\right] \\
&= \lim_{h\to 0^-} \left[\frac{f_i(\mathbf{a}+h\mathbf{e}_j) - f_i(\mathbf{a})}{-h} + (\text{row } i \text{ of } A)\mathbf{e}_j\right] \\
&= \lim_{h\to 0^-} -\frac{f_i(a_1, \ldots, a_j + h, \ldots, a_n) - f_i(a_1, \ldots, a_n)}{h} + a_{ij}.
\end{aligned}$$

Taking both cases together, we have shown that

$$\lim_{h\to 0} \frac{f_i(a_1, \ldots, a_j + h, \ldots, a_n) - f_i(a_1, \ldots, a_n)}{h} = a_{ij}.$$

This last limit is precisely the definition of the partial derivative. Hence we have shown that $a_{ij} = \dfrac{\partial f_i}{\partial x_j}(\mathbf{a})$, as desired.

## 2.4 Properties; Higher-Order Partial Derivatives

*In Exercises 1–4 there isn't much to show . . . the students just need to verify that the sum of the derivative is the derivative of the sum (Proposition 4.1).*

**1.** $f(x, y) = xy + \cos x$, and $g(x, y) = \sin(xy) + y^3$, so $Df = [y - \sin x, x]$, $Dg = [y\cos xy, x\cos xy + 3y^2]$, and $D(f+g) = [y - \sin x + y\cos xy, x + x\cos xy + 3y^2]$.

**2.** $\mathbf{f}(x, y) = (e^{x+y}, xe^y)$, and $\mathbf{g}(x, y) = (\ln(xy), ye^x)$, so

$$D\mathbf{f} = \begin{bmatrix} e^{x+y} & e^{x+y} \\ e^y & xe^y \end{bmatrix}, \quad D\mathbf{g} = \begin{bmatrix} \frac{y}{xy} & \frac{x}{xy} \\ ye^x & e^x \end{bmatrix}$$

and

$$D(\mathbf{f}+\mathbf{g}) = \begin{bmatrix} e^{x+y} + \frac{y}{xy} & e^{x+y} + \frac{x}{xy} \\ e^y + ye^x & xe^y + e^x \end{bmatrix}.$$

*Note the use of the product rule in Exercise 3 when calculating* $(g_1)_x$.

**3.** $\mathbf{f}(x,y,z) = (x\sin y + z, ye^x - 3x^2)$ and $\mathbf{g}(x,y,z) = (x^3\cos x, xyz)$, so

$$D\mathbf{f} = \begin{bmatrix} \sin y & x\cos y & 1 \\ -6x & e^z & ye^z \end{bmatrix}, D\mathbf{g} = \begin{bmatrix} 3x^2\cos x - x^3\sin x & 0 & 0 \\ yz & xz & xy \end{bmatrix} \quad \text{and}$$

$$D(\mathbf{f}+\mathbf{g}) = \begin{bmatrix} \sin y + 3x^2\cos x - x^3\sin x & x\cos y & 1 \\ -6x + yz & e^z + xz & ye^z + xy \end{bmatrix}.$$

**4.** $\mathbf{f}(x,y,z) = (xyz^2, xe^{-y}, y\sin xz)$ and $\mathbf{g}(x,y,z) = (x-y, x^2+y^2+z^2, \ln(xz+2))$, so

$$D\mathbf{f} = \begin{bmatrix} yz^2 & xz^2 & 2xyz \\ e^{-y} & -xe^{-y} & 0 \\ zy\cos xz & \sin xz & xy\cos xz \end{bmatrix}, D\mathbf{g} = \begin{bmatrix} 1 & -1 & 0 \\ 2x & 2y & 2z \\ z/(xz+2) & 0 & x/(xz+2) \end{bmatrix} \quad \text{and}$$

$$D(\mathbf{f}+\mathbf{g}) = \begin{bmatrix} 1 + yz^2 & -1 + xz^2 & 2xyz \\ e^{-y} + 2x & -xe^{-y} + 2y & 2z \\ zy\cos xz + z/(xz+2) & \sin xz & xy\cos xz + x/(xz+2) \end{bmatrix}.$$

*Exercises 5–8 are again mainly calculations to convince the students of the formulas given in Proposition 4.2; we hope that they remember to apply them when confronted with a product or quotient. In Exercises 6 and 7 we notice that we just get the quotient rule in each component which factors into the quotient rule given in the proposition (and we drop the argument when convenient and clear).*

**5.** $f(x,y) = x^2y + y^3, g(x,y) = x/y, f(x,y)g(x,y) = x^3 + xy^2$, and $\dfrac{f(x,y)}{g(x,y)} = xy^2 + y^4/x$.

$$\begin{aligned} \text{So} \quad Df &= [2xy, x^2+3y^2], \quad \text{and} \quad Dg = [1/y, -x/y^2], \\ D(fg) &= [3x^2+y^2, 2xy] \\ &= (x^2y+y^3)[1/y, -x/y^2] + (x/y)[2xy, x^2+3y^2] \\ &= fD(g) + gD(f), \text{ and} \\ D\left(\frac{f}{g}\right) &= [y^2 - y^4/x^2, 2xy + 4y^3/x] \\ &= (y/x)[2xy, x^2+3y^2] - (y^2/x^2)(x^2y+y^3)[1/y, -x/y^2] \\ &= \frac{gDf - fDg}{g^2}. \end{aligned}$$

**6.** $f(x,y) = e^{xy}, g(x,y) = x\sin 2y, f(x,y)g(x,y) = xe^{xy}\sin 2y$, and $\dfrac{f(x,y)}{g(x,y)} = \dfrac{e^{xy}}{x\sin 2y}$.

$$\begin{aligned} \text{So} \quad Df &= [ye^{xy}, xe^{xy}], \quad \text{and} \quad Dg = [\sin 2y, 2x\cos 2y], \\ D(fg) &= [\sin 2y(e^{xy} + xy\,e^{xy}), x(xe^{xy}\sin 2y + 2e^{xy}\cos 2y)] \\ &= e^{xy}[\sin 2y, 2x\cos 2y] + x\sin 2y[ye^{xy}, xe^{xy}] \\ &= fD(g) + gD(f), \text{and} \\ D\left(\frac{f}{g}\right) &= \left[\frac{xye^{xy}\sin 2y - e^{xy}\sin 2y}{x^2\sin^2 2y}, \frac{x^2e^{xy}\sin 2y - 2xe^{xy}\cos 2y}{x^2\sin^2 2y}\right] \\ &= \frac{x\sin 2y[ye^{xy}, xe^{xy}] - e^{xy}[\sin 2y, 2x\cos 2y]}{x^2\sin^2 2y} \\ &= \frac{gDf - fDg}{g^2}. \end{aligned}$$

**7.** $f(x,y) = 3xy + y^5$, $g(x,y) = x^3 - 2xy^2$, $f(x,y)g(x,y) = 3x^4y + x^3y^5 - 6x^2y^3 - 2xy^7$, and $\dfrac{f(x,y)}{g(x,y)} = \dfrac{3xy + y^5}{x^3 - 2xy^2}$. So

$$\begin{aligned}
Df &= [3y, 3x + 5y^4], \quad \text{and} \quad Dg = [3x^2 - 2y^2, -4xy], \\
D(fg) &= [12x^3y + 3x^2y^5 - 12xy^3 - 2y^7, 3x^4 + 5x^3y^4 - 18x^2y^2 - 14xy^6] \\
&= (3xy + y^5)[3x^2 - 2y^2, -4xy] + (x^3 - 2xy^2)[3y, 3x + 5y^4] \\
&= fD(g) + gD(f), \text{ and} \\
D\left(\frac{f}{g}\right) &= \left[\frac{g(x,y)f_x(x,y) - f(x,y)g_x(x,y)}{[g(x,y)]^2}, \frac{g(x,y)f_y(x,y) - f(x,y)g_y(x,y)}{[g(x,y)]^2}\right] \\
&= \frac{gDf - fDg}{g^2}.
\end{aligned}$$

**8.** $f(x,y,z) = x\cos(yz)$, $g(x,y,z) = x^2 + x^9y^2 + y^2z^3 + 2$, $f(x,y)g(x,y) = x^3\cos(yz) + x^{10}y^2\cos(yz) + xy^2z^3\cos(yz) + 2x\cos(yz)$, and $\dfrac{f(x,y)}{g(x,y)} = \dfrac{x\cos(yz)}{x^2 + x^9y^2 + y^2z^3 + 2}$.

$$\text{So} \quad Df = [\cos(yz), -xz\sin(yz), -xy\sin(yz)], \quad \text{and} \quad Dg = [2x + 9x^8y^2, 2x^9y + 2yz^3, 3y^2z^2],$$

$$\begin{aligned}
D(fg) &= \begin{bmatrix} 3x^2\cos yz + 10x^9y^2\cos yz + y^2z^3\cos yz + 2\cos yz \\ -x^3z\sin yz + 2x^{10}y\cos yz - x^{10}y^2z\sin yz + 2xyz^3\cos yz - xy^2z^4\sin yz - 2xz\sin yz \\ -x^3y\sin yz - x^{10}y^3\sin yz + 3xy^2z^2\cos xy - xy^3z^3\sin yz - 2xy\sin yz \end{bmatrix}^T \\
&= (x\cos yz)\begin{bmatrix} 2x + 9x^8y^2 \\ 2x^9y + 2yz^3 \\ 3y^2z^2 \end{bmatrix}^T + (x^2 + x^9y^2 + y^2z^3 + 2)\begin{bmatrix} \cos yz \\ -xz\sin yz \\ -xy\sin yz \end{bmatrix}^T \\
&= fDg + gDf, \text{ and} \\
D\left(\frac{f}{g}\right) &= \left[\frac{gf_x - fg_x}{g^2}, \frac{gf_y - fg_y}{g^2}, \frac{gf_z - fg_z}{g^2}\right] \\
&= \frac{gDf - fDg}{g^2}.
\end{aligned}$$

*In Exercises 9–21, students should verify that $f_{xy} = f_{yx}$. The fact that in these problems the derivative with respect to $y$ of $f_x$ is equal to the derivative with respect to $x$ of $f_y$ is not trivial. Problem 22 explicitly asks them to examine the mixed partials.*

**9.** $f(x,y) = x^3y^7 + 3xy^2 - 7xy$ so $f_x(x,y) = 3x^2y^7 + 3y^2 - 7y$ and $f_y(x,y) = 7x^3y^6 + 6xy - 7x$. The second order partials are:

$$\begin{aligned}
f_{xx}(x,y) &= 6xy^7, \\
f_{xy}(x,y) = f_{yx}(x,y) &= 21x^2y^6 + 6y - 7, \text{ and} \\
f_{yy}(x,y) &= 42x^3y^5 + 6x.
\end{aligned}$$

**10.** $f(x,y) = \cos(xy)$ so $f_x(x,y) = -y\sin(xy)$ and $f_y(x,y) = -x\sin(xy)$. The second order partials are:

$$\begin{aligned}
f_{xx}(x,y) &= -y^2\cos xy, \\
f_{xy}(x,y) = f_{yx}(x,y) &= -xy\cos xy - \sin xy, \text{ and} \\
f_{yy}(x,y) &= -x^2\cos xy.
\end{aligned}$$

**11.** $f(x,y) = e^{y/x} - ye^{-x}$ so $f_x(x,y) = \dfrac{-y}{x^2}e^{y/x} + ye^{-x}$ and $f_y(x,y) = \dfrac{1}{x}e^{y/x} - e^{-x}$. The second order partials are:

$$\begin{aligned} f_{xx}(x,y) &= \frac{2y}{x^3}e^{y/x} + \frac{y^2}{x^4}e^{y/x} - ye^{-x}, \\ f_{xy}(x,y) = f_{yx}(x,y) &= \frac{-1}{x^2}e^{y/x} - \frac{y}{x^3}e^{y/x} + e^{-x}, \text{ and} \\ f_{yy}(x,y) &= \frac{1}{x^2}e^{y/x}. \end{aligned}$$

**12.** $f(x,y) = \sin\sqrt{x^2+y^2}$ so

$$f_x(x,y) = \frac{x\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \quad \text{and} \quad f_y(x,y) = \frac{y\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}.$$

The second order partials are:

$$\begin{aligned} f_{xx}(x,y) &= \frac{\sqrt{x^2+y^2}\left[\cos\sqrt{x^2+y^2} + (x)\frac{-x\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\right] - (x\cos\sqrt{x^2+y^2})\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} \\ &= \frac{y^2\cos\sqrt{x^2+y^2} - x^2\sqrt{x^2+y^2}\sin\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}}, \text{ and by symmetry} \\ f_{yy}(x,y) &= \frac{x^2\cos\sqrt{x^2+y^2} - y^2\sqrt{x^2+y^2}\sin\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}}, \text{ and} \\ f_{xy}(x,y) = f_{yx}(x,y) &= \frac{-xy\sqrt{x^2+y^2}\sin\sqrt{x^2+y^2} - xy\cos\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}}. \end{aligned}$$

**13.** $f(x,y) = \dfrac{1}{\sin^2 x + 2e^y}$ so

$$f_x(x,y) = \frac{-2\sin x\cos x}{(\sin^2 x + 2e^y)^2} = \frac{-\sin 2x}{(\sin^2 x + 2e^y)^2} \quad \text{and} \quad f_y(x,y) = \frac{-2e^y}{(\sin^2 x + 2e^y)^2}.$$

The second order partials are:

$$\begin{aligned} f_{xx}(x,y) &= \frac{(\sin^2 x + 2e^y)^2(-2\cos 2x) + \sin 2x \cdot 2(\sin^2 x + 2e^y)\sin 2x}{(\sin^2 x + 2e^y)^4} \\ &= \frac{(\sin^2 x + 2e^y)(-2\cos 2x) + 2\sin^2 2x}{(\sin^2 x + 2e^y)^3}, \\ f_{xy}(x,y) = f_{yx}(x,y) &= \frac{4e^y\sin 2x}{(\sin^2 x + 2e^y)^3}, \text{ and} \\ f_{yy}(x,y) &= \frac{2e^y(2e^y - \sin^2 x)}{(\sin^2 x + 2e^y)^3}. \end{aligned}$$

**14.** $f(x,y) = e^{x^2+y^2}$ so $f_x(x,y) = 2xe^{x^2+y^2}$ and $f_y(x,y) = 2ye^{x^2+y^2}$. The second order partials are:

$$\begin{aligned} f_{xx}(x,y) &= 2e^{x^2+y^2} + 2x \cdot 2xe^{x^2+y^2} \\ &= e^{x^2+y^2}(2+4x^2), \\ f_{xy}(x,y) = f_{yx}(x,y) &= 4xye^{x^2+y^2}, \text{ and} \\ f_{yy}(x,y) &= e^{x^2+y^2}(2+4y^2). \end{aligned}$$

**15.** $f(x,y) = y\sin x - x\cos y$, so

$$f_x(x,y) = y\cos x - \cos y \qquad \text{and} \qquad f_y(x,y) = \sin x + x\sin y.$$

The second order partial derivatives are:

$$\begin{aligned} f_{xx}(x,y) &= -y\sin x, \\ f_{xy}(x,y) = f_{yx}(x,y) &= \cos x + \sin y, \quad \text{and} \\ f_{yy}(x,y) &= x\cos y. \end{aligned}$$

**16.** $f(x,y) = \ln\left(\dfrac{x}{y}\right)$, so

$$f_x(x,y) = \frac{y}{x}\cdot\frac{1}{y} = \frac{1}{x} \qquad \text{and} \qquad f_y(x,y) = \left(\frac{y}{x}\right)\left(-\frac{x}{y^2}\right) = -\frac{1}{y}.$$

The second order partial derivatives are:

$$\begin{aligned} f_{xx}(x,y) &= -\frac{1}{x^2}, \\ f_{xy}(x,y) = f_{yx}(x,y) &= 0, \quad \text{and} \\ f_{yy}(x,y) &= \frac{1}{y^2}. \end{aligned}$$

**17.** $f(x,y,z) = x^2e^y + e^{2z}$, so $f_x(x,y,z) = 2xe^y$, $f_y(x,y,z) = x^2e^y$, and $f_z(x,y,z) = 2e^{2z}$. The second order partial derivatives are:

$$\begin{aligned} f_{xx}(x,y,z) &= 2e^y \\ f_{yy}(x,y,z) &= x^2e^y \\ f_{zz}(x,y,z) &= 4e^{2z} \\ f_{xy}(x,y,z) = f_{yx}(x,y,z) &= 2xe^y \\ f_{xz}(x,y,z) = f_{zx}(x,y,z) &= 0 \\ f_{yz}(x,y,z) = f_{zy}(x,y,z) &= 0 \end{aligned}$$

**18.** $f(x,y,z) = \dfrac{x-y}{y+z}$, so

$$\begin{aligned} f_x(x,y,z) &= \frac{1(y+z) - 0(x-y)}{(y+z)^2} = \frac{1}{y+z} \\ f_y(x,y,z) &= \frac{-1(y+z) - 1(x-y)}{(y+z)^2} = -\frac{x+z}{(y+z)^2} \\ f_z(x,y,z) &= \frac{0(y+z) - 1(x-y)}{(y+z)^2} = \frac{y-x}{(y+z)^2} \end{aligned}$$

The second order partial derivatives are:

$$f_{xx}(x,y,z) = 0$$
$$f_{yy}(x,y,z) = \frac{2(x+z)}{(y+z)^3}$$
$$f_{zz}(x,y,z) = \frac{2(x-y)}{(y+z)^3}$$
$$f_{xy}(x,y,z) = f_{yx}(x,y,z) = -\frac{1}{(y+z)^2}$$
$$f_{xz}(x,y,z) = f_{zx}(x,y,z) = -\frac{1}{(y+z)^2}$$
$$f_{yz}(x,y,z) = f_{zy}(x,y,z) = -\frac{1(y+z)^2 - 2(y+z)(x+z)}{(y+z)^4} = \frac{2x-y+z}{(y+z)^3}$$

**19.** $f(x,y,z) = x^2yz + xy^2z + xyz^2$ so $f_x(x,y,z) = 2xyz + y^2z + yz^2$, $f_y(x,y,z) = x^2z + 2xyz + xz^2$, and $f_z(x,y,z) = x^2y + xy^2 + 2xyz$. The second order partials are:

$$f_{xx}(x,y,z) = 2yz$$
$$f_{yy}(x,y,z) = 2xz$$
$$f_{zz}(x,y,z) = 2xy$$
$$f_{xy}(x,y,z) = f_{yx}(x,y,z) = 2xz + 2yz + z^2$$
$$f_{xz}(x,y,z) = f_{zx}(x,y,z) = 2xy + y^2 + 2yz$$
$$f_{yz}(x,y,z) = f_{zy}(x,y,z) = x^2 + 2xy + 2xz$$

**20.** $f(x,y,z) = e^{xyz}$ so $f_x(x,y,z) = yze^{xyz}$, $f_y(x,y,z) = xze^{xyz}$, and $f_z(x,y,z) = xye^{xyz}$. The second order partials are:

$$f_{xx}(x,y,z) = y^2z^2e^{xyz}$$
$$f_{yy}(x,y,z) = x^2z^2e^{xyz}$$
$$f_{zz}(x,y,z) = x^2y^2e^{xyz}$$
$$f_{xy}(x,y,z) = f_{yx}(x,y,z) = ze^{xyz}(1+xyz)$$
$$f_{xz}(x,y,z) = f_{zx}(x,y,z) = ye^{xyz}(1+xyz)$$
$$f_{yz}(x,y,z) = f_{zy}(x,y,z) = xe^{xyz}(1+xyz)$$

**21.** $f(x,y,z) = e^{ax}\sin y + e^{bx}\cos z$ so $f_x(x,y,z) = ae^{ax}\sin y + be^{bx}\cos z$, $f_y(x,y,z) = e^{ax}\cos y$, and $f_z(x,y,z) = -e^{bx}\sin z$. The second order partials are:

$$f_{xx}(x,y,z) = a^2e^{ax}\sin y + b^2e^{bx}\cos z$$
$$f_{yy}(x,y,z) = -e^{ax}\sin y$$
$$f_{zz}(x,y,z) = -e^{bx}\cos z$$
$$f_{xy}(x,y,z) = f_{yx}(x,y,z) = ae^{ax}\cos y$$
$$f_{xz}(x,y,z) = f_{zx}(x,y,z) = -be^{bx}\sin z$$
$$f_{yz}(x,y,z) = f_{zy}(x,y,z) = 0$$

**22.** $F(x,y,z) = 2x^3y + xz^2 + y^3z^5 - 7xyz$ so $F_x(x,y,z) = 6x^2y + z^2 - 7yz$, $F_y(x,y,z) = 2x^3 + 3y^2z^5 - 7xz$, and $F_z(x,y,z) = 2xz + 5y^3z^4 - 7xy$.

**(a)** $F_{xx}(x,y,z) = 12xy$, $F_{yy}(x,y,z) = 6yz^5$, and $F_{zz}(x,y,z) = 20y^3z^3 + 2x$.
**(b)** $F_{xy}(x,y,z) = 6x^2 - 7z = F_{yx}(x,y,z)$, $F_{xz}(x,y,z) = 2z - 7y = F_{zx}(x,y,z)$, and $F_{yz}(x,y,z) = 15y^2z^4 - 7x = F_{zy}(x,y,z)$.
**(c)** $F_{xyx}(x,y,z) = 12x = F_{xxy}(x,y,z)$. We knew that these would be equal because they are the mixed partials of $F_x$ (i.e., $(F_x)_{yx} = (F_x)_{xy}$).
**(d)** $F_{xyz}(x,y,z) = -7 = F_{yzx}(x,y,z)$.

**23.** For $f(x,y) = ye^{3x}$, we have $f_x(x,y) = 3ye^{3x}$; that is, the differentiation with respect to $x$ causes a factor of 3 to arise. Hence it follows that

$$\frac{\partial^n f}{\partial x^n} = 3^n ye^{3x}.$$

Moreover, this result is valid for $n \geq 1$. On the other hand, $f_y(x,y) = e^{3x}$; note that $y$ does not appear in the derivative. Therefore,

$$\frac{\partial^n f}{\partial y^n} = 0 \quad \text{for } n \geq 2.$$

**24.** For $f(x,y,z) = xe^{2y} + ye^{3z} + ze^{-x}$, we have $f_x(x,y,z) = e^{2y} - ze^{-x}$; note that $x$ only appears in the second term of the derivative (and a negative sign has arisen). Therefore,

$$\frac{\partial^n f}{\partial x^n} = \begin{cases} e^{2y} - ze^{-x} & n = 1 \\ (-1)^n ze^{-x} & n \geq 2 \end{cases}.$$

Similarly, $f_y(x,y,z) = 2xe^{2y} + e^{3z}$; note that $y$ does not appear in the second term of this derivative. Therefore,

$$\frac{\partial^n f}{\partial y^n} = \begin{cases} 2xe^{2y} + e^{3z} & n = 1 \\ 2^n xe^{2y} & n \geq 2 \end{cases}.$$

Finally, $f_z(x,y,z) = 3ye^{3z} + e^{-x}$. In the same manner, we have

$$\frac{\partial^n f}{\partial z^n} = \begin{cases} 3ye^{3z} + e^{-x} & n = 1 \\ 3^n ye^{3z} & n \geq 2 \end{cases}.$$

**25.** First, for $f(x,y,z) = \ln\left(\frac{xy}{z}\right)$, we have

$$f_x(x,y,z) = \left(\frac{z}{xy}\right)\left(\frac{y}{z}\right) = \frac{1}{x},$$
$$f_y(x,y,z) = \left(\frac{z}{xy}\right)\left(\frac{x}{z}\right) = \frac{1}{y},$$
$$f_z(x,y,z) = \left(\frac{z}{xy}\right)\left(-\frac{xy}{z^2}\right) = -\frac{1}{z}.$$

From this, we see that, for $n \geq 1$.

$$\frac{\partial^n f}{\partial x^n} = \frac{(-1)^{n-1}(n-1)!}{x^n}, \quad \frac{\partial^n f}{\partial y^n} = \frac{(-1)^{n-1}(n-1)!}{y^n}, \quad \text{and} \quad \frac{\partial^n f}{\partial z^n} = \frac{(-1)^n(n-1)!}{z^n}.$$

Note that all mixed partials of this function are zero, since the first-order partial derivatives each involve just a single variable.

**26.** Note that the function $f$ is of class $C^\infty$, so we may differentiate in any order we wish.
**(a)** If we differentiate first with respect to $y$ and $z$, we obtain

$$\frac{\partial^2 f}{\partial y \partial z} = 6x^7yz^2 - 2x^4.$$

Differentiating this result with respect to $x$ twice gives our answer:

$$\frac{\partial^4 f}{\partial x^2 \partial y \partial z} = 252x^5yz^2 - 24x^2.$$

**(b)** We may take our answer in part (a) and differentiate once more with respect to $x$:

$$\frac{\partial^5 f}{\partial x^3 \partial y \partial z} = 1260x^4yz^2 - 48x.$$

**(c)** Every additional time we differentiate $f$ with respect to $x$, the power in $x$ drops. Since the highest power of $x$ that appears is 7, once we differentiate with respect to $x$ seven times, the partial derivative will be constant with respect to $x$. Hence any higher derivatives with respect to $x$ will be zero and so $\partial^{15} f/\partial x^{13} \partial y \partial z = 0$.

**27.** We will denote the degree of $f$ by $deg(f)$ in this solution.

**(a)** $deg(p_x) = 16$, $deg(p_y) = 16$, $deg(p_{xx}) = 15$, $deg(p_{yy}) = 15$, and $deg(p_{yx}) = 15$.

**(b)** $deg(p_x) = 3$, $deg(p_y) = 3$, $deg(p_{xx}) = 2$, $deg(p_{yy})$ is undefined, and $deg(p_{yx}) = 2$.

**(c)** This is difficult because the term of highest degree can switch during the process of taking a derivative. For example consider $f(x, y) = xy^2 + x^3y$. Take the derivative with respect to $y$ and the degree has decreased by one as we would expect: $f_y(x, y) = 2xy + x^3$ so $deg(f_y) = 3$. Now take another derivative with respect to $y$: $f_{yy}(x, y) = 2x$ and so the degree is now one.

For a polynomial $f(x_1, x_2, \ldots, x_n)$ which has degree $d = d_1 + d_2 + \cdots + d_n$ because of a term $cx_1^{d_1} x_x^{d_2} \ldots x_n^{d_n}$, $\partial^k f/\partial x_{i_1} \ldots \partial x_{i_k}$ has degree $d - k$ if $x_j$ occurs at most $d_j$ times in the partial derivative—otherwise we must look for the highest degree of any other surviving terms. If no terms survive, (i.e., $\partial^k f/\partial x_{i_1} \ldots \partial x_{i_k} = 0$) then the degree is undefined.

*Exercises 28 and 29 have the students verify that certain functions are solutions to the given differential equations. When the students studied exponential equations in first semester calculus they may have seen that $f(x) = ce^{kx}$ solves the differential equation $y' = ky$. Here is a nice way to introduce the idea of a partial differential equation.*

**28. (a)** For the first function, $f(x, y, z) = x^2 + y^2 - 2z^2$, $f_x(x, y, z) = 2x$, $f_y(x, y, z) = 2y$, and $f_x(x, y, z) = -4z$.

This means that $f_{xx}(x, y, z) = 2$, $f_{yy}(x, y, z) = 2$, and $f_{zz}(x, y, z) = -4$. We see that $f_{xx} + f_{yy} + f_{zz} = 0$ and conclude that $f$ is harmonic.

For the second function, $f(x, y, z) = x^2 - y^2 + z^2$, $f_x(x, y, z) = 2x$, $f_y(x, y, z) = -2y$, and $f_z(x, y, z) = 2z$.

This means that $f_{xx}(x, y, z) = 2$, $f_{yy}(x, y, z) = -2$, and $f_{zz}(x, y, z) = 2$. We see that $f_{xx} + f_{yy} + f_{zz} \neq 0$ and conclude that $f$ is not harmonic.

**(b)** One possible example is $f(x_1, x_2, \ldots, x_n) = x_1^2 - x_2^2 + 3x_3 + 4x_4 + 5x_5 + \cdots + nx_n$.

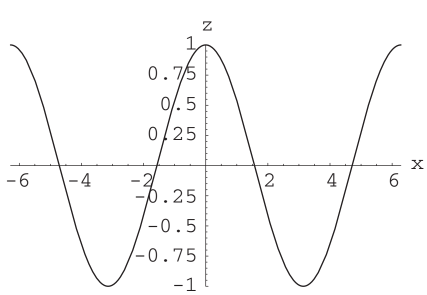
Here $f_{x_i x_i} = \begin{cases} 2 & \text{if } i = 1, \\ -2 & \text{if } i = 2, \\ 0 & \text{if } i > 2. \end{cases}$ and we see that $\sum_{i=1}^{n} f_{x_i x_i} = 0$ so $f$ is harmonic.

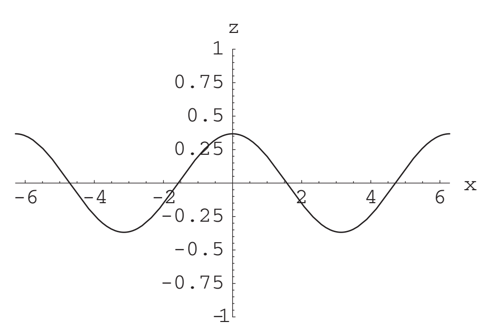
**29. (a)** To show that $T(x, t) = e^{-kt} \cos x$ satisfies the differential equation $kT_{xx} = T_t$ we calculate the derivatives:

$$T_x(x, t) = -e^{-kt} \sin x$$
$$T_{xx}(x, t) = -e^{-kt} \cos x$$
$$T_t(x, t) = -ke^{-kt} \cos x$$

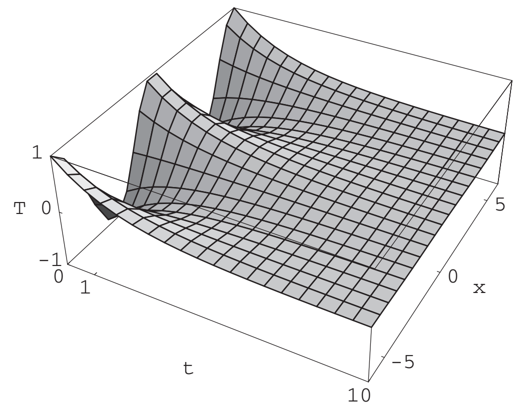
so $kT_{xx} = T_t$.

For $t_0 = 0$ and $t_0 = 1$ the graphs are:




For $t_0 = 10$ the graph is further damped. The graph of the surface $z = T(x, t)$ is:

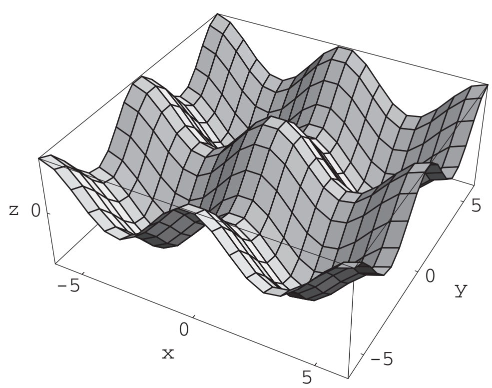


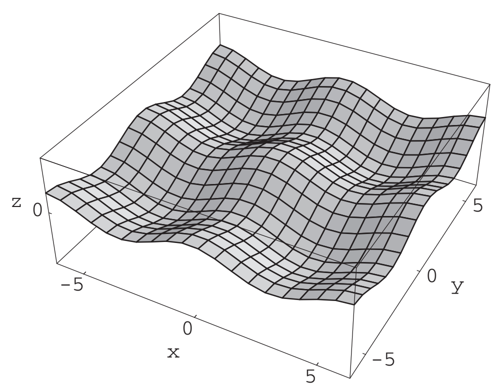

**(b)** To show that $T(x, y, t) = e^{-kt}(\cos x + \cos y)$ satisfies the differential equation $k(T_{xx} + T_{yy}) = T_t$ we calculate the derivatives:

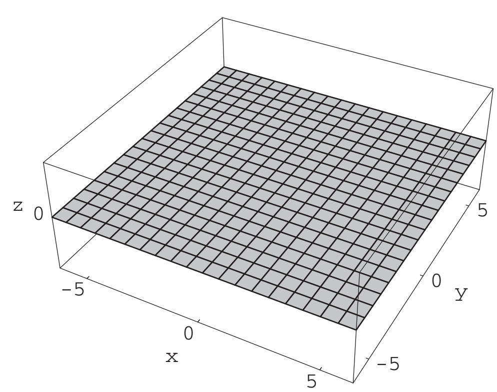
$$T_x(x, y, t) = -e^{-kt}\sin x$$
$$T_{xx}(x, y, t) = -e^{-kt}\cos x$$
$$T_y(x, y, t) = -e^{-kt}\sin y$$
$$T_{yy}(x, y, t) = -e^{-kt}\cos y$$
$$T_t(x, y, t) = -ke^{-kt}(\cos x + \cos y)$$

so $k(T_{xx} + T_{yy}) = T_t$.

The graphs of the surfaces given by $z = T(x, y, t_0)$ for $t_0 = 0, 1$, and 10 are:



**(c)** Finally, to show that $T(x,y,z,t) = e^{-kt}(\cos x + \cos y + \cos z)$ satisfies the differential equation $k(T_{xx} + T_{yy} + T_{xx}) = T_t$ we calculate the derivatives:

$$\begin{aligned}
T_x(x,y,z,t) &= -e^{-kt}\sin x\\
T_{xx}(x,y,z,t) &= -e^{-kt}\cos x\\
T_y(x,y,z,t) &= -e^{-kt}\sin y\\
T_{yy}(x,y,z,t) &= -e^{-kt}\cos y\\
T_z(x,y,z,t) &= -e^{-kt}\sin z\\
T_{zz}(x,y,z,t) &= -e^{-kt}\cos z\\
T_t(x,y,z,t) &= -ke^{-kt}(\cos x + \cos y + \cos z)
\end{aligned}$$

so $k(T_{xx} + T_{yy} + T_{zz}) = T_t$.

**30. (a)** For $(x,y) \neq (0,0)$, compute the partial derivatives:

$$\begin{aligned}
f_x(x,y) &= y\left(\frac{x^2-y^2}{x^2+y^2}\right) + xy\left(\frac{[x^2+y^2](2x) - [x^2-y^2](2x)}{(x^2+y^2)^2}\right)\\
&= \frac{y(x^2-y^2)(x^2+y^2) + xy(4xy^2)}{(x^2+y^2)^2}\\
&= \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} \text{ and similarly}\\
f_y(x,y) &= \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}
\end{aligned}$$

**(b)** We use part (a):

$$\begin{aligned}
f_x(0,y) &= \frac{y(-y^4)}{(y^2)^2}\\
&= -y \text{ for } y \neq 0, \text{ and}\\
f_y(x,0) &= x \text{ for } x \neq 0.
\end{aligned}$$

**(c)** From part (b), $f_{xy}(0,y) = -1$ while $f_{yx}(x,0) = 1$ and $f_x(0,y)$ and $f_y(x,0)$ are continuous at the origin so you can conclude that $f_{xy}(0,0) = -1$ while $f_{yx}(0,0) = 1$. Why aren't the mixed partials equal? The answer is that the second partials are not continuous at the origin. We can see this by calculating

$$f_{xy}(x,y) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}.$$

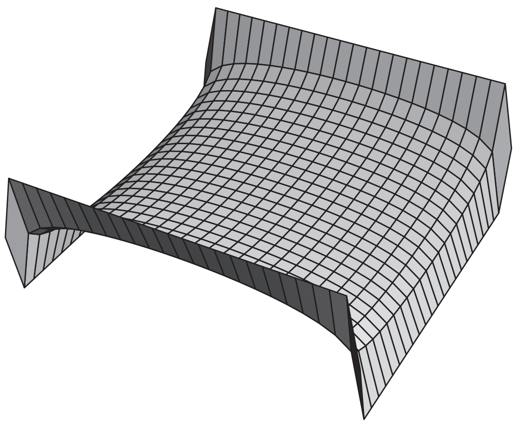
$$\begin{aligned}
\text{Therefore}\quad f_{xy}(x,0) &= 1 \text{ and}\\
f_{xy}(0,y) &= -1.
\end{aligned}$$

$$\text{Hence} \lim_{(x,y)\to(0,0)} f_{xy}(x,y) \quad \text{does not exist.}$$

In other words, $f_{xy}$ is not continuous at the origin.

**31.** An equation of a plane in the form $z = f(x,y)$ is $z = Ax + By + C$. Here $z_x = A$, $z_y = B$ and the second derivatives are all 0. The partial differential equation for minimal surfaces is therefore trivially satisfied and a plane is seen to be a minimal surface.

**32. (a)** Here's an image of Scherk's surface.

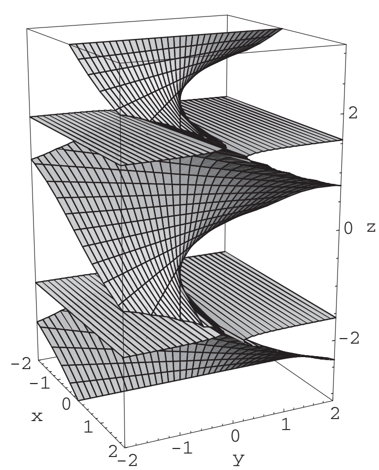


**(b)** In this case $z = \ln(\cos x/\cos y)$. So $z_x = -\tan x$, $z_y = \tan y$, $z_{xy} = 0$, $z_{xx} = -\sec^2 x$, and $z_{yy} = \sec^2 y$. So

$$\begin{aligned}(1 + z_y^2)z_{xx} + (1 + z_x^2)z_{yy} &= (1 + \tan^2 y)(-\sec^2 x) + (1 + \tan^2 x)(\sec^2 y) \\ &= -\sec^2 x \sec^2 y + \sec^2 x \sec^2 y = 0.\end{aligned}$$

This agrees with the right side of the equation as $z_{xy} = 0$.

**33. (a)** Here's an image of the helicoid:




**(b)** There's no reason not to think of this surface as $z = x \tan y$. Then $z_x = \tan y$, $z_y = x \sec^2 y$, $z_{xx} = 0$, $z_{xy} = \sec^2 y$, and $z_{yy} = 2 \tan y \sec^2 y$. So

$$\begin{aligned}(1 + z_y^2)z_{xx} + (1 + z_x^2)z_{yy} &= (1 + x^2 \sec^4 y)(0) + (1 + \tan^2 y)(2 \tan y \sec^2 y) \\ &= (\sec^2 y)(2 \tan y \sec^2 y) = 2(\tan y)(x \sec^2 y)(\sec^2 y) \\ &= 2z_x z_y z_{xy}\end{aligned}$$

## 2.5 The Chain Rule

*In Exercises 1– 3 students see that if you have a composite function you can take the derivative either by substituting or by using the chain rule.*

**1.** $f(x, y, z) = x^2 - y^3 + xyz$, $x = 6t + 7$, $y = \sin 2t$, and $z = t^2$.

**Substitution**:

$$\begin{aligned} f(x(t), y(t), z(t)) &= (6t+7)^2 - (\sin 2t)^3 + (6t+7)(\sin 2t)(t^2) \\ &= (6t+7)^2 - (\sin 2t)^3 + (6t^3 + 7t^2)(\sin 2t) \quad \text{and so} \\ \frac{df}{dt} &= 2(6t+7)6 - 3(\sin 2t)^2(2\cos 2t) + (18t^2 + 14t)\sin 2t + (6t^3 + 7t^2)(2\cos 2t) \end{aligned}$$

**Chain Rule**:

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= (2x + yz)(6) + (-3y^2 + xz)(2\cos 2t) + (xy)(2t) \\ &= [2(6t+7) + (\sin 2t)(t^2)](6) + [-3\sin^2 2t + (6t+7)t^2](2\cos 2t) + [(6t+7)\sin 2t](2t) \end{aligned}$$

**2.** $f(x, y) = \sin(xy)$, $x = s + t$, and $y = s^2 + t^2$.

**(a)** $f(x(t), y(t)) = \sin(x(t)y(t)) = \sin[(s+t)(s^2+t^2)]$.

$$\begin{aligned} \frac{\partial f}{\partial s} &= \cos[(s+t)(s^2+t^2)][(s^2+t^2) + (s+t)(2s)] \\ \frac{\partial f}{\partial t} &= \cos[(s+t)(s^2+t^2)][(s^2+t^2) + (s+t)(2t)] \end{aligned}$$

**(b)**

$$\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ &= y\cos(xy) + x\cos(xy)2s \\ &= \cos[(s+t)(s^2+t^2)][(s^2+t^2) + (s+t)(2s)] \text{ and} \\ \frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\ &= y\cos(xy) + x\cos(xy)2t \\ &= \cos[(s+t)(s^2+t^2)][(s^2+t^2) + (s+t)(2t)] \end{aligned}$$

**3. (a)** We want

$$\begin{aligned} \frac{dP}{dt} &= \frac{\partial P}{\partial x}\frac{dx}{dt} + \frac{\partial P}{\partial y}\frac{dy}{dt} + \frac{\partial P}{\partial z}\frac{dz}{dt} \\ &= \frac{12xz}{y}(-2\sin t) - \frac{6x^2 z}{y^2}(2\cos t) + \frac{6x^2}{y}(3) \\ &= \frac{12(2\cos t)(3t)}{2\sin t}(-2\sin t) - \frac{6(4\cos^2 t)3t}{4\sin^2 t}(2\cos t) + \frac{6(4\cos^2 t)}{2\sin t}(3) \\ &= -72t\cos t - 36t\frac{\cos^3 t}{\sin^2 t} + \frac{36\cos^2 t}{\sin t}. \end{aligned}$$

Therefore,

$$\left.\frac{dP}{dt}\right|_{t=\pi/4} = \frac{(36 - 27\pi)}{\sqrt{2}}.$$

**(b)** $P(x(t), y(t), z(t)) = \frac{6(2\cos t)^2(3t)}{2\sin t} = \frac{36t\cos^2 t}{\sin t}$, so

$$\frac{dP}{dt} = \frac{\sin t(36\cos^2 t - 36t \cdot 2\cos t \sin t) - 36t\cos^2 t(\cos t)}{\sin^2 t}.$$

Therefore,

$$\left.\frac{dP}{dt}\right|_{t=\pi/4} = \frac{(36 - 27\pi)}{\sqrt{2}}.$$

**(c)** Using differentials,

$$\Delta P \approx \left(\left.\frac{dP}{dt}\right|_{t=\pi/4}\right)(dt) = \left(\frac{36 - 27\pi}{\sqrt{2}}\right)(.01) \approx -.34523.$$

So (writing $P$ as a function of $t$),

$$P(\pi/4 + .01) \approx P(\pi/4) + \Delta P \approx \frac{9\pi}{\sqrt{2}} - .34523 \approx 19.6477.$$

**4.** We are thinking of $z = z(s,t) = [x(s,t)]^2 + [y(s,t)]^3$. So

$$\frac{\partial z}{\partial t}(2,1) = \left.\frac{\partial z}{\partial x}\right|_{(2,1)} \cdot \left.\frac{\partial x}{\partial t}\right|_{(2,1)} + \left.\frac{\partial z}{\partial y}\right|_{(2,1)} \cdot \left.\frac{\partial y}{\partial t}\right|_{(2,1)} = 2x|_{(2,1)} \cdot s|_{(2,1)} + 0 = 8.$$

**5.** Here $V = LWH$, so

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial L}\frac{dL}{dt} + \frac{\partial V}{\partial W}\frac{dW}{dt} + \frac{\partial V}{\partial H}\frac{dH}{dt}\\ &= WH\left(\frac{dL}{dt}\right) + LH\left(\frac{dW}{dt}\right) + LW\left(\frac{dH}{dt}\right)\\ &= 5 \cdot 4(.75) + 7 \cdot 4(.5) + 7 \cdot 5(-1)\\ &= -6\ in^3/min.\end{aligned}$$

Since $\frac{dV}{dt} < 0$, the volume of the dough is decreasing at this instant.

**6.** Let the length of the butter be $y$ and the length of an edge of the cross section be $x$. Then the volume $V = x^2 y$. The rate at which the volume is changing is

$$\frac{dV}{dt} = 2xy\frac{dx}{dt} + x^2\frac{dy}{dt} = 2(1.5)(6)(-.125) + (1.5)^2(-.25) = -2.8125\ in^3/min.$$

**7.** Note that in 6 months:

$$x = 1 + .6 - \cos\pi = 2.6$$
$$y = 200 + 12\sin\pi = 200$$

The chain rule gives

$$\begin{aligned}\left.\frac{dP}{dt}\right|_{t=6} &= \left.\frac{\partial P}{\partial x}\right|_{\substack{x=2.6\\y=200}} \left.\frac{dx}{dt}\right|_{t=6} + \left.\frac{\partial P}{\partial y}\right|_{\substack{x=2.6\\y=200}} \left.\frac{dy}{dt}\right|_{t=6}\\ &= 10(0.1x+10)^{-\frac{1}{2}}(0.1)|_{x=2.6}\left.\left(0.1 - \frac{\pi}{6}\sin\frac{\pi t}{6}\right)\right|_{t=6}\\ &\quad - 4y^{-\frac{2}{3}}|_{y=200}\left.\left(2\sin\frac{\pi t}{6} + \frac{2\pi t}{6}\cos\frac{\pi t}{6}\right)\right|_{t=6}\\ &= (10.26)^{-\frac{1}{2}}(0.1) - 4(200^{-\frac{2}{3}})(-2\pi)\\ &= 0.031219527 + 0.734885812 = 0.766105339 \text{ units/month (demand in rising slightly).}\end{aligned}$$

**8.** **(a)** The chain rule gives

$$\frac{d}{dt}(\text{BMI}) = \frac{\partial(\text{BMI})}{\partial w}\frac{dw}{dt} + \frac{\partial(\text{BMI})}{\partial h}\frac{dh}{dt}$$
$$= \frac{10{,}000}{h^2}\frac{dw}{dt} - \frac{20{,}000\, w}{h^3}\frac{dh}{dt}$$

On the child's 10th birthday: $w = 33$ kg, $h = 140$ cm,

$$\frac{dw}{dt} = 0.4, \quad \frac{dh}{dt} = 0.6.$$

So

$$\frac{d(\text{BMI})}{dt} = \frac{10{,}000}{140^2}(0.4) - \frac{20{,}000 \cdot 33}{140^3}(0.6)$$
$$\approx 0.0598 \text{ points/month.}$$

**(b)** The rate we found in part (a) is greater than the typical rate by about 49%. I'd monitor the situation monthly so that it doesn't persist for too long, but I wouldn't be very concerned, since the current BMI is roughly 16.84, which is quite low.

**9.** If we let $h$ denote the height of the pile and $r$ the base radius, then we have the volume $V$ given by $V = \frac{\pi}{3}r^2h$. If we differentiate with respect to time $t$ and use the chain rule, we obtain

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right).$$

We wish to find $dr/dt$ when $h = 30$, $r = 12$, $dh/dt = 1$, and $dV/dt = 320$. Using this numerical information, we have

$$320 = \frac{\pi}{3}\left(720\frac{dr}{dt} + 144 \cdot 1\right) = \pi\left(240\frac{dr}{dt} + 48\right).$$

Now we solve for $dr/dt$:

$$\frac{dr}{dt} = \frac{1}{240}\left(\frac{320}{\pi} - 48\right) = \frac{4}{3\pi} - \frac{1}{5} \approx 0.2244 \text{ cm/min.}$$

**10.** Bearing in mind that $c$ is a constant (i.e., 330 m/sec), the frequency Hermione hears when $f = 440$ and $v = 4$ is

$$\phi(f, v) = \left(\frac{330+4}{330}\right)440 = 445.\overline{3} \text{ Hz.}$$

Now we wish to find $d\phi/dt$ when $f = 440$ and $v = 4$. To do this, we use the chain rule:

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial f}\frac{df}{dt} + \frac{\partial\phi}{\partial v}\frac{dv}{dt} = \frac{c+v}{c}\frac{df}{dt} + \frac{f}{c}\frac{dv}{dt}.$$

The numerical information tells us that when $f = 440$ and $v = 4$:

$$\frac{df}{dt} = 100, \qquad \frac{dv}{dt} = -2.$$

Therefore,

$$\frac{d\phi}{dt} = 1.0\overline{12}(100) + 1.\overline{3}(-2) = 98.\overline{54} \text{ Hz/sec.}$$

Since this result is positive, the perceived frequency is **increasing**, so that Hermione hears the clarinet as sounding **higher**.

**11.** Since $x = e^r\cos\theta$ and $y = e^r\sin\theta$ we can write

$$\frac{\partial z}{\partial r} = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial x}{\partial r}\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial r}\right) = \left(\frac{\partial z}{\partial x}\right)(e^r\cos\theta) + \left(\frac{\partial z}{\partial y}\right)(e^r\sin\theta).$$

Similarly,

$$\frac{\partial z}{\partial \theta} = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial \theta}\right) = \left(\frac{\partial z}{\partial x}\right)(-e^r\sin\theta) + \left(\frac{\partial z}{\partial y}\right)(e^r\cos\theta).$$

Therefore,

$$\left(\frac{\partial z}{\partial r}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2=e^{2r}\left[(\cos^2\theta+\sin^2\theta)\left(\frac{\partial z}{\partial x}\right)^2\right.$$
$$\left.+(\cos^2\theta+\sin^2\theta)\left(\frac{\partial z}{\partial y}\right)^2+(2\cos\theta\sin\theta-2\cos\theta\sin\theta)\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\right]$$
$$=e^{2r}\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right].$$

The result follows.

*Exercises 12–18 are fun exercises. You may want to stress that we are showing that the partial differential equations are true without even knowing the "outside" function.*

**12.** By the chain rule, we have

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=2v\,\frac{\partial z}{\partial x}+2u\,\frac{\partial z}{\partial y},$$
$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=2u\,\frac{\partial z}{\partial x}+2v\,\frac{\partial z}{\partial y}.$$

Hence

$$\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}=4uv\left(\frac{\partial z}{\partial x}\right)^2+\left(4u^2+4v^2\right)\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+4uv\left(\frac{\partial z}{\partial y}\right)^2$$
$$=4uv\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]+4\left(u^2+v^2\right)\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$
$$=2x\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]+4y\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

since $x=2uv$ and $y=u^2+v^2$.

**13.** First we calculate

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial u}=2u\,\frac{\partial w}{\partial x}-2u\,\frac{\partial w}{\partial y},$$
$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial v}=-2v\,\frac{\partial w}{\partial x}+2v\,\frac{\partial w}{\partial y}.$$

Hence

$$v\,\frac{\partial w}{\partial u}+u\,\frac{\partial w}{\partial v}=v\left(2u\,\frac{\partial w}{\partial x}-2u\,\frac{\partial w}{\partial y}\right)+u\left(-2v\,\frac{\partial w}{\partial x}+2v\,\frac{\partial w}{\partial y}\right)$$
$$=2uv\,\frac{\partial w}{\partial x}-2uv\,\frac{\partial w}{\partial y}-2uv\,\frac{\partial w}{\partial x}+2uv\,\frac{\partial w}{\partial y}=0.$$

**14.** We'll start by calculating the components on the left side:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\\ &= \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(1)\\ &= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \text{and}\\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\\ &= \frac{\partial z}{\partial u}(1) + \frac{\partial z}{\partial v}(-1)\\ &= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \quad \text{so}\\ \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} &= \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\\ &= \left(\frac{\partial z}{\partial u}\right)^2 - \left(\frac{\partial z}{\partial v}\right)^2.\end{aligned}$$

**15.** First calculate:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{y(y^2-x^2)}{(x^2+y^2)^2} \text{ and}\\ \frac{\partial u}{\partial y} &= \frac{x(x^2-y^2)}{(x^2+y^2)^2}\end{aligned}$$

Now

$$\begin{aligned}x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} &= x\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + y\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}\\ &= \left(\frac{\partial w}{\partial u}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\\ &= \left(\frac{\partial w}{\partial u}\right)\left(x\frac{y(y^2-x^2)}{(x^2+y^2)^2} + y\frac{x(x^2-y^2)}{(x^2+y^2)^2}\right)\\ &= 0.\end{aligned}$$

**16.** First calculate:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{4xy^2}{(x^2+y^2)^2} \text{ and}\\ \frac{\partial u}{\partial y} &= \frac{-4x^2y}{(x^2+y^2)^2}\end{aligned}$$

Now

$$\begin{aligned}x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} &= x\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + y\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}\\ &= \left(\frac{\partial w}{\partial u}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\\ &= \left(\frac{\partial w}{\partial u}\right)\left(x\frac{4xy^2}{(x^2+y^2)^2} + y\frac{-4x^2y}{(x^2+y^2)^2}\right)\\ &= 0.\end{aligned}$$

**17.** $\frac{\partial u}{\partial x} = \frac{-1}{x^2}$, $\frac{\partial u}{\partial y} = \frac{1}{y^2}$, and $\frac{\partial u}{\partial z} = 0$. Also $\frac{\partial v}{\partial x} = \frac{-1}{x^2}$, $\frac{\partial v}{\partial y} = 0$, and $\frac{\partial v}{\partial z} = \frac{1}{z^2}$. Now it is just a matter of using the chain rule and plugging in:

$$\begin{aligned} x^2\frac{\partial w}{\partial x} + y^2\frac{\partial w}{\partial y} + z^2\frac{\partial w}{\partial z} &= x^2\left[\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}\right] + y^2\left[\frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}\right] + z^2\left[\frac{\partial w}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial z}\right] \\ &= \frac{\partial w}{\partial u}\left[x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}\right] + \frac{\partial w}{\partial v}\left[x^2\frac{\partial v}{\partial x} + y^2\frac{\partial v}{\partial y} + z^2\frac{\partial v}{\partial z}\right] \\ &= \frac{\partial w}{\partial u}\left[x^2\left(\frac{-1}{x^2}\right) + y^2\left(\frac{1}{y^2}\right) + 0\right] + \frac{\partial w}{\partial v}\left[x^2\left(\frac{-1}{x^2}\right) + 0 + z^2\left(\frac{1}{z^2}\right)\right] \\ &= 0. \end{aligned}$$

**18.** $\frac{\partial u}{\partial x} = \frac{1}{y}$, $\frac{\partial u}{\partial y} = \frac{-x}{y^2}$, and $\frac{\partial u}{\partial z} = 0$. Also $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = \frac{-z}{y^2}$, and $\frac{\partial v}{\partial z} = \frac{1}{y}$. Again, it is just a matter of using the chain rule and plugging in:

$$\begin{aligned} x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} &= x\left[\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}\right] + y\left[\frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}\right] + z\left[\frac{\partial w}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial z}\right] \\ &= \frac{\partial w}{\partial u}\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right] + \frac{\partial w}{\partial v}\left[x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z}\right] \\ &= \frac{\partial w}{\partial u}\left[x\left(\frac{1}{y}\right) + y\left(\frac{-x}{y^2}\right) + 0\right] + \frac{\partial w}{\partial v}\left[0 + y\left(\frac{-z}{y^2}\right) + z\left(\frac{1}{y}\right)\right] \\ &= 0. \end{aligned}$$

**19. (a)** $\mathbf{f}\circ g = (3(s-7t)^5, e^{2s-14t})$ so

$$D(\mathbf{f}\circ g) = \begin{bmatrix} 15(s-7t)^4 & -105(s-7t)^4 \\ 2e^{2s-14t} & -14e^{2x-14t} \end{bmatrix}$$

**(b)**

$$D\mathbf{f} = \begin{bmatrix} 15x^4 \\ 2e^{2x} \end{bmatrix} = \begin{bmatrix} 15(s-7t)^4 \\ 2e^{2s-14t} \end{bmatrix} \quad \text{and} \quad Dg = \begin{bmatrix} 1 & -7 \end{bmatrix}$$

We can easily see that $D\mathbf{f}\,Dg = D(\mathbf{f}\circ g)$.

**20. (a)** $\mathbf{f}\circ g = \left((s+t^2+u^3)^2, \cos 3(s+t^2+u^3), \ln(s+t^2+u^3)\right)$ so

$$D(\mathbf{f}\circ g) = \begin{bmatrix} 2(s+t^2+u^3) & 4t(s+t^2+u^3) & 3u^2(s+t^2+u^3) \\ -3\sin 3(s+t^2+u^3) & -6t\sin 3(s+t^2+u^3) & -9u^2\sin 3(s+t^2+u^3) \\ \frac{1}{s+t^2+u^3} & \frac{2t}{s+t^2+u^3} & \frac{3u^2}{s+t^2+u^3} \end{bmatrix}$$

**(b)**

$$D\mathbf{f} = \begin{bmatrix} -3\sin 3x \\ 1/x \\ 1/x + 3\sin 3x \end{bmatrix} = \begin{bmatrix} -3\sin 3(s+t^2+u^3) \\ 1/(s+t^2+u^3) \\ 1/(s+t^2+u^3) + 3\sin 3(s+t^2+u^3) \end{bmatrix}$$

and

$$Dg = \begin{bmatrix} 1 & 2t & 3u^2 \end{bmatrix},$$

so that $D\mathbf{f}\,Dg = D(\mathbf{f}\circ g)$.

**21. (a)** $f\circ\mathbf{g} = (s+t)e^{s-t}$ so

$$D(f\circ\mathbf{g}) = \begin{bmatrix} (s+t)e^{s-t} + e^{s-t} & -(s+t)e^{s-t} + e^{s-t} \end{bmatrix}$$

**(b)**

$$Df = \begin{bmatrix} ye^x & e^x \end{bmatrix} = \begin{bmatrix} (s+t)e^{s-t} & e^{s-t} \end{bmatrix}$$

and

$$D\mathbf{g} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

so that

$$Df\,D\mathbf{g} = \begin{bmatrix} (s+t)e^{s-t} & e^{s-t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = D(f\circ\mathbf{g}).$$

**22.** **(a)** $f \circ \mathbf{g} = (st)^2 - 3(s+t^2)^2 = s^2t^2 - 3s^2 - 6st^2 - 3t^4$, so

$$D(f \circ \mathbf{g}) = [2st^2 - 6s - 6t^2 \quad 2s^2t - 12st - 12t^3].$$

**(b)** $Df = \begin{bmatrix} 2x & -6y \end{bmatrix} = \begin{bmatrix} 2st & -6s - 6t^2 \end{bmatrix}$, and $D\mathbf{g} = \begin{bmatrix} t & s \\ 1 & 2t \end{bmatrix}$, so

$$Df\, D\mathbf{g} = \begin{bmatrix} 2st & -6s - 6t^2 \end{bmatrix} \begin{bmatrix} t & s \\ 1 & 2t \end{bmatrix} = [2st^2 - 6s - 6t^2 \quad 2s^2t - 12st - 12t^3].$$

**23.** **(a)** $\mathbf{f} \circ \mathbf{g} = \left((s/t)s^2t - \dfrac{s^2t}{s/t}, \dfrac{s/t}{s^2t} + s^6t^3\right) = \left(s^3 - st^2, \dfrac{1}{st^2} + s^6t^3\right)$, so

$$D(\mathbf{f} \circ \mathbf{g}) = \begin{bmatrix} 3s^2 - t^2 & -2st \\ -1/(s^2t^2) + 6s^5t^3 & -2/(st^3) + 3s^6t^2 \end{bmatrix}$$

**(b)**

$$D\mathbf{f} = \begin{bmatrix} y + \frac{y}{x^2} & x - \frac{1}{x} \\ \frac{1}{y} & \frac{-x}{y^2} + 3y^2 \end{bmatrix} = \begin{bmatrix} s^2t + \frac{s^2t}{s^2/t^2} & \frac{s}{t} - \frac{t}{s} \\ \frac{1}{s^2t} & \frac{-s/t}{s^4t^2} + 3s^4t^2 \end{bmatrix} = \begin{bmatrix} s^2t + t^3 & \frac{s^2-t^2}{st} \\ \frac{1}{s^2t} & -\frac{1}{s^3t^3} + 3s^4t^2 \end{bmatrix}$$

and $\quad D\mathbf{g} = \begin{bmatrix} \frac{1}{t} & -\frac{s}{t^2} \\ 2st & s^2 \end{bmatrix}$ so

$$D\mathbf{f}\, D\mathbf{g} = \begin{bmatrix} s^2t + t^3 & \frac{s^2-t^2}{st} \\ \frac{1}{s^2t} & -\frac{1}{s^3t^3} + 3s^4t^2 \end{bmatrix} \begin{bmatrix} \frac{1}{t} & -\frac{s}{t^2} \\ 2st & s^2 \end{bmatrix} = \begin{bmatrix} 3s^2 - t^2 & -2st \\ \frac{-1}{s^2t^2} + 6s^5t^3 & \frac{-2}{st^3} + 3s^6t^2 \end{bmatrix}.$$

**24.** **(a)** $\mathbf{f} \circ \mathbf{g} = ((t-2)^2(3t+7) + (3t+7)^2t^3, (t-2)(3t+7)t^3, e^{t^3})$ so

$$D(\mathbf{f} \circ \mathbf{g}) = \begin{bmatrix} 45t^4 + 168t^3 + 156t^2 - 10t - 16 \\ 15t^4 + 4t^3 - 42t^2 \\ 3t^2e^{t^3} \end{bmatrix}.$$

**(b)**

$$D(\mathbf{f}) = \begin{bmatrix} 2xy & x^2 + 2yz & y^2 \\ yz & xz & xy \\ 0 & 0 & e^z \end{bmatrix}$$

$$= \begin{bmatrix} 2(t-2)(3t+7) & (t-2)^2 + 2(3t+7)t^3 & (3t+7)^2 \\ (3t+7)t^3 & (t-2)t^3 & (t-2)(3t+7) \\ 0 & 0 & e^{t^3} \end{bmatrix}$$

and $\quad D(\mathbf{g}) = \begin{bmatrix} 1 \\ 3 \\ 3t^2 \end{bmatrix} \quad$ so $\quad D(\mathbf{f})D(\mathbf{g}) = \begin{bmatrix} 45t^4 + 168t^3 + 156t^2 - 10t - 16 \\ 15t^4 + 4t^3 - 42t^2 \\ 3t^2e^{t^3} \end{bmatrix}.$

**25.** **(a)** $\mathbf{f} \circ \mathbf{g} = \left(e^{2t}\sin t, e^t \sin^2 t, \sin^3 t + e^{3t},\right)$ so

$$D(\mathbf{f} \circ \mathbf{g}) = \begin{bmatrix} 2e^{2t}\sin t + e^{2t}\cos t \\ e^t \sin^2 t + 2e^t \sin t \cos t \\ 3\sin^2 t \cos t + 3e^{3t} \end{bmatrix}$$

**(b)**

$$D\mathbf{f} = \begin{bmatrix} y^2 & 2xy \\ 2xy & x^2 \\ 3x^2 & 3y^2 \end{bmatrix} = \begin{bmatrix} e^{2t} & 2e^t \sin t \\ 2e^t \sin t & \sin^2 t \\ 3\sin^2 t & 3e^{2t} \end{bmatrix}$$

and

$$D\mathbf{g} = \begin{bmatrix} \cos t \\ e^t \end{bmatrix},$$

so that

$$Df\,D\mathbf{g} = \begin{bmatrix} e^{2t} & 2e^t \sin t \\ 2e^t \sin t & \sin^2 t \\ 3\sin^2 t & 3e^{2t} \end{bmatrix} \begin{bmatrix} \cos t \\ e^t \end{bmatrix} = D(\mathbf{f} \circ \mathbf{g}).$$

**26.** **(a)** $\mathbf{f} \circ \mathbf{g} = \left((s+2t+3u)^2 - stu, stu/(s+2t+3u), e^{stu}\right)$ so

$$D(\mathbf{f} \circ \mathbf{g}) = \begin{bmatrix} 2(s+2t+3u) - tu & 4(s+2t+3u) - su & 6(s+2t+3u) - st \\ \frac{tu(2t+3u)}{(s+2t+3u)^2} & \frac{su(s+3u)}{(s+2t+3u)^2} & \frac{st(s+2t)}{(s+2t+3u)^2} \\ tue^{stu} & sue^{stu} & ste^{stu} \end{bmatrix}$$

**(b)**

$$D\mathbf{f} = \begin{bmatrix} 2x & -1 \\ -y/x^2 & 1/x \\ 0 & e^y \end{bmatrix} = \begin{bmatrix} 2(s+2t+3u) & -1 \\ -stu/(s+2t+3u)^2 & 1/(s+2t+3u) \\ 0 & e^{stu} \end{bmatrix}$$

and

$$D\mathbf{g} = \begin{bmatrix} 1 & 2 & 3 \\ tu & su & st \end{bmatrix}.$$

Again, we can see that $D\mathbf{f}\,D\mathbf{g} = D(\mathbf{f} \circ \mathbf{g})$.

**27.** **(a)** $\mathbf{f} \circ \mathbf{g} = (st + tu + su, s^3t^3 - e^{stu^2})$ so

$$D(\mathbf{f} \circ \mathbf{g}) = \begin{bmatrix} t+u & s+u & s+t \\ 3s^2t^3 - tu^2e^{stu^2} & 3s^3t^2 - su^2e^{stu^2} & -2stue^{stu^2} \end{bmatrix}.$$

**(b)**

$$D\mathbf{f} = \begin{bmatrix} 1 & 1 & 1 \\ 3x^2 & -ze^{yz} & -ye^{yz} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3s^2t^2 & -sue^{stu^2} & -tue^{stu^2} \end{bmatrix}$$

and $D\mathbf{g} = \begin{bmatrix} t & s & 0 \\ 0 & u & t \\ u & 0 & s \end{bmatrix}$ so $D\mathbf{f}D\mathbf{g} = \begin{bmatrix} t+u & s+u & s+t \\ 3s^2t^3 - tu^2e^{stu^2} & 3s^3t^2 - su^2e^{stu^2} & -2stue^{stu^2} \end{bmatrix}$.

**28.** This is a matter of seeing what we have to determine and which formula to use. We calculate $D(\mathbf{f} \circ \mathbf{g})(1, -1, 3)$ as $D\mathbf{f}(\mathbf{g}(1, -1, 3))D(\mathbf{g}(1, -1, 3))$. The second piece is given in the exercise. For the first we calculate

$$D\mathbf{f}(\mathbf{g}(1, -1, 3)) = \begin{bmatrix} 2y & 2x \\ 3 & -1 \end{bmatrix}\Bigg|_{\mathbf{g}(1,-1,3)} = \begin{bmatrix} 2y & 2x \\ 3 & -1 \end{bmatrix}\Bigg|_{(2,5)} = \begin{bmatrix} 10 & 4 \\ 3 & -1 \end{bmatrix}.$$

Then we can multiply the matrices to get the result

$$D(\mathbf{f} \circ \mathbf{g})(1, -1, 3) = \begin{bmatrix} 10 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 4 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 26 & -10 & 28 \\ -1 & -3 & -7 \end{bmatrix}.$$

**29.** **(a)** This is similar to Exercise 28.

$$\begin{aligned} D(\mathbf{f} \circ \mathbf{g})(1, 2) &= D\mathbf{f}(\mathbf{g}(1, 2))D\mathbf{g}(1, 2) = D\mathbf{f}(3, 5)D\mathbf{g}(1, 2) \\ &= \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 31 & 44 \end{bmatrix} \end{aligned}$$

**(b)**

$$\begin{aligned} D(\mathbf{g} \circ \mathbf{f})(4, 1) &= D\mathbf{g}(\mathbf{f}(4, 1))D\mathbf{f}(4, 1) = D\mathbf{g}(1, 2)D\mathbf{f}(4, 1) \\ &= \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 13 \\ 2 & 31 \end{bmatrix} \end{aligned}$$

**30.** We'll start with the right hand side of the equation because we can easily calculate the partials of $x$ and $y$ with respect to $r$ and $\theta$.

$$\begin{aligned}\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2&=\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}\right)^2\\
&=\left(\frac{\partial z}{\partial x}\right)^2\left[\left(\frac{\partial x}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial x}{\partial \theta}\right)^2\right]+\left(\frac{\partial z}{\partial y}\right)^2\left[\left(\frac{\partial y}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial y}{\partial \theta}\right)^2\right]\\
&\quad+2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\left[\frac{\partial y}{\partial r}\frac{\partial x}{\partial r}+\frac{1}{r^2}\frac{\partial x}{\partial \theta}\frac{\partial y}{\partial \theta}\right]\\
&=\left(\frac{\partial z}{\partial x}\right)^2\left[\cos^2\theta+\frac{1}{r^2}(r^2\sin^2\theta)\right]+\left(\frac{\partial z}{\partial y}\right)^2\left[\sin^2\theta+\frac{1}{r^2}(r^2\cos^2\theta)\right]\\
&\quad+2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\left[\sin\theta\cos\theta+\frac{1}{r^2}(-r\sin\theta)(r\cos\theta)\right]=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\end{aligned}$$

**31. (a)** From formula (10) in Section 2.5, we have

$$\begin{aligned}\frac{\partial}{\partial x}&=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\quad\text{and}\\
\frac{\partial}{\partial y}&=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\end{aligned}$$

Hence if $z=f(x,y)$, then

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}&=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\cos\theta\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right)\\
&=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial z}{\partial r}-\frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial z}{\partial r}-\frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}\right)\end{aligned}$$

Now use the product rule:

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}&=\cos\theta\left(\cos\theta\frac{\partial^2 z}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial z}{\partial \theta}-\frac{\sin\theta}{r}\frac{\partial^2 z}{\partial r\partial \theta}\right)\\
&\quad-\frac{\sin\theta}{r}\left(-\sin\theta\frac{\partial z}{\partial r}+\cos\theta\frac{\partial^2 z}{\partial \theta\partial r}-\frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}-\frac{\sin\theta}{r}\frac{\partial^2 z}{\partial \theta^2}\right)\\
&=\cos^2\theta\frac{\partial^2 z}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial z}{\partial \theta}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 z}{\partial r\partial \theta}+\frac{\sin^2\theta}{r}\frac{\partial z}{\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 z}{\partial \theta^2}.\end{aligned}$$

Follow the same steps to calculate

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2}&=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\sin\theta\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right)\\
&=\sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial z}{\partial r}+\frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial z}{\partial r}+\frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}\right)\\
&=\sin^2\theta\frac{\partial^2 z}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial z}{\partial \theta}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 z}{\partial \theta\partial r}+\frac{\cos^2\theta}{r}\frac{\partial z}{\partial r}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 z}{\partial \theta^2}.\end{aligned}$$

**(b)** Adding the two equations above we easily see that

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

**32.** Given Exercise 31, this is easy: We know $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial}{r^2}\frac{\partial^2}{\partial \theta^2}$. Since the $z$-coordinate means the same thing in both Cartesian and cylindrical coordinates, the result follows.

**33.** **(a)** The chain rule gives $\dfrac{\partial w}{\partial \rho} = \dfrac{\partial w}{\partial r}\dfrac{\partial r}{\partial \rho} + \dfrac{\partial w}{\partial \theta}\dfrac{\partial \theta}{\partial \rho} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial \rho}$ for any appropriately differentiable function $w$. Now (6) of §1.7 gives $z = \rho\cos\varphi, r = \rho\sin\varphi$. Hence

$$\frac{\partial w}{\partial \rho} = \sin\varphi\frac{\partial w}{\partial r} + 0 + \cos\varphi\frac{\partial w}{\partial z} = \sin\varphi\frac{\partial w}{\partial r} + \cos\varphi\frac{\partial w}{\partial z}.$$

Also

$$\frac{\partial w}{\partial \varphi} = \rho\cos\varphi\frac{\partial w}{\partial r} - \rho\sin\varphi\frac{\partial w}{\partial z} \quad \text{from a similar chain rule computation.}$$

From this, we have

$$\begin{aligned}\rho\sin\varphi\frac{\partial w}{\partial \rho} + \cos\varphi\frac{\partial w}{\partial \varphi} &= \left(\rho\sin^2\varphi\frac{\partial w}{\partial r} + \rho\sin\varphi\cos\varphi\frac{\partial w}{\partial z}\right) + \left(\rho\cos^2\varphi\frac{\partial w}{\partial r} - \rho\cos\varphi\sin\varphi\frac{\partial w}{\partial z}\right)\\ &= \rho\frac{\partial w}{\partial r}.\end{aligned}$$

Thus

$$\frac{\partial w}{\partial r} = \sin\varphi\frac{\partial w}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial w}{\partial \varphi} \quad \text{or} \quad \frac{\partial}{\partial r} = \sin\varphi\frac{\partial}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial \varphi}.$$

(Alternatively, consider formula (10) in this section with $x = z$, $y = r$, $\theta$ replaced by $\varphi$, and $r$ replaced by $\rho$.)

**(b)** The cylindrical Laplacian is $\dfrac{\partial^2}{\partial r^2} + \dfrac{\partial^2}{\partial z^2} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r}$. From $z = \rho\cos\varphi$, $r = \rho\sin\varphi$, we may treat $z$ and $r$ as if they are Cartesian coordinates, so that

$$\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} \quad (\text{Cartesian}/\textit{cylindrical})$$

Now we know $\dfrac{\partial}{\partial r}$ from part (a). So, with $r = \rho\sin\varphi$, we have

$$\begin{aligned}\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r} &= \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\right)\\ &+ \frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2}{\partial \theta^2} + \frac{1}{\rho\sin\varphi}\left(\sin\varphi\frac{\partial}{\partial \rho} + \frac{\cos\varphi}{\rho}\frac{\partial}{\partial \varphi}\right)\\ &= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{2}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\varphi}{\rho^2}\frac{\partial}{\partial \varphi} \quad \text{as desired.}\end{aligned}$$

*Exercises 34–36 puts the implicit differentiation techniques which the students learned in a previous course in the context of the current discussion. This is one of those problems where it would be immediately clear if we were able to talk to each other. The problem is explaining to you which derivative with respect to $x$ is being considered. One solution is to introduce another variable. You might want to use this as an example of why the author introduces the notation she does for Exercises 39–43. One other note is that the results hold also for $F(x, y)$ or $F(x, y, z)$ being constant (not necessarily 0).*

**34.** **(a)** View $x$ and $y$ as functions of $t$, where $x = x(t) = t$ and $y = y(t)$. Since $F(x, y) = 0$ we know that $F_t(x, y) = 0$. This means that we know:

$$0 = \frac{dF}{dt} = F_x(x, y)\frac{dx}{dt} + F_y(x, y)\frac{dy}{dt}.$$

But $\dfrac{dx}{dt} = 1$ and $\dfrac{dy}{dt} = \dfrac{dy}{dx}$ so $\dfrac{dy}{dx} = -\dfrac{F_x(x, y)}{F_y(x, y)}$.

**(b-i)** If $F(x, y) = x^3 - y^2$ then $F_x(x, y) = 3x^2$ and $F_y(x, y) = -2y$ so $\dfrac{dy}{dx} = -\dfrac{3x^2}{-2y} = \dfrac{3x^2}{2y}$.

**(b-ii)** $y^2 = x^3$ so $y = x^{3/2}$ so $\dfrac{dy}{dx} = \dfrac{3}{2}x^{1/2}$. Multiply numerator and denominator by $x^{3/2}$ to get the answer in

**(b-ii)** (b-i).

**35.** Here we'll just use the formula from Exercise 34(a) where here $F(x, y) = \sin(xy) - x^2y^7 + e^y$.

$$\frac{dy}{dx} = -\frac{y\cos(xy) - 2xy^7}{x\cos(xy) - 7x^2y^6 + e^y}.$$

*The results of Exercise 36 are used in Exercises 41 and 43 in a nice way. None of them is very time consuming—it is worth assigning all three.*

**36. (a)** We have the same problem here with ambiguity about what is meant by the derivative with respect to $x$ and $y$. Let $x = x(s,t) = s, y = y(s,t) = t$, and $z = z(s,t)$. Then

$$0 = \frac{\partial F}{\partial s} = F_x(x,y,z)\frac{\partial x}{\partial s} + F_y(x,y,z)\frac{\partial y}{\partial s} + F_z(x,y,z)\frac{\partial z}{\partial s} = F_x(x,y,z) + F_z(x,y,z)\frac{\partial z}{\partial x}.$$

Solving we get

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)}.$$

An analogous calculation gives

$$\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}.$$

**(b-i)** $F(x,y,z) = xyz - 2$ so by part (a):

$$\frac{\partial z}{\partial x} = -\frac{yz}{xy} = -\frac{z}{x} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{xz}{xy} = -\frac{z}{y}.$$

**(b-ii)** $z = 2/xy$ so

$$\frac{\partial z}{\partial x} = \frac{-2}{x^2 y} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-2}{xy^2}.$$

**37.** Use the equations from Exercise 36(a) for $F(x,y,z) = x^3 z + y\cos z + (\sin y)/z = 0$:

$$\frac{\partial z}{\partial x} = \frac{-3x^2 z}{x^3 - y\sin z - (\sin y)/z^2} = \frac{-3x^2 z^3}{x^3 z^2 - yz^2 \sin z - \sin y} \quad \text{and}$$

$$\frac{\partial z}{\partial y} = \frac{-\cos z - (\cos y)/z}{x^3 - y\sin z - (\sin y)/z^2} = \frac{-z^2\cos z - z\cos y}{x^3 z^2 - yz^2\sin z - \sin y}.$$

*Exercise 38 is a good example of why you can not just blindly apply formulas such as the chain rule without first checking that all of the hypotheses are met.*

**38. (a)** By definition

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0, \quad \text{and}$$

$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

**(b)**

$$f \circ \mathbf{x} = \begin{cases} \dfrac{at}{1+a^2} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

therefore $f \circ \mathbf{x} = \dfrac{at}{1+a^2}$ and so $D(f \circ \mathbf{x})(0) = \dfrac{a}{1+a^2}$.

**(c)** By definition, $D(f)(0,0) = [f_x(0,0), f_y(0,0)]$. We calculated these in part (a) to be 0 so

$$Df(0,0)D\mathbf{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = 0.$$

The function $f$ is not differentiable at the origin and so not all of the assumptions of the chain rule are met.

**39. (a)** $\left(\dfrac{\partial w}{\partial x}\right)_{y,z} = 1$, $\left(\dfrac{\partial w}{\partial y}\right)_{x,z} = 7$, $\left(\dfrac{\partial w}{\partial z}\right)_{x,y} = -10$, $\left(\dfrac{\partial w}{\partial x}\right)_{y} = 1 - 10(2x) = 1 - 20x$, and $\left(\dfrac{\partial w}{\partial y}\right)_{x} = 7 - 10(2y) = 7 - 20y$.

**(b)** $\left(\dfrac{\partial w}{\partial x}\right)_{y} = \left(\dfrac{\partial w}{\partial x}\right)_{y,z}\left(\dfrac{\partial x}{\partial x}\right) + \left(\dfrac{\partial w}{\partial y}\right)_{x,z}\left(\dfrac{\partial y}{\partial x}\right) + \left(\dfrac{\partial w}{\partial z}\right)_{x,y}\left(\dfrac{\partial z}{\partial x}\right)$. But $\dfrac{\partial x}{\partial x} = 1$ and $\dfrac{\partial y}{\partial x} = 0$ so $\left(\dfrac{\partial w}{\partial x}\right)_{y} = \left(\dfrac{\partial w}{\partial x}\right)_{y,z} + \left(\dfrac{\partial w}{\partial z}\right)_{x,y}\left(\dfrac{\partial z}{\partial x}\right)$.

**40.** $\left(\dfrac{\partial w}{\partial x}\right)_{y,z} = 3x^2$, $\left(\dfrac{\partial w}{\partial y}\right)_{x,z} = 3y^2$, $\left(\dfrac{\partial w}{\partial z}\right)_{x,y} = 3z^2$, $\left(\dfrac{\partial w}{\partial x}\right)_{y} = 3x^2 + 3z^2(2) = 3x^2 + 6(2x-3y)^2$, and $\left(\dfrac{\partial w}{\partial y}\right)_{x} = 3y^2 + 3z^2(-3) = 3y^2 - 9(2x-3y)^2$.

**41.** $\left(\dfrac{\partial s}{\partial z}\right)_{x,y,w} = xw - 2z$, so $\left(\dfrac{\partial s}{\partial z}\right)_{x,w} = \left(\dfrac{\partial s}{\partial z}\right)_{x,y,w} + \left(\dfrac{\partial s}{\partial y}\right)_{x,z,w}\left(\dfrac{\partial y}{\partial z}\right)_{x,w}$.

To calculate $\left(\dfrac{\partial y}{\partial z}\right)_{x,w}$ we can use the results of Exercise 36 with $F(x,y,z,w) = xyw - y^3z + xz$:

$$\left(\frac{\partial y}{\partial z}\right)_{x,w} = -\frac{F_z(x,y,z,w)}{F_y(x,y,z,w)} = -\frac{-y^3+x}{xw - 3y^2z}.$$

So $\left(\dfrac{\partial s}{\partial z}\right)_{x,w} = xw - 2z + (x^2)\left(\dfrac{y^3-x}{xw-3y^2}\right)$.

**42.** $U = F(P,V,T)$ and $PV = kT$.

**(a)** $\left(\dfrac{\partial U}{\partial T}\right)_P = \left(\dfrac{\partial U}{\partial T}\right)_{P,V} + \left(\dfrac{\partial U}{\partial V}\right)_{P,T}\left(\dfrac{\partial V}{\partial T}\right) = F_T(P,V,T) + F_V(P,V,T)\left(\dfrac{k}{P}\right)$.

**(b)** $\left(\dfrac{\partial U}{\partial T}\right)_V = \left(\dfrac{\partial U}{\partial T}\right)_{P,V} + \left(\dfrac{\partial U}{\partial P}\right)_{V,T}\left(\dfrac{\partial P}{\partial T}\right) = F_T(P,V,T) + F_P(P,V,T)\left(\dfrac{k}{V}\right)$.

**(c)** $\left(\dfrac{\partial U}{\partial P}\right)_V = \left(\dfrac{\partial U}{\partial P}\right)_{V,T} + \left(\dfrac{\partial U}{\partial T}\right)_{P,V}\left(\dfrac{\partial T}{\partial P}\right) = F_P(P,V,T) + F_T(P,V,T)\left(\dfrac{V}{k}\right)$.

**43.** $\left(\dfrac{\partial x}{\partial y}\right)_z\left(\dfrac{\partial y}{\partial z}\right)_x\left(\dfrac{\partial z}{\partial x}\right)_y = \left(-\dfrac{F_y(x,y,z)}{F_x(x,y,z)}\right)\left(-\dfrac{F_z(x,y,z)}{F_y(x,y,z)}\right)\left(-\dfrac{F_x(x,y,z)}{F_z(x,y,z)}\right) = -1.$

**44.** In this case $P = kT/V$ so $(\partial P/\partial T)_V = k/V$. Similarly, $V = kT/P$ so $(\partial V/\partial P)_T = -kT/P^2$ and $T = PV/k$ so $(\partial T/\partial V)_P = P/k$. So

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = \left(\frac{k}{V}\right)\left(\frac{-kt}{P^2}\right)\left(\frac{P}{k}\right) = \frac{-kTP}{VP^2} = \frac{-kT}{VP} = -1.$$

The last equality holds since $PV = kT$.

**45.** It is easiest to use implicit differentiation and solve. For example, for the equation $ax^2 + by^2 + cz^2 - d = 0$, hold $z$ constant and take the derivative with respect to $y$. You get $2ax(\partial x/\partial y)_z + 2by = 0$. Solve this and get $(\partial x/\partial y)_z = -by/ax$. Similarly we get that $(\partial y/\partial z)_x = -cz/by$ and $(\partial z/\partial x)_y = -ax/cz$. So

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = \left(\frac{-by}{ax}\right)\left(\frac{-cz}{by}\right)\left(\frac{-ax}{cz}\right) = -1.$$

## 2.6 Directional Derivatives and the Gradient

**1. (a)** $\nabla f(x,y,z)\cdot(-\mathbf{k})$ is the directional derivative of $f(x,y,z)$ in the direction $-\mathbf{k}$ (i.e., the negative $z$ direction).

**(b)** $\nabla f(x,y,z)\cdot(-\mathbf{k}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\cdot(0,0,-1) = -\frac{\partial f}{\partial z}$.

*In Exercises 2–8, the students should notice that the given vector* **u** *is not always a unit vector and that they may have to normalize it first.*

**2.** $\nabla f(x,y) = (e^y\cos x, e^y\sin x)$ so $\nabla f(\pi/3, 0) = (1/2, \sqrt{3}/2)$.

$$D_{\mathbf{u}}f(\pi/3,0) = \nabla f(\pi/3,0)\cdot(3,-1)/\sqrt{10} = \frac{3-\sqrt{3}}{2\sqrt{10}}.$$

**3.** $\nabla f(x,y) = (2x - 6x^2y, -2x^3 + 6y^2)$, so $\nabla f(2,-1) = (28,-10)$ and

$$D_{\mathbf{u}}f(2,-1) = (28,-10)\cdot\frac{(1,2)}{\sqrt{5}} = \frac{8}{\sqrt{5}}.$$

**4.** As noted above, here we have to normalize **u** so $D_{\mathbf{u}}(f(\mathbf{a})) = \nabla f(\mathbf{a}) \cdot \dfrac{\mathbf{u}}{\|\mathbf{u}\|}$.

$$\nabla f(x,y) = \left( \frac{-2x}{(x^2+y^2)^2}, \frac{-2y}{(x^2+y^2)^2} \right) \text{ so } \nabla f(3,-2) = (1/169)(-6,4) \text{ and}$$

$$D_{\mathbf{u}} f(\mathbf{a}) = \left( \frac{-6}{169}, \frac{4}{169} \right) \cdot \frac{(1,-1)}{\sqrt{2}} = \frac{-10}{169\sqrt{2}}.$$

**5.** $\nabla f(x,y) = (e^x - 2x, 0)$ so $\nabla f(1,2) = (e-2, 0)$ and

$$D_{\mathbf{u}} f(\mathbf{a}) = (e-2,0) \cdot \frac{(2,1)}{\sqrt{5}} = \frac{2e-4}{\sqrt{5}}.$$

**6.** $\nabla f(x,y,z) = (yz, xz, xy)$ so $\nabla f(-1,0,2) = (0,-2,0)$ and

$$D_{\mathbf{u}} f(\mathbf{a}) = (0,-2,0) \cdot \frac{(-1,0,2)}{\sqrt{5}} = 0.$$

**7.** $\nabla f(x,y,z) = -e^{-(x^2+y^2+z^2)}(2x,2y,2z)$ so $\nabla f(1,2,3) = -e^{-14}(2,4,6)$ and

$$D_{\mathbf{u}} f(\mathbf{a}) = -e^{-14}(2,4,6) \cdot \frac{(1,1,1)}{\sqrt{3}} = -4\sqrt{3}e^{-14}.$$

**8.** $\nabla f(x,y,z) = \left( \frac{e^y}{3z^2+1}, \frac{xe^y}{3z^2+1}, \frac{-6xe^y z}{(3z^2+1)^2} \right)$ so $\nabla f(2,-1,0) = (e^{-1}, 2e^{-1}, 0)$ and

$$D_{\mathbf{u}} f(\mathbf{a}) = (e^{-1}, 2e^{-1}, 0) \cdot \frac{(1,-2,3)}{\sqrt{14}} = \frac{-3}{e\sqrt{14}}.$$

**9.** **(a)**

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

**(b)**

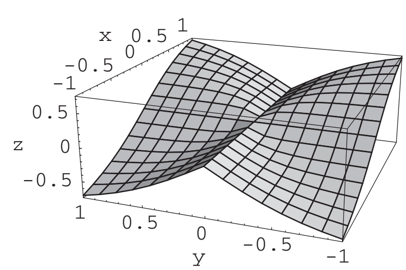
$$D_{(u,v)} f(0,0) = \lim_{h\to 0} \frac{f(hu,hv)-0}{h} = \lim_{h\to 0} \frac{hu|hv|}{h\sqrt{h^2u^2+h^2v^2}}$$

But $(u, v)$ is a unit vector so this

$$= \lim_{h\to 0} \frac{hu|h||v|}{h|h|(1)} = u|v|$$

for all unit vectors $(u, v)$.

**(c)** The graph is shown below.




**10.** **(a)**

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

**(b)**

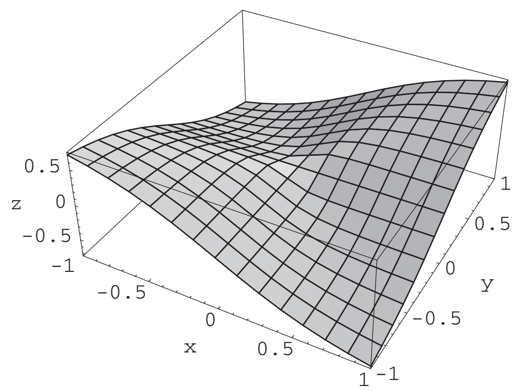
$$D_{(u,v)}f(0,0) = \lim_{h\to 0} \frac{f(hu,hv)-0}{h} = \lim_{h\to 0} \frac{(hu)(hv)}{h\sqrt{h^2u^2+h^2v^2}}$$

But $(u, v)$ is a unit vector so this

$$= \lim_{h\to 0} \frac{h^2uv}{h|h|} = uv(\text{sgn}(h))$$

for all unit vectors $(u, v)$ where $\text{sgn}(h)$ is 1 for $h \geq 0$ and $-1$ for $h < 0$. Unless $u$ or $v$ are zero, this limit doesn't exist.

**(c)** The graph is shown below.




**11.** The gradient direction for the function $h$ is $\nabla h = (-6xy^2, -6x^2y)$.

**(a)** Head in the direction $\nabla h(1,-2) = (-24, 12)$. If you prefer your directions given by a unit vector, we normalize to obtain:

$$\frac{\nabla h(1,-2)}{\|\nabla h(1,-2)\|} = \frac{(-24,12)}{\sqrt{24^2+12^2}} = \frac{(-2,1)}{\sqrt{5}}.$$

**(b)** Head in a direction orthogonal to your answer for part (a): $\pm\frac{(1,2)}{\sqrt{5}}$.

**12.** $f_x(3,7) = 3$ and $f_y(3,7) = -2$ so the gradient is $\nabla f(3,7) = (3,-2)$.

**(a)** To warm up we head in the direction of the gradient; this is the unit vector $(3,-2)/\sqrt{13}$.

**(b)** To cool off we head in the opposite direction; this is the unit vector $(-3,2)/\sqrt{13}$.

**(c)** To maintain temperature we head in a direction orthogonal to the gradient, namely $\pm(2,3)/\sqrt{13}$.

**13.** We begin by heading east and keep heading towards lower levels while intersecting each level curve orthogonally. See the solution given in the text.

**14.** We're looking at the top half of this ellipsoid. The equation is $f(x,y) = z = \sqrt{4-x^2-y^2/4}$. For the path of steepest descent, we look at the negative gradient

$$-\nabla f(x,y) = (1/2)(4-x^2-y^2/4)^{-1/2}(2x, y/2).$$

This means that

$$\frac{dy}{dx} = \frac{y/2}{2x} = \frac{y}{4x}.$$

This is the separable differential equation $(4/y)\,dy = (1/x)\,dx$ or $4\ln y = \ln x + c$. Work the usual magic and get $y^4 = kx$. So the raindrops will follow curves of that form where $z$ is constrained by the surface of the ellipsoid.

**15.** We want to head in the direction of the negative gradient. Since $M(x,y) = 3x^2 + y^2 + 5000$, the negative gradient is $-\nabla M(x,y) = (-6x, -2y)$. This means that

$$\frac{dy}{dx} = \frac{-2y}{-6x} = \frac{y}{3x}.$$

This is the separable differential equation $(3/y)\,dy = (1/x)\,dx$ or $3\ln y = \ln x + c$. Work the usual magic and get $y^3 = kx$. Substitute in the point (8, 6) to solve for $k$ to end up with the path $y^3 = 27x$.

*For Exercises 16–22 we can use equations (5) and (6) from Section 2.6 in the text.*

**16.** $f(x,y,z) = x^3 + y^3 + z^3 = 7$ so $\nabla f(x,y,z) = (3x^2, 3y^2, 3z^2)$ and $\nabla f(0,-1,2) = (0,3,12)$. So the equation of the tangent plane is:

$$0 = (0,3,12)\cdot(x-0, y+1, z-2) \quad \text{or} \quad y + 4z = 7.$$

**17.** $f(x,y,z) = ze^y \cos x = 1$ so $\nabla f(x,y,z) = (-ze^y \sin x, ze^y \cos x, e^y \cos x)$ and $\nabla f(\pi,0,-1) = (0,1,-1)$. So the equation of the tangent plane is:

$$0 = (0,1,-1)\cdot(x-\pi, y, z+1) \quad \text{or} \quad y - z = 1.$$

**18.** $f(x,y,z) = 2xz + yz - x^2y + 10 = 0$ so $\nabla f(x,y,z) = (2x - 2xy, z - x^2, 2x + y)$ and $\nabla f(1,-5,5) = (20,4,-3)$. So the equation of the tangent plane is:

$$0 = (20,4,-3)\cdot(x-1, y+5, z-5) \quad \text{or} \quad 20x + 4y - 3z = -15.$$

**19.** $f(x,y,z) = 2xy^2 - 2z^2 + xyz$ so $\nabla f(x,y,z) = (2y^2 + yz, 4xy + xz, xy - 4z)$ and $\nabla f(2,-3,3) = (9,-18,-18)$. So the equation of the tangent plane is:

$$0 = (9,-18,-18)\cdot(x-2, y+3, z-3) \quad \text{or} \quad x - 2y - 2z = 2.$$

**20.** **(a)** First we use the formula (4) from Section 2.3 in the text: $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$. If $x^2 - 2y^2 + 5xz = 7$ then $z = \frac{7+2y^2-x^2}{5x} = f(x,y)$. Calculate the two partial derivatives:

$$f_x(x,y) = \frac{-7-2y^2-x^2}{5x^2} \quad \text{so} \quad f_x(-1,0) = \frac{-8}{5}$$
$$\text{and} \quad f_y(x,y) = \frac{4y}{5x} \quad \text{so} \quad f_y(-1,0) = 0.$$

At $(-1,0,-6/5)$ formula (4) gives the equation of the tangent plane as

$$z = \frac{-6}{5} + \frac{-8}{5}(x+1).$$

**(b)** Now we'll use formula (6) from this section and calculate the gradient of $f(x,y,z) = x^2 - 2y^2 + 5xz$ as $\nabla f(x,y,z) = (2x + 5z, -4y, 5x)$ so $\nabla f(-1,0,-6/5) = (-8,0,-5)$ and so the equation for the plane is

$$0 = (-8,0,-5)\cdot(x+1, y, z+6/5) \quad \text{or} \quad -8x - 5z = 14.$$

This agrees with the answer we found in part (a).

**21.** **(a)** First we use the formula (4) from Section 2.3 in the text: $x = f(a,b) + f_y(a,b)(y-a) + f_z(a,b)(z-b)$. If $x \sin y + xz^2 = 2e^{yz}$ then $x = \frac{2e^{yz}}{\sin y + z^2} = f(y,z)$. Calculate the two partial derivatives:

$$f_y(y,z) = 2e^{yz}\frac{z \sin y + z^3 - \cos z}{(\sin y + z^2)^2} \quad \text{so} \quad f_y(\pi/2, 0) = 0$$
$$\text{and} \quad f_z(y,z) = 2e^{yz}\frac{y \sin y + yz^2 - 2z}{(\sin y + z^2)^2} \quad \text{so} \quad f_z(\pi/2, 0) = \pi.$$

At $(2, \pi/2, 0)$ formula (4) gives the equation of the tangent plane as

$$x = 2 + \pi z.$$

**(b)** Now we'll use formula (6) from this section and calculate the gradient of $f(x,y,z) = x \sin y + xz^2 - 2e^{yz}$ as $\nabla f(x,y,z) = (\sin y + z^2, -x\cos y - 2ze^{yz}, 2xz - 2ye^{yz})$ so $\nabla f(2, \pi/2, 0) = (1, 0, -\pi)$ and so the equation for the plane is

$$0 = (1,0,-\pi)\cdot(x-2, y-\pi/2, z) \quad \text{or} \quad x - 2 - \pi z = 0.$$

This agrees with the answer we found in part (a).

**22.** Using formula (6) we get that the gradient of $f(x,y,z) = x^3 - 2y^2 + z^2$ at $(x_0, y_0, z_0)$ is $\nabla f(x_0,y_0,z_0) = (3x_0^2, -4y_0, 2z_0)$. For this to be perpendicular to the given line, $(3x_0^2, -4y_0, 2z_0) = k(3, 2, -\sqrt{2})$. This means that $x_0^2 = -2y_0$ and $z_0 = -(\sqrt{2}/2)x_0^2$. Substituting this back into the equation of the surface, we get that $x_0^3 - 2x_0^4/4 + x_0^4/2 = 27$ or $x_0 = 3$. Our point is, therefore $(3, -9/2, -9\sqrt{2}/2)$.

**23.** The tangent plane to the surface at a point $(x_0, y_0, z_0)$ is

$$0 = 18x_0(x - x_0) - 90y_0(y - y_0) + 10z_0(z - z_0).$$

For this to be parallel to $x + 5y - 2z = 7$, the vector

$$(18x_0, -90y_0, 10z_0) = k(1, 5, -2).$$

This means that $y_0 = -x_0$ and $z_0 = (-18/5)x_0$. Substitute these back into the equation of the hyperboloid: $9x^2 - 45y^2 + 5z^2 = 45$ to get:

$$45 = 9x_0^2 - 45x_0^2 + 5(18^2/5^2)x_0^2 \quad \text{therefore} \quad x_0 = \pm 5/4.$$

This means that the points are $(5/4, -5/4, -9/2)$ and $(-5/4, 5/4, 9/2)$.

**24.** First note that $(2, 1, -1)$ lies on both surfaces: $7 \cdot 2^2 - 12 \cdot 2 - 5 \cdot 1 = -1$, $2 \cdot 1(-1)^2 = 2$. The normal to the first surface at $(2, 1, -1)$ is given by $(f_x(2, 1), f_y(2, 1), -1)$ where $f(x, y) = 7x^2 - 12x - 5y^2$. This is $((14x - 12)|_{(2,1)}, -10y|_{(2,1)}, -1) = (16, -10, -1)$. The normal to the second surface at $(2, 1, -1)$ is $\nabla F(2, 1, -1)$ where $F(x, y, z) = xyz^2$. This is $(yz^2, xz^2, 2xyz)|_{(2,1,-1)} = (1, 2, -4)$. We have

$$(16, -10, -1) \cdot (1, 2, -4) = 16 - 20 + 4 = 0.$$

Since the normals are orthogonal, the tangent planes must be so as well.

**25.** The two surfaces are tangent at $(x_0, y_0, z_0) \Leftrightarrow$ the tangent planes at $(x_0, y_0, z_0)$ are the same $\Leftrightarrow$ normal vectors at $(x_0, y_0, z_0)$ are parallel (since the surfaces intersect at $(x_0, y_0, z_0)$) $\Leftrightarrow \nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0) = \mathbf{0}$.

**26. (a)** $S$ is the level set at height 0 of $f(x, y, z) = x^2 + 4y^2 - z^2$ so $\nabla f = (2x, 8y, -2z) \Rightarrow$ $\nabla f(3, -2, -5) = (6, -16, 10)$. Thus formula (6) gives the equation of the tangent plane as $6(x - 3) - 16(y + 2) + 10(z + 5) = 0$ or $3x - 8y + 5z = 0$.

**(b)** $\nabla f(0, 0, 0) = (0, 0, 0)$ so formula (6) cannot be used. Note that there's no tangent plane at the origin, which is the vertex of the cone (i.e., the surface is not "locally flat" there).

**27. (a)** For $f(x, y, z) = x^3 - x^2y^2 + z^2$, $\nabla f(x, y, z) = (3x^2 - 2xy^2, -2x^2y, 2z)$ so $\nabla f(2, -3/2, 1) = (3, 12, 2)$. Thus the equation of the tangent plane is

$$3(x - 2) + 12(y + 3/2) + 2(z - 1) = 0 \quad \text{or} \quad 3x + 12y + 2z + 10 = 0.$$

**(b)** $\nabla f(0, 0, 0) = (0, 0, 0)$ so the gradient cannot be used as a normal vector. If we solve $z = \pm\sqrt{y^2x^2 - x^3} = \pm x\sqrt{y^2 - x}$, we see that $g(x, y) = x\sqrt{y^2 - x}$ fails to be differentiable at (0, 0)—so there is no tangent plane there.

**28. (a)** $2x + 2y\dfrac{dy}{dx} = 0$ so

$$\left.\frac{dy}{dx}\right|_{(-\sqrt{2},\sqrt{2})} = \left.\frac{-x}{y}\right|_{(-\sqrt{2},\sqrt{2})} = \frac{-\sqrt{2}}{-\sqrt{2}} = 1.$$

The equation of the line is $y - \sqrt{2} = x + \sqrt{2}$.

**(b)** The equation of the tangent line is $0 = \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0)$. Here $f(x, y) = x^2 + y^2 = 4$ so $\nabla f(x, y) = (2x, 2y)$ or $\nabla f(-\sqrt{2}, \sqrt{2}) = (-2\sqrt{2}, 2\sqrt{2})$. The equation of the tangent line is

$$0 = (-2\sqrt{2}, 2\sqrt{2}) \cdot (x + \sqrt{2}, y - \sqrt{2}) \quad \text{or} \quad x - y = -2\sqrt{2}.$$

**29. (a)** $3y^2\dfrac{dy}{dx} = 2x + 3x^2$ so $\left.\dfrac{dy}{dx}\right|_{(1,\sqrt[3]{2})} = \dfrac{5}{(3)2^{2/3}}$. The equation of the tangent line is

$$y - \sqrt[3]{2} = \frac{5}{(3)2^{2/3}}(x - 1).$$

**(b)** $f(x, y) = y^3 - x^2 - x^3$ so $\nabla f(x, y) = (-2x - 3x^2, 3y^2)$ so $\nabla f(1, \sqrt[3]{2}) = (-5, (3)2^{2/3})$. The equation of the tangent line is

$$0 = (-5, (3)2^{2/3}) \cdot (x - 1, y - \sqrt[3]{2}) \quad \text{or} \quad -5x + (3)2^{2/3}y = 1.$$

**30. (a)** $5x^4 + 2y + 2x\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 0$ so $\left.\dfrac{dy}{dx}\right|_{(2,-2)} = \dfrac{-76}{16} = \dfrac{-19}{4}$. The equation of the tangent line is

$$y + 2 = \frac{-19}{4}(x - 2).$$

**(b)** $f(x,y) = y^3 - x^2 - x^3$ so $\nabla f(x,y) = (5x^4 + 2y, 2x + 3y^2)$ so $\nabla f(2,-2) = (76,16)$. The equation of the tangent line is

$$0 = (76,16)\cdot(x-2, y+2) \quad \text{or} \quad 19x + 4y = 30.$$

**31.** If $f(x,y) = x^2 - y^2$ then $\nabla f(5,-4) = (10,8)$ so the equations of the normal line are

$$x(t) = 10t + 5 \quad \text{and} \quad y(t) = 8t - 4 \quad \text{or} \quad 8x - 10y = 80.$$

**32.** If $f(x,y) = x^2 - x^3 - y^2$ then $\nabla f(-1,\sqrt{2}) = (5, 2\sqrt{2})$ so the equations of the normal line are

$$x(t) = 5t - 1 \quad \text{and} \quad y(t) = 2\sqrt{2}t - \sqrt{2} \quad \text{or} \quad 2\sqrt{2}x - 5y = -7\sqrt{2}.$$

**33.** If $f(x,y) = x^3 - 2xy + y^5$ then $\nabla f(2,-1) = (14,1)$ so the equations of the normal line are

$$x(t) = 14t + 2 \quad \text{and} \quad y(t) = t - 1 \quad \text{or} \quad x - 14y = 16.$$

**34.** If $f(x,y,z) = x^3 z + x^2 y^2 + \sin(yz)$ then

$$\nabla f(x,y,z) = (3x^2 + 2xy^2, 2x^2 y + z\cos(yz), x^3 + y\cos(yz)).$$

**(a)** The plane is given by $0 = \nabla f(-1,0,3)\cdot(x+1, y, z-3) = 9(x+1) + 3y - (z-3)$ or $9x + 3y - z = -12$.
**(b)** The normal line to the surface at $(-1,0,3)$ is given by

$$\begin{cases} x = 9t - 1 \\ y = 3t \\ z = -t + 3. \end{cases}$$

**35.** Using the method above for $f(x,y,z) = e^{xy} + e^{zx} - 2e^{yz}$, we find that $\nabla f = (ye^{xy} + ze^{xz}, xe^{xy} - 2ze^{yz}, xe^{xz} - 2ye^{yz})$ so $\nabla f(-1,-1,-1) = e(-2,1,1)$. So

$$\begin{cases} x = -2et - 1 \\ y = et - 1 \\ z = et - 1 \end{cases} \quad \text{or, factoring out } e, \quad \begin{cases} x = -2t - 1 \\ y = t - 1 \\ z = t - 1. \end{cases}$$

**36.** Remember in the equation of a plane $0 = \mathbf{v}\cdot(x - x_0, y - y_0, z - z_0)$ that $\mathbf{v}$ is a vector orthogonal to the plane. We saw in this section that we can use $\nabla f(x_0, y_0, z_0)$ for $\mathbf{v}$. This means that the equation of the line normal to a surface given by the equation $F(x,y,z) = 0$ at a given point $(x_0, y_0, z_0)$ is

$$(x,y,z) = \nabla F(x_0, y_0, z_0)t + (x_0, y_0, z_0).$$

**37.** The hypersurface is the level set at height $-1$ of the function $f(x_1, \ldots, x_5) = \sin x_1 + \cos x_2 + \sin x_3 + \cos x_4 + \sin x_5$. We find $\nabla f\left(\pi, \pi, \frac{3\pi}{2}, 2\pi, 2\pi\right) = (-1,0,0,0,1)$. Hence the tangent hyperplane has equation

$$-1(x_1 - \pi) + 1(x_5 - 2\pi) = 0 \quad \text{or} \quad x_5 - x_1 = \pi.$$

**38.** The surface is the level set at height $\frac{n(n+1)}{2}$ of the function $f(x_1, \ldots, x_n) = x_1^2 + 2x_2^2 + \cdots + nx_n^2$. We have $\nabla f = (2x_1, 4x_2, 6x_3, \ldots, 2nx_n) \Rightarrow \nabla f(-1, \ldots, -1) = -2(1,2,3,\ldots,n)$. An equation for the tangent hyperplane is thus

$$1(x_1 + 1) + 2(x_2 + 1) + 3(x_3 + 1) + \cdots + n(x_n + 1) = 0$$

or

$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n + \frac{n(n+1)}{2} = 0$$

**39.** Here $f(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2$ so $\nabla f(x_1, x_2, \ldots, x_n) = (2x_1, 2x_2, \ldots, 2x_n)$. Using the techniques of this section, the tangent hyperplane to the $(n-1)$-dimensional sphere $f(x_1, x_2, \ldots, x_n) = 1$ at $(1/\sqrt{n}, 1/\sqrt{n}, \ldots, 1/\sqrt{n}, -1/\sqrt{n})$ is

$$\begin{aligned} 0 &= \nabla f(1/\sqrt{n}, \ldots, 1/\sqrt{n}, -1/\sqrt{n})\cdot(x_1 - 1/\sqrt{n}, x_2 - 1/\sqrt{n}, \ldots, x_{n-1} - 1/\sqrt{n}, x_n + 1/\sqrt{n}) \\ &= \frac{2}{\sqrt{n}}\left(x_1 - \frac{1}{\sqrt{n}}\right) + \frac{2}{\sqrt{n}}\left(x_2 - \frac{1}{\sqrt{n}}\right) + \cdots + \frac{2}{\sqrt{n}}\left(x_{n-1} - \frac{1}{\sqrt{n}}\right) + \frac{-2}{\sqrt{n}}\left(x_n + \frac{1}{\sqrt{n}}\right) \quad \text{or} \end{aligned}$$

$$0 = (x_1 - 1/\sqrt{n}) + (x_2 - 1/\sqrt{n}) + \cdots + (x_{n-1} - 1/\sqrt{n}) - (x_n + 1/\sqrt{n}) \quad \text{so}$$

$$\sqrt{n} = x_1 + x_2 + \cdots + x_{n-1} - x_n.$$

**40.** $F(x, y, z) = z^2y^3 + x^2y = 2$.

**(a)** We can write $z = f(x, y)$ when $F_z \neq 0$. $F_z(x, y, z) = 2zy^3$ is not 0 when both $z \neq 0$ and $y \neq 0$.

**(b)** We can write $x = f(y, z)$ when $F_x \neq 0$. $F_x(x, y, z) = 2xy$ is not 0 when both $x \neq 0$ and $y \neq 0$.

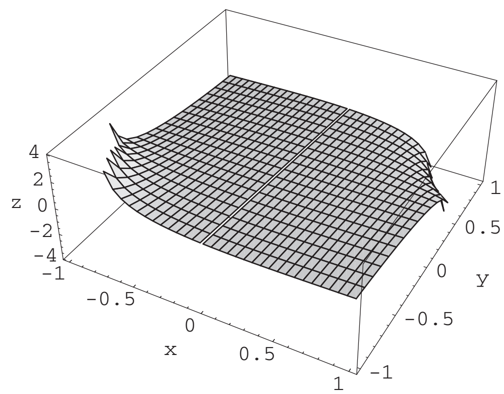
**(c)** We can write $y = f(x, z)$ when $F_y \neq 0$. $F_y(x, y, z) = 3z^2y^2 + x^2$ is not 0 everywhere but on the $y$- or $z$-axis (i.e., except when $x = 0$ at the same time that either $y = 0$ or $z = 0$).

**41. (a)** $\dfrac{\partial F}{\partial z} = xe^{xz}$. This is non-zero whenever $x \neq 0$. There we can solve for $z$ to get

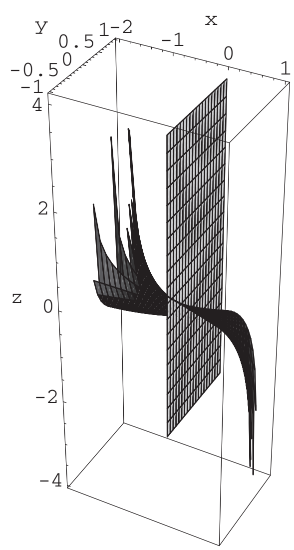
$$z = \frac{\ln(1 - \sin xy - x^3y)}{x}.$$

**(b)** Looking only at points in $S$ we only need to stay away from points in $yz$-plane (i.e., where $x = 0$).

**(c)** You shouldn't then make the leap from your answer to part (b) that you can graph $z = \dfrac{\ln(1 - \sin xy - x^3y)}{x}$ for any values of $x$ and $y$ just so $x \neq 0$. Your other restriction is that $1 - \sin xy - x^3y > 0$ as it is the argument of the natural logarithm. A sketch that gives you an idea of the surface is:




Now the actual surface $S$ includes the plane $x = 0$ since $x = 0$ satisfies the original equation: $\sin xy + e^{xz} + x^3y = 1$. $S$ will actually look a bit like:



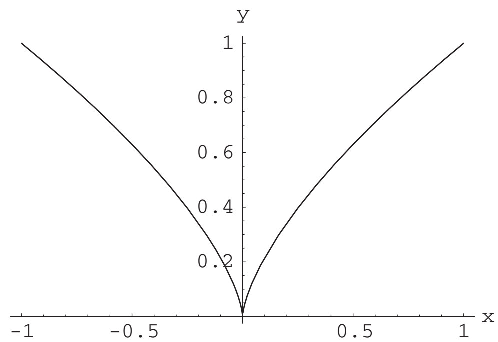

**42.** The point of this problem is that since $F(x, y) = c$ defines a curve $C$ in $\mathbf{R}^2$ such that either $f_x(x_0, y_0) \neq 0$ or $f_y(x_0, y_0) \neq 0$ then by the implicit function theorem we can represent the curve near $(x_0, y_0)$ as either the graph of a function $x = g(y)$ or a function $y = g(x)$.

*Exercise 43 poses a bit of a puzzle. Here we can write the equation of $C$ as $y = f(x)$ even though $F_y$ is zero at the origin. Why doesn't this contradict the implicit function theorem? What "goes bad" in Exercise 43 is that we have a corner at the origin.*

*You may also want to assign the students the same problem for the function* $F(x, y) = x - y^3$.

**43. (a)** $F(0,0) = 0$ so the origin lies on the curve $C$. $F_y(x, y) = 3y^2$ and so $F_y(0,0) = 0$.

**(b)** We can write $C$ as the graph of $y = x^{2/3}$. The graph of $C$ is




**(c)** So here we are with $F_y(0,0) = 0$ but we can express the graph of $C$ everywhere as $y = x^{2/3}$. On second look we see that $C$ is not a $C^1$ function—it has a corner at the origin—and so the implicit function theorem doesn't apply.

**44. (a)** $F(x, y) = xy + 1$ so $F_y(x, y) = x$ and so we cannot solve $F(x, y) = c$ for $y$ when $x = 0$ or when $c = 0(y) + 1 = 1$. In other words, level sets are unions of smooth curves in $\mathbf{R}^2$ except for $c = 1$.

**(b)** Here the function is $F(x, y, z) = xyz + 1$. Using a similar argument to that in part (a), $F_z(x, y, z) = xy$ and this is only 0 when $xy = 0$. This means that we cannot solve $F(x, y, z) = c$ for $z$ when $xy = 0$ or when $c = z(0) + 1 = 1$. So level sets of this family are unions of smooth surfaces in $\mathbf{R}^3$ except for level $c = 1$.

**45. (a)** $G(-1, 1, 1) = F(-1 - 2 + 1, -1 - 1 + 3) = F(-2, 1) = 0$.

**(b)** To invoke the implicit function theorem, we need to show that $G_z(-1, 1, 1) \neq 0$.

$$\begin{aligned} G_z(-1,1,1) &= F_u(-2,1)\frac{\partial(x^3 - 2y^2 + z^5)}{\partial z}\bigg|_{(-1,1,1)} + F_v(-2,1)\frac{\partial(xy - x^2 z + 3)}{\partial z}\bigg|_{(-1,1,1)} \\ &= (7)(5) + (5)(1) = 40 \neq 0. \end{aligned}$$

**46.** Let $F_1 = x_2 y_2 - x_1 \cos y_1 = 5$ and $F_2 = x_2 \sin y_1 + x_1 y_2 = 2$. Solving for $y$ in terms of $x$ means that we have to look at the determinant

$$\det \begin{bmatrix} \dfrac{\partial F_1}{\partial y_1} & \dfrac{\partial F_1}{\partial y_2} \\ \dfrac{\partial F_2}{\partial y_1} & \dfrac{\partial F_2}{\partial y_2} \end{bmatrix} = \det \begin{bmatrix} x_1 \sin y_1 & x_2 \\ x_2 \cos y_1 & x_1 \end{bmatrix} = x_1^2 \sin y_1 + x_2^2 \cos y_1.$$

To see that you can solve for $y_1$ and $y_2$ in terms of $x_1$ and $x_2$ near $(x_1, x_2, y_1, y_2) = (2, 3, \pi, 1)$, evaluate the determinant at that point. We get $-9$. This is not 0 so you can, at least in theory, solve for the $y$'s in terms of the $x$'s.

To see that you can solve for $y_1$ and $y_2$ as functions of $x_1$ and $x_2$ near $(x_1, x_2, y_1, y_2) = (0, 2, \pi/2, 5/2)$, evaluate the determinant at that point. We get 0. We can not solve for the $y$'s in terms of the $x$'s.

**47. (a)** Let $F_1 = x_1^2 y_2^2 - 2x_2 y_3 = 1$, $F_2 = x_1 y_1^5 + x_2 y_2 - 4y_2 y_3 = -9$, and $F_3 = x_2 y_1 + 3x_1 y_3^2 = 12$. Solving for $y$'s in terms of $x$'s means that we have to look at the determinant

$$\det \begin{bmatrix} \dfrac{\partial F_1}{\partial y_1} & \dfrac{\partial F_1}{\partial y_2} & \dfrac{\partial F_1}{\partial y_3} \\ \dfrac{\partial F_2}{\partial y_1} & \dfrac{\partial F_2}{\partial y_2} & \dfrac{\partial F_2}{\partial y_3} \\ \dfrac{\partial F_3}{\partial y_1} & \dfrac{\partial F_3}{\partial y_2} & \dfrac{\partial F_3}{\partial y_3} \end{bmatrix} = \det \begin{bmatrix} 0 & 2x_1^2 y_2 & -2x_2 \\ 5x_1 y_1^4 & x_2 - 4y_3 & -4y_2 \\ x_2 & 0 & 6x_1 y_3 \end{bmatrix}$$

$$= -60x_1^4 y_1^4 y_2 y_3 - 8x_1^2 x_2 y_2^2 + 2x_2^3 - 8x_2^2 y_3.$$

Evaluating this at the point $(x_1, x_2, y_1, y_2, y_3) = (1, 0, -1, 1, 2)$ results in $-120 \neq 0$. This means that we can solve for $y_1, y_2$, and $y_3$ in terms of $x_1$ and $x_2$.

**(b)** Take the partials of the three equations with respect to $x_1$ to get

$$\begin{cases} y_2^2 + 2x_1y_2\dfrac{\partial y_2}{\partial x_1} - 2x_2\dfrac{\partial y_3}{\partial x_1} = 0 \\ y_1^5 + 5x_1y_1^4\dfrac{\partial y_1}{\partial x_1} + x_2\dfrac{\partial y_2}{\partial x_1} - 4y_3\dfrac{\partial y_2}{\partial x_1} - 4y_2\dfrac{\partial y_3}{\partial x_1} = 0 \\ x_2\dfrac{\partial y_1}{\partial x_1} + 3y_3^2 + 6x_1y_3\dfrac{\partial y_3}{\partial x_1} = 0. \end{cases}$$

At the point $(1, 0, -1, 1, 2)$ this system of equations becomes:

$$\begin{cases} 1 + 2\dfrac{\partial y_2}{\partial x_1} = 0 \\ -1 + 5\dfrac{\partial y_1}{\partial x_1} - 8\dfrac{\partial y_2}{\partial x_1} - 4\dfrac{\partial y_3}{\partial x_1} = 0 \\ 12 + 12\dfrac{\partial y_3}{\partial x_1} = 0. \end{cases}$$

Solving, we find that

$$\frac{\partial y_1}{\partial x_1} = \frac{-7}{5}, \frac{\partial y_2}{\partial x_1} = -\frac{1}{2}, \quad \text{and} \quad \frac{\partial y_3}{\partial x_1} = -1.$$

**48. (a)** We need to consider where the following determinant is non-zero.

$$\begin{vmatrix} \partial F_1/\partial r & \partial F_1/\partial \theta & \partial F_1/\partial z \\ \partial F_2/\partial r & \partial F_2/\partial \theta & \partial F_2/\partial z \\ \partial F_3/\partial r & \partial F_3/\partial \theta & \partial F_3/\partial z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

In other words, for any points for which $r \neq 0$.

**(b)** This makes complete sense. When the radius is 0 then $r$ and $z$ completely determine the point. You get no extra information from the $\theta$ component. Without the $z$ coordinate, this is the standard problem when using polar coordinates in the plane.

**49. (a)** As with Exercise 48, we need to consider where the same determinant is non-zero. In this case the determinant is

$$\begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix} = \rho^2\sin\varphi\cos^2\varphi + \rho^2\sin^3\varphi = \rho^2\sin\varphi.$$

In other words, for any points for which $\rho \neq 0$ and for which $\sin\varphi \neq 0$.

**(b)** Again, this makes complete sense. When the radius is 0, then $\rho$ completely determines the point as being the origin. When $\sin\varphi = 0$ you are on the $z$-axis so $\theta$ no longer contributes any information.

## 2.7 Newton's Method

**1.** We begin by defining the function $\mathbf{f}(x, y) = (y^2e^x - 3, 2ye^x + 10y^4)$. Then we have

$$D\mathbf{f}(x, y) = \begin{bmatrix} y^2e^x & 2ye^x \\ 2ye^x & 2e^x + 40y^3 \end{bmatrix}.$$

The inverse of this matrix is

$$[D\mathbf{f}(x, y)]^{-1} = \begin{bmatrix} \dfrac{2e^x + 40y^3}{40y^5e^x - 2y^2e^{2x}} & \dfrac{1}{ye^x - 20y^4} \\ \dfrac{1}{ye^x - 20y^4} & \dfrac{1}{40y^3 - 2e^x} \end{bmatrix}.$$

Hence the iteration expression

$$\mathbf{x}_k = \mathbf{x}_{k-1} - [D\mathbf{f}(\mathbf{x}_{k-1})]^{-1}\mathbf{f}(\mathbf{x}_{k-1})$$

becomes (after some simplification using *Mathematica*)

$$x_k = \frac{x_{k-1}y_{k-1}^2 e^{x_{k-1}} - y_{k-1}^2 e^{x_{k-1}} + 10y_{k-1}^5 - 60e^{-x_{k-1}}y_{k-1}^3 - 20x_{k-1}y_{k-1}^5 - 3}{y_{k-1}^2 e^{x_{k-1}} - 20y_{k-1}^5}$$

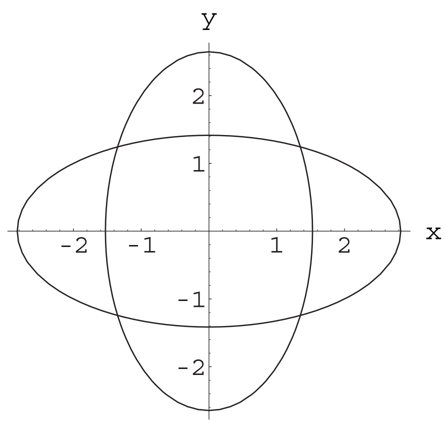
$$y_k = \frac{y_{k-1}^2 e^{x_{k-1}} - 15y_{k-1}^5 + 3}{y_{k-1}e^{x_{k-1}} - 20y_{k-1}^4}.$$

Using initial vector $(x_0, y_0) = (1, -1)$ and iterating the formulas above we obtain the following results:

| $k$ | $x_k$ | $y_k$ |
|---|---|---|
| 0 | 1 | −1 |
| 1 | 1.279707977 | −0.911965173 |
| 2 | 1.302659547 | −0.902966291 |
| 3 | 1.302942519 | −0.902880458 |
| 4 | 1.302942538 | −0.902880451 |
| 5 | 1.302942538 | −0.902880451 |

Since the result appears to be stable to nine decimal places, we conclude that the approximate solution is $(1.302942538, -0.902880451)$.

**2. (a)** We obtain the following graph for the ellipses:




From this graph, we can estimate an intersection point in the first quadrant to be near to the point $(1, 1)$.

**(b)** If $(X, Y)$ is an intersection point, then we must have

$$\begin{cases} 3X^2 + Y^2 = 7 \\ X^2 + 4Y^2 = 8 \end{cases}.$$

Because only even exponents appear, we also conclude that

$$\begin{cases} 3(\pm X)^2 + (\pm Y)^2 = 7 \\ (\pm X)^2 + 4(\pm Y)^2 = 8 \end{cases}.$$

Hence if $(X, Y)$ is an intersection point, then so are $(-X, Y)$, $(X, -Y)$, and $(-X, -Y)$.

**(c)** Using the function $\mathbf{f}(x, y) = (3x^2 + y^2 - 7, x^2 + 4y^2 - 8)$, we have

$$D\mathbf{f}(x,y) = \begin{bmatrix} 6x & 2y \\ 2x & 8y \end{bmatrix} \implies [D\mathbf{f}(x,y)]^{-1} = \begin{bmatrix} \dfrac{2}{11x} & -\dfrac{1}{22x} \\ -\dfrac{1}{22x} & \dfrac{3}{22y} \end{bmatrix}.$$

Hence the iteration expression in formula (6) becomes

$$x_k = \frac{11x_{k-1}^2 + 20}{22x_{k-1}}, \qquad y_k = \frac{11y_{k-1}^2 + 17}{22y_{k-1}}.$$

Using initial vector $(x_0, y_0) = (1, 1)$ and iterating the formulas above we obtain the following results:

| $k$ | $x_k$ | $y_k$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1.409090909 | 1.272727273 |
| 2 | 1.349706745 | 1.243506494 |
| 3 | 1.348400358 | 1.243163168 |
| 4 | 1.348399725 | 1.243163121 |
| 5 | 1.348399725 | 1.243163121 |

This result appears to be stable to nine decimal places, so we conclude that the intersection point in the first quadrant is approximately $(1.348399725, 1.243163121)$. In view of part (b), the other intersection points must be (approximately) $(-1.348399725, 1.243163121)$, $(1.348399725, -1.243163121)$, and $(-1.348399725, -1.243163121)$.

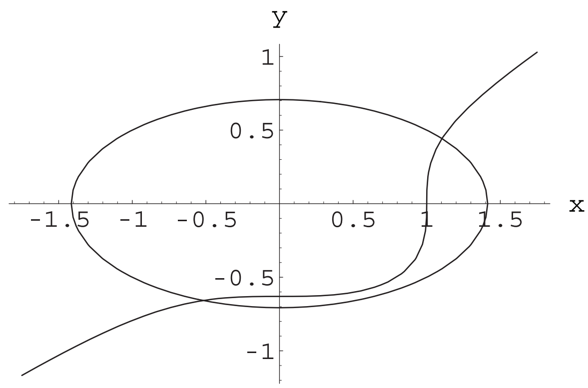
**(d)** From the equation $3x^2 + y^2 = 7$, we must have $y^2 = 7 - 3x^2$. Substituting for $y^2$ in the equation $x^2 + 4y^2 = 8$, we find that

$$x^2 + 4(7 - 3x^2) = 8 \iff 11x^2 = 20 \iff x = \pm\sqrt{\frac{20}{11}} \approx \pm 1.348399725$$

and

$$y = \pm\sqrt{\frac{17}{11}} \approx \pm 1.243163121.$$

**3. (a)** The graphs of the curves are as follows:




From the graph, we estimate one intersection point near $(1, 1/2)$, and a second near $(-1/2, -3/4)$.

**(b)** Using the function $\mathbf{f}(x, y) = (x^3 - 4y^3 - 1, x^2 + 4y^2 - 2)$, we have

$$D\mathbf{f}(x, y) = \begin{bmatrix} 3x^2 & -12y^2 \\ 2x & 8y \end{bmatrix} \implies [D\mathbf{f}(x, y)]^{-1} = \begin{bmatrix} \dfrac{1}{3x^2 + 3xy} & \dfrac{y}{2x^2 + 2xy} \\ -\dfrac{1}{12xy + 12y^2} & \dfrac{x}{8xy + 8y^2} \end{bmatrix}.$$

Then the iteration expression in formula (6) becomes

$$x_k = \frac{4x_{k-1}^3 + 3x_{k-1}^2 y_{k-1} - 4y_{k-1}^3 + 6y_{k-1} + 2}{6x_{k-1}(x_{k-1} + y_{k-1})}$$

$$y_k = \frac{-x_{k-1}^3 + 12x_{k-1}y_{k-1}^2 + 16y_{k-1}^3 + 6x_{k-1} - 2}{24y_{k-1}(x_{k-1} + y_{k-1})}.$$

Using initial vector $(x_0, y_0) = (1, 0.5)$ and iterating the formulas above we obtain the following results:

| $k$ | $x_k$ | $y_k$ |
|---|---|---|
| 0 | 1 | 0.5 |
| 1 | 1.111111111 | 0.444444444 |
| 2 | 1.103968254 | 0.441964286 |
| 3 | 1.103931712 | 0.441965716 |
| 4 | 1.103931711 | 0.441965716 |
| 5 | 1.103931711 | 0.441965716 |

The data imply that to nine decimal places there is an intersection point at $(1.103931711, 0.441965716)$. Using initial vector $(x_0, y_0) = (-0.5, -0.75)$ and iterating, we find

| $k$ | $x_k$ | $y_k$ |
|---|---|---|
| 0 | $-0.5$ | $-0.75$ |
| 1 | $-0.5$ | $-0.666666667$ |
| 2 | $-0.518518519$ | $-0.657986111$ |
| 3 | $-0.518214436$ | $-0.65792361$ |
| 4 | $-0.518214315$ | $-0.657923613$ |
| 5 | $-0.518214315$ | $-0.657923613$ |

Thus it appears that to nine decimal places there is an second intersection point at $(-0.518214315, -0.657923613)$.

**4.** Let $\mathbf{L}$ denote $\lim_{k\to\infty} \mathbf{x}_k$. Then $\lim_{k\to\infty} \mathbf{x}_{k-1} = \mathbf{L}$ and taking limits in (6), we have

$$\mathbf{L} = \mathbf{L} - [D\mathbf{f}(\mathbf{L})]^{-1}\mathbf{f}(\mathbf{L}).$$

Hence $[D\mathbf{f}(\mathbf{L})]^{-1}\mathbf{f}(\mathbf{L}) = \mathbf{0}$. Now multiply by $D\mathbf{f}(\mathbf{L})$ on the left to obtain $D\mathbf{f}(\mathbf{L})([D\mathbf{f}(\mathbf{L})]^{-1}\mathbf{f}(\mathbf{L}))D\mathbf{f}(\mathbf{L})\mathbf{0} = \mathbf{0} \Leftrightarrow I_n\mathbf{f}(\mathbf{L}) = \mathbf{0} \Leftrightarrow \mathbf{f}(\mathbf{L}) = \mathbf{0}$.

**5. (a)**

| $k$ | $x_k$ | $y_k$ |
|---|---|---|
| 0 | $-1$ | 1 |
| 1 | $-1.3$ | 1.7 |
| 2 | $-1.2653846$ | 1.55588235 |
| 3 | $-1.2649112$ | 1.54920772 |
| 4 | $-1.2649111$ | 1.54919334 |
| 5 | $-1.2649111$ | 1.54919334 |

This table suggests that $\mathbf{x}_k \to (-1.2649111, 1.54919334) \approx (-\sqrt{8/5}, \sqrt{12/5})$.

**(b)**

| $k$ | $x_k$ | $y_k$ |
|---|---|---|
| 0 | 1 | $-1$ |
| 1 | 1.3 | $-1.7$ |
| 2 | 1.26538462 | $-1.558824$ |
| 3 | 1.2649115 | $-1.5492077$ |
| 4 | 1.26491106 | $-1.5491933$ |
| 5 | 1.26491106 | $-1.5491933$ |

Here $\mathbf{x}_k \to (\sqrt{8/5}, -\sqrt{12/5})$ it seems.

| $k$ | $x_k$ | $y_k$ |
|---|---|---|
| 0 | $-1$ | $-1$ |
| 1 | $-1.3$ | $-1.7$ |
| 2 | $-1.2653846$ | $-1.555824$ |
| 3 | $-1.2649112$ | $-1.5492077$ |
| 4 | $-1.2649111$ | $-1.5491933$ |
| 5 | $-1.2649111$ | $-1.5491933$ |

Here $\mathbf{x}_k \to (-\sqrt{8/5}, -\sqrt{12/5})$.

**(c)** The results don't seem too strange; each initial vector is in a different quadrant and the limit is an intersection point in the same quadrant.

**6.** **(a)**

| $k$ | $x_k$ | $y_k$ |
|---|---|---|
| 0 | 1.4 | 10 |
| 1 | 54.7 | −317.452 |
| 2 | 28.0832917 | −75583.381 |
| 3 | 14.8412307 | −9364.2812 |
| 4 | 8.35050251 | −1128.3294 |
| 5 | 5.34861164 | −119.96986 |
| 6 | 4.2264792 | −4.8602841 |
| 7 | 4.00886454 | 4.73583325 |
| 8 | 4.0001468 | 4.99959722 |
| 9 | 4 | 5 |
| 10 | 4 | 5 |

**(b)**

| $k$ | $x_k$ | $y_k$ |
|---|---|---|
| 0 | 1.3 | 10 |
| 1 | −105.35 | 641.7815 |
| 2 | −52.041661 | 606283.635 |
| 3 | −25.420779 | 75622.9747 |
| 4 | −12.17662 | 9372.7823 |
| 5 | −5.6848239 | 1132.95037 |
| 6 | −2.6823677 | 124.108919 |
| 7 | −1.5599306 | 8.94078154 |
| 8 | −1.3422068 | −0.6614827 |
| 9 | −1.333348 | −0.9255223 |
| 10 | −1.3333333 | −0.9259259 |
| 11 | −1.3333333 | −0.9259259 |

**(c)** (1.3, 10) is a good deal closer to (4, 5) than it is to (−1.3333333, −0.9259259).

**(d)** It seems surprising that, beginning with $\mathbf{x}_0 = (1.3, 10)$, we found the limit we did, especially when $\mathbf{x}_0 = (1.4, 10)$ causes things to converge to (4, 5). This suggests that, when there are multiple solutions, it can be difficult to know to which solution the initial vector will converge.

**7.** Formula (6) says $\mathbf{x}_{k+1} = \mathbf{x}_k - [D\mathbf{f}(\mathbf{x}_k)]^{-1}\mathbf{f}(\mathbf{x}_k)$. But if $\mathbf{x}_k$ solves (2) exactly, then $\mathbf{f}(\mathbf{x}_k) = \mathbf{0}$. Thus $\mathbf{x}_{k+1} = \mathbf{x}_k - [D\mathbf{f}(\mathbf{x}_k)]^{-1}\mathbf{0} = \mathbf{x}_k$. By the same argument $\mathbf{x}_k = \mathbf{x}_{k+2} = \mathbf{x}_{k+3} = \cdots$.

**8.** $D\mathbf{f}(x, y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$. By Exercise 36 of §1.6, $[D\mathbf{f}(x, y)]^{-1} = \frac{1}{f_x g_y - f_y g_x}\begin{bmatrix} g_y & -f_y \\ -g_x & f_x \end{bmatrix}$. If we evaluate at $(x_{k-1}, y_{k-1})$ and calculate, we find that formula (6) tells us that

$$\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} - \underbrace{\frac{1}{f_x g_y - f_y g_x}\begin{bmatrix} g_y & -f_y \\ -g_x & f_x \end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix}}_{\text{all evaluated at } (x_{k-1}, y_{k-1})}.$$

Expanding and taking entries we obtain the desired formulas.

**9.** $DF(x, y) = [4y\cos(xy) + 3x^2, 4x\cos(xy) + 3y^2]$, so we want to solve $\begin{cases} 4y\cos xy + 3x^2 = 0 \\ 4x\cos xy + 3y^2 = 0 \end{cases}$. Using the result of Exercise 8, we have

$$x_k = \frac{\begin{array}{c} 6y_{k-1}^2\cos(x_{k-1}y_{k-1}) + x_{k-1}(6(x_{k-1}^3 + 3y_{k-1}^3)\sin(x_{k-1}y_{k-1}) - \\ x_{k-1}y_{k-1}(9 + 8\sin 2x_{k-1}y_{k-1})) \end{array}}{\begin{array}{c} 2(2 - 9x_{k-1}y_{k-1} + 2\cos(2x_{k-1}y_{k-1}) + 6(x_{k-1}^3 + y_{k-1}^3)\sin(x_{k-1}y_{k-1}) - \\ 4x_{k-1}y_{k-1}\sin(2x_{k-1}y_{k-1})) \end{array}}$$

$$y_k = \frac{\begin{array}{c} 6x_{k-1}^2\cos(x_{k-1}y_{k-1}) + y_{k-1}(6(3x_{k-1}^3 + y_{k-1}^3)\sin(x_{k-1}y_{k-1}) - \\ x_{k-1}y_{k-1}(9 + 8\sin(2x_{k-1}y_{k-1}))) \end{array}}{\begin{array}{c} 2(2 - 9x_{k-1}y_{k-1} + 2\cos(2x_{k-1}y_{k-1}) + 6(x_{k-1}^3 + y_{k-1}^3)\sin(x_{k-1}y_{k-1}) \\ -4x_{k-1}y_{k-1}\sin(2x_{k-1}y_{k-1})) \end{array}}$$

(This was obtained using *Mathematica* to simplify.)

Using initial vector $(x_0, y_0) = (-1, -1)$ and iterating the formulas above we find

| $k$ | $x_k$ | $y_k$ | |
|---|---|---|---|
| 0 | $-1$ | $-1$ | |
| 1 | $-0.9206484$ | $-0.9206484$ | |
| 2 | $-0.9073724$ | $-0.9073724$ | |
| 3 | $-0.9070156$ | $-0.9070156$ | |
| 4 | $-0.9070154$ | $-0.9070154$ | |
| 5 | $-0.9070154$ | $-0.9070154$ | $\leftarrow$ Here's the approximate root. |

**10. (a)** Here we're trying to solve the system $\begin{cases} x^2+y^2+z^2=4 \\ x^2+y^2=1 \\ 4y^2+z^2=4. \end{cases}$ Hence we define $\mathbf{f}(x,y,z) = (x^2+y^2+z^2-4,\ x^2+y^2-1,\ 4y^2+z^2-4)$.

Thus $D\mathbf{f}(x,y,z) = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & 0 \\ 0 & 8y & 2z \end{bmatrix}$. It follows (see Exercise 37 of §1.6) that

$$[D\mathbf{f}(x,y,z)]^{-1} = \begin{bmatrix} \frac{1}{8x} & \frac{3}{8x} & -\frac{1}{8x} \\ -\frac{1}{8y} & \frac{1}{8y} & \frac{1}{8y} \\ \frac{1}{2z} & -\frac{1}{2z} & 0 \end{bmatrix}.$$

Thus $\begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} = \begin{bmatrix} x_{k-1} \\ y_{k-1} \\ z_{k-1} \end{bmatrix} - [D\mathbf{f}(x_{k-1}, y_{k-1}, z_{k-1})]^{-1} \begin{bmatrix} x_{k-1}^2 + y_{k-1}^2 + z_{k-1}^2 - 4 \\ x_{k-1}^2 + y_{k-1}^2 - 1 \\ 4y_{k-1}^2 + z_{k-1}^2 - 4 \end{bmatrix}$.

This simplifies to give

$$x_k = \frac{x_{k-1}}{2} + \frac{3}{8x_{k-1}}$$

$$y_k = \frac{y_{k-1}}{2} + \frac{1}{8y_{k-1}}$$

$$z_k = \frac{z_{k-1}}{2} + \frac{3}{2z_{k-1}}$$

Newton's method with $\mathbf{x}_0 = (1,1,1)$ gives the following set of results

| $k$ | $x_k$ | $y_k$ | $z_k$ |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | 0.875 | 0.625 | 2 |
| 2 | 0.86607143 | 0.5125 | 1.75 |
| 3 | 0.86602541 | 0.50015244 | 1.73214286 |
| 4 | 0.8660254 | 0.50000002 | 1.73205081 |
| 5 | 0.8660254 | 0.5 | 1.73205081 |
| 6 | 0.8660254 | 0.5 | 1.73205081 |

With $\mathbf{x}_0 = (1,-1,1)$, we find

| $k$ | $x_k$ | $y_k$ | $z_k$ |
|---|---|---|---|
| 0 | 1 | $-1$ | 1 |
| 1 | 0.875 | $-0.625$ | 2 |
| 2 | 0.86607143 | $-0.5125$ | 1.75 |
| 3 | 0.86602541 | $-0.5001524$ | 1.73214286 |
| 4 | 0.8660254 | $-0.5$ | 1.73205081 |
| 5 | 0.8660254 | $-0.5$ | 1.73205081 |

**(b)** We solve $\begin{cases} x^2+y^2+z^2=4 \\ x^2+y^2=1 \\ 4y^2+z^2=4 \end{cases}$ by hand. First insert the second equation into the first: $1+z^2=4 \Leftrightarrow z = \pm\sqrt{3}$. Use this in the third equation $4y^2+3=4 \Leftrightarrow y = \pm\frac{1}{2}$.

Now use this in the second equation: $x^2 + \frac{1}{4} = 1 \Leftrightarrow x = \pm\frac{\sqrt{3}}{2}$.
So we have 8 solutions:

$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}\right), \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}\right), \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}\right)$$
$$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}\right), \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}\right), \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}\right)$$

We found two of them above.

## True/False Exercises for Chapter 2

**1.** False.
**2.** True.
**3.** False. (The range also requires $v \neq 0$.)
**4.** False. (Note that $\mathbf{f}(\mathbf{i}) = \mathbf{f}(\mathbf{j})$.)
**5.** True.
**6.** False. (It's a paraboloid.)
**7.** False. (The graph of $x^2 + y^2 + z^2 = 0$ is a single point.)
**8.** True.
**9.** False.
**10.** False. (The limit does not exist.)
**11.** False. ($\lim_{(x,y)\to(0,0)} f(x, y) = 0 \neq 2$.)
**12.** False.
**13.** False.
**14.** True.
**15.** False. ($\nabla f(x, y, z) = (0, \cos y, 0)$.)
**16.** False. (It's a $4 \times 3$ matrix.)
**17.** True.
**18.** False.
**19.** False. (The partial derivatives must be continuous.)
**20.** True.
**21.** False. ($f_{xy} \neq f_{yx}$.)
**22.** False. ($f$ must be of class $C^2$.)
**23.** True. (Write the chain rule for this situation.)
**24.** True.
**25.** False. (The correct equation is $4x + y + 4z = 0$.)
**26.** False. (The plane is *normal* to the given vector.)
**27.** True.
**28.** False. (The directional derivative equals $-\partial f/\partial z$.)
**29.** False.
**30.** True.

## Miscellaneous Exercises for Chapter 2

**1. (a)** Calculate the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ x_1 & x_2 & x_3 \end{vmatrix} = (-x_2, x_1 - x_3, x_2).$$

More explicitly, the component functions are $f_1(x_1, x_2, x_3) = -x_2$, $f_2(x_1, x_2, x_3) = x_1 - x_3$, and $f_3(x_1, x_2, x_3) = x_2$.

**(b)** The domain is all of $\mathbf{R}^3$ while the range restricts the first component to be the opposite of the last component. In other words the range is the set of all vectors $(a, b, -a)$.

**2. (a)** It might help to see **f** explicitly first as

$$\left(\frac{(3,-2,1)\cdot(x,y,z)}{(3,-2,1)\cdot(3,-2,-1)}\right)(3,-2,1) = \frac{3x-2y+z}{14}(3,-2,1).$$

**(b)** The domain is all of $\mathbf{R}^3$ and the range are vectors of the form $(3a,-2a,a)$.

**3. (a)** The domain of $f$ is $\{(x,y)|x \geq 0 \text{ and } y \geq 0\} \cup \{(x,y)|x \leq 0 \text{ and } y \leq 0\}$. The range is all real numbers greater than or equal to 0.

**(b)** The domain is closed. The quarter planes are closed on two sides because they include the axes.

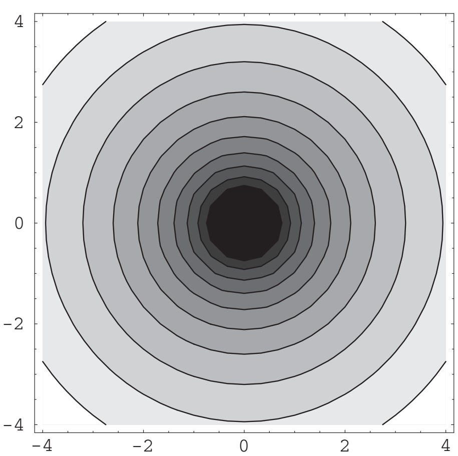
**4. (a)** The domain of $f$ is $\{(x,y)|x \geq 0 \text{ and } y > 0\} \cup \{(x,y)|x \leq 0 \text{ and } y < 0\}$. The range is all real numbers greater than or equal to 0.

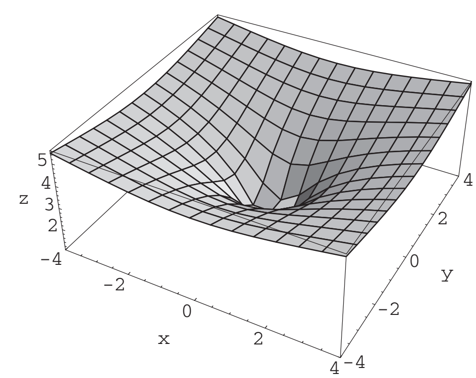
**(b)** The domain is neither open nor closed. The quarter planes are closed on one side because they include the $y$-axis but they don't include the $x$-axis and so aren't closed.

**5.**

| $f(x, y)$ | Graph | Level curves |
|---|---|---|
| $1/(x^2+y^2+1)$ | D | d |
| $\sin\sqrt{x^2+y^2}$ | B | e |
| $(3y^2-2x^2)e^{-x^2-2y^2}$ | A | b |
| $y^3-3x^2y$ | E | c |
| $x^2y^2e^{-x^2-y^2}$ | F | a |
| $ye^{-x^2-y^2}$ | C | f |

**6. (a)** See below left.




**(b)** See above right.

**7.** First we'll substitute $x = r\cos\theta$ and $y = r\sin\theta$ while noting that $(x,y)\to(0,0)$ is equivalent to $r \to 0$.

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{yx^2-y^3}{x^2+y^2} &= \lim_{r\to 0}\frac{(r\sin\theta)(r^2\cos^2\theta)-(r^3\sin^3\theta)}{r^2\cos^2\theta+r^2\sin^2\theta}\\ &= \lim_{r\to 0}\frac{r^3(\cos^2\theta-\sin^2\theta)\sin\theta}{r^2}\\ &= \lim_{r\to 0} r\cos 2\theta\sin\theta = 0\end{aligned}$$

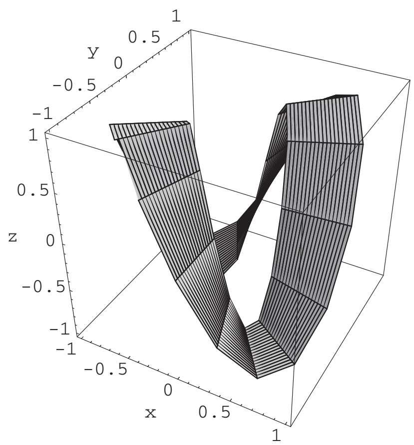
**8. (a)** $\dfrac{2xy}{x^2+y^2} = \dfrac{2r^2\cos\theta\sin\theta}{r^2} = 2\cos\theta\sin\theta = \sin 2\theta$. So

$$f(x,y) = \begin{cases}\sin 2\theta & \text{if } r \neq 0\\ 0 & \text{if } r = 0\end{cases}.$$

**(b)** We're looking for $(x, y)$ such that $f(x,y) = c$. For $-1 < c < 1$ the level sets are pairs of radial lines symmetric about $\theta = \pi/4$.

For example, if $c = 1/2$ then we are looking for $\theta$ such that $\sin 2\theta = 1/2$. In this case, $\theta = \pi/12, 5\pi/12, 13\pi/12$, and $17\pi/12$. So the level sets are the lines $\theta = \pi/12$ and $\theta = 5\pi/12$. These could also be written as $\theta = \pi/4 \pm \pi/6$.
For $c = 1$ the level set is the line $\theta = \pi/4$, for $c = -1$ the level set is the line $\theta = 3\pi/4$ and for $|c| > 1$ the level set is the empty set.

**(c)** $f$ is constant along radial lines, so the figure below just shows a ribbon corresponding to $.4 < r < 1$.




**(d)** $\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{r\to 0} \sin 2\theta$ which doesn't exist.

**(e)** Since the limit doesn't exist at the origin, $f$ couldn't be continuous there. Also, $f$ takes on every value between 1 and $-1$ in every open neighborhood of the origin.

*Before assigning Exercise 9 you may want to ask the students if it is true that if a function F(x, y) is continuous in each variable separately it is continuous. The calculations in Exercise 9 are fairly routine but the conclusion is very important.*

**9.** $g(x) = F(x, 0) \equiv 0$ and so is continuous at $x = 0$ and $h(y) = F(0, y) \equiv 0$ and so is continuous at $y = 0$. Consider $p(x) = F(x, x) = 1$ when $x \neq 0$ and $F(0, 0) = 0$. Clearly, $p(x)$ is not continuous at 0 so $F(x, y)$ is not continuous at $(0, 0)$.

**10. (a)** You can see as $x$ gets closer and closer to 0 that $1/x^2$ gets larger and larger. More formally, for any $N > 0$, if $0 < |x| < 1/\sqrt{N}$ then $1/x^2 > N$.

**(b)** Here $\|(x, y) - (1, 3)\| = \sqrt{(x-1)^2 + (y-3)^2}$ so for any $N > 0$, if $0 < \|(x, y) - (1, 3)\| < \sqrt{(2/N)}$, then

$$\frac{2}{(x-1)^2 + (y-3)^2} = \frac{2}{\|(x, y) - (1, 3)\|^2} > \frac{2}{2/N} = N.$$

**(c)** The definition is analogous to that for above: $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = -\infty$ means that given any $N < 0$ there is some $\delta > 0$ such that if $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$ then $f(\mathbf{a}) < N$.

**(d)** We are considering $\lim_{(x,y)\to(0,0)}$ so let's restrict our attention to $|x| < 1$ and $|y| < 1$. For $|x| < 1$ we have

$$\frac{1-x}{xy^4 - y^4 + x^3 - x^2} = \frac{-1}{y^4 + x^2}.$$

For $|y| < 1$ we have $y^4 < y^2$ so

$$\frac{-1}{y^4 + x^2} < \frac{-1}{y^2 + x^2} = \frac{-1}{\|(x, y)\|^2}.$$

So for any $N < 0$ if $0 < \|(x, y)\| < \min\{1, 1/\sqrt{-N}\}$ then $\dfrac{1-x}{xy^4 - y^4 + x^3 - x^2} < N$.

**11.** We read right from the table in the text:

**(a)** $15^\circ$ F.

**(b)** $5^\circ$ F.

**12. (a)** If the temperature of the air is $10^\circ$ F we read off the chart that when the windspeed is 10 mph the windchill is $-4$; when the windspeed is 15 mph the windchill is $-7$. Since we are looking to estimate when the windchill is $-5$ you might be tempted to stop here and just conclude that the answer is between 10 mph and 15 mph (and you'd be correct) but we want to say more. Our first estimate will just use linear interpolation (similar triangles) to get $\frac{x}{2} = \frac{5}{3}$ or the distance from 15 is $x = 10/3$. We would then conclude that, to the nearest degree, the windspeed is $15 - 10/3 \approx 12$ mph.

**(b)** Before you feel too good about your answer to part (a) you should notice further that when the windspeed is 20 the windchill is $-9$ and when the windspeed is 25 the windchill is $-11$. In other words, the rate at which the windchill is dropping is slowing slightly. In calculus terms, for the function $f(s) = W(s \text{ mph}, 30°)$, $f''(s)$ seems to be positive so the curve is concave up. The line used to estimate in part (a) then probably lies above the curve and our guess of 12 mph is, most likely, too high.

**13.** For the function $W(s \text{ mph}, t°)$, we want to estimate

$$\left.\frac{\partial W}{\partial t}\right|_{(30 \text{ mph}, 35°)} = \lim_{h\to 0} \frac{W(30, 35+h) - W(30, 35)}{h}.$$

We will use the slopes of the two secant lines:

$$\frac{W(30,40) - W(30,35)}{5} = \frac{28-22}{5} = 1.2$$
$$\frac{W(30,30) - W(30,35)}{-5} = \frac{15-22}{-5} = 1.4$$

We average them to get an estimate of 1.3.

**14.** We will use the same technique as in Exercise 13 and estimate the derivative with respect to windspeed by averaging the slopes of the two secant lines.

$$\frac{W(20,25) - W(15,25)}{5} = \frac{11-13}{5} = -0.4$$
$$\frac{W(10,25) - W(15,25)}{-5} = \frac{15-13}{-5} = -0.4$$

so we average them to get an estimate of $-0.4$.

**15. (a)** Comparison with Exercise 11: With an air temperature of 25° F, windspeed of 10 mph,

$$\begin{aligned} W(10,25) &= 91.4 + (25 - 91.4)(0.474 + 0.304\sqrt{10} - 0.203) \\ &\approx 9.573 \quad \text{or} \quad 10° \text{ F} \end{aligned}$$

(as compared to 15° F in 11(a)).
If $s = 20$ mph, then $W = -15°$ F if

$$91.4 + (t - 91.4)(0.474 + 0.304\sqrt{20} - 0.406) = -15.$$

Hence $t = 91.4 - \dfrac{15 + 91.4}{(0.474 + 0.304\sqrt{20} - 0.406)} \approx 16.866$ or 17° F (as compared to 5° F in 11(b)).

Comparison with Exercise 12: With $W(s,t) = 91.4 + (t - 91.4)(.474 + .304\sqrt{s} - .0203s)$, we must solve

$$-5 = 91.4 + (10 - 91.4)(.474 + .304\sqrt{s} - .0203s)$$

or

$$\begin{aligned} \frac{-5 - 91.4}{10 - 91.4} &= .474 + .304\sqrt{s} - .0203s \quad \text{so that} \\ 1.18428 &\approx .474 + .304\sqrt{s} - .0203s \quad \text{or} \\ 0 &\approx .0203s - .304\sqrt{s} + .7102752 \end{aligned}$$

Now solve the quadratic: $\sqrt{s} \approx \dfrac{.304 \pm \sqrt{.186391}}{.0406}$. The two solutions are 8.39128 and 145.893.

**(b)** The windchill effect of windspeed appears to be greater in the Siple formula than that which may be inferred from the table.

**(c)** For temperatures greater than 91.4 the model has the wind actually making the apparent temperature warmer than air temperature. Physically, the model probably falls apart because between 91.4 and 106 you are too close to body temperature for the wind to have much effect and if you are in temperatures much greater than 106 a breeze won't replace a frosty beverage. For winds below 4 mph, the effect is negligible and won't be reflected in the model.

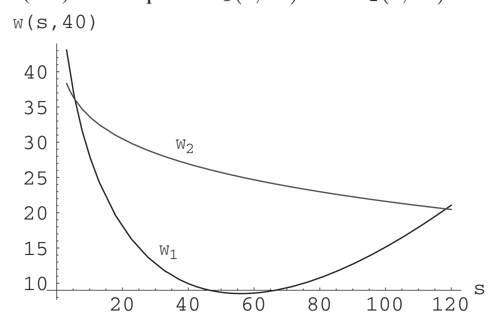
**16.** Comparison with Exercise 13: We want to calculate $W_t(30,35)$. $\partial W/\partial t = 0.621 + 0.4275s^{0.16}$, so $W_t(30,35) = 0.621 + (0.4275)30^{0.16} \approx 1.358$ (this is close).

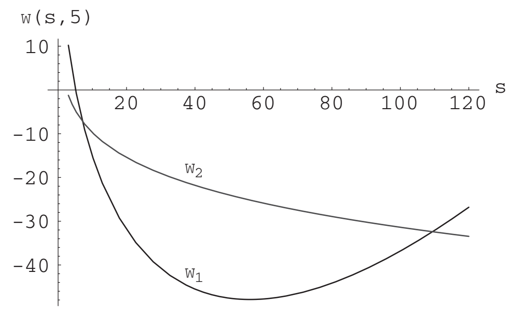
Comparison with Exercise 14: We want $W_s(15,25)$.

$$\begin{aligned}\partial W/\partial s &= -35.75(0.16)s^{-0.84} + 0.4275(0.16)ts^{-0.84}\\ &= (-5.72 + 0.0684t)s^{-0.84}\end{aligned}$$

$W_s(15,25) = (-5.72 + 0.0684 \cdot 25)15^{-0.84} \approx -0.412$ (again close).

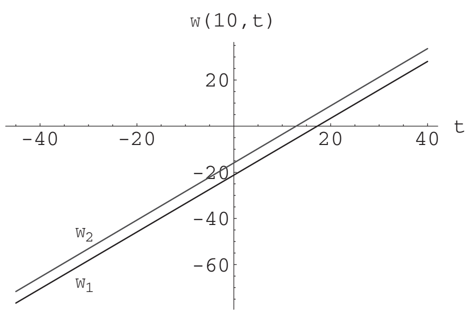
**17.** **(a)** Pictured (left) are the pairs $W_1(s,40)$ and $W_2(s,40)$ and, on the right, the pairs $W_1(s,5)$ and $W_2(s,5)$.

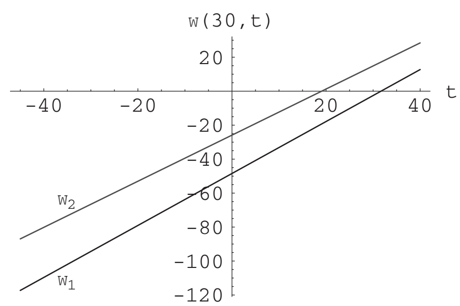


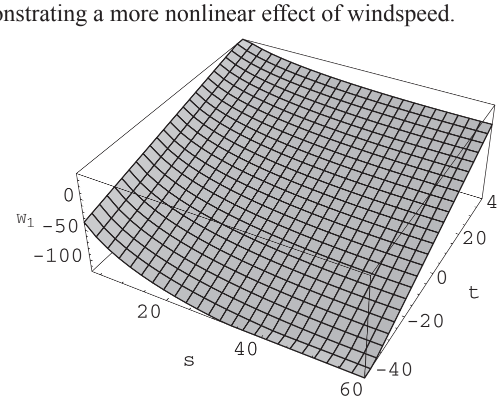

From these graphs, we see that windspeed depresses apparent temperature in the Siple formula much more than in the National Weather Service Formula.

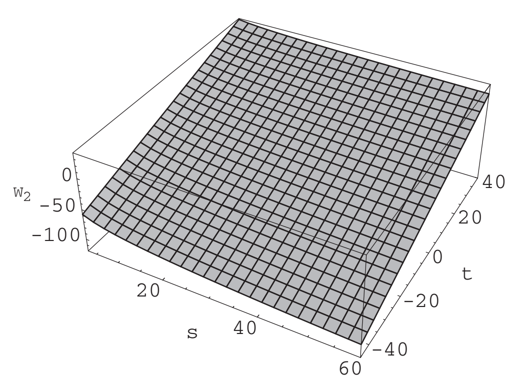
**(b)** Pictured (left) are the pairs $W_1(10,t)$ and $W_2(10,t)$ and, on the right, the pairs $W_1(30,t)$; and $W_2(30,t)$. Again we see that the Siple formula results in lower apparent temperatures predicted, only the effect appears to be more of a constant difference.




**(c)** The surfaces $z = W_1(s,t)$ and $z = W_2(s,t)$ are pictured. Note that the Siple surface determined by $W_1$ is more curved, demonstrating a more nonlinear effect of windspeed.




**18.** The equation of the sphere is $F(x,y,z) = x^2 + y^2 + z^2 = 9$ so $\nabla F = (2x, 2y, 2z)$ and the plane tangent to the sphere at (1, 2, 2) is $0 = (2,4,4) \cdot (x-1, y-2, z-2)$ or $x + 2y + 2z = 9$. This intersects the $x$-axis when $y = 0$ and $z = 0$ so $x = 9$.

**19.** Without loss of generality we can locate the center of the sphere at the origin and so the equation of the sphere is $F(x, y, z) = x^2 + y^2 + z^2 = r^2$ so $\nabla F = (2x, 2y, 2z)$ and the equation of the plane tangent to the sphere at $P = (x_0, y_0, z_0)$ is $0 = (2x_0, 2y_0, 2z_0) \cdot (x - x_0, y - y_0, z - z_0)$ or $x_0 x + y_0 y + z_0 z = x_0^2 + y_0^2 + z_0^2$. This is orthogonal to the vector $(x_0, y_0, z_0)$ which is the vector from the center of the sphere to $P$.

**20.** Because we're looking at a curve in the plane $2x - y = 1$ we know the $x$ and $y$ components of the parametric equations. What is left to determine is $z$. Substitute in $2x - 1$ for $y$ in the equation of the surface to get $z = 3x^2 + x^3/6 - x^4/8 - 4(2x-1)^2 = -5x^2 + x^3/6 - x^4/8 + 4$. We can now calculate the derivative $\partial z/\partial x = -10x + x^2/2 - x^3/2$ and evaluate it at the point $(1, 1, -23/24)$ to get $-10$. Because the value of $z$ is $-23/24$ when $x = 1$, this component of the tangent line is derived by looking at $z + 23/24 = -10(x - 1)$. So the parametric equations for the tangent line are $(t, 2t - 1, -10t + 192/24)$.

**21.** **(a)** For the function $f(x, y, z) = x^2 + y^2 - z^2 = 0$ we consider $\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$. Here we get $(2(3), 2(-4), -2(5)) \cdot (x - 3, y + 4, z - 5) = 0$ or the equation is $6(x - 3) - 8(y + 4) - 10(z - 5) = 0$.

**(b)** In general we get $(2(a), 2(b), -2(c)) \cdot (x - a, y - b, z - c) = 0$. This amounts to $2a(x - a) + 2b(y - b) - 2c(z - c) = 0$.

**(c)** Note that (0, 0, 0) is a solution so the plane passes through the origin.

**22.** Show that the two surfaces

$$S_1\colon z = xy \text{ and } S_2\colon z = \frac{3}{4}x^2 - y^2$$

intersect perpendicularly at the point (2, 1, 2). First we see that $2 = 1(2)$ and $2 = (3/4)(4) - 1$ so (2, 1, 2) is a point on both surfaces. Rewrite the surfaces so that they are level sets of functions:

$$F_1(x, y, z) = xy - z \text{ and } F_2(x, y, z) = z + y^2 - \frac{3}{4}x^2.$$

The gradients are normal to the tangent planes (see Section 2.6, Exercise 36), so we calculate the two gradients at the given point: $\nabla F_1(2, 1, 2) = (1, 2, -1)$ and $\nabla F_2(2, 1, 2) = (-3, 2, 1)$ so

$$\nabla F_1(2, 1, 2) \cdot \nabla F_2(2, 1, 2) = 0.$$

So the two surfaces intersect perpendicularly at (2, 1, 2).

**23.** **(a)** As we have done before we find the plane tangent to the surface given by $F(x, y, z) = z - x^2 - 4y^2 = 0$ by formula (6):

$$0 = \nabla F(1, -1, 5) \cdot (x - 1, y + 1, z - 5) = (-2, 8, 1) \cdot (x - 1, y + 1, z - 5)$$
$$\text{or } -2x + 8y + z = -5.$$

**(b)** The line is parallel to a vector which is orthogonal to $\nabla F(1, -1, 5) = (-2, 8, 1)$ and with no component in the $x$ direction. So it is of the form (0, *a, b*) with $(0, a, b) \cdot (-2, 8, 1) = 0$ so the line has the direction $(0, 1, -8)$ and passes through $(1, -1, 5)$. The equations are $\begin{cases} x = 1 \\ y = t - 1 \\ z = -8t + 5. \end{cases}$

**24.** We are assuming that the collar is fairly rigid so that it is maintaining a cylindrical shape throughout this process. We want $\dfrac{\partial V}{\partial t}$ at $t = t_0$. Since $V = \pi r^2 h$, $\dfrac{\partial V}{\partial t} = 2\pi r h \dfrac{dr}{dt} + \pi r^2 \dfrac{dh}{dt}$. We are given that the rate of change of the circumference at $t = t_0$ is $-.2$ *in/min*. This means

$$-.2 = \left.\frac{\partial C}{\partial t}\right|_{t_0} = \left.\frac{\partial(2\pi r)}{\partial t}\right|_{t_0} = 2\pi \left.\frac{dr}{dt}\right|_{t_0}.$$

We also know that at $t = t_0$, $2\pi r = 18$, $h = 3$, and $\dfrac{dh}{dt} = .1$. Substituting into the equation above, we get:

$$\left.\frac{\partial V}{\partial t}\right|_{t_0} = (18)(3)\left(\frac{-.2}{2\pi}\right) + \pi\left(\frac{18}{2\pi}\right)^2 (.1) = \frac{-5.4}{\pi} + \frac{8.1}{\pi} = \frac{2.7}{\pi}.$$

So the volume is increasing at $t = t_0$.

**25.** First note that $0.2 \deg \mathrm{C/day} = 0.2 \cdot 24 = 4.8 \deg \mathrm{C/month}$. Then, with time measured in months, the chain rule tells us

$$\frac{dP}{dt} = \frac{\partial P}{\partial S}\frac{dS}{dt} + \frac{\partial P}{\partial T}\frac{dT}{dt}.$$

Here $\dfrac{dS}{dt} = -2$, $\dfrac{dT}{dt} = 4.8$. With $P(S,T) = 330S^{2/3}T^{4/5}$, we have

$$\begin{aligned}\left.\frac{dP}{dt}\right|_{(S=75,T=15)} &= (220S^{-1/3}T^{4/5})|_{(75,15)}(-2) + 264S^{2/3}T^{-1/5}|_{(75,15)}(4.8)\\ &= 220(75)^{-1/3}(15)^{4/5}(-2) + 264(75)^{2/3}(15)^{-1/5}(4.8)\\ &= 12{,}201.4 \text{ units/month}\\ &\quad (\text{or } 508.392 \text{ units/day})\end{aligned}$$

**26.** We want to know $\dfrac{du}{dt}$ ($t$ in weeks) when $x = 80$, $y = 240$, given that $\dfrac{dx}{dt} = 5$ and $\dfrac{dy}{dt} = -15$. The chain rule tells us

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \qquad = (0.002xe^{-0.001x^2-0.00005y^2})\frac{dx}{dt} + (0.0001ye^{-0.001x^2-0.00005y^2})\frac{dy}{dt}$$

Thus

$$\begin{aligned}\left.\frac{du}{dt}\right|_{x=80,y=240} &= e^{(-0.001)80^2-0.00005(240)^2}[(0.002)80\cdot 5 - (0.0001)240\cdot 15]\\ &\approx 0.000041.\end{aligned}$$

So the utility function is increasing ever so slightly.

**27.**

$$\begin{aligned} w &= x^2 + y^2 + z^2,\\ x &= \rho\cos\theta\sin\varphi,\\ y &= \rho\sin\theta\sin\varphi \quad \text{and}\\ z &= \rho\cos\varphi\end{aligned}$$

**(a)**

$$\begin{aligned}\frac{\partial w}{\partial \rho} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \rho}\\ &= 2x\cos\theta\sin\varphi + 2y\sin\theta\sin\varphi + 2z\cos\varphi\\ &= 2\rho\cos^2\theta\sin^2\varphi + 2\rho\sin^2\theta\sin^2\varphi + 2\rho\cos^2\varphi\\ &= 2\rho,\\ \frac{\partial w}{\partial \varphi} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \varphi} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \varphi}\\ &= 2x\rho\cos\theta\cos\varphi + 2y\rho\sin\theta\cos\varphi - 2z\rho\sin\varphi\\ &= 2\rho^2\cos^2\theta\cos\varphi\sin\varphi + 2\rho^2\sin^2\theta\cos\varphi\sin\varphi - 2\rho^2\cos\varphi\sin\varphi\\ &= 0, \quad \text{and}\\ \frac{\partial w}{\partial \theta} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta}\\ &= -2x\rho\sin\theta\sin\varphi + 2y\rho\cos\theta\sin\varphi\\ &= -2\rho^2\cos\theta\sin\theta\sin^2\varphi + 2\rho^2\cos\theta\sin\theta\sin^2\varphi\\ &= 0.\end{aligned}$$

**(b)** First substitute: $w = x^2 + y^2 + z^2 = (\rho\cos\theta\sin\varphi)^2 + (\rho\sin\theta\sin\varphi)^2 + (\rho\cos\varphi)^2 = \rho^2$. Now taking the derivatives from part (a) is trivial: $w_\rho = 2\rho$, $w_\varphi = 0$, and $w_\theta = 0$.

**28.** If $w = f\left(\frac{x+y}{xy}\right)$, let $u = \frac{x+y}{xy}$. So

$$\begin{aligned} x^2\frac{\partial w}{\partial x} - y^2\frac{\partial w}{\partial y} &= x^2\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} - y^2\frac{\partial w}{\partial u}\frac{\partial u}{\partial y} \\ &= x^2\frac{\partial w}{\partial u}\left(\frac{-y^2}{x^2y^2}\right) - y^2\frac{\partial w}{\partial u}\left(\frac{-x^2}{x^2y^2}\right) \\ &= 0. \end{aligned}$$

**29.** **(a)** First use the chain rule to find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$:

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial z}{\partial x}(e^r\cos\theta) + \frac{\partial z}{\partial y}(e^r\sin\theta), \quad \text{and} \\ \frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} \\ &= \frac{\partial z}{\partial x}(-e^r\sin\theta) + \frac{\partial z}{\partial y}(e^r\cos\theta). \end{aligned}$$

Now solve for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$\begin{aligned} \frac{\partial z}{\partial x} &= e^{-r}\cos\theta\frac{\partial z}{\partial r} - e^{-r}\sin\theta\frac{\partial z}{\partial \theta}, \quad \text{and} \\ \frac{\partial z}{\partial y} &= e^{-r}\sin\theta\frac{\partial z}{\partial r} + e^{-r}\cos\theta\frac{\partial z}{\partial \theta}. \end{aligned}$$

**(b)** Given the results for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in part (a), we compute:

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = e^{-r}\cos\theta\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) - e^{-r}\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial x}\right) \\ &= e^{-r}\cos\theta\frac{\partial}{\partial r}\left(e^{-r}\cos\theta\frac{\partial z}{\partial r} - e^{-r}\sin\theta\frac{\partial z}{\partial \theta}\right) - e^{-r}\sin\theta\frac{\partial}{\partial \theta}\left(e^{-r}\cos\theta\frac{\partial z}{\partial r} - e^{-r}\sin\theta\frac{\partial z}{\partial \theta}\right) \\ &= e^{-r}\cos\theta\left(-e^{-r}\cos\theta\frac{\partial z}{\partial r} + e^{-r}\cos\theta\frac{\partial^2 z}{\partial r^2} + e^{-r}\sin\theta\frac{\partial z}{\partial \theta} - e^{-r}\sin\theta\frac{\partial^2 z}{\partial r\partial \theta}\right) \\ &\quad - e^{-r}\sin\theta\left(-e^{-r}\sin\theta\frac{\partial z}{\partial r} + e^{-r}\cos\theta\frac{\partial^2 z}{\partial \theta\partial r} - e^{-r}\cos\theta\frac{\partial z}{\partial \theta} - e^{-r}\sin\theta\frac{\partial^2 z}{\partial \theta^2}\right) \\ &= e^{-2r}\left[(\sin^2\theta - \cos^2\theta)\frac{\partial z}{\partial r} + \cos^2\theta\frac{\partial^2 z}{\partial r^2} + 2\sin\theta\cos\theta\frac{\partial z}{\partial \theta} - 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial r\partial \theta} + \sin^2\theta\frac{\partial^2 z}{\partial \theta^2}\right] \end{aligned}$$

A similar calculation gives:

$$\begin{aligned} \frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = e^{-r}\sin\theta\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) + e^{-r}\cos\theta\frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial y}\right) \\ &= e^{-r}\sin\theta\frac{\partial}{\partial r}\left(e^{-r}\sin\theta\frac{\partial z}{\partial r} + e^{-r}\cos\theta\frac{\partial z}{\partial \theta}\right) + e^{-r}\cos\theta\frac{\partial}{\partial \theta}\left(e^{-r}\sin\theta\frac{\partial z}{\partial r} + e^{-r}\cos\theta\frac{\partial z}{\partial \theta}\right) \\ &= e^{-2r}\left[(\cos^2\theta - \sin^2\theta)\frac{\partial z}{\partial r} + \sin^2\theta\frac{\partial^2 z}{\partial r^2} - 2\sin\theta\cos\theta\frac{\partial z}{\partial \theta} + 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial r\partial \theta} + \cos^2\theta\frac{\partial^2 z}{\partial \theta^2}\right]. \end{aligned}$$

Now add these to get:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-2r}[(\cos^2\theta + \sin^2\theta)z_{\theta\theta} + (\cos^2\theta + \sin^2\theta)z_{rr}] = e^{-2r}[z_{\theta\theta} + z_{rr}].$$

**30. (a)** Consider $w = f(x, y) = x^y = e^{y \ln x}$. Then $\frac{d}{du}(u^u)$ can be calculated by taking the derivative and evaluating at the point $(u, u)$.

$$\frac{dw}{du} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} = yx^{y-1} + \ln x e^{y \ln x} = u \cdot u^{u-1} + (\ln u)\, u^u = u^u(1 + \ln u).$$

**(b)** Here $x = \sin t$ and $y = \cos t$. So

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = yx^{y-1}\cos t + (\ln x)\, e^{y \ln x}(-\sin t) = \cos^2 t(\sin t)^{\cos t - 1} - \sin t \ln(\sin t) \sin t^{\cos t}.$$

**31.** This is an extension of the preceding exercise. This time $w = f(x, y, z) = x^{y^z}$. If $x = u$, $y = u$, and $z = u$ we again calculate

$$\begin{aligned} \frac{dw}{du} &= \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} + \frac{\partial w}{\partial z}\frac{dz}{du} = y^z x^{y^z - 1} + e^{y^z \ln x}(z \ln x) y^{z-1} + e^{e^{z \ln y} \ln x}(\ln x)\, e^{z \ln y} \ln y \\ &= u^u u^{(u^u - 1)} + u^{u^u}(u \ln u) u^{u-1} + u^{u^u}(\ln u)^2 u^u = u^u u^{u^u}(1/u + \ln u + (\ln u)^2). \end{aligned}$$

**32.** With

$$r = \|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}, \quad \frac{\partial r}{\partial x_i} = \frac{x_i}{\sqrt{x_1^2 + \cdots + x_n^2}} = \frac{x_i}{r}.$$

The chain rule gives $\dfrac{\partial f}{\partial x_i} = \dfrac{dg}{dr}\dfrac{\partial r}{\partial x_i} = g'(r)\dfrac{x_i}{r}$
By the product and chain rules:

$$\begin{aligned} \frac{\partial^2 f}{\partial x_i^2} &= \frac{\partial}{\partial x_i}\left(g'(r)\frac{x_i}{r}\right) = \frac{g'(r)}{r} + x_i \frac{d}{dr}\left(\frac{g'(r)}{r}\right)\frac{\partial r}{\partial x_i} \\ &= \frac{1}{r}g'(r) + x_i\left(\frac{rg''(r) - g'(r)}{r^2}\right)\frac{x_i}{r} \\ &= \frac{1}{r}g'(r) + x_i^2\left(\frac{g''(r)}{r^2} - \frac{g'(r)}{r^3}\right). \end{aligned}$$

Add these to find

$$\begin{aligned} \nabla^2 f = \sum_{i=1}^n \frac{\partial^2 f}{\partial {x_i}^2} &= \frac{n}{r}g'(r) + \left(\frac{g''(r)}{r^2} - \frac{g'(r)}{r^3}\right)\underbrace{(x_1^2 + \cdots + x_n^2)}_{=r^2} \\ &= \frac{n}{r}g'(r) + g''(r) - \frac{g'(r)}{r} \\ &= \frac{1}{r}(n-1)g'(r) + g''(r). \end{aligned}$$

**33. (a)**

$$\begin{aligned} \nabla^2(\nabla^2 f(x, y)) &= \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) \\ &= \frac{\partial^4 f}{\partial x^4} + \underbrace{\frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^2 \partial x^2}}_{\text{these are equal} - f \text{ is of class } C^4} + \frac{\partial^4 f}{\partial y^4} \\ &= \text{desired expression.} \end{aligned}$$

**(b)** Similar:

$$\begin{aligned} \nabla^2(\nabla^2 f) &= \frac{\partial^2}{\partial x_1^2}\left(\sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}\right) + \cdots + \frac{\partial^2}{\partial x_n^2}\left(\sum_{j=1}^n \frac{\partial^2 f}{\partial x_j}\right) \\ &= \sum_{i=1}^n \frac{\partial^2}{\partial {x_i}^2}\left(\sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}\right) = \sum_{i,j=1}^n \frac{\partial^4 f}{\partial x_i^2 \partial x_j^2}. \end{aligned}$$

**34.** Livinia is at (0, 0, 1) and $T(x, y, z) = 10(xe^{-y^2} + ze^{-x^2})$

**(a)** The unit vector in the direction from (0, 0, 1) to (2, 3, 1) is $\mathbf{u} = (2, 3, 0)/\sqrt{13}$.

$$D_{\mathbf{u}}T = \nabla T(0, 0, 1) \cdot \mathbf{u} = 10(1, 0, 1) \cdot (2, 3, 0)/\sqrt{13} = 20/\sqrt{13}\, deg/cm.$$

**(b)** She should head in the direction of the negative gradient: $(-1, 0, -1)/\sqrt{2}$.

**(c)** $(3)10(1, 0, 1) \cdot (-1, 0, -1)/\sqrt{2} = -30\sqrt{2}\, deg/sec$.

**35.** $z = r\cos 3\theta$

**(a)** $z = r[\cos\theta\cos 2\theta - \sin\theta\sin 2\theta] = r[\cos\theta(\cos^2\theta - \sin^2\theta) - \sin\theta(2\sin\theta\cos\theta)]$ so

$$z = \frac{r^3[\cos^3\theta - \cos\theta\sin^2\theta - 2\sin^2\theta\cos\theta]}{r^2} = \frac{x^3 - 3xy^2}{x^2 + y^2}.$$

**(b)** Note that $\lim_{r\to 0} r\cos 3\theta = 0$ which is the value of the function at the origin. So yes, $f(x, y) = z$ is continuous at the origin.

**(c)** **(i)** $f_x = \dfrac{(x^2 + y^2)(3x^2 - 3y^2) - (x^3 - 3xy^2)2x}{(x^2 + y^2)^2} = \dfrac{x^4 - 3y^4 + 6x^2y^2}{(x^2 + y^2)^2}$.

**(ii)** $f_y = \dfrac{(x^2 + y^2)(-6xy) - (x^3 - 3xy^2)2y}{(x^2 + y^2)^2} = \dfrac{-8x^3y}{(x^2 + y^2)^2}$.

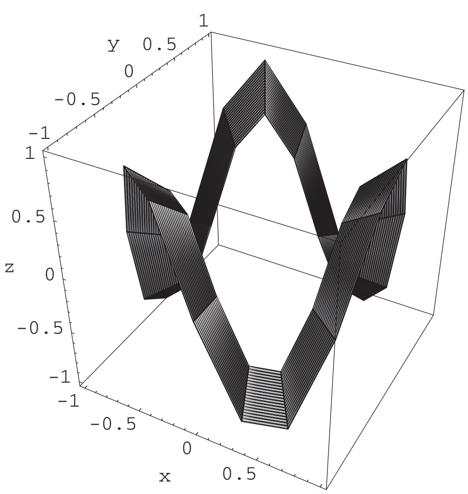
**(iii)** $f_x(0, 0) = \lim_{h\to 0} \dfrac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \dfrac{h - 0}{h} = 1$.

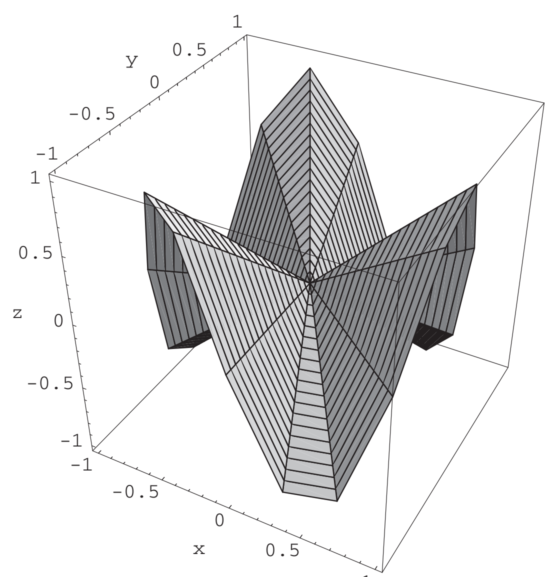
**(iv)** $f_y(0, 0) = \lim_{h\to 0} \dfrac{f(0, h) - f(0, 0)}{h} = \lim_{h\to 0} \dfrac{0 - 0}{h} = 0$.

**(d)** $g(r, \theta) = r\cos 3\theta$ so $g_r(r, \theta) = \cos 3\theta$. This is the directional derivative $D_{\mathbf{u}}f$.

**(e)** When $(x, y) \neq (0, 0)$, $f_y(x, y) = \dfrac{-8x^3y}{x^2 + y^2}$. In particular, when $y = x$, $f_y = -2$. From part (c) $f_y(0, 0) = 0$ so $f_y$ is not continuous at the origin.

**(f)** Below are two sketches; the one on the left just shows a ribbon of the surface:




**36.** **(a)** $u = \cos(x - t) + \sin(x + t) - 2e^{z+t} - (y - t)^3$ so

- $u_x = -\sin(x - t) + \cos(x + t)$ and $u_{xx} = -\cos(x - t) - \sin(x + t)$.
- $u_y = -3(y - t)^2$ and $u_{yy} = -6(y - t)$.
- $u_z = -2e^{z+t}$ and $u_{zz} = -2e^{z+t}$.
- $u_t = \sin(x - t) + \cos(x + t) - 2e^{z+t} + 3(y - t)^2$ and $u_{tt} = -\cos(x - t) - \sin(x + t) - 2e^{z+t} - 6(y - t)$.

We have, therefore, the result: $u_{xx} + u_{yy} + u_{zz} = u_{tt}$.

**(b)** $u(x, y, z, t) = f_1(x - t) + f_2(x + t) + g_1(y - t) + g_2(y + t) + h_1(z - t) + h_2(z + t)$ so

- $u_x = (f_1)_{x-t}\dfrac{\partial(x-t)}{\partial x} + (f_2)_{x+t}\dfrac{\partial(x+t)}{\partial x} = (f_1)_{x-t} + (f_2)_{x+t}$ so
- $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 f_1}{\partial(x-t)^2} + \dfrac{\partial^2 f_2}{\partial(x+t)^2}$
- $u_y = (g_1)_{y-t}\dfrac{\partial(y-t)}{\partial y} + (g_2)_{y+t}\dfrac{\partial(y+t)}{\partial y} = (g_1)_{y-t} + (g_2)_{y+t}$ so
- $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 g_1}{\partial(y-t)^2} + \dfrac{\partial^2 g_2}{\partial(y+t)^2}.$
- $u_z = (h_1)_{z-t}\dfrac{\partial(z-t)}{\partial z} + (h_2)_{z+t}\dfrac{\partial(z+t)}{\partial z} = (h_1)_{z-t} + (h_2)_{z+t}$ so
- $\dfrac{\partial^2 u}{\partial z^2} = \dfrac{\partial^2 h_1}{\partial(z-t)^2} + \dfrac{\partial^2 h_2}{\partial(z+t)^2}.$
- $u_t = (f_1)_{x-t}\dfrac{\partial(x-t)}{\partial t} + (f_2)_{x+t}\dfrac{\partial(x+t)}{\partial t} + (g_1)_{y-t}\dfrac{\partial(y-t)}{\partial t} + (g_2)_{y+t}\dfrac{\partial(y+t)}{\partial t} +$ $(h_1)_{z-t}\dfrac{\partial(z-t)}{\partial t} + (h_2)_{z+t}\dfrac{\partial(z+t)}{\partial t}$ so $u_t = -(f_1)_{x-t} + (f_2)_{x+t} - (g_1)_{y-t} + (g_2)_{y+t} - (h_1)_{z-t}\dfrac{\partial(z-t)}{\partial t} +$ $(h_2)_{z+t}\dfrac{\partial(z+t)}{\partial t}$ and
- $u_{tt} = u_{xx} + u_{yy} + u_{zz}.$

**37.** $F(tx, ty) = t^3x^3 + t^3xy^2 - 6t^3y^3 = t^3F(x, y)$ so $F$ is homogeneous of degree 3.

**38.** $F(tx, ty, tz) = t^3x^3y - t^4x^2z^2 + t^8z^8$ so, no, $F$ is not homogeneous.

**39.** $F(tx, ty, tz) = t^3zy^2 - t^3x^3 + t^3x^2z = t^3F(x, y, z)$ so yes $F$ is homogeneous of degree 3.

**40.** $F(tx, ty) = e^{ty/tx} = e^{y/x} = F(x, y)$ so $F$ is homogeneous of degree 0.

**41.** $F(tx, ty, tz) = \dfrac{t^3x^3 + t^3x^2y - t^3yz^2}{t^3xyz + 7t^3xz^2} = F(x, y, z)$ so $F$ is homogeneous of degree 0.

**42.** Make sure that the students realize (as in Exercises 40 and 41) that a function can be homogeneous and not be a polynomial. In the special case that $F$ is a polynomial, $F$ is homogeneous when all of the terms are of the same degree.

**43.** $F(tx_1, tx_2, \ldots, tx_n) = t^dF(x_1, x_2, \ldots, x_n)$ so that, by differentiating both sides with respect to $t$:

$$x_1\frac{\partial F}{\partial x_1}(tx_1, \ldots, tx_n) + \cdots + x_n\frac{\partial F}{\partial x_n}(tx_1, \ldots, tx_n) = dt^{d-1}F(x_1, \ldots, x_n).$$

Now let $t = 1$ and we get the result:

$$x_1\frac{\partial F}{\partial x_1} + \cdots + x_n\frac{\partial F}{\partial x_n} = dF.$$

**44.** The conjecture is:

$$\sum_{i_1,\ldots,i_k=1}^{n} = x_{i_1}x_{i_2}\cdots x_{i_k}F_{x_{i_1}x_{i_2}\cdots x_{i_k}} = \frac{d!}{(d-k)!}F.$$

Although not asked in the text, a good exercise is to ask the students to establish the formula given in this exercise. Show that

$$\frac{\partial}{\partial x_i}[dF] = \sum_{j=1}^{n} x_j\frac{\partial^2 F}{\partial x_i \partial x_j} + \frac{\partial F}{\partial x_i}.$$

Then you can show

$$d^2F = d\sum_{i=1}^{n} x_i\frac{\partial F}{\partial x_i} = \sum_{i,j=1}^{n} x_ix_j\frac{\partial^2 F}{\partial x_i \partial x_j} + dF.$$

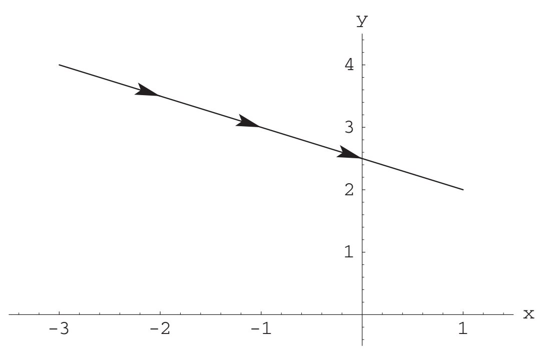
You can finish from there.

## Chapter 3

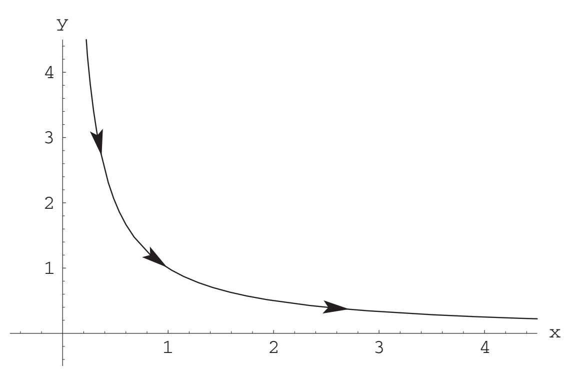
# Vector-Valued Functions

### 3.1 Parametrized Curves and Kepler's Laws

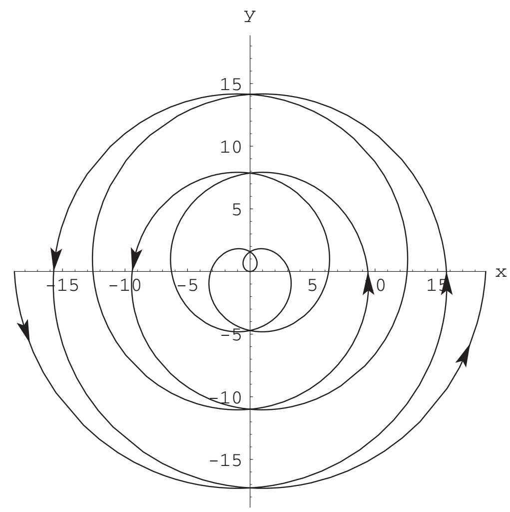
**1.** The graph is a line segment with slope $-1/2$ and $y$-intercept 3:



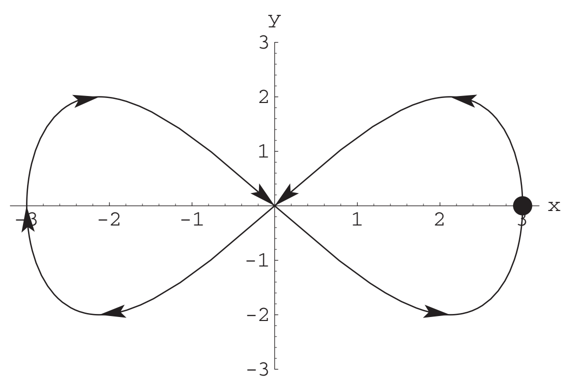

**2.** In this case $y = 1/x$ and both $x$ and $y$ are positive:



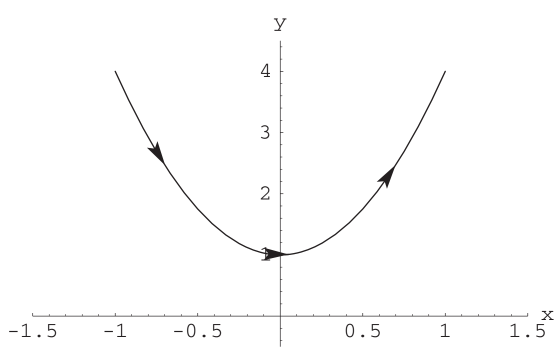
**3.** This is the spiral $r = \theta$ (note $x = r\cos\theta$ and $y = r\sin\theta$):



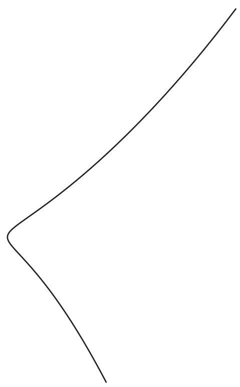

**4.** This is a lemniscate beginning and ending at the point (3, 0):

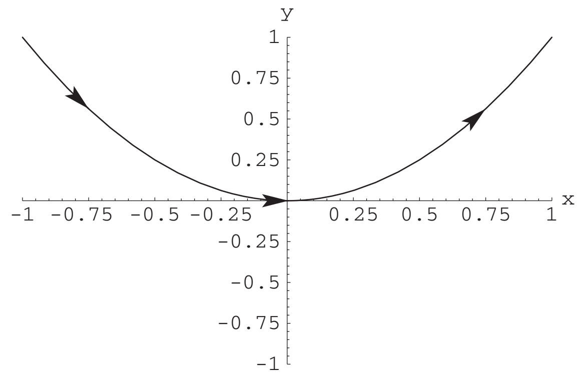



**5.** Although this is a curve in $\mathbf{R}^3$, because $z \equiv 0$ the curve lives in the $xy$-plane. It is the parabola $y = 3x^2 + 1$:

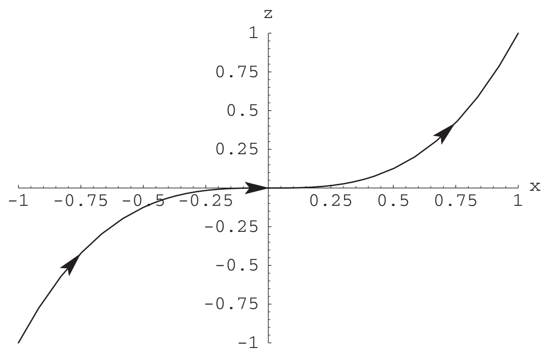


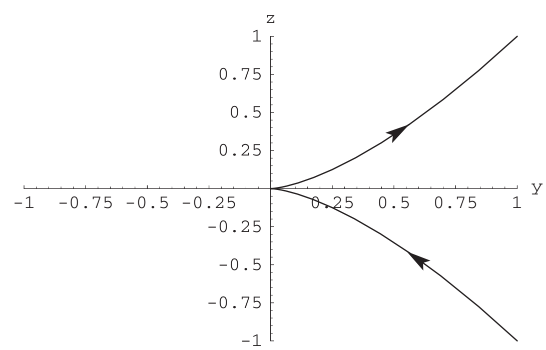
**6.** It's hard to see what this curve looks like in $\mathbf{R}^3$ (below left):






so I have also projected it onto the three coordinate planes:




*For Exercises 7–10, the velocity is the derivative of position, the speed is the length of the velocity vector and the acceleration is the derivative of the velocity vector.*

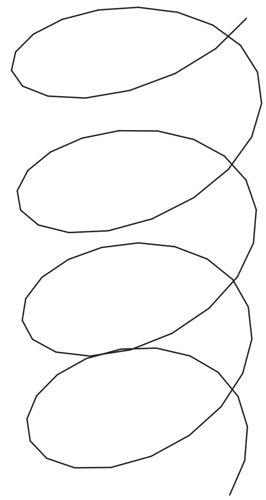
**7.** $\mathbf{x}(t) = (3t-5, 2t+7)$ so velocity $= \mathbf{v}(t) = \mathbf{x}'(t) = (3,2)$ and speed $= \|\mathbf{v}(t)\| = \sqrt{3^2+2^2} = \sqrt{13}$. Finally, acceleration $= \mathbf{a}(t) = \mathbf{x}''(t) = (0,0)$.

**8.** $\mathbf{x}(t) = (5\cos t, 3\sin t)$ so velocity $= \mathbf{v}(t) = \mathbf{x}'(t) = (-5\sin t, 3\cos t)$ and speed $= \|\mathbf{v}(t)\| = \sqrt{(-5\sin t)^2 + (3\cos t)^2} = \sqrt{9+16\sin^2 t}$. Acceleration $= \mathbf{a}(t) = \mathbf{x}''(t) = (-5\cos t, -3\sin t) = -\mathbf{x}(t)$.

**9.** $\mathbf{x}(t) = (t\sin t, t\cos t, t^2)$ so velocity $= \mathbf{v}(t) = \mathbf{x}'(t) = (\sin t + t\cos t, \cos t - t\sin t, 2t)$ and speed $= \|\mathbf{v}(t)\| = \sqrt{\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + 4t^2} = \sqrt{1+5t^2}$. Finally, acceleration $= \mathbf{a}(t) = \mathbf{x}''(t) = (2\cos t - t\sin t, -2\sin t - t\cos t, 2)$.

**10.** $\mathbf{x}(t) = (e^t, e^{2t}, 2e^t)$ so velocity $= \mathbf{v}(t) = \mathbf{x}'(t) = (e^t, 2e^{2t}, 2e^t)$ and speed $= \|\mathbf{v}(t)\| = \sqrt{5e^{2t}+4e^{4t}} = e^t\sqrt{5+4e^{2t}}$. Finally, acceleration $= \mathbf{a}(t) = \mathbf{x}''(t) = (e^t, 4e^{2t}, 2e^t)$.

**11. (a)**

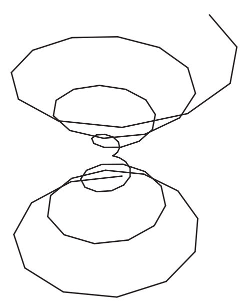


**(b)** To verify that the curve lies on the surface check that

$$\frac{x^2}{9} + \frac{y^2}{16} = \frac{9\cos^2 \pi t}{9} + \frac{16\sin^2 \pi t}{16} = \cos^2 \pi t + \sin^2 \pi t = 1.$$

The $z$ component just determines the speed traveling up the cylinder.

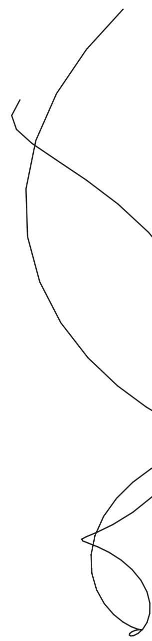
**12. (a)**



**(b)** To verify that the curve lies on the surface check that

$$x^2 + y^2 = t^2 \cos^2 t + t^2 \sin^2 t = t^2(\cos^2 t + \sin^2 t) = t^2 = z^2.$$

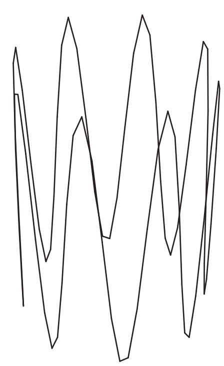
**13. (a)**



**(b)** To verify that the curve lies on the surface check that

$$x^2 + y^2 = t^2 \sin^2 2t + t^2 \cos^2 2t = t^2(\sin^2 2t + \cos^2 2t) = t^2 = z.$$

**14. (a)**



**(b)** To verify that the curve lies on the surface check that

$$x^2 + y^2 = 4\cos^2 t + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4.$$

*In Exercises 15–18 use formulas (2) and (3) from the text. In each case we will need to calculate the position and velocity at the given time.*

**15.** $\mathbf{x}(t) = (te^{-t}, e^{3t})$ so $\mathbf{x}(0) = (0, 1)$ and $\mathbf{x}'(t) = (e^{-t} - te^{-t}, 3e^{3t})$ so $\mathbf{x}'(0) = (1, 3)$. The equation of the tangent line at $t = 0$ is $\mathbf{l}(t) = (0, 1) + (1, 3)t = (t, 1 + 3t)$.

**16.** $\mathbf{x}(t) = (4\cos t, -3\sin t, 5t)$ so $\mathbf{x}(\pi/3) = (2, -3\sqrt{3}/2, 5\pi/3)$ and $\mathbf{x}'(t) = (-4\sin t, -3\cos t, 5)$ so $\mathbf{x}'(\pi/3) = (-2\sqrt{3}, -3/2, 5)$. The equation of the tangent line at $t = \pi/3$ is

$$\mathbf{l}(t) = (2, -3\sqrt{3}/2, 5\pi/3) + (-2\sqrt{3}, -3/2, 5)(t - \pi/3).$$

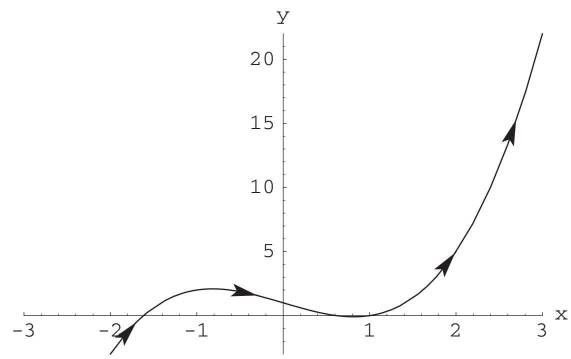
**17.** $\mathbf{x}(t) = (t^2, t^3, t^5)$ so $\mathbf{x}(2) = (4, 8, 32)$ and $\mathbf{x}'(t) = (2t, 3t^2, 5t^4)$ so $\mathbf{x}'(2) = (4, 12, 80)$. The equation of the tangent line at $t = 2$ is

$$\mathbf{l}(t) = (4, 8, 32) + (4, 12, 80)(t - 2) = (4t - 4, 12t - 16, 80t - 128).$$

**18.** $\mathbf{x}(t) = (\cos(e^t), 3 - t^2, t)$ so $\mathbf{x}(1) = (\cos e, 2, 1)$ and $\mathbf{x}'(t) = (-e^t \sin(e^t), -2t, 1)$. Therefore, $\mathbf{x}'(1) = (-e \sin e, -2, 1)$. The equation of the tangent line at $t = 1$ is

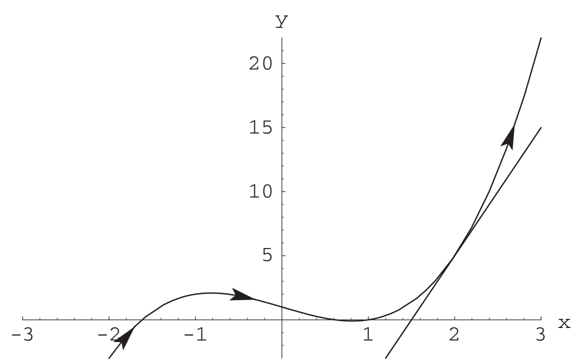
$$\mathbf{l}(t) = (\cos e, 2, 1) + (-e \sin e, -2, 1)(t - 1) = (\cos e + e \sin e - (e \sin e)t, 4 - 2t, t).$$

**19.** **(a)** The sketch of $\mathbf{x}(t) = (t, t^3 - 2t + 1)$ is:




**(b)** $\mathbf{x}(2) = (2, 5)$ and since $\mathbf{x}'(t) = (1, 3t^2 - 2)$ we get $\mathbf{x}'(2) = (1, 10)$. The equation of the line is then

$$\mathbf{l}(t) = (2, 5) + (1, 10)(t - 2) = (t, 10t - 15).$$




**(c)** Since $x = t$ we see that $y = f(x) = x^3 - 2x + 1$.

**(d)** So the equation of the tangent line at $x = 2$ is $y - f(2) = f'(2)(x - 2)$, where $f$ is as in part (c). Substituting, we get $y - 5 = 10(x - 2)$ or $y = 10x - 15$. This is consistent with our answer for part (b).

**20.** From the first equation $t = x/(v_0 \cos\theta)$. Substitute this into the second equation to get

$$y = \frac{(v_0 \sin\theta)x}{v_0 \cos\theta} - \frac{1}{2} g \frac{x^2}{(v_0 \cos\theta)^2} = (\tan\theta)x - \frac{g}{2(v_0 \cos\theta)^2} x^2.$$

This is of the form $y = ax^2 + bx$ and the graph is a parabola.

**21.** We know from the text that Roger is on the ground at $t = 0$ and $t = 2v_0 \sin\theta/g$. By symmetry, Roger is at his maximum height at $t = v_0 \sin\theta/g$. For this exercise this is at time $t = 100 \sin 60°/(32) = 25\sqrt{3}/16$. The maximum height is found by substituting into the equation for $y$:

$$y = (v_0 \sin\theta)\left(\frac{25\sqrt{3}}{16}\right) - \frac{1}{2}(32)\left(\frac{25\sqrt{3}}{16}\right)^2 = (50\sqrt{3})\left(\frac{25\sqrt{3}}{16}\right) - \frac{(625)(3)}{16} = \frac{(625)(3)}{16}$$

Roger's maximum height is 117.1875 feet.

**22.** By formula (5) from the text, $x = v_0^2 \sin 2\theta/g$. In this case we can say that $2640 = v_0^2 \sin 120°/32$. Solve this for

$$v_0 = \sqrt{\frac{(2640)(32)}{\sqrt{3}/2}} = 32\sqrt{55\sqrt{3}} \approx 312.329.$$

**23.** We use the same formula as in Exercise 22 but now solve for $\theta$. So, $x = v_0^2 \sin 2\theta/g$ becomes $1500 = 250^2 \sin 2\theta/32$ or

$$\sin 2\theta = \frac{(1500)(32)}{62500} = \frac{96}{125}.$$

There are two values of $\theta$ with $0 \le \theta \le \pi/2$ that satisfy this last equation. One is

$$\theta = (1/2)\sin^{-1}(96/125) \approx 0.43786 \approx 25.088^\circ,$$

and the other is

$$\theta = \pi/2 - (1/2)\sin^{-1}(96/125) \approx 1.13294 \approx 64.913^\circ.$$

**24.** This is similar to Example 6 from the text. We have the equation:

$$\mathbf{x}(t) = -(1/2)gt^2\mathbf{j} + tv_0 + x_0\mathbf{j}.$$

**(a)** Here the angle is given as $45^\circ$ and the initial speed of the water is 7 *m/s*, therefore, $\mathbf{v}_0 = 7(\sqrt{2}/2, \sqrt{2}/2)$. Also $x_0$ is the initial height of 1 $m$ and gravity is about $-9.8\,m/s^2$. This means that

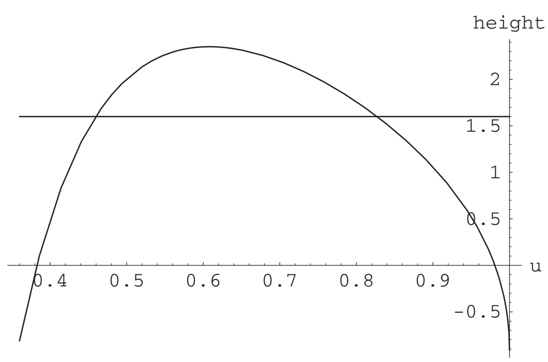
$$\mathbf{x}(t) = -4.9t^2\mathbf{j} + 7(\sqrt{2}/2, \sqrt{2}/2)t + \mathbf{j} = \left(\frac{7\sqrt{2}}{2}t, -4.9t^2 + \frac{7\sqrt{2}}{2}t + 1\right).$$

We want to know the height when the $x$ distance is 5 so first solve $\frac{7\sqrt{2}}{2}t = 5$ for $t$ to get $t = 10/(7\sqrt{2})$. Substitute this into our vertical equation to find that the height would be 1 so the answer is yes, Egbert gets wet.

**(b)** Here the idea is the same as in part (a). The initial speed of the water is 8 *m/s* and we don't know the direction so $v_0 = 8(u, \sqrt{1-u^2})$ for some $u$ between 0 and 1. So

$$\mathbf{x}(t) = -4.9t^2\mathbf{j} + 8(u, \sqrt{1-u^2})t + \mathbf{j} = (8ut, -4.9t^2 + \sqrt{1-u^2}t + 1).$$

We want the height when the horizontal distance is 5 or when $t = 5/(8u)$. In that case, the height is $-4.9(5/(8u))^2 + 8\sqrt{1-u^2}(5/(8u)) + 1$. For what values of $u$ is this between 0 and 1.6? Consider the figure:




Explore with *Mathematica* or a graphing calculator and you will find the $u$ values in the two intervals which correspond to the correct heights. This gives the two approximate ranges for $\alpha$ as between $11.2^\circ$ and $34.2^\circ$ and as between $62.6^\circ$ and $67.5^\circ$.

**25.** We have $\mathbf{x}(2) = \left(e^4, 8, \frac{3}{2}\right)$ and $\mathbf{x}'(2) = \left(2e^4, 10, \frac{5}{4}\right)$. If the rocket's engines cease when $t = 2$, then the rocket will follow the tangent line path

$$\mathbf{l}(t) = \mathbf{x}(2) + (t-2)\mathbf{x}'(2) = \left(e^4(2t-3), 10t - 12, \tfrac{5}{4}t - 1\right).$$

For this path to reach the space station, we must have

$$\left(e^4(2t-3), 10t-12, \frac{5}{4}t - 1\right) = (7e^4, 35, 5).$$

Thus, in particular

$$e^4(2t-3) = 7e^4 \Leftrightarrow 2t - 3 = 7 \Leftrightarrow t = 5.$$

However $\mathbf{l}(5) = \left(7e^4, 38, \frac{21}{4}\right) \ne (7e^4, 35, 5)$. Hence the rocket does *not* reach the repair station.

**26.** **(a)** We set $\mathbf{x}(t) = \mathbf{y}(t)$ and solve for $t$:

$$\left(t^2 - 2, \frac{t^2}{2} - 1\right) = (t, 5 - t^2).$$

Comparing first components, we have $t^2 - 2 = t \Leftrightarrow t^2 - t - 2 = 0 \Leftrightarrow t = -1, 2$. Now $\mathbf{x}(-1) = \left(-1, -\frac{1}{2}\right)$ and $\mathbf{y}(-1) = (-1, 4)$, so this is not a collision point. However, $\mathbf{x}(2) = (2, 1) = \mathbf{y}(2)$. So the balls collide when $t = 2$ at the point (2, 1).

**(b)** We have $\mathbf{x}'(2) = (4, 2), \mathbf{y}'(2) = (1, -4)$. The angle between the paths is the angle between these tangent vectors, which is

$$\cos^{-1}\left[\frac{\mathbf{x}'(2) \cdot \mathbf{y}'(2)}{\|\mathbf{x}'(2)\|\|\mathbf{y}'(2)\|}\right] = \cos^{-1}\frac{-4}{\sqrt{20}\sqrt{17}} = \cos^{-1}\frac{-2}{\sqrt{5}\sqrt{17}}.$$

**27.** The calculation is fairly straightforward:

$$\begin{aligned}
\frac{d}{dt}(\mathbf{x} \cdot \mathbf{y}) &= \frac{d}{dt}(x_1(t)y_1(t) + x_2(t)y_2(t) + \cdots + x_n(t)y_n(t)) \\
&= x_1'(t)y_1(t) + x_1(t)y_1'(t) + x_2'(t)y_2(t) + x_2(t)y_2'(t) + \cdots + x_n'(t)y_n(t) + x_n(t)y_n'(t) \\
&= [x_1'(t)y_1(t) + x_2'(t)y_2(t) + \cdots + x_n'(t)y_n(t)] + [x_1(t)y_1'(t) + x_2(t)y_2'(t) + \cdots + x_n(t)y_n'(t)] \\
&= \mathbf{y} \cdot \frac{d\mathbf{x}}{dt} + \mathbf{x} \cdot \frac{d\mathbf{y}}{dt}.
\end{aligned}$$

**28.** This is similar to Exercise 27:

$$\begin{aligned}
\frac{d}{dt}(\mathbf{x} \times \mathbf{y}) &= \frac{d}{dt}[(x_2y_3 - x_3y_2)\mathbf{i} - (x_1y_3 - x_3y_1)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}] \\
&= (x_2'y_3 - x_3'y_2 + x_2y_3' - x_3y_2')\mathbf{i} - (x_1'y_3 - x_3'y_1 + x_1y_3' - x_3y_1')\mathbf{j} \\
&\quad + (x_1'y_2 - x_2'y_1 + x_1y_2' - x_2y_1')\mathbf{k} \\
&= ([x_2'y_3 - x_3'y_2] + [x_2y_3' - x_3y_2'])\mathbf{i} - ([x_1'y_3 - x_3'y_1] + [x_1y_3' - x_3y_1'])\mathbf{j} \\
&\quad + ([x_1'y_2 - x_2'y_1] + [x_1y_2' - x_2y_1'])\mathbf{k} \\
&= \frac{d\mathbf{x}}{dt} \times \mathbf{y} + \mathbf{x} \times \frac{d\mathbf{y}}{dt}.
\end{aligned}$$

**29.** You're asked to show that if $\|\mathbf{x}(t)\|$ is constant, then $\mathbf{x}$ is perpendicular to $d\mathbf{x}/dt$. If $\|\mathbf{x}(t)\|$ is constant, then $\frac{d}{dt}\|\mathbf{x}(t)\| \equiv 0$. So

$$0 = \frac{d}{dt}\|\mathbf{x}(t)\| = \frac{d}{dt}\sqrt{\mathbf{x} \cdot \mathbf{x}} = \left(\frac{1}{2\sqrt{\mathbf{x} \cdot \mathbf{x}}}\right)\left(2\frac{d\mathbf{x}}{dt} \cdot \mathbf{x}\right).$$

This means that $\frac{d\mathbf{x}}{dt} \cdot \mathbf{x} = 0$.

**30.** **(a)** $\|\mathbf{x}(t)\|^2 = \cos^2 t + \cos^2 t \sin^2 t + \sin^4 t = \cos^2 t + \sin^2 t(\cos^2 t + \sin^2 t) = 1.$

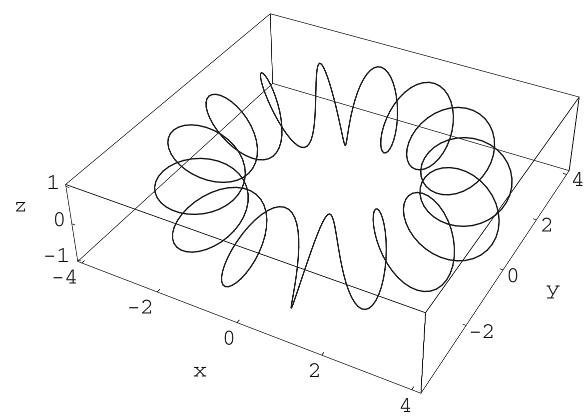
**(b)** This follows from Proposition 1.7 since $\|\mathbf{x}(t)\| \equiv 1$. The exercise really wants you to calculate the velocity vector: $\mathbf{v} = (-\sin t, -\sin^2 t + \cos^2 t, 2\sin t\cos t)$. Then $\mathbf{v} \cdot \mathbf{x} = 0$.

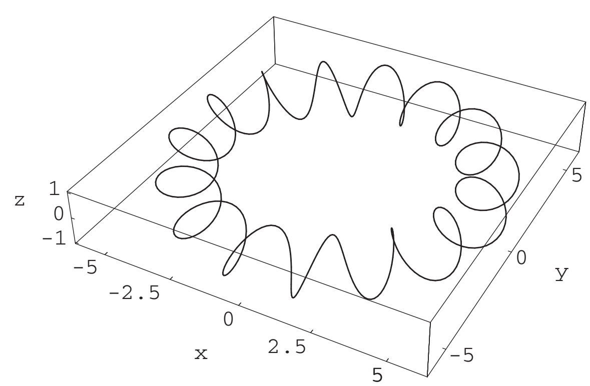
**(c)** If $\mathbf{x}(t)$ is a path on the unit sphere, $\|\mathbf{x}(t)\| \equiv 1$ so by Proposition 1.7 the position vector is perpendicular to its velocity vector.

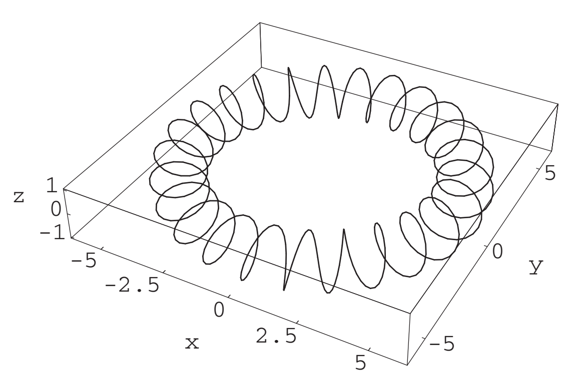
**31. (a)** Computer graphs are shown for (i), (ii), (iii). The constant $a$ affects the size (radius) of the rings; the constant $b$ affects the size (radius) of the coils; the constant $\omega$ affects the number of coils going around the ring.

**(b)** If $x = (a + b\cos\omega t)\cos t, y = (a + b\cos\omega t)\sin t$, then $x^2 + y^2 = (a + b\cos\omega t)^2$, so that $(\sqrt{x^2 + y^2} - a)^2 = (a + b\cos\omega t - a)^2 = b^2\cos^2\,\omega t$. (Note $a > b > 0$.) Hence

$$(\sqrt{x^2 + y^2} - a)^2 + z^2 = b^2\cos^2\,\omega t + b^2\sin^2\,\omega t = b^2.$$



**32.** The angle between $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ is given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{x}(t)\cdot\mathbf{x}'(t)}{\|\mathbf{x}(t)\|\,\|\mathbf{x}'(t)\|}\right).$$

Thus we calculate

$$\begin{aligned}
\mathbf{x}'(t) &= \left(e^t(\cos t - \sin t), e^t(\sin t + \cos t)\right);\\
\mathbf{x}(t)\cdot\mathbf{x}'(t) &= e^{2t}\cos t(\cos t - \sin t) + e^{2t}\sin t(\sin t + \cos t) = e^{2t};\\
\|\mathbf{x}(t)\| &= \sqrt{e^{2t}\cos^2 t + e^{2t}\sin^2 t} = e^t;\\
\|\mathbf{x}'(t)\| &= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2};\\
&= e^t\sqrt{\cos^2 t - 2\cos t\sin t + \sin^2 t + \sin^2 t + 2\cos t\sin t + \cos^2 t} = \sqrt{2}\,e^t.
\end{aligned}$$

Thus

$$\theta = \cos^{-1}\left(\frac{e^{2t}}{(e^t)(\sqrt{2}e^t)}\right) = \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}.$$

**33. (a)** To have $\mathbf{x}(t_1) = (t_1^2, t_1^3 - t_1) = (t_2^2, t_2^3 - t_2) = \mathbf{x}(t_2)$, we must have $t_1^2 = t_2^2$, so if $t_1 \neq t_2$, then $t_1 = -t_2$. Then, comparing the second components: $t_1^3 - t_1 = -t_1^3 + t_1 \iff 2t_1^3 = 2t_1$. Since $t_1 \neq 0$ (otherwise $t_2 = 0$ as well), we must have $t_1^2 = 1$. Thus $\mathbf{x}(1) = \mathbf{x}(-1) = (1, 0)$.

**(b)** The velocity vector of the path is $\mathbf{x}'(t) = (2t, 3t^2 - 1)$. Therefore, the corresponding tangent vectors at $t = \pm 1$ are $\mathbf{x}'(-1) = (-2, 2)$ and $\mathbf{x}'(1) = (2, 2)$. Note that $\mathbf{x}'(-1)\cdot\mathbf{x}'(1) = 0$. Since these tangent vectors are parallel to the corresponding tangent lines, we see that the tangent lines must be perpendicular—so the angle they make is $\pi/2$.

**34. (a)** The slope is

$$t = \frac{y - 0}{x - (-1)} = \frac{y}{x + 1}.$$

Thus $y = t(x + 1)$ is the equation for the line.

**(b)** Since we have $y = t(x + 1)$, we may substitute this expression for $y$ into the equation $x^2 + y^2 = 1$ for the circle. This gives

$$x^2 + t^2(x + 1)^2 = 1 \iff (1 + t^2)x^2 + 2t^2x + (t^2 - 1) = 0.$$

We may use the quadratic formula with this last equation to solve for $x$ in terms of $t$:

$$x = \frac{-2t^2 \pm \sqrt{4t^4 - 4(t^2 + 1)(t^2 - 1)}}{2(t^2 + 1)} = \frac{-t^2 \pm 1}{t^2 + 1}.$$

Hence the two solutions are $x = -1$ (which was to be expected) and

$$x = \frac{-t^2 + 1}{t^2 + 1} = \frac{1 - t^2}{1 + t^2}.$$

**(c)** From $y = t(x + 1)$ in part (a), we see that when $x = -1$, $y = 0$, and when $x = (1 - t^2)/(1 + t^2)$,

$$y = t\left(\frac{1 - t^2}{1 + t^2} + 1\right) = t\left(\frac{(1 - t^2) + (1 + t^2)}{1 + t^2}\right) = \frac{2t}{1 + t^2}.$$

Hence the parametric equations are

$$x = \frac{1 - t^2}{1 + t^2},\ y = \frac{2t}{1 + t^2}.$$

**(d)** The parametrization misses the point $(-1, 0)$, since to have $y = 0$ $t$ must be zero, but then $x = 1$, not $-1$.

**35.** The distance between a point on the image and the origin is $\|\mathbf{x}(t)\|$ and this is minimized when $t = t_0$. Thus the function

$$f(t) = \|\mathbf{x}(t)\|^2 = \mathbf{x}(t)\cdot\mathbf{x}(t)$$

is also minimized when $t = t_0$. Hence

$$\begin{aligned}
0 = f'(t_0) &= \frac{d}{dt}(\mathbf{x}(t)\cdot\mathbf{x}(t))\Big|_{t=t_0} = \mathbf{x}(t_0)\cdot\mathbf{x}'(t_0) + \mathbf{x}'(t_0)\cdot\mathbf{x}(t_0)\\
&= 2\mathbf{x}(t_0)\cdot\mathbf{x}'(t_0).
\end{aligned}$$

Thus $\mathbf{x}(t_0)$ and $\mathbf{x}'(t_0)$ are orthogonal.

## 3.2 Arclength and Differential Geometry

*In Exercises 1–6 we are using Definition 2.1 to calculate the length of the given paths.*

**1.** $\mathbf{x}(t) = (2t+1, 7-3t)$ so $\mathbf{x}'(t) = (2, -3)$. The length of the path is then

$$L(\mathbf{x}) = \int_{-1}^{2} \|\mathbf{x}'\|\, dt = \int_{-1}^{2} \sqrt{2^2 + (-3)^2}\, dt = \int_{-1}^{2} \sqrt{13}\, dt = \sqrt{13}t\Big|_{-1}^{2} = 3\sqrt{13}.$$

**2.** $\mathbf{x}(t) = (t^2, 2/3(2t+1)^{3/2})$ so $\mathbf{x}'(t) = (2t, 2(2t+1)^{1/2})$. The length of the path is then

$$L(\mathbf{x}) = \int_0^4 \sqrt{(2t)^2 + 4(2t+1)}\, dt = \int_0^4 \sqrt{4t^2 + 8t + 4}\, dt = 2\int_0^4 |t+1|\, dt$$
$$= 2\int_0^4 (t+1)\, dt = 2(t^2/2 + t)\Big|_0^4 = 24.$$

**3.** $\mathbf{x}(t) = (\cos 3t, \sin 3t, 2t^{3/2})$ so $\mathbf{x}'(t) = (-3\sin 3t, 3\cos 3t, 3t^{1/2})$. The length of the path is then

$$L(\mathbf{x}) = \int_0^2 \sqrt{9\sin^2 3t + 9\cos^2 3t + 9t}\, dt = 3\int_0^2 \sqrt{1+t}\, dt = 3\int_1^3 \sqrt{u}\, du = 2u^{3/2}\Big|_1^3 = 6\sqrt{3} - 2.$$

**4.** $\mathbf{x}(t) = (7, t, t^2)$ so $\mathbf{x}'(t) = (0, 1, 2t)$. The length of the path is then

$$L(\mathbf{x}) = \int_1^3 \sqrt{1+4t^2}\, dt = 2\int_1^3 \sqrt{1/4 + t^2}\, dt = \left[t\sqrt{1/4+t^2} + (1/4)\ \ln(t + \sqrt{1/4+t^2})\right]\Bigg|_1^3$$
$$= 3\sqrt{\frac{37}{4}} - \sqrt{\frac{5}{4}} + \left(\frac{1}{4}\right)\left[\ln(3+\sqrt{37/4}) - \ln(1+\sqrt{5/4})\right]$$
$$= \frac{3\sqrt{37} - \sqrt{5}}{2} + \left(\frac{1}{4}\right)\left[\ln\left(\frac{6+\sqrt{37}}{2+\sqrt{5}}\right)\right] \approx 8.2681459.$$

**5.** $\mathbf{x}(t) = (t^3, 3t^2, 6t)$ so $\mathbf{x}'(t) = (3t^2, 6t, 6)$. The length of the path is then

$$L(\mathbf{x}) = \int_{-1}^{2} \sqrt{9t^4 + 36t^2 + 36}\, dt = \int_{-1}^{2} \sqrt{9(t^2+2)^2}\, dt$$
$$= \int_{-1}^{2} 3(t^2+2)\, dt = \left(t^3 + 6t\right)\Big|_{-1}^{2} = 27.$$

**6.** $\mathbf{x}(t) = (\ln(\cos t), \cos t, \sin t)$ so $\mathbf{x}'(t) = \left(-\dfrac{\sin t}{\cos t}, \sin t, \cos t\right)$. The length of the path is then

$$L(\mathbf{x}) = \int_{\pi/6}^{\pi/3} \sqrt{\frac{\sin^2 t}{\cos^2 t} + \sin^2 t + \cos^2 t}\, dt = \int_{\pi/6}^{\pi/3} \sqrt{\frac{\sin^2 t}{\cos^2 t} + 1}\, dt$$
$$= \int_{\pi/6}^{\pi/3} \sqrt{\frac{\sin^2 t + \cos^2 t}{\cos^2 t}}\, dt = \int_{\pi/6}^{\pi/3} \frac{1}{\cos t}\, dt = \int_{\pi/6}^{\pi/3} \sec t\, dt$$
$$= \ln|\sec t + \tan t|\Big|_{\pi/6}^{\pi/3} = \ln(2+\sqrt{3}) - \ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) = \ln\left(\frac{2\sqrt{3}+3}{3}\right).$$

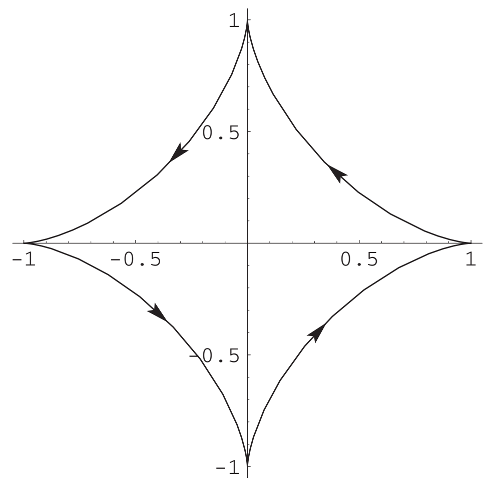
**7.** $\mathbf{x}(t) = (\ln t, t^2/2, \sqrt{2}t)$ so $\mathbf{x}'(t) = (1/t, t, \sqrt{2})$. The length of the path is then

$$L(\mathbf{x}) = \int_1^4 \sqrt{1/t^2 + t^2 + 2}\, dt = \int_1^4 \sqrt{(1/t + t)^2}\, dt = \int_1^4 (1/t + t)\, dt$$
$$= [\ln t + t^2/2]\Big|_1^4 = \ln 4 + 8 - 1/2 = \ln 4 + \tfrac{15}{2}.$$

**8.** $\mathbf{x}(t) = (2t\cos t, 2t\sin t, 2\sqrt{2}t^2)$ so $\mathbf{x}'(t) = (2\cos t - 2t\sin t, 2\sin t + 2t\cos t, 4\sqrt{2}t)$. The length of the path is then

$$\begin{aligned} L(\mathbf{x}) &= \int_0^3 \sqrt{4\cos^2 t - 8t\cos t\sin t + 4t^2\sin^2 t + 4\sin^2 t + 8t\sin t\cos t + 4t^2\cos^2 t + 32t^2}\,dt \\ &= \int_0^3 \sqrt{4 + 4t^2 + 32t^2}\,dt = \int_0^3 \sqrt{4 + 36t^2}\,dt \\ &= [t\sqrt{1+9t^2} + \sinh^{-1}(3t)/3]\Big|_0^3 = 3\sqrt{82} + \sinh^{-1}(9)/3. \end{aligned}$$

**9.** A sketch of the curve $\mathbf{x}(t) = (a\cos^3 t, a\sin^3 t)$ for $0 \le t \le 2\pi$ is:




Because of the obvious symmetries we will compute the length of the portion of the curve in the first quadrant and multiply it by 4:

$$\begin{aligned} L(\mathbf{x}) &= 4\int_0^{\pi/2} \|(-3a\cos^2 t\sin t, 3a\sin^2 t\cos t)\|\,dt = 4\int_0^{\pi/2} \sqrt{9a^2(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t)}\,dt \\ &= 4\int_0^{\pi/2} \sqrt{9a^2\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)}\,dt = 4\int_0^{\pi/2} 3a\sin t\cos t\,dt = 6a\sin^2 t\Big|_0^{\pi/2} \\ &= 6a. \end{aligned}$$

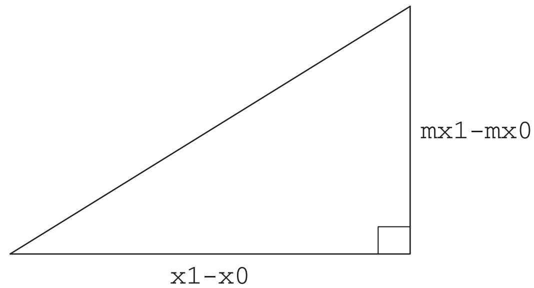
**10.** If $f$ is a continuously differentiable function then we can calculate the length of the curve $y = f(x)$ between $(a, f(a))$ and $(b, f(b))$ by viewing the curve as the path $\mathbf{y}(x) = (x, f(x))$ so $\mathbf{y}'(x) = (1, f'(x))$, and so by Definition 2.1 the length is

$$L(\mathbf{y}) = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

**11.** Here $f(x) = mx + b$ and $f'(x) = m$ so by Exercise 10, the length of the curve is

$$L = \int_{x_0}^{x_1} \sqrt{1+m^2}\,dt = (x_1 - x_0)\sqrt{1+m^2}.$$

A quick look at a sketch shows why this should be the case:




If $x_1 > x_0$ then the horizontal distance is $|x_1 - x_0| = x_1 - x_0$ and the vertical distance is $|mx_1 - mx_0| = |m|(x_1 - x_0)$. By the Pythagorean theorem, the length of the hypotenuse is

$$\sqrt{(x_1 - x_0)^2 + (|m|(x_1 - x_0))^2} = \sqrt{(x_1 - x_0)^2(m^2 + 1)} = (x_1 - x_0)\sqrt{m^2 + 1}.$$

**12. (a)** $\mathbf{x}(t) = (a_1 t + b, a_2 t + b)$ so $\mathbf{x}'(t) = (a_1, a_2)$ so

$$L(\mathbf{x}) = \int_{t_0}^{t_1} \sqrt{a_1^2 + a_2^2}\, dt = (t_1 - t_0)\sqrt{a_1^2 + a_2^2}.$$

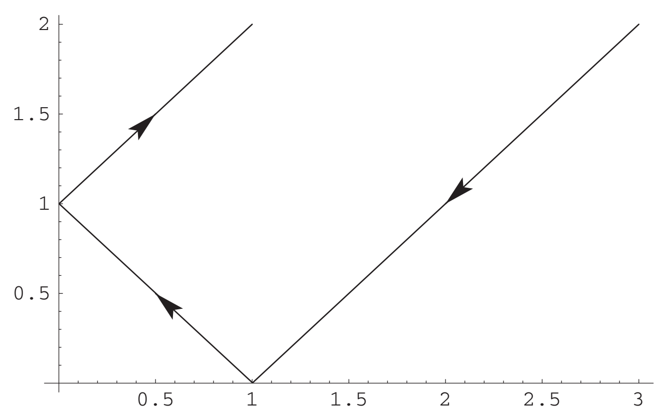
**(b)** The equation of the line in Exercise 11 could be given as $\mathbf{x}(t) = (t, mt + b)$ in which case part (a) would tell us that the length is $(x_1 - x_0)\sqrt{1 + m^2}$.

**(c)** $\mathbf{x}(t) = \mathbf{a}t + \mathbf{b}$ so $\mathbf{x}'(t) = \mathbf{a}$. Then

$$L(\mathbf{x}) = \int_{t_0}^{t_1} \|\mathbf{a}\|\, dt = \|\mathbf{a}\|(t_1 - t_0).$$

This, of course, is the same as our answer in part (a).

**13. (a)** A sketch of $\mathbf{x} = |t - 1|\mathbf{i} + |t|\mathbf{j}, -2 \le t \le 2$ is:




**(b)** Except for two points, the path is smooth (more than $C^1$). In fact, the path is comprised of three line segments joined end to end. In other words, on the open intervals $-2 \le t < 0, 0 < t < 1$, and $1 < t \le 2$, the path $\mathbf{x}$ is $C^1$. We say that the path $\mathbf{x}$ is piecewise $C^1$.

**(c)** We could figure out the length of each piece and add them together. In the process we will find that we're working too hard.

$$\mathbf{x}(t) = \begin{cases} (1 - t)\mathbf{i} - t\mathbf{j} & -2 \le t \le 0 \\ (1 - t)\mathbf{i} + t\mathbf{j} & 0 < t \le 1 \\ (t - 1)\mathbf{i} + t\mathbf{j} & 1 < t \le 2 \end{cases} \quad \text{so} \quad \mathbf{x}'(t) = \begin{cases} (-1, -1) & -2 \le t \le 0 \\ (-1, 1) & 0 < t \le 1 \\ (1, 1) & 1 < t \le 2 \end{cases}.$$

So we see that $\|\mathbf{x}'(t)\| \equiv \sqrt{2}$. This means that to calculate the length of the curve we don't have to break up the integral into three pieces:

$$L(\mathbf{x}) = \int_{-2}^{2} \sqrt{2}\, dt = 4\sqrt{2}.$$

**14. (a)** We have that

$$\|\mathbf{x}(t)\|^2 = e^{-2t}\cos^2 t + e^{-2t}\sin^2 t = e^{-2t}.$$

Thus

$$\lim_{t\to+\infty}\|\mathbf{x}(t)\| = \lim_{t\to+\infty} e^{-t} = 0.$$

Hence $\lim_{t\to+\infty}\mathbf{x}(t) = \mathbf{0}$.

**(b)** We compute that $\mathbf{x}(t) = (-e^{-t}\cos t - e^{-t}\sin t, e^{-t}\cos t - e^{-t}\sin t)$. Hence

$$\begin{aligned}\|\mathbf{x}'(t)\| &= \sqrt{e^{-2t}(-\cos t - \sin t)^2 + e^{-2t}(\cos t - \sin t)^2}\\ &= e^{-t}\sqrt{\cos^2 t + 2\cos t\sin t + \sin^2 t + \cos^2 t - 2\cos t\sin t + \sin^2 t}\\ &= \sqrt{2}e^{-t}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_a^\infty \|\mathbf{x}'(t)\|\,dt &= \lim_{t\to\infty}\int_a^t \|\mathbf{x}'(\tau)\|\,d\tau = \lim_{t\to\infty}\int_a^t \sqrt{2}e^{-\tau}\,d\tau\\ &= \lim_{t\to\infty}\left(-\sqrt{2}e^{-t} + \sqrt{2}e^{-a}\right) = \sqrt{2}e^{-a}.\end{aligned}$$

**(c)** The integral in part (b) represents the length of the path that spirals into $(0,0)$ from the point $\mathbf{x}(a)$. The result of part (b) shows that this arclength is always finite, regardless of $a$.

**15.** We use the polar/rectangular conversion equations $x = r\cos\theta$, $y = r\sin\theta$ to define a path $\mathbf{x}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. Then

$$\mathbf{x}'(\theta) = (f'(\theta)\cos\theta - f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta),$$

which implies

$$\begin{aligned}\|\mathbf{x}'(\theta)\| &= \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\\ &= \sqrt{f'(\theta)^2 + f(\theta)^2},\end{aligned}$$

after expansion and simplification. Hence $L = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\,d\theta$, as desired.

**16. (a)** We'll use the equation: $s(t) = \int_0^t \|\mathbf{x}'(\tau)\|\,d\tau$. For $\mathbf{x}(\tau) = e^{a\tau}\cos b\tau\,\mathbf{i} + e^{a\tau}\sin b\tau\,\mathbf{j} + e^{a\tau}\mathbf{k}$ the derivative is $\mathbf{x}'(\tau) = ae^{a\tau}(\cos b\tau, \sin b\tau, 1) + e^{a\tau}(-b\sin b\tau, b\cos b\tau, 0)$. Therefore the speed is given by $\|\mathbf{x}'(\tau)\| = \sqrt{a^2e^{2a\tau}(2) + b^2e^{2a\tau}} = e^{a\tau}\sqrt{2a^2+b^2}$. This means that

$$s(t) = \int_0^t e^{a\tau}\sqrt{2a^2+b^2}\,d\tau = \frac{\sqrt{2a^2+b^2}}{a}e^{a\tau}\bigg|_0^t = \frac{\sqrt{2a^2+b^2}}{a}(e^{at}-1).$$

**(b)** Just solve the above for $t$:

$$t = \left(\ln\left[\frac{as}{\sqrt{2a^2+b^2}} + 1\right]\right)\bigg/ a.$$

*For Problems 17–20, we'll use*

$$\mathbf{T} = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}, \quad \mathbf{N} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}, \quad \text{and} \quad \mathbf{B} = \mathbf{T}\times\mathbf{N}.$$

*Also*

$$\frac{ds}{dt} = \|\mathbf{x}'(t)\|, \quad \kappa(t) = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \left\|\frac{d\mathbf{T}}{ds}\right\|, \quad \text{and} \quad \frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}/dt}{ds/dt} = -\tau\mathbf{N}.$$

*You may want to ask your students to make a guess about $\tau$ before they do Exercises 18 and 20. The curves are planar—what might that suggest about $\tau$? Also see Section 3.6, Exercise 28.*

**17.** $\mathbf{x}(t) = (5\cos 3t, 6t, 5\sin 3t)$ so $\mathbf{x}'(t) = (-15\sin 3t, 6, 15\cos 3t)$ and $\|\mathbf{x}'(t)\| = \sqrt{225+36} = \sqrt{261}$.

$$\begin{aligned}\mathbf{T} &= (1/\sqrt{261})(-15\sin 3t, 6, 15\cos 3t)\\ &= (1/\sqrt{29})(-5\sin 3t, 2, 5\cos 3t).\end{aligned}$$

$d\mathbf{T}/dt = (1/\sqrt{29})(-15\cos 3t, 0, -15\sin 3t)$ so

$$\mathbf{N} = (-\cos 3t, 0, -\sin 3t),$$

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = (1/\sqrt{29})(-2\sin 3t, -5, 2\cos 3t), \text{ and}$$

$$\kappa = \left\| \frac{(1/\sqrt{29})(-15\cos 3t, 0, -15\sin 3t)}{\sqrt{261}} \right\| = \frac{\|(-5\cos 3t, 0, -5\sin 3t)\|}{29} = \frac{5}{29}.$$

Finally,

$$\tau\mathbf{N} = \frac{(1/\sqrt{29})(-6\cos 3t, 0, -6\sin 3t)}{\sqrt{261}} = \frac{(-2\cos 3t, 0, -2\cos 3t)}{29} \quad \text{so}$$

$$\tau = -\frac{2}{29}.$$

**18.** $\mathbf{x}(t) = (\sin t - t\cos t, \cos t + t\sin t, 2)$ with $t \geq 0$. So $\mathbf{x}'(t) = (t\sin t, t\cos t, 0)$ and $\|\mathbf{x}'(t)\| = |t| = t$.

$$\mathbf{T} = \frac{(t\sin t, t\cos t, 0)}{t} = (\sin t, \cos t, 0), \text{ and}$$

$$\mathbf{N} = (\cos t, -\sin t, 0), \mathbf{B} = (0, 0, -1), \text{ and}$$

$$\kappa = \frac{\|(\cos t, -\sin t, 0)\|}{t} = \frac{1}{t}.$$

Finally, $d\mathbf{B}/dt = 0$ so $\tau = 0$.

**19.** $\mathbf{x}(t) = (t, (1/3)(t+1)^{3/2}, (1/3)(1-t)^{3/2})$ so $\mathbf{x}'(t) = (1, (1/2)(t+1)^{1/2}, -(1/2)(1-t)^{1/2})$, and $\|\mathbf{x}'(t)\| = \sqrt{3/2}$.

$$\mathbf{T} = \sqrt{\frac{2}{3}}\left(1, \tfrac{1}{2}\sqrt{t+1}, -\tfrac{1}{2}\sqrt{1-t}\right), \text{ and}$$

$$\mathbf{N} = \frac{\sqrt{2/3}(0, (1/4)(t+1)^{-1/2}, (1/4)(1-t)^{-1/2})}{\sqrt{(2/3)(1/16)\left(\frac{1}{t+1} + \frac{1}{1-t}\right)}}$$

$$= \frac{1}{\sqrt{2}}(0, \sqrt{1-t}, \sqrt{t+1}), \text{ and}$$

$$\mathbf{B} = \sqrt{\frac{1}{3}}\left(\tfrac{1}{2}(t+1) + \tfrac{1}{2}(1-t), -\sqrt{t+1}, \sqrt{1-t}\right) = \frac{1}{\sqrt{3}}(1, -\sqrt{t+1}, \sqrt{1-t}).$$

Also,

$$\kappa = \frac{\|\sqrt{2/3}(0, (1/4)(t+1)^{-1/2}, (1/4)(1-t)^{-1/2})\|}{\sqrt{3/2}} = \frac{1}{3\sqrt{2(1-t^2)}}.$$

Finally,

$$\frac{d\mathbf{B}}{dt} = \frac{1}{\sqrt{3}}\left(0, -\frac{1}{2\sqrt{t+1}}, -\frac{1}{2\sqrt{1-t}}\right) \text{ so}$$

$$\frac{d\mathbf{B}}{ds} = \frac{1}{\sqrt{3}}\left(0, -\frac{1}{2\sqrt{t+1}}, -\frac{1}{2\sqrt{1-t}}\right) \Big/ \sqrt{3/2} = -\tau\mathbf{N}.$$

Solving,

$$\tau = \frac{1}{3\sqrt{(1-t^2)}}.$$

**20.** $\mathbf{x}(t) = (e^{2t}\sin t, e^{2t}\cos t, 1)$ so $\mathbf{x}'(t) = e^{2t}(2\sin t + \cos t, 2\cos t - \sin t, 0)$, and $\|\mathbf{x}'(t)\| = e^{2t}\sqrt{5}$.

$$\mathbf{T} = \frac{(2\sin t + \cos t, 2\cos t - \sin t, 0)}{\sqrt{5}},$$

$$\mathbf{N} = \frac{(2\cos t - \sin t, -2\sin t - \cos t, 0)}{\sqrt{5}}, \text{ and}$$

$$\mathbf{B} = (0, 0, -1).$$

Also,

$$\kappa = \frac{\|(2\cos t - \sin t, -2\sin t - \cos t, 0)\|}{e^{2t}\sqrt{5}} = \frac{1}{e^{2t}\sqrt{5}}.$$

Finally, again we see that $d\mathbf{B}/dt = \mathbf{0}$ so $d\mathbf{B}/ds = \mathbf{0}$ and hence $\tau = 0$.

**21. (a)** By formula (17): $\kappa = \frac{\|\mathbf{x}' \times \mathbf{x}''\|}{\|\mathbf{x}'\|^3}$. Let $y = f(x)$ and view the problem as sitting inside of $\mathbf{R}^3$. Then $\mathbf{x} = (x, f(x), 0)$, $\mathbf{x}' = (1, f'(x), 0)$, and $\mathbf{x}'' = (0, f''(x), 0)$. We calculate the cross product $\mathbf{x}' \times \mathbf{x}'' = (0, 0, f''(x))$ so

$$\kappa = \frac{\|(0, 0, f''(x))\|}{\|(1, f'(x), 0)\|^3} = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

**(b)** If $y = \ln(\sin x)$, then $y' = \cos x / \sin x$ and $y'' = -1/\sin^2 x$. By our results for part (a),

$$\kappa = \frac{|-1/\sin^2 x|}{[1 + (\cos^2 x/\sin^2 x)]^{3/2}} = |\sin x|.$$

**22. (a)** Formula (17) requires the use of the cross product, so we view this problem as sitting inside of $\mathbf{R}^3$. Let $\mathbf{x} = (x(s), y(s), 0)$. Then $\mathbf{x}' = (x'(s), y'(s), 0)$, and $\mathbf{x}'' = (x''(s), y''(s), 0)$. By formula (17):

$$\kappa = \frac{\|\mathbf{x}' \times \mathbf{x}''\|}{\|\mathbf{x}'\|^3} = \frac{\|(0, 0, x'y'' - x''y')\|}{\|(x'(s), y'(s), 0)\|^3}.$$

But the curve is parametrized by arclength so $\|(x'(s), y'(s), 0)\| = 1$ so $\kappa = |x'y'' - x''y'|$.

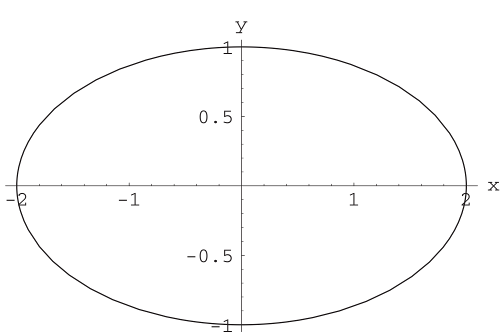
**(b)** Here $x(s) = (1/2)(1 - s^2)$ and $y(s) = (1/2)(\cos^{-1} s - s\sqrt{1 - s^2})$ so $x'(s) = -s$ and $y'(s) = -\sqrt{1 - s^2}$ so $(x'(s))^2 + (y'(s))^2 = 1$. So the curve is parametrized by arclength. We can then compute its curvature using the formula from part (a):

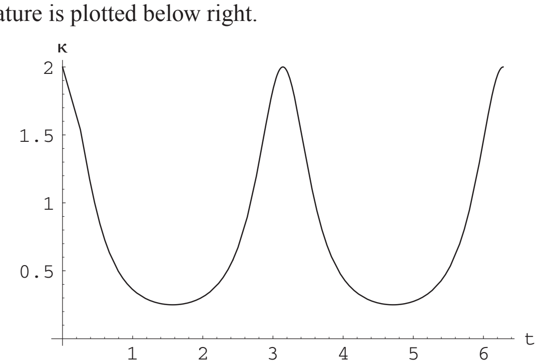
$$\kappa = |x'y'' - x''y'| = \left|(-s)\left(\frac{-s}{\sqrt{1 - s^2}}\right) - (-1)\sqrt{1 - s^2}\right| = \frac{1}{\sqrt{1 - s^2}}.$$

**23. (a)** The curvature is calculated to be

$$\frac{2}{(\cos^2 t + 4\sin^2 t)^{3/2}}.$$

**(b)** The path is pictured below left while the corresponding curvature is plotted below right.

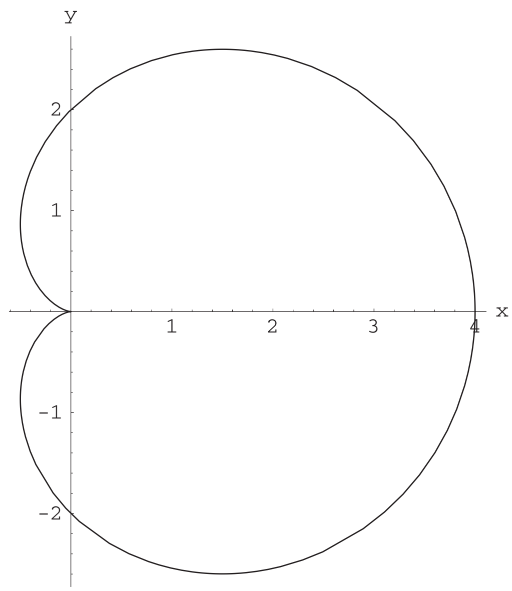


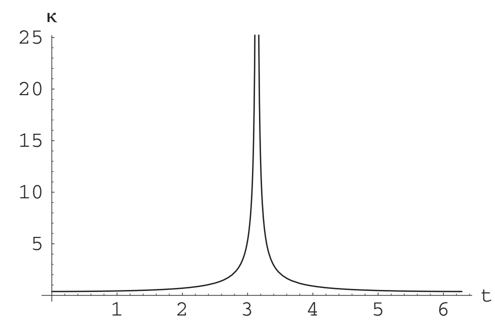




**24. (a)** The curvature is calculated to be (with some simplification)

$$\frac{3(1 + \cos t)}{16\cos^3(t/2)}.$$

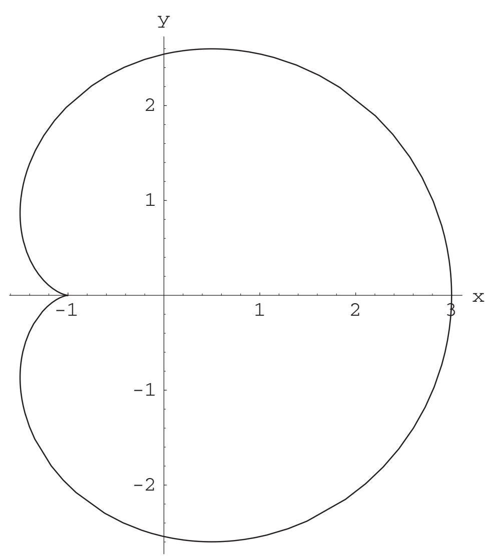
**(b)** The path is pictured below left while the corresponding curvature is plotted below right.

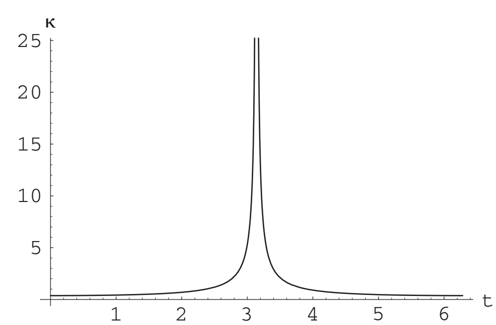



**25.** **(a)** The curvature is the same as in Exercise 24.

**(b)** The path is pictured below left while the corresponding curvature is plotted below right.

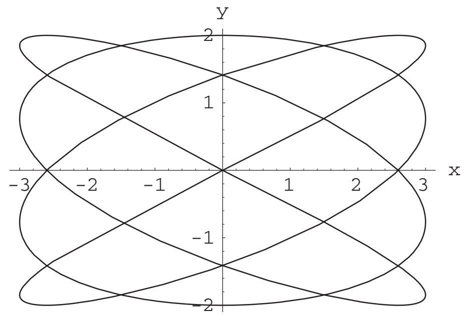


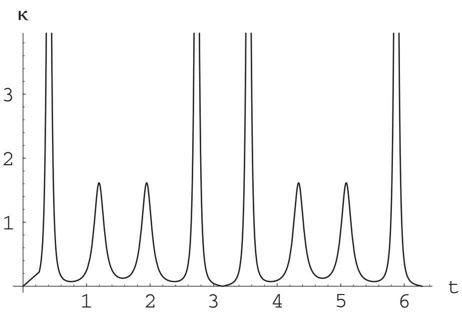




**26.** **(a)** The curvature is calculated to be

$$\frac{\sqrt{2}|7\sin t + \sin 7t|}{3(5 + \cos 6t + 4\cos 8t)^{3/2}}.$$

**(b)** The path is pictured below left while the corresponding curvature is plotted below right.




*For Exercises 27–32, calculate the tangential component $\ddot{s}$ and then subtract it from the length of the acceleration to obtain the normal component.*

**27.** $\mathbf{x}(t) = (t^2, t)$ so $\mathbf{x}'(t) = (2t, 1)$ and $\mathbf{x}''(t) = (2, 0)$. The speed is then $\|\mathbf{x}'(t)\| = \sqrt{1+4t^2}$ and so the tangential component of acceleration is $\ddot{s} = 4t/\sqrt{1+4t^2}$. Since $\|\mathbf{a}\| = 2$, $\|\mathbf{a}\|^2 - \ddot{s}^2 = 4/(1+4t^2)$, so the normal component of acceleration is $2/\sqrt{1+4t^2}$.

**28.** $\mathbf{x}(t) = (2t, e^{2t})$ so $\mathbf{x}'(t) = (2, 2e^{2t})$ and $\mathbf{x}''(t) = (0, 4e^{2t})$. The speed is then $\|\mathbf{x}'(t)\| = 2\sqrt{1+e^{4t}}$ and so the tangential component of acceleration is $\ddot{s} = 4e^{4t}/\sqrt{1+e^{4t}}$. Since $\|\mathbf{a}\| = 16e^{4t}$, $\|\mathbf{a}\|^2 - \ddot{s}^2 = 16e^{4t}/(1+e^{4t})$, so the normal component of acceleration is $4e^{2t}/\sqrt{1+e^{4t}}$.

**29.** $\mathbf{x}(t) = (e^t\cos 2t, e^t\sin 2t)$ so $\mathbf{x}'(t) = (e^t(\cos 2t - 2\sin 2t), e^t(\sin 2t + 2\cos 2t))$ and $\mathbf{x}''(t) = (e^t(-3\cos 2t - 4\sin 2t),$ $e^t(4\cos 2t - 3\sin 2t))$. The speed is then $\|\mathbf{x}'(t)\| = e^t\sqrt{5}$ and so the tangential component of acceleration is $\ddot{s} = e^t\sqrt{5}$. Since $\|\mathbf{a}\| = 5e^t$, $\|\mathbf{a}\|^2 - \ddot{s}^2 = 25e^{2t} - 5e^{2t}$, so the normal component of acceleration is $2\sqrt{5}e^t$.

**30.** $\mathbf{x}(t) = (4\cos 5t, 5\sin 4t, 3t)$ so $\mathbf{x}'(t) = (-20\sin 5t, 20\cos 4t, 3)$ and we also have that $\mathbf{x}''(t) = (-100\cos 5t, -80\sin 4t, 0)$. The speed is then $\|\mathbf{x}'(t)\| = \sqrt{400\sin^2 5t + 400\cos^2 4t + 9}$ and so the tangential component of acceleration is

$$\ddot{s} = \frac{(-3200\cos 4t\sin 4t + 4000\cos 5t\sin 5t)}{\sqrt{400\sin^2 5t + 400\cos^2 4t + 9}}.$$

Since $\|\mathbf{a}\| = 20\sqrt{25\cos^2 5t + 16\sin^2 4t}$,

$$\|\mathbf{a}\|^2 - \ddot{s}^2 = 10000\cos^2 5t + 6400\sin^2 4t - \frac{(3200\cos 4t\sin 4t + 4000\cos 5t\sin 5t)^2}{4(400\sin^2 5t + 400\cos^2 4t + 9)},$$

so the normal component of acceleration is the square root of this last quantity.

**31.** $\mathbf{x}(t) = (t, t, t^2)$ so $\mathbf{x}'(t) = (1, 1, 2t)$ and $\mathbf{x}''(t) = (0, 0, 2)$. The speed is then $\|\mathbf{x}'(t)\| = \sqrt{2+4t^2}$ and so the tangential component of acceleration is $\ddot{s} = 4t/\sqrt{2+4t^2}$. Since $\|\mathbf{a}\| = 2$, $\|\mathbf{a}\|^2 - \ddot{s}^2 = 4/(1+2t^2)$, so the normal component of acceleration is $2/\sqrt{1+2t^2}$.

**32.** $\mathbf{x}(t) = ((3/5)(1-\cos t), \sin t, (4/5)\cos t)$ so $\mathbf{x}'(t) = ((3/5)\sin t, \cos t, (-4/5)\sin t)$ and $\mathbf{x}''(t) =$ $((3/5)\cos t, -\sin t, (-4/5)\cos t)$. The speed is then $\|\mathbf{x}'(t)\| = 1$ and so the tangential component of acceleration is $\ddot{s} = 0$. Since $\|\mathbf{a}\| = 1$, $\|\mathbf{a}\|^2 - \ddot{s}^2 = 1$, so the normal component of acceleration is 1.

**33.** **(a)** Tangential component:

$$\ddot{s} = \frac{d\dot{s}}{dt} = \frac{d\|\mathbf{x}'\|}{dt} = \frac{d\sqrt{\mathbf{x}'\cdot\mathbf{x}'}}{dt} = \left(\frac{1}{2\sqrt{\mathbf{x}'\cdot\mathbf{x}'}}\right)(2\mathbf{x}'\cdot\mathbf{x}'') = \frac{\mathbf{x}'\cdot\mathbf{x}''}{\|\mathbf{x}'\|}.$$

Normal component (using formula (17)):

$$\kappa\dot{s}^2 = \left(\frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3}\right)\|\mathbf{v}\|^2 = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \frac{\|\mathbf{x}'\times\mathbf{x}''\|}{\|\mathbf{x}'\|}.$$

**(b)** $\mathbf{x}(t) = (t+2, t^2, 3t)$ so $\mathbf{x}'(t) = (1, 2t, 3)$ and $\mathbf{x}''(t) = (0, 2, 0)$. So by part (a), the tangential component of acceleration is $4t/\sqrt{10+4t^2}$, and the normal component of acceleration is $2\sqrt{10}/\sqrt{10+4t^2}$.

**34.** Here $\mathbf{x} = (x, f(x), 0)$, $\mathbf{x}' = (1, f'(x), 0)$, and $\mathbf{x}'' = (0, f''(x), 0)$. Further, you need to calculate $\|\mathbf{x}'\| = \sqrt{1+[f'(x)]^2}$, $\mathbf{x}'\cdot\mathbf{x}'' = f'(x)f''(x)$, and $\|\mathbf{x}'\times\mathbf{x}''\| = \|(0, 0, f''(x))\| = |f''(x)|$. Substituting into the formulas from Exercise 33 gives us:

$$a_{\text{tang}} = \frac{f'(x)f''(x)}{\sqrt{1+[f'(x)]^2}}, \quad \text{and} \quad a_{\text{norm}} = \frac{|f''(x)|}{\sqrt{1+[f'(x)]^2}}.$$

**35.** To establish the formula, first note that $\mathbf{v} \times \mathbf{a} = \kappa \dot{s}^3 \mathbf{B}$ (see, for example, the calculation leading up to formula (17)) and $\|\mathbf{v} \times \mathbf{a}\| = \kappa \dot{s}^3 = \kappa \|\mathbf{v}\|^3$. So

$$\frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{\|\mathbf{v} \times \mathbf{a}\|^2} = \frac{\kappa \dot{s}^3 \mathbf{B} \cdot \mathbf{a}'}{\kappa^2 \dot{s}^6} = \frac{\mathbf{B} \cdot \mathbf{a}'}{\kappa \dot{s}^3}.$$

Now, $\mathbf{a}(t) = \ddot{s}\mathbf{T} + \kappa \dot{s}^2 \mathbf{N}$ and by the Frenet equations $\mathbf{N}'(s) = -\kappa \mathbf{T} + \tau \mathbf{B}$. Since we are calculating the dot product of $\mathbf{a}'$ with $\mathbf{B}$, the only piece that will survive is the coefficient of $\mathbf{B}$, so

$$\begin{aligned}\mathbf{a}'(t) &= (\text{something without } \mathbf{B}) + \kappa \dot{s}^2 \mathbf{N}'(s) \frac{ds}{dt} \\ &= (\text{something else without } \mathbf{B}) + \kappa \dot{s}^3 \tau \mathbf{B}\end{aligned}$$

and so, putting it all together,

$$\frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{\|\mathbf{v} \times \mathbf{a}\|^2} = \frac{\kappa \dot{s}^3 \mathbf{B} \cdot \mathbf{a}'}{\kappa^2 \dot{s}^6} = \frac{\mathbf{B} \cdot \mathbf{a}'}{\kappa \dot{s}^3} = \frac{\mathbf{B} \cdot \kappa \dot{s}^3 \tau \mathbf{B}}{\kappa \dot{s}^3} = \tau.$$

**36.** By equations (11) and (13) we have $\mathbf{T}' = \kappa \mathbf{N}$ and $\mathbf{B}' = -\tau \mathbf{N}$
Hence

$$-\mathbf{T}' \cdot \mathbf{B}' = -(\kappa \mathbf{N}) \cdot (-\tau \mathbf{N}) = \kappa \tau\, \mathbf{N} \cdot \mathbf{N} = \kappa \tau,$$

since $\mathbf{N}$ is a unit vector.

**37.** From formula (17) $\kappa = \dfrac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \|\mathbf{x}' \times \mathbf{x}''\|$ since $\mathbf{x}$ must be a unit speed path as it is parametrized by arclength.
By Exercise 35,

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{\|\mathbf{v} \times \mathbf{a}\|^2} = \frac{(\mathbf{x}' \times \mathbf{x}'') \cdot \mathbf{x}'''}{\|\mathbf{x}' \times \mathbf{x}''\|^2}$$

Thus

$$\kappa^2 \tau = \|\mathbf{x}' \times \mathbf{x}''\|^2 \cdot \left( \frac{(\mathbf{x}' \times \mathbf{x}'') \cdot \mathbf{x}'''}{\|\mathbf{x}' \times \mathbf{x}''\|^2} \right) = (\mathbf{x}' \times \mathbf{x}'') \cdot \mathbf{x}'''.$$

**38.** **(a)** Really, there's nothing much to show in this part—but it really helps you solve part (b). $\mathbf{B}$ is $\mathbf{T} \times \mathbf{N}$ so it is perpendicular to the plane determined by them. In this case, we interpret that as $\mathbf{B}$ is perpendicular to the osculating plane. Make the analagous observations for the other two cases.

**(b)** Example 9 gives us the formulas for $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$. Using the result of part (a) we can use the perpendicular vector to write down the equation of the plane. First, $\mathbf{B}$ is perpendicular to the osculating plane. So at $t = t_0$ the osculating plane must be of the form $b \sin t_0 (x - a \cos t_0) - b \cos t_0 (y - a \sin t_0) + a(z - bt_0) = 0$. Similarly the rectifying plane can be obtained from $\mathbf{N}$ as $-\cos t_0 (x - a \cos t_0) - \sin t_0 (y - a \sin t_0) = 0$. Finally, the normal plane is obtained from $\mathbf{T}$ as $-a \sin t_0 (x - a \cos t_0) + a \cos t_0 (y - a \sin t_0) + b(z - bt_0) = 0$.

**39.** We have $\|\mathbf{x} - \mathbf{x}_0\|^2 = (\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = a^2$. Thus $\|\mathbf{x} - \mathbf{x}_0\| = a$, so $\mathbf{x}(t)$ lies on a sphere of radius $a$.

**40.** The normal plane to $\mathbf{x}$ at any point $\mathbf{x}(t)$ is the plane passing through $\mathbf{x}(t)$ and perpendicular to $\mathbf{T}(t)$. Thus the plane has equation $(\mathbf{x} - \mathbf{x}(t)) \cdot \mathbf{T}(t) = 0$ (Here $\mathbf{x}(t)$ and $\mathbf{T}(t)$ are used as "constant" vectors.) Thus, using the product rule,

$$\frac{d}{dt}(\mathbf{x}(t) - \mathbf{x}_0) \cdot (\mathbf{x}(t) - \mathbf{x}_0) = 2(\mathbf{x}(t) - \mathbf{x}_0) \cdot \mathbf{x}'(t).$$

Hence

$$0 = (\mathbf{x}_0 - \mathbf{x}(t)) \cdot \mathbf{T} = -(\mathbf{x}(t) - \mathbf{x}_0) \cdot \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = -\frac{1}{\|\mathbf{x}'(t)\|}(\mathbf{x}(t) - \mathbf{x}_0) \cdot \mathbf{x}'(t)$$

Thus $(\mathbf{x}(t) - \mathbf{x}_0) \cdot \mathbf{x}'(t) = 0$ for all $t$. Hence $(\mathbf{x}(t) - \mathbf{x}_0) \cdot (\mathbf{x}(t) - \mathbf{x}_0) =$ constant, which implies that we have a sphere curve.

**41.** We have $\mathbf{T}(t) = \dfrac{(-2 \sin 2t, -2 \cos 2t, -2 \sin t)}{\sqrt{4 + 4 \sin^2 t}}$.
Now we check that $(\mathbf{x}(t) - (1, 0, 0)) \cdot \mathbf{T}(t) = 0$. This equation is

$$\begin{aligned}&(\cos 2t - 1, -\sin 2t, 2 \cos t) \cdot \frac{(-2 \sin 2t, -2 \cos 2t, -2 \sin t)}{\sqrt{4 + 4 \sin^2 t}} \\ &= \frac{1}{\sqrt{4 + 4 \sin^2 t}}(-2 \cos 2t \sin 2t + 2 \sin 2t + 2 \sin 2t \cos 2t + 4 \cos t \sin t) \\ &= 0.\end{aligned}$$

**42.** By Exercise 27 of §1.4: $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$, so

$$\mathbf{T}\times\mathbf{B} = \mathbf{T}\times(\mathbf{T}\times\mathbf{N}) = -(\mathbf{T}\times\mathbf{N})\times\mathbf{T} = -[(\mathbf{T}\cdot\mathbf{T})\mathbf{N} - (\mathbf{N}\cdot\mathbf{T})\mathbf{T}] = -\mathbf{N}$$
$$\mathbf{N}\times\mathbf{B} = \mathbf{N}\times(\mathbf{T}\times\mathbf{N}) = -(\mathbf{T}\times\mathbf{N})\times\mathbf{N} = -[(\mathbf{T}\cdot\mathbf{N})\mathbf{N} - (\mathbf{N}\cdot\mathbf{N})\mathbf{T}] = \mathbf{T}$$

**43.**
$$\|\mathbf{w}\|^2 = \mathbf{w}\cdot\mathbf{w} = (\tau\mathbf{T}+\kappa\mathbf{B})\cdot(\tau\mathbf{T}+\kappa\mathbf{B}) = \tau^2\mathbf{T}\cdot\mathbf{T} + \kappa\tau\mathbf{B}\cdot\mathbf{T} + \kappa\tau\mathbf{T}\cdot\mathbf{B} + \kappa^2\mathbf{B}\cdot\mathbf{B} = \tau^2+\kappa^2$$

**44. (a)**
$$\begin{aligned}
\mathbf{w}\times\mathbf{T} &= (\tau\mathbf{T}+\kappa\mathbf{B})\times\mathbf{T} = \tau(\mathbf{T}\times\mathbf{T}) + \kappa(\mathbf{B}\times\mathbf{T})\\
&= \kappa(\mathbf{B}\times\mathbf{T}) = \kappa\mathbf{N} \text{ by Exercise 42}\\
&= \mathbf{T}' \text{ by Frenet–Serret}\\
\mathbf{w}\times\mathbf{N} &= (\tau\mathbf{T}+\kappa\mathbf{B})\times\mathbf{N} = \tau(\mathbf{T}\times\mathbf{N}) + \kappa(\mathbf{B}\times\mathbf{N})\\
&= \tau\mathbf{B} - \kappa\mathbf{T} \text{ by Exercise 42}\\
&= \mathbf{N}' \text{ by Frenet–Serret}\\
\mathbf{w}\times\mathbf{B} &= (\tau\mathbf{T}+\kappa\mathbf{B})\times\mathbf{B} = \tau(\mathbf{T}\times\mathbf{B}) = -\tau\mathbf{N} \text{ by Exercise 42}\\
&= \mathbf{B}' \text{ by Frenet–Serret}
\end{aligned}$$

**(b)** $\mathbf{T}' = \mathbf{w}\times\mathbf{T} = (\tau\mathbf{T}+\kappa\mathbf{B})\times\mathbf{T} = \kappa\mathbf{N}$ by manipulations and Exercise 42. The other equations are similar.

**45.** $\mathbf{w}$ is a constant vector $\Leftrightarrow \mathbf{w}'(s) = \mathbf{0}$. So

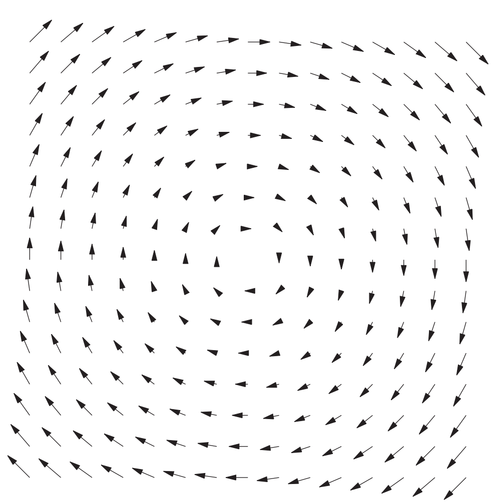
$$\begin{aligned}
\mathbf{0} = \mathbf{w}'(s) &= \tau'\mathbf{T} + \tau\mathbf{T}' + \kappa'\mathbf{B} + \kappa\mathbf{B}'\\
&= \tau'\mathbf{T} + \kappa'\mathbf{B} + \tau\kappa\mathbf{N} - \kappa\tau\mathbf{N} \text{ using Frenet–Serret}\\
&= \tau'\mathbf{T} + \kappa'\mathbf{B}.
\end{aligned}$$

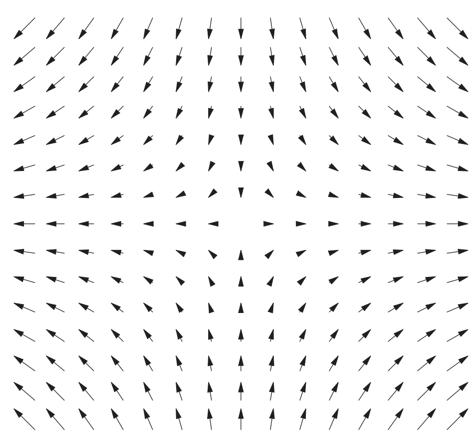
$\mathbf{T}$ and $\mathbf{B}$ are always perpendicular—hence we can never have $\mathbf{T} = c\,\mathbf{B}$ (or vice versa). Thus $\tau' = \kappa' = 0$ so $\tau, \kappa$ are constant and nonzero because $\mathbf{x}'\times\mathbf{x}'' \neq \mathbf{0}$. Thus by Theorem 2.5 the path must be a helix. Conversely, having a helix implies constant $\tau, \kappa$ so $\mathbf{w}' \equiv \mathbf{0}$. Thus $\mathbf{w}$ must be constant.

## 3.3 Vector Fields: An Introduction

*The figures can be generated using Mathematica or Maple. The axes are in the 'usual' positions with the origin at the center. The relative length of the shaft of the arrows corresponds to the length of the vectors. The students should then compare the results in Exercises 1–3 and Exercises 4–6. The differences between the equations for the vector fields should be compared to the differences in the resulting sketches.*

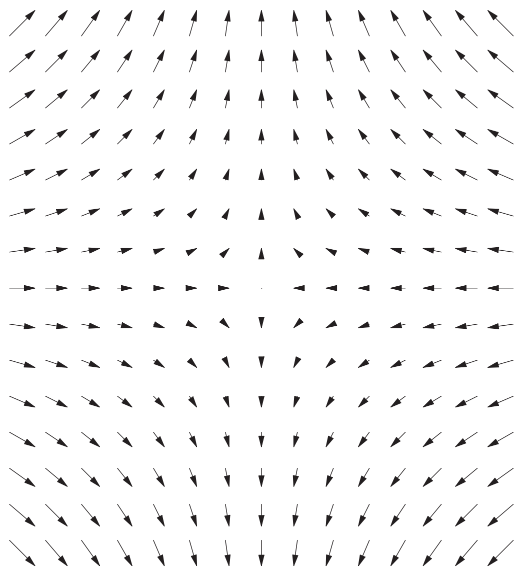
**1.** $\mathbf{F} = y\mathbf{i} - x\mathbf{j} = (y, -x)$ is shown below left.

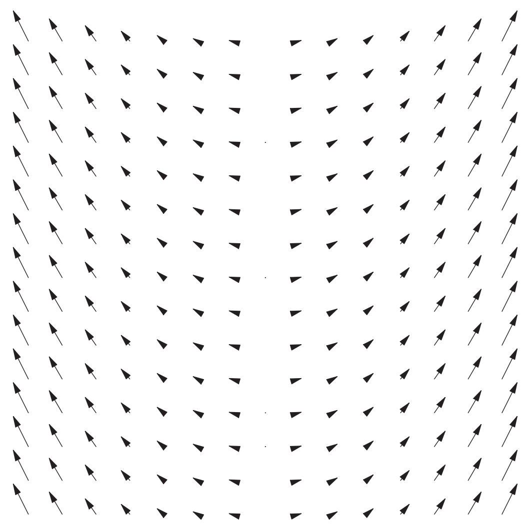




**2.** $\mathbf{F} = x\mathbf{i} - y\mathbf{j} = (x, -y)$ is shown above right.

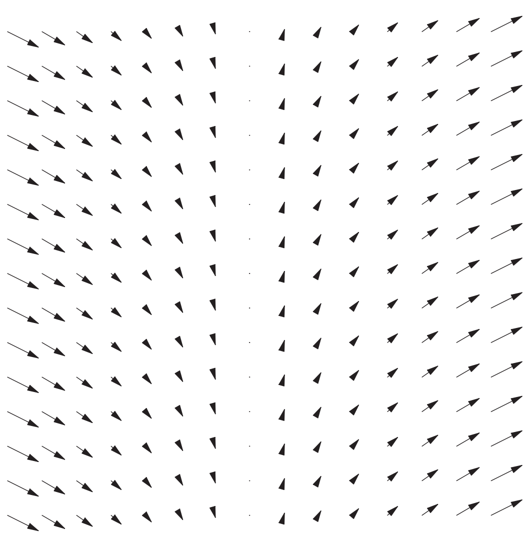
**3.** $\mathbf{F} = (-x, y)$ is shown below left.

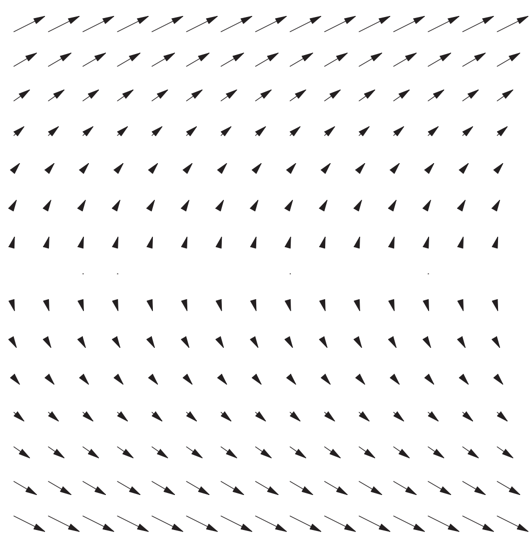




**4.** $\mathbf{F} = (x, x^2)$ is shown above right.

**5.** $\mathbf{F} = (x^2, x)$ is shown below left.

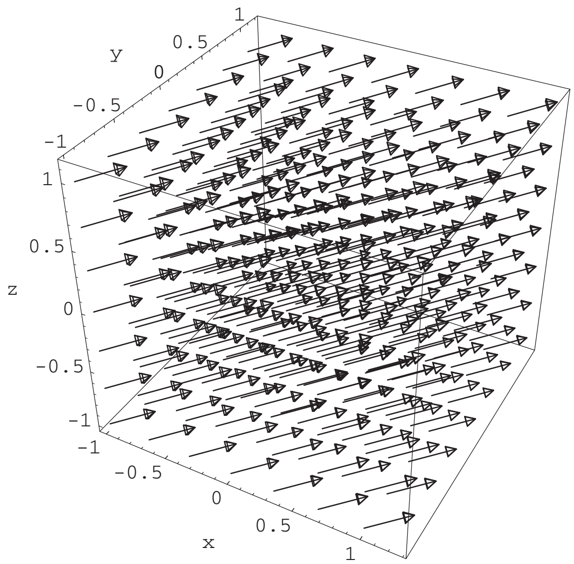


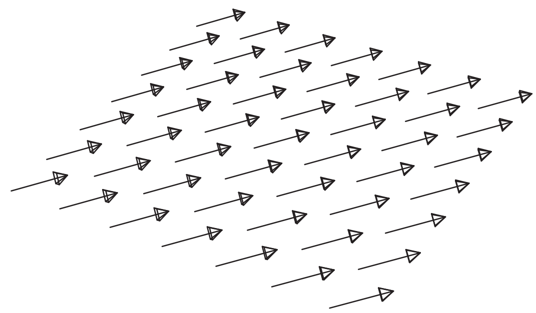


**6.** $\mathbf{F} = (y^2, y)$ is shown above right.

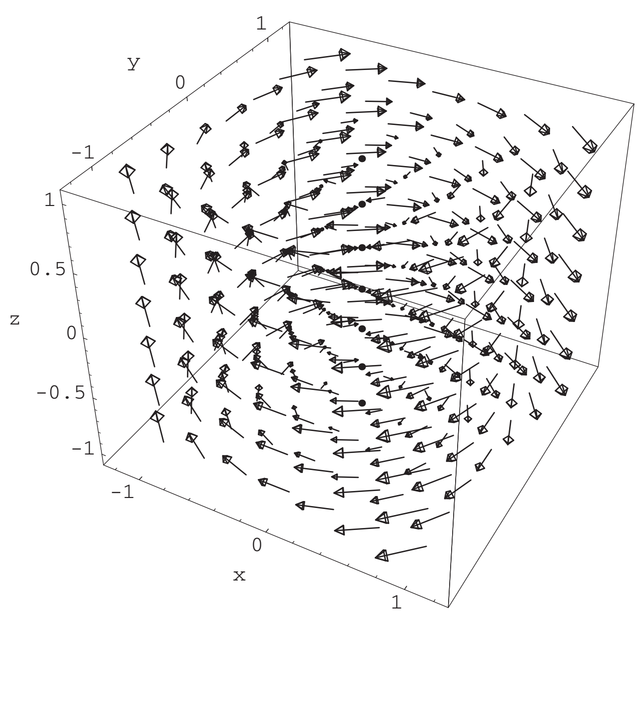
*Now we are looking at sketches of vector fields in* $\mathbf{R}^3$*. These are harder to see. In most cases, I have also included a sketch of a slice.*

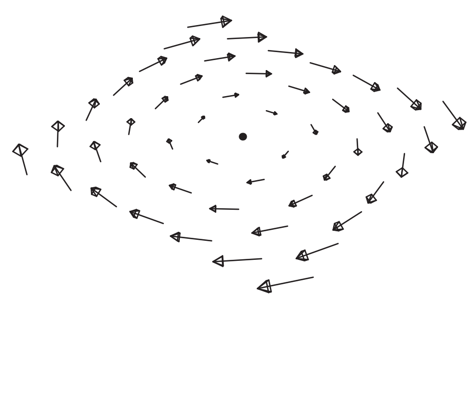
**7.** $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} = (3, 2, 1)$ is constant. The figure on the right shows the slice in the $xy$-plane:



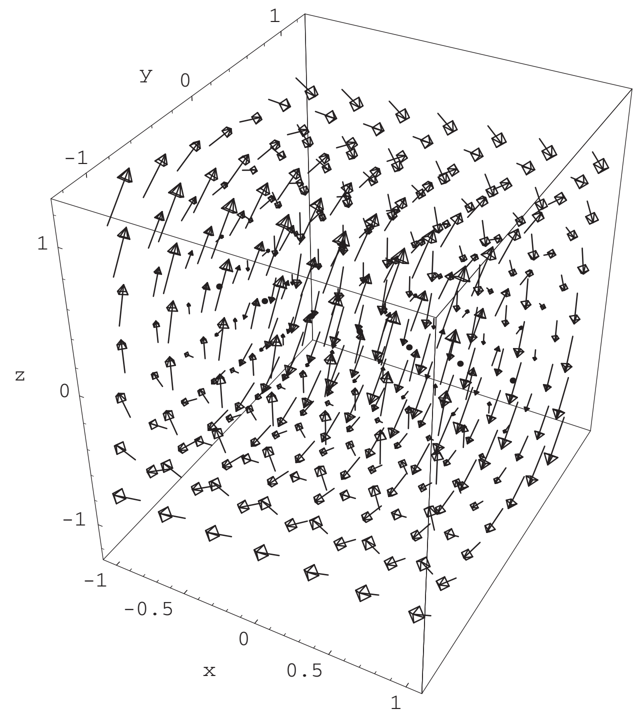



**8.** $\mathbf{F} = (y, -x, 0)$. The figure on the right shows the slice in the $xy$-plane—compare this to Exercise 1:

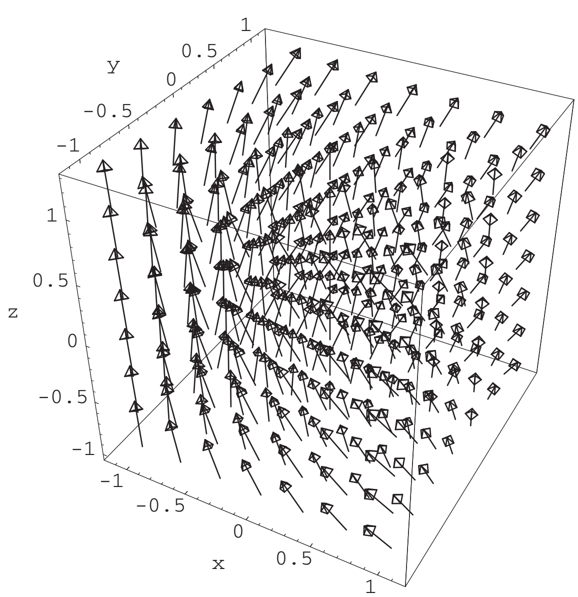


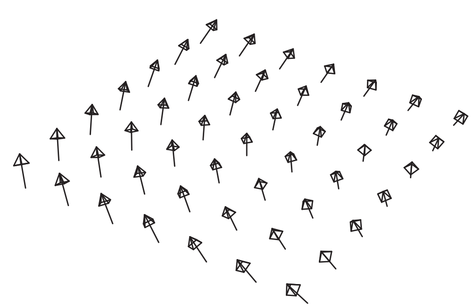



**9.** $\mathbf{F} = (0, z, -y)$; compare this to Exercise 8:

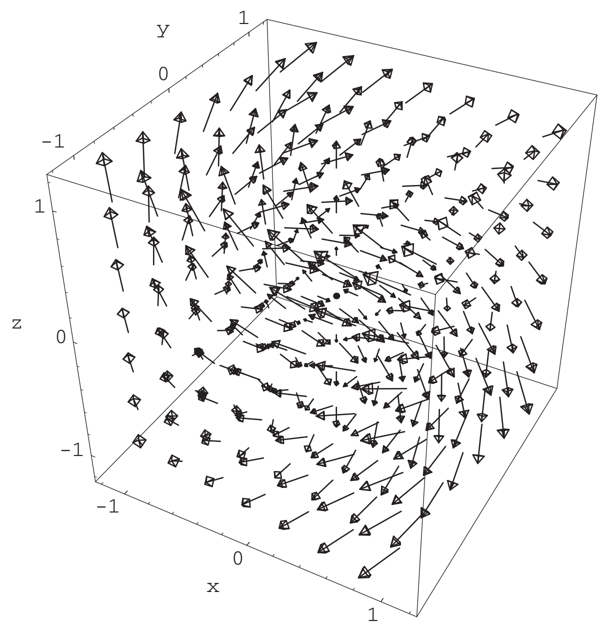


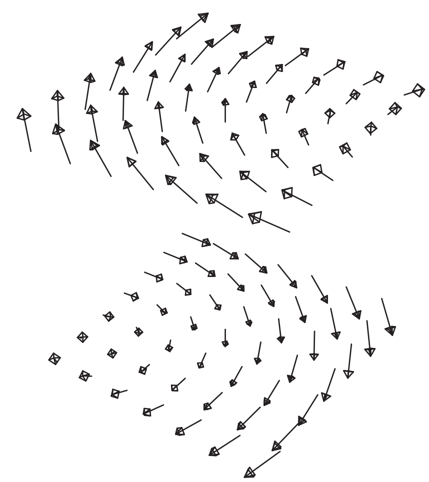

**10.** $\mathbf{F} = (y, -x, 2)$. The figure on the right shows the slice in the $xy$-plane—compare this to Exercise 8:



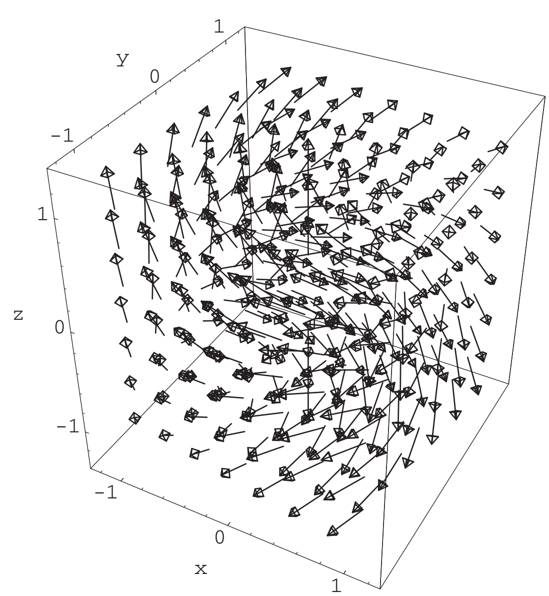



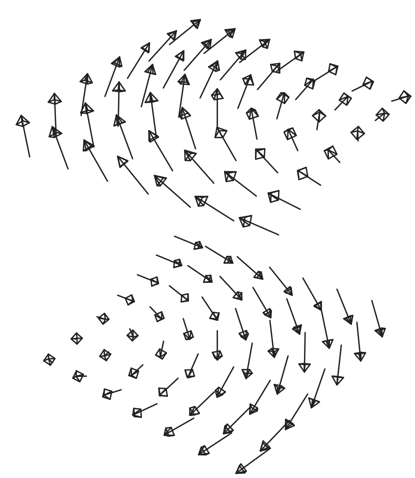
**11.** $\mathbf{F} = (y, -x, z)$. The figure on the right shows the slices in the $z = 1$ and $z = -1$ planes—compare this to Exercises 8 and 10:



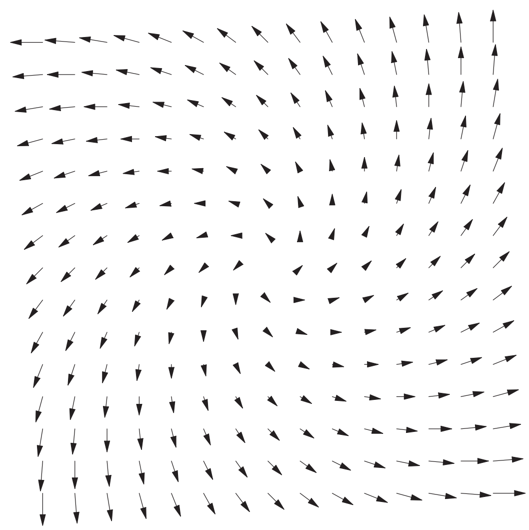



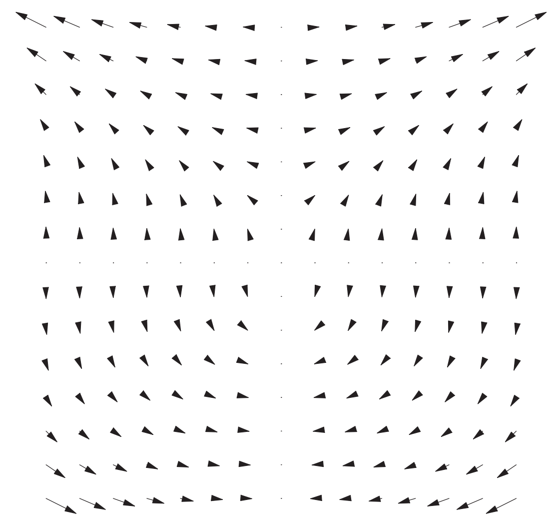
**12.** $\mathbf{F} = (y, -x, z)/\sqrt{x^2 + y^2 + z^2}$ except at the origin. The figure on the right shows the slices in the $z = 1$ and $z = -1$ planes—compare this to Exercise 11 (they are the same except the vectors in this problem are all unit vectors—they may not look like unit vectors because of the vertical components):



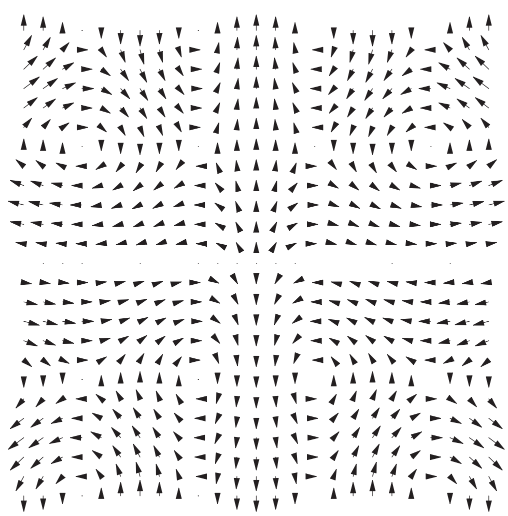



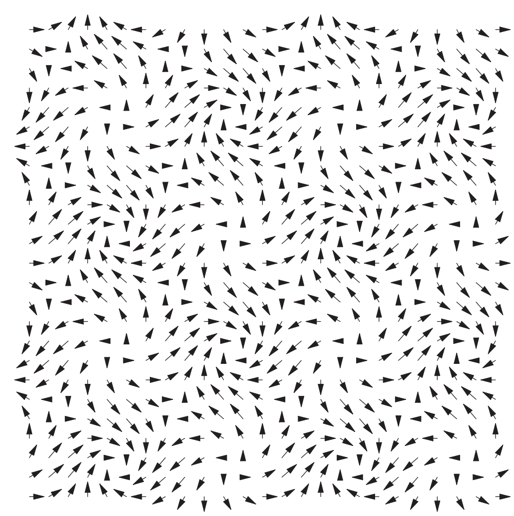
**13.** The figure is below left.





**14.** The figure is above right.
**15.** The figure is below left.

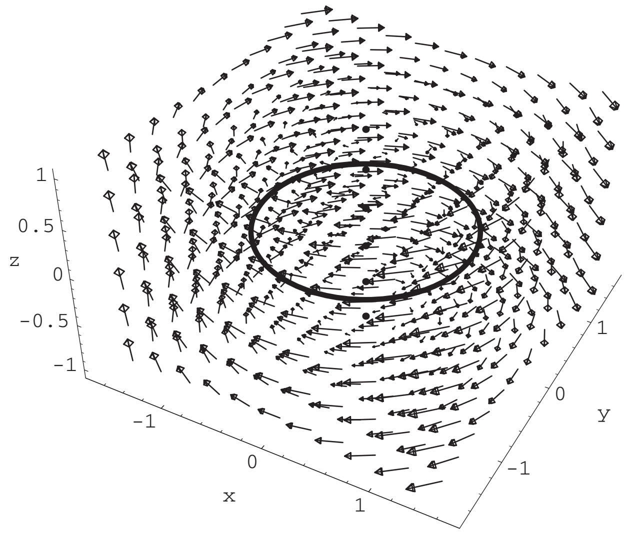


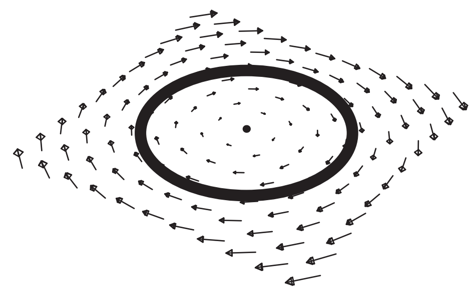


**16.** The figure is above right.

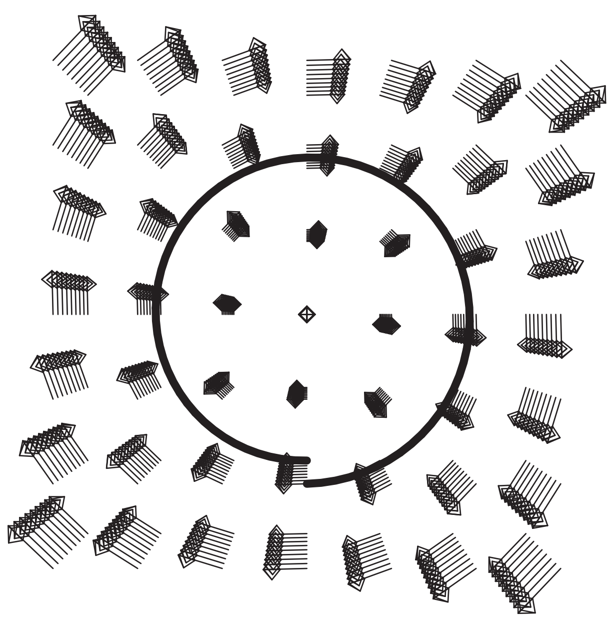
*In Exercises 17–19 we will show that* $\mathbf{x}$ *is a flow line of* $\mathbf{F}$ *using Definition 3.2, by showing* $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$.

**17.** $\mathbf{x}(t) = (x, y, z) = (\sin t, \cos t, 0)$ so $\mathbf{x}'(t) = (\cos t, -\sin t, 0) = (y, -x, 0) = \mathbf{F}(\mathbf{x}(t))$. We can see below how the path, in bold, is a flow line for the vector field we saw above in Exercise 8. The figure on the right is the $xy$-plane slice of the figure on the left.

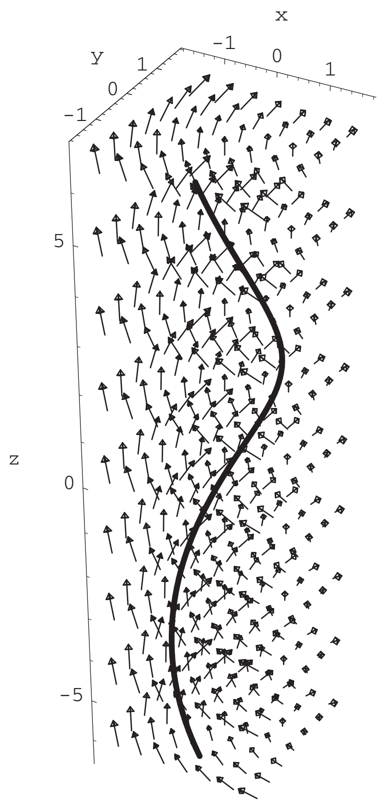


**18.** $\mathbf{x}(t) = (x, y, z) = (\sin t, \cos t, 2t)$ so $\mathbf{x}'(t) = (\cos t, -\sin t, 2) = (y, -x, 2) = \mathbf{F}(\mathbf{x}(t))$. Below we see the view from almost directly above one "period" of the path.



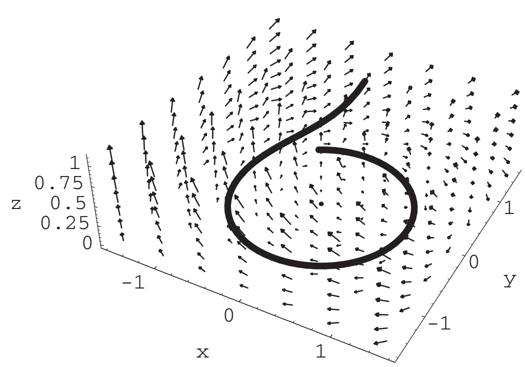
The path, below in bold, is a flow line of the vector field we saw above in Exercise 10.




**19.** $\mathbf{x}(t) = (x, y, z) = (\sin t, \cos t, e^{2t})$ so

$$\mathbf{x}'(t) = (\cos t, -\sin t, 2e^{2t}) = (y, -x, 2z) = \mathbf{F}(\mathbf{x}(t)).$$

The projection of this path onto the $xy$-plane is the same as that of the path in Exercise 18. The difference is that the rate at which the path climbs is changing:




**20.** If $\mathbf{x}(t) = (x, y)$ then $\mathbf{x}'(t) = \mathbf{F}(x, y) = (-x, y)$. Consider $x$ for a moment. This says that $dx/dt = -x$. The solution to this is $x = ce^{-t}$. Our initial condition is that $x(0) = 2$ so $c = 2$. Similarly, $dy/dt = y$ so $y = ke^t$. The initial condition $y(0) = 1$ tells us that $k = 1$. The equation of the flow line is $\mathbf{x}(t) = (2e^{-t}, e^t)$.

**21.** If $\mathbf{x}(t) = (x, y)$ then $\mathbf{x}'(t) = \mathbf{F}(x, y) = (x^2, y)$. As in Exercise 20, we know that $y = ke^t$ and $y(1) = e$ tells us that $k = 1$. As for $x$, $dx/dt = x^2$. This is a separable differential equation $dx/x^2 = dt$. Integrating and solving for $x$ gives us $x = -1/(t + c)$. From the initial condition $x(1) = 1$ we find that $1 = -1/(1 + c)$ or $c = -2$. The equation, therefore, of the flow line is $\mathbf{x}(t) = (1/(2 - t), e^t)$.

**22.** If $\mathbf{x}(t) = (x, y, z)$ then $\mathbf{x}'(t) = \mathbf{F}(x, y, z) = (2, -3y, z^3)$. We see immediately that the $x$ coordinate function must be linear and of the form $2t + c$. From the initial condition, this constant is 3 so $x = 2t + 3$. As in Exercise 20, we know that $y = ke^{-3t}$

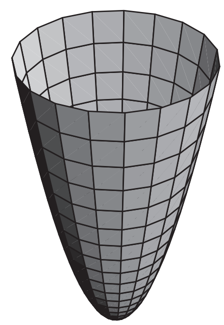
and $y(0) = 5$ tells us that $k = 5$. As for $z$, $dz/dt = z^3$. This is a separable differential equation $dz/z^3 = dt$. Integrating and solving for $z$ gives us $z = 1/\sqrt{-2(t+c)}$. From the initial condition $z(0) = 7$ we find that $7 = 1/\sqrt{(-2c)}$ or $c = -1/98$. The equation, therefore, of the flow line is $\mathbf{x}(t) = (2t+3, 5e^{-3t}, 7/\sqrt{1-98t})$.

**23.** **(a)** For the function $f(x, y, z) = 3x - 2y + z$, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is a gradient field.

**(b)** The equipotiential surfaces are those for which $f(x, y, z)$ is constant. $3x - 2y + z = c$. These are planes with normal vector $(3, -2, 1)$.

**24.** **(a)** For the function $f(x, y, z) = x^2 + y^2 - 3z$, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is a gradient field.

**(b)** The equipotiential surfaces are those for which $f(x, y, z)$ is constant. $x^2 + y^2 - 3z = c$ is equivalent to $z = (1/3)(x^2 + y^2 - c)$. These are paraboloids with $z$ intercept $(0, 0, -c/3)$. A typical surface is:



**25.** Let $\mathbf{x}$ be a flow line of a gradient vector field $\mathbf{F} = \nabla f$ and let $G(t) = f(\mathbf{x}(t))$. We will show that $G$ is an increasing function of $t$ by showing $G'(t) \geq 0$. First, $G'(t) = \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)$ since $\mathbf{F} = \nabla f$.
Now we use the fact that $\mathbf{x}$ is a flow line of $\mathbf{F}$:

$$\mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = \mathbf{x}'(t) \cdot \mathbf{x}'(t) = \|\mathbf{x}'(t)\|^2 \geq 0.$$

*For Exercises 26–28, verify that* $\frac{\partial}{\partial t}\phi(\mathbf{x}, t) = \mathbf{F}(\phi(\mathbf{x}, t))$ *and* $\phi(\mathbf{x}, 0) = \mathbf{x}$.

**26.** First we see that $\phi(x, y, 0) = \left(\frac{x+y}{2}e^0 + \frac{x-y}{2}e^0, \frac{x+y}{2}e^0 + \frac{y-x}{2}e^0\right) = (x, y)$. Next,

$$\begin{aligned}\frac{\partial}{\partial t}\phi(x, y, t) &= \left(\frac{x+y}{2}e^t - \frac{x-y}{2}e^{-t}, \frac{x+y}{2}e^t - \frac{y-x}{2}e^{-t}\right)\\ &= \phi(y, x, t) = \mathbf{F}(\phi(x, y, t)).\end{aligned}$$

**27.** First we see that $\phi(x, y, 0) = (y \sin 0 + x \cos 0, y \cos 0 - x \sin 0) = (x, y)$. Next,

$$\begin{aligned}\frac{\partial}{\partial t}\phi(x, y, t) &= (y \cos t - x \sin t, -y \sin t - x \cos t)\\ &= \phi(y, -x, t) = \mathbf{F}(\phi(x, y, t)).\end{aligned}$$

**28.** First we see that $\phi(x, y, z, 0) = (x \cos 0 - y \sin 0, y \cos 0 + x \sin 0, ze^0) = (x, y, z)$. Next,

$$\begin{aligned}\frac{\partial}{\partial t}\phi(x, y, z, t) &= (-2x \sin 2t - 2y \cos 2t, -2y \sin 2t + 2x \cos 2t, -ze^{-t})\\ &= \phi(-2y, 2x, -z, t) = \mathbf{F}(\phi(x, y, z, t)).\end{aligned}$$

**29.** We are assuming that $\phi$ is a flow of $\mathbf{F}$ and that $\mathbf{x}(t) = \phi(x_0, t)$. Then

$$\mathbf{x}'(t) = \frac{\partial}{\partial t}\phi(\mathbf{x}_0, t) = \mathbf{F}(\phi(\mathbf{x}_0, t)) = \mathbf{F}(\mathbf{x}(t)).$$

The middle equality holds because $\phi$ is a flow of $\mathbf{F}$.

**30.** Using the hint, we can apply the results of Exercise 29. If $\phi$ is a flow of the vector field $\mathbf{F}$ then for any fixed point $\mathbf{x}_0$ in $X$, the map $\mathbf{x}(t) = \phi(\mathbf{x}_0, t)$ is a flow line of $\mathbf{F}$.

So $\phi(\mathbf{x}_0, s + t)$ is where we are if we flow for $t + s$ seconds while $\phi(\phi(\mathbf{x}_0, t), s)$ is where we are if we first flow for $t$ seconds and then we flow for $s$ seconds. It should be clear that we end up the same place in either case. It is worth checking that your students understand the idea behind the problem—the author of the text has taken great care to make sure that these symbols make some physical sense to them.

**31.** We know that $\frac{\partial}{\partial t}\phi(\mathbf{x},t) = \mathbf{F}(\phi(\mathbf{x},t))$. So

$$\frac{\partial}{\partial t} D_{\mathbf{x}}\phi(\mathbf{x},t) = D_{\mathbf{x}}\left(\frac{\partial}{\partial t}\phi(\mathbf{x},t)\right) = D_{\mathbf{x}}\mathbf{F}(\phi(\mathbf{x},t)).$$

Now by the chain rule (Theorem 5.3):

$$D_{\mathbf{x}}\mathbf{F}(\phi(\mathbf{x},t)) = D\mathbf{F}(\phi(\mathbf{x},t))D_{\mathbf{x}}\phi(\mathbf{x},t).$$

## 3.4 Gradient, Divergence, Curl, and The Del Operator

*For Exercises 1–6 calculate the divergence of* $\mathbf{F}$: *div* $\mathbf{F} = \nabla \cdot \mathbf{F}$.

**1.** $\mathbf{F} = (x^2, y^2)$, so div $\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 2x + 2y$.

**2.** $\mathbf{F} = (y^2, x^2)$, so div $\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0 + 0 = 0$.

**3.** $\mathbf{F} = (x+y, y+z, x+z)$, so div $\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1 + 1 + 1 = 3$.

**4.** $\mathbf{F} = (z\cos(e^{y^2}), x\sqrt{z^2+1}, e^{2y}\sin 3x)$, so div $\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 + 0 + 0 = 0$.

**5.** $\mathbf{F} = (x_1^2, 2x_2^2, \ldots, nx_n^2)$, so div $\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n} = 2x_1 + 4x_2 + \cdots + 2nx_n$.

**6.** $\mathbf{F} = (x_1, 2x_1, \ldots, nx_1)$, so div $\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n} = 1 + 0 + \cdots + 0 = 1$.

*For Exercises 7–11 calculate the curl of* $\mathbf{F}$: *curl* $\mathbf{F} = \nabla \times \mathbf{F}$.

**7.** curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -xe^y & 2xyz \end{vmatrix} = (2xz, -2yz, -e^y)$.

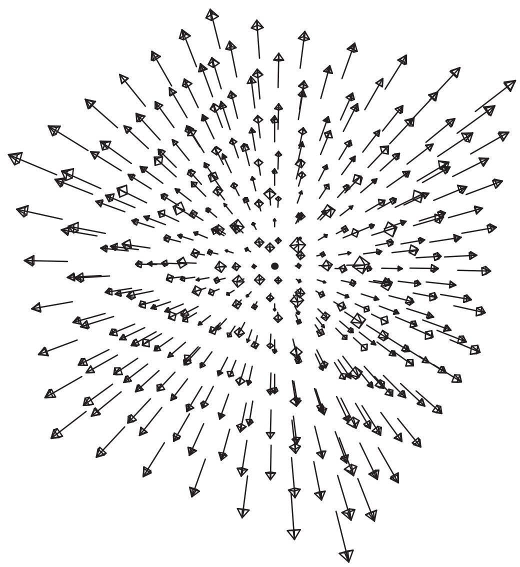
**8.** curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0,0,0)$.

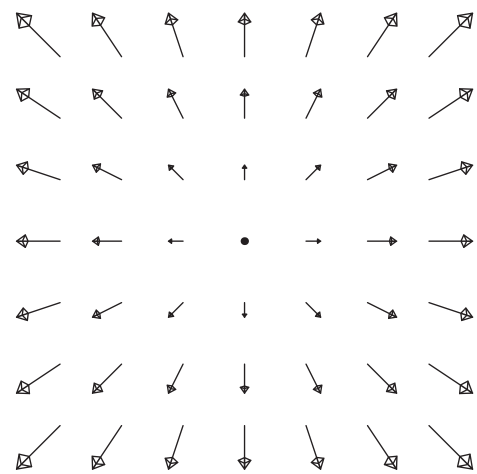
**9.** curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+yz & y+xz & z+xy \end{vmatrix} = (x-x, -y+y, z-z) = (0,0,0)$.

**10.** curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos yz - x & \cos xz - y & \cos xy - z \end{vmatrix}$
$= (x(\sin xz - \sin xy), y(\sin xy - \sin yz), z(\sin yz - \sin xz))$.

**11.** curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & e^{xyz} & x^2 y \end{vmatrix} = (x^2 - xye^{xyz}, y^2 - 2xy, yze^{xyz} - 2yz)$.

**12. (a)** We denote the vector field from Exercise 8 by $\mathbf{F}_8$ and sketch it below on the left. The figure on the right represents any planar slice through the origin. Every point is being pushed outwards. If you imagine a twig caught in this body of water and you think in terms of spherical coordinates, the change in position is an increase in $\rho$ with no change to $\varphi$ or $\theta$.





**(b)** Note that $\mathbf{F} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{\mathbf{F}_8}{\sqrt{x^2+y^2+z^2}}$. At each point the direction of $\mathbf{F}$ is the same as that of $\mathbf{F}_8$ but $\mathbf{F}$ is made up of unit vectors. As in part (a) we would argue that the motion of each point is in the direction of increasing $\rho$ and so the curl is again 0.

**(c)** In Exercise 24 below we'll show that $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$. If you don't like citing a future problem, you can follow through the steps in Exercise 24 for this exercise. Note that we know from Exercise 8 that $\nabla \times \mathbf{F}_8 = (0, 0, 0)$.

$$\begin{aligned}
\nabla \times \mathbf{F} &= \nabla \times \left(\frac{1}{\sqrt{x^2+y^2+z^2}}\mathbf{F}_8\right) \\
&= \left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\nabla \times \mathbf{F}_8 + \left[\nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\right] \times \mathbf{F}_8 \\
&= (0,0,0) - \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}} \times \mathbf{F}_8 \\
&= -\frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}} \times (x,y,z) = (0,0,0).
\end{aligned}$$

**13. (a)** At each point "more is moving away than towards" so div $\mathbf{F} > 0$ on all $\mathbf{R}^2$.

**(b)** At each point "more is moving towards than away" so div $\mathbf{F} < 0$ on all $\mathbf{R}^2$.

**(c)** Here we have a mixed bag. At each point to the left of the $y$-axis "more is moving towards than away" and at each point to the right of the $y$-axis "more is moving away than towards" so div $\mathbf{F} < 0$ for $x < 0$, div $\mathbf{F} > 0$ for $x > 0$, and div $\mathbf{F} = 0$ for $x = 0$.

**(d)** Again we have a mixed bag. At each point the above the $x$-axis "more is moving towards than away" and at each point below the $x$-axis "more is moving away than towards" so div $\mathbf{F} < 0$ for $y > 0$, div $\mathbf{F} > 0$ for $y < 0$, and div $\mathbf{F} = 0$ for $y = 0$.

*In Exercises 14 and 15, the student is asked to work examples of the results of Theorems 4.3 and 4.4. Exercise 16 has the student prove Theorem 4.4.*

**14.** $f(x,y,z) = x^2 \sin y + y^2 \cos z$ so $\nabla f = (2x \sin y, x^2 \cos y + 2y \cos z, -y^2 \sin z)$.

$$\nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x \sin y & x^2 \cos y + 2y \cos z & -y^2 \sin z \end{vmatrix}$$
$$= (-2y \sin z + 2y \sin z, 0 - 0, 2x \cos y - 2x \cos y) = (0,0,0).$$

**15.** $\mathbf{F}(x,y,z) = xyz\mathbf{i} + e^z \cos x \mathbf{j} + xy^2 z^3 \mathbf{k}$ so

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & e^z \cos x & xy^2 z^3 \end{vmatrix} = (2xyz^3 + e^z \cos x, -y^2 z^3 + xy, e^z \sin x - xz).$$

Finally we calculate

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}(2xyz^3 + e^z \cos x) + \frac{\partial}{\partial y}(-y^2 z^3 + xy) + \frac{\partial}{\partial z}(e^z \sin x - xz)$$
$$= 2yz^3 - e^z \sin x - 2yz^3 + x + e^z \sin x - x = 0.$$

**16.** We want to show that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$
$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}.$$

Finally, because **F** is of class $C^2$, the mixed partials are equal and so this last quantity is 0.

**17.** This is a good warm-up.

$$\nabla r^n = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{n/2} = \left(\frac{n}{2}\right)(x^2+y^2+z^2)^{(n-2)/2}(2x,2y,2z)$$
$$= nr^{n-2}(x,y,z) = nr^{n-2}\mathbf{r}.$$

**18.** This is similar to Exercise 17 as most of the derivative of the ln pulls out.

$$\nabla(\ln r) = \nabla(\ln(x^2+y^2+z^2)^{1/2}) = \tfrac{1}{2}\nabla(\ln(x^2+y^2+z^2))$$
$$= \tfrac{1}{2}\left(\frac{1}{x^2+y^2+z^2}\right)(2x,2y,2z) = \left(\frac{1}{r^2}\right)(x,y,z) = \frac{\mathbf{r}}{r^2}$$

**19.** In this exercise and the next we'll need to know that $r^n\mathbf{r} = (x^2+y^2+z^2)^{n/2}(x,y,z)$.

$$\nabla \cdot (r^n \mathbf{r}) = \frac{\partial}{\partial x}[x(x^2+y^2+z^2)^{n/2}] + \frac{\partial}{\partial y}[y(x^2+y^2+z^2)^{n/2}] + \frac{\partial}{\partial z}[z(x^2+y^2+z^2)^{n/2}]$$
$$= \left[r^n + x\left(\frac{n}{2}\right)2x(x^2+y^2+z^2)^{(n-2)/2}\right] + \left[r^n + y\left(\frac{n}{2}\right)2y(x^2+y^2+z^2)^{(n-2)/2}\right]$$
$$+ \left[r^n + z\left(\frac{n}{2}\right)2z(x^2+y^2+z^2)^{(n-2)/2}\right]$$
$$= [r^n + nx^2 r^{n-2}] + [r^n + ny^2 r^{n-2}] + [r^n + nz^2 r^{n-2}]$$
$$= 3r^n + n(x^2+y^2+z^2)r^{n-2} = 3r^n + nr^2 r^{n-2} = 3r^n + nr^n = (n+3)r^n$$

**20.** Here

$$\nabla \times (r^n \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(x^2+y^2+z^2)^{n/2} & y(x^2+y^2+z^2)^{n/2} & z(x^2+y^2+z^2)^{n/2} \end{vmatrix}$$

Let's begin by calculating the coefficient of **i**:

$$\begin{aligned}&\frac{\partial}{\partial y}[z(x^2+y^2+z^2)^{n/2}] - \frac{\partial}{\partial z}[y(x^2+y^2+z^2)^{n/2}]\\ &\quad = z(x^2+y^2+z^2)^{(n-2)/2}(2y) - y(x^2+y^2+z^2)^{(n-2)/2}(2z) = 0.\end{aligned}$$

The calculation is the same for the coefficients of **j** and **k**.

*Exercises 21 and 22 follow quickly from properties we explored in Chapter 1. The $\nabla$ seems to distribute over the sum because the derivative of a sum is the sum of the derivatives. Exercises 23–25 are product rules.*

**21.**

$$\nabla \cdot (\mathbf{F}+\mathbf{G}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(F_i + G_i) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(F_i) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(G_i) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}.$$

**22.** You can expand the first matrix below and see the result pretty quickly. On the other hand, you can use the result of Exercise 28 from Section 1.6 and the fact that $\frac{d}{dx_i}(F+G) = \frac{d}{dx_i}(F) + \frac{d}{dx_i}(G)$.

$$\begin{aligned}\nabla \times (\mathbf{F}+\mathbf{G}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1+G_1 & F_2+G_2 & F_3+G_3 \end{vmatrix}\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix} = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}.\end{aligned}$$

**23.**

$$\begin{aligned}\nabla \cdot (f\mathbf{F}) &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(fF_i) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}F_i + f\frac{\partial F_i}{\partial x_i}\right)\\ &= \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}F_i\right) + \sum_{i=1}^{n}\left(f\frac{\partial F_i}{\partial x_i}\right) = \nabla(f)\cdot\mathbf{F} + f(\nabla\cdot\mathbf{F})\\ &= f\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla f.\end{aligned}$$

**24.**

$$\begin{aligned}\nabla \times (f\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix}\\ &= \left[\frac{\partial}{\partial y}(fF_3) - \frac{\partial}{\partial z}(fF_2)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(fF_3) - \frac{\partial}{\partial z}(fF_1)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(fF_2) - \frac{\partial}{\partial y}(fF_1)\right]\mathbf{k}\\ &= \left[\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2 + \frac{\partial F_3}{\partial y}f - \frac{\partial F_2}{\partial z}f\right]\mathbf{i} - \left[\frac{\partial f}{\partial x}F_3 - \frac{\partial f}{\partial z}F_1 + \frac{\partial F_3}{\partial x}f - \frac{\partial F_1}{\partial z}f\right]\mathbf{j}\\ &\quad + \left[\frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1 + \frac{\partial F_2}{\partial x}f - \frac{\partial F_1}{\partial y}f\right]\mathbf{k} = f\nabla\times\mathbf{F} + \nabla f\times\mathbf{F}.\end{aligned}$$

**25.**

$$
\begin{aligned}
\nabla\cdot(\mathbf{F}\times\mathbf{G}) &= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(F_2G_3-F_3G_2, F_3G_1-F_1G_3, F_1G_2-F_2G_1)\\
&= \frac{\partial F_2}{\partial x}G_3+F_2\frac{\partial G_3}{\partial x}-\frac{\partial F_3}{\partial x}G_2-F_3\frac{\partial G_2}{\partial x}+\frac{\partial F_3}{\partial y}G_1+F_3\frac{\partial G_1}{\partial y}-\frac{\partial F_1}{\partial y}G_3-F_1\frac{\partial G_3}{\partial y}\\
&\quad+\frac{\partial F_1}{\partial z}G_2+F_1\frac{\partial G_2}{\partial z}-\frac{\partial F_2}{\partial z}G_1-F_2\frac{\partial G_1}{\partial z}\\
&= G_1\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)-G_2\left(\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right)+G_3\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)-F_1\left(\frac{\partial G_3}{\partial y}-\frac{\partial G_2}{\partial z}\right)\\
&\quad+F_2\left(\frac{\partial G_3}{\partial x}-\frac{\partial G_1}{\partial z}\right)-F_3\left(\frac{\partial G_2}{\partial x}-\frac{\partial G_1}{\partial y}\right)\\
&= \mathbf{G}\cdot\nabla\times\mathbf{F}-\mathbf{F}\cdot\nabla\times\mathbf{G}.
\end{aligned}
$$

**26.** We will use formulas (6) and (7) from the text. First we establish formula (3):

$$
\begin{aligned}
\nabla f &= \frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\\
&= \left(\cos\theta\frac{\partial f}{\partial r}-\frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right)\mathbf{i}+\left(\sin\theta\frac{\partial f}{\partial r}+\frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}\right)\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\\
&= \frac{\partial f}{\partial r}(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})+\left(\frac{1}{r}\right)\frac{\partial f}{\partial\theta}(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})+\frac{\partial f}{\partial z}\mathbf{k}\\
&= \frac{\partial f}{\partial r}\mathbf{e}_r+\left(\frac{1}{r}\right)\frac{\partial f}{\partial\theta}\mathbf{e}_\theta+\frac{\partial f}{\partial z}\mathbf{e}_z.
\end{aligned}
$$

Now we establish formula (5). Again we need formulas (6) and (7). First use (6) to obtain: $F_r\mathbf{e}_r+F_\theta\mathbf{e}_\theta=(F_r\cos\theta-F_\theta\sin\theta)\mathbf{i}+(F_r\sin\theta+F_\theta\cos\theta)\mathbf{j}$.

$$
\begin{aligned}
\nabla\times\mathbf{F} &= \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ F_x&F_y&F_z\end{vmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)&\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)&\frac{\partial}{\partial z}\\ (F_r\cos\theta-F_\theta\sin\theta)&(F_r\sin\theta+F_\theta\cos\theta)&F_z\end{vmatrix}\\
&= \left[\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)F_z-\frac{\partial}{\partial z}(F_r\sin\theta+F_\theta\cos\theta)\right]\mathbf{i}\\
&\quad-\left[\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)F_z-\frac{\partial}{\partial z}(F_r\cos\theta-F_\theta\sin\theta)\right]\mathbf{j}\\
&\quad+\left[\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)(F_r\sin\theta+F_\theta\cos\theta)\right.\\
&\qquad\left.-\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)(F_r\cos\theta-F_\theta\sin\theta)\right]\mathbf{k}\\
&= \left[\frac{1}{r}\frac{\partial}{\partial\theta}F_z-\frac{\partial}{\partial z}F_\theta\right](\cos\theta\mathbf{i}+\sin\theta\mathbf{j})-\left[\frac{\partial}{\partial r}F_z-\frac{\partial}{\partial z}F_r\right](-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})\\
&\quad+\left[\frac{\partial}{\partial r}F_\theta-\left(\frac{1}{r}\right)\frac{\partial}{\partial\theta}F_r\right]\mathbf{k}
\end{aligned}
$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial\theta}F_z - \frac{\partial}{\partial z}F_\theta\right]\mathbf{e}_r - \left[\frac{\partial}{\partial r}F_z - \frac{\partial}{\partial z}F_r\right]\mathbf{e}_\theta + \left[\frac{\partial}{\partial r}F_\theta - \left(\frac{1}{r}\right)\frac{\partial}{\partial\theta}F_r\right]\mathbf{e}_z$$

$$= \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \left(\frac{1}{r}\right)\frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_r & F_\theta & F_z \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}.$$

**27.** We will need formula (9) from Section 1.7:

$$\begin{cases} \mathbf{e}_\rho = \sin\varphi\cos\theta\mathbf{i} + \sin\varphi\sin\theta\mathbf{j} + \cos\varphi\mathbf{k} \\ \mathbf{e}_\varphi = \cos\varphi\cos\theta\mathbf{i} + \cos\varphi\sin\theta\mathbf{j} - \sin\varphi\mathbf{k} \\ \mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}. \end{cases}$$

From the chain rule, we have the following relations between rectangular and spherical differential operators:

$$\begin{cases} \dfrac{\partial}{\partial\rho} = \sin\varphi\cos\theta\dfrac{\partial}{\partial x} + \sin\varphi\sin\theta\dfrac{\partial}{\partial y} + \cos\varphi\dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial\varphi} = \rho\cos\varphi\cos\theta\dfrac{\partial}{\partial x} + \rho\cos\varphi\sin\theta\dfrac{\partial}{\partial y} - \rho\sin\varphi\dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial\theta} = -\rho\sin\varphi\sin\theta\dfrac{\partial}{\partial x} + \rho\sin\varphi\cos\theta\dfrac{\partial}{\partial y}. \end{cases}$$

Solving for $\partial/\partial x, \partial/\partial y$, and $\partial/\partial z$:

$$\begin{cases} \dfrac{\partial}{\partial x} = \sin\varphi\cos\theta\dfrac{\partial}{\partial\rho} + \dfrac{\cos\varphi\cos\theta}{\rho}\dfrac{\partial}{\partial\varphi} - \dfrac{\sin\theta}{\rho\sin\varphi}\dfrac{\partial}{\partial\theta} \\ \dfrac{\partial}{\partial y} = \sin\varphi\sin\theta\dfrac{\partial}{\partial\rho} + \dfrac{\cos\varphi\sin\theta}{\rho}\dfrac{\partial}{\partial\varphi} + \dfrac{\cos\theta}{\rho\sin\varphi}\dfrac{\partial}{\partial\theta} \\ \dfrac{\partial}{\partial z} = \cos\varphi\dfrac{\partial}{\partial\rho} - \dfrac{\sin\varphi}{\rho}\dfrac{\partial}{\partial\varphi}. \end{cases}$$

Now we calculate the gradient:

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \\ &= \left(\sin\varphi\cos\theta\frac{\partial f}{\partial\rho} + \frac{\cos\varphi\cos\theta}{\rho}\frac{\partial f}{\partial\varphi} - \frac{\sin\theta}{\rho\sin\varphi}\frac{\partial f}{\partial\theta}\right)\mathbf{i} \\ &\quad + \left(\sin\varphi\sin\theta\frac{\partial f}{\partial\rho} + \frac{\cos\varphi\sin\theta}{\rho}\frac{\partial f}{\partial\varphi} + \frac{\cos\theta}{\rho\sin\varphi}\frac{\partial f}{\partial\theta}\right)\mathbf{j} + \left(\cos\varphi\frac{\partial f}{\partial\rho} - \frac{\sin\varphi}{\rho}\frac{\partial f}{\partial\varphi}\right)\mathbf{k} \\ &= \frac{\partial f}{\partial\rho}(\sin\varphi\cos\theta\mathbf{i} + \sin\varphi\sin\theta\mathbf{j} + \cos\varphi\mathbf{k}) + \left(\frac{1}{\rho}\right)\frac{\partial f}{\partial\varphi}(\cos\varphi\cos\theta\mathbf{i} + \cos\varphi\sin\theta\mathbf{j} - \sin\varphi\mathbf{k}) \\ &\quad + \left(\frac{1}{\rho\sin\varphi}\right)\frac{\partial f}{\partial\theta}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = \frac{\partial f}{\partial\rho}\mathbf{e}_r + \left(\frac{1}{\rho}\right)\frac{\partial f}{\partial\varphi}\mathbf{e}_\varphi + \left(\frac{1}{\rho\sin\varphi}\right)\frac{\partial f}{\partial\theta}\mathbf{e}_\theta. \end{aligned}$$

**28.** $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$.

**(a)** $\nabla\cdot\nabla = \left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)\cdot\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right) = \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}\dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\dfrac{\partial}{\partial z} = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} = \nabla^2$.

**(b)** We use ideas from Exercise 23:

$$\begin{aligned} \nabla^2(fg) &= \nabla\cdot\nabla(fg) = \nabla\cdot((\nabla f)g + f\nabla g) = (\nabla^2 f)g + \nabla f\cdot\nabla g + \nabla f\cdot\nabla g + f\nabla^2 g \\ &= f\nabla^2 g + g\nabla^2 f + 2(\nabla f\cdot\nabla g). \end{aligned}$$

**(c)** Again we use Exercise 23:

$$\nabla\cdot(f\nabla g - g\nabla f) = \nabla f\cdot\nabla g + f\nabla^2 g - \nabla g\cdot\nabla f - g\nabla^2 f = f\nabla^2 g - g\nabla^2 f.$$

**29.** $f\nabla f = \left(f\frac{\partial f}{\partial x}, f\frac{\partial f}{\partial y}, f\frac{\partial f}{\partial z}\right)$ hence

$$\begin{aligned}\nabla\cdot(f\nabla f) &= \frac{\partial}{\partial x}\left(f\frac{\partial f}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\frac{\partial f}{\partial y}\right)+\frac{\partial}{\partial z}\left(f\frac{\partial f}{\partial z}\right)\\ &= \left(\frac{\partial f}{\partial x}\right)^2+f\frac{\partial^2 f}{\partial x^2}+\left(\frac{\partial f}{\partial y}\right)^2+f\frac{\partial^2 f}{\partial y^2}+\left(\frac{\partial f}{\partial z}\right)^2+f\frac{\partial^2 f}{\partial z^2}\\ &= \left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2+f\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)\\ &= \|\nabla f\|^2+f\nabla^2 f.\end{aligned}$$

**30.** Write $\mathbf{F} = M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$. Then

$$\nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ M & N & P\end{vmatrix} = (P_y-N_z)\mathbf{i}+(M_z-P_x)\mathbf{j}+(N_x-M_y)\mathbf{k}$$

and thus

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{F}) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P_y-N_z & M_z-P_x & N_x-M_y\end{vmatrix}\\ &= (N_{xy}-M_{yy}-M_{zz}+P_{xz})\mathbf{i}+(P_{yz}-N_{zz}-N_{xx}+M_{yx})\mathbf{j}\\ &\quad+(M_{zx}-P_{xx}-P_{yy}+N_{zy})\mathbf{k}.\end{aligned}$$

On the other hand,

$$\nabla(\nabla\cdot\mathbf{F}) = \nabla(M_x+N_y+P_z) = (M_{xx}+N_{yx}+P_{zx})\mathbf{i}+(M_{xy}+N_{yy}+P_{zy})\mathbf{j}+(M_{xz}+N_{yz}+P_{zz})\mathbf{k}$$

and

$$\begin{aligned}\nabla^2\mathbf{F} = (\nabla\cdot\nabla)\mathbf{F} &= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\mathbf{F}\\ &= (M_{xx}+M_{yy}+M_{zz})\mathbf{i}+(N_{xx}+N_{yy}+N_{zz})\mathbf{j}+(P_{xx}+P_{yy}+P_{zz})\mathbf{k}\end{aligned}$$

Hence,

$$\nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F} = (N_{yx}+P_{zx}-M_{yy}-M_{zz})\mathbf{i}+(M_{xy}+P_{zy}-N_{xx}-N_{zz})\mathbf{j}+(M_{xz}+N_{yz}-P_{xx}-P_{yy})\mathbf{k}$$

By assumption, $\mathbf{F}$ is of class $C^2$ so $M_{xy} = M_{yx}$, etc.

Thus we have shown: $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F}$, as desired.

**31. (a)** Let $\mathbf{G}(t) = \mathbf{F}(\mathbf{a}+t\mathbf{v})$. Then

$$\begin{aligned}D_{\mathbf{v}}\mathbf{F}(\mathbf{a}) = \lim_{t\to 0}\frac{1}{t}(\mathbf{F}(\mathbf{a}+t\mathbf{v})-\mathbf{F}(\mathbf{a})) &= \lim_{t\to 0}\frac{1}{t}(\mathbf{G}(t)-\mathbf{G}(0))\\ &= \mathbf{G}'(0).\end{aligned}$$

Thus

$$D_{\mathbf{v}}\mathbf{F}(\mathbf{a}) = \left.\frac{d}{dt}\mathbf{F}(\mathbf{a}+t\mathbf{v})\right|_{t=0}.$$

**(b)** $\frac{d}{dt}\mathbf{F}(\mathbf{a}+t\mathbf{v}) = D\mathbf{F}(\mathbf{a}+t\mathbf{v})\frac{d}{dt}(\mathbf{a}+t\mathbf{v}) = D\mathbf{F}(\mathbf{a}+t\mathbf{v})\mathbf{v}$. Now evaluate at $t = 0$ to get $D_{\mathbf{v}}\mathbf{F}(\mathbf{a}) = D\mathbf{F}(\mathbf{a})\mathbf{v}$.

**32.** By definition,

$$\begin{aligned} D_{\mathbf{v}}\mathbf{F}(\mathbf{a}) &= \lim_{h\to 0}\frac{1}{h}(\mathbf{F}(\mathbf{a}+h\mathbf{v})-\mathbf{F}(\mathbf{a})) \\ &= \lim_{h\to 0}\frac{1}{h}(F_1(\mathbf{a}+h\mathbf{v})-F_1(\mathbf{a}),\ldots,F_n(\mathbf{a}+h\mathbf{v})-F_n(\mathbf{a})) \\ &= \lim_{h\to 0}\left(\frac{F_1(\mathbf{a}+h\mathbf{v})-F_1(\mathbf{a})}{h},\ldots,\frac{F_n(\mathbf{a}+h\mathbf{v})-F_n(\mathbf{a})}{h}\right) \\ &= \left(\lim_{h\to 0}\frac{F_1(\mathbf{a}+h\mathbf{v})-F_1(\mathbf{a})}{h},\ldots,\lim_{h\to 0}\frac{F_n(\mathbf{a}+h\mathbf{v})-F_n(\mathbf{a})}{h}\right) = (D_{\mathbf{v}}F_1(\mathbf{a}),\ldots,D_{\mathbf{v}}F_n(\mathbf{a})) \end{aligned}$$

using Definition 6.1 of Chapter 2.

**33.** We use part (b) of Exercise 31 since **F** is evidently differentiable.

$$D\mathbf{F}(x,y,z) = \begin{bmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix} \text{ so } D\mathbf{F}(3,2,1) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

$$D_{(\mathbf{i}-\mathbf{j}+\mathbf{k})/\sqrt{3}}\mathbf{F}(3,2,1) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 4/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$

**34.**

$$D_{\mathbf{v}}\mathbf{F}(\mathbf{a}) = D\mathbf{F}(\mathbf{a})\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \mathbf{v}.$$

In general if $\mathbf{F} = (x_1,\ldots,x_n)$, then $D\mathbf{F}(\mathbf{a}) = I_n$ ($n\times n$ identity matrix), so $D_{\mathbf{v}}\mathbf{F}(\mathbf{a}) = D\mathbf{F}(\mathbf{a})\mathbf{v} = I_n\mathbf{v} = \mathbf{v}$.

## True/False Exercises for Chapter 3

**1.** True.
**2.** False. (The path has unit speed.)
**3.** True.
**4.** False.
**5.** False. (There should be a negative sign in the second term on the right.)
**6.** False. ($\kappa = \|d\mathbf{T}/ds\|$, where $s$ is arclength.)
**7.** True.
**8.** True.
**9.** False.
**10.** False. ($d\mathbf{T}/ds$ must be normalized to give **N**.)
**11.** True.
**12.** True.

**13.** True.
**14.** False. (It's a vector field.)
**15.** False. (It's a scalar field.)
**16.** True.
**17.** True.
**18.** False. (It's a scalar field.)
**19.** False. (It's a meaningless expression.)
**20.** False. ($\mathbf{F}(\mathbf{x}(t)) \neq \mathbf{x}'(t)$.)
**21.** True. (Check that $\mathbf{F}(\mathbf{x}(t)) = \mathbf{x}'(t)$.)
**22.** True. (Verify that $\nabla \cdot \mathbf{F} = 0$.)
**23.** False. ($\nabla \times \mathbf{F} \neq \mathbf{0}$.)
**24.** False. ($f$ must be of class $C^2$.)
**25.** False. (Consider $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$.)
**26.** False. (The first term on the right needs a negative sign.)
**27.** True.
**28.** False. ($\nabla \times \mathbf{F} \neq \mathbf{0}$.)
**29.** False. ($\nabla \cdot (\nabla \times \mathbf{F}) \neq 0$.)
**30.** True.

## Miscellaneous Exercises for Chapter 3

**1.** Here are the answers: (a) D (b) F (c) A
(d) B (e) C (f) E

Here's some explanation: The formulas in (a) and in (f) are the only ones that keep $x$ and $y$ bounded (between $-1$ and 1), so they must correspond to D and E. Note that in (a) $\mathbf{x}(0) = (0, 0)$, but the graph in E does not pass through the origin. Note that in (c) $x \geq 1$ and the only graph with that property is A. In (b) we see that $\mathbf{x}(-t) = (-t - \sin 5t, t^2 + \cos 6t) = (-x(t), y(t))$. This means that the graph will be symmetric about the $y$-axis and the only plot that remains with this property is F. What remains is to match the formulas in (d) and (e) with the graphs in B and C. This is easy: in (d) large positive values of $t$ give points in the first quadrant. The graph in C has no points in the first quadrant.

**2.** *Answers*: (a) E (b) F (c) C (d) B (e) A (f) D

*Explanation*: (a) Must have $z$ between $-1$ and 1, but $y$ can be arbitrarily large and positive.

(b) All three coordinates should be bounded (making the only choices B or F). The projection of the curve into the $xy$-plane should be an astroid—giving choice F.

(c) This is an elliptical helix—so choice C.

(d) All three coordinates are between $-1$ and 1, so graph B.

(e) Note that $x^2 + y^2 = 4t^2 = \frac{1}{4}z^2$—thus the graph lies on a cone (so A).

(f) Only D remains, but note that we must have $x \geq 1$.

**3.** First note that $\frac{d}{dt}\|\mathbf{x}'(t)\| = \frac{d}{dt}\sqrt{\mathbf{x}'(t) \cdot \mathbf{x}'(t)} = (\mathbf{x}'(t) \cdot \mathbf{x}''(t))/\|\mathbf{x}'(t)\|$. So $\mathbf{x}$ has constant (non-zero) speed if and only if $\frac{d}{dt}\|\mathbf{x}'(t)\| = 0$ if and only if $\mathbf{x}'(t) \cdot \mathbf{x}''(t) = 0$ (i.e., its velocity and acceleration vectors are perpendicular).

**4. (a)** If we forget about gravity, the glasses travel along the tangent line to $\mathbf{x}$ at $t = 90$. We need the position along this tangent line two seconds after we lose our glasses:

$$\begin{aligned} \mathbf{1}(t) &= \mathbf{x}(90) + 2(\mathbf{x}'(90)) = (-e^{3/2}, 0, 80) + 2(-e^{3/2}/60, -\pi e^{3/2}/30, 0) \\ &= (-31e^{3/2}/30, -\pi e^{3/2}/15, 80). \end{aligned}$$

**(b)** The only component that changes when we factor in gravity is the height $h(t)$ of the glasses at time $t$. We know that gravity is $h''(t) = -32\, ft/sec^2$. The initial vertical velocity is zero so $h'(t) = -32t$. We know that when the glasses fall off they are 80 feet off the ground, so $h(t) = -16t^2 + 80$ so $h(2) = 16$ and the position of the glasses two seconds after they fall off is $(-31e^{3/2}/30, -\pi e^{3/2}/15, 16)$.

**5.** The velocity is $\mathbf{x}'(t) = \left(-\sin(t-1), 3t^2, -\frac{1}{t^2}\right)$ so $\mathbf{x}'(1) = (0, 3, -1)$. At $t = 1$ the position is $\mathbf{x}(1) = (1, 0, -1)$. If we define a surface by the equation $f(x, y, z) = x^3 + y^3 + z^3 - xyz = 0$, then $\nabla f(x, y, z) = (3x^2 - yz, 3y^2 - xz, 3z^2 - xy)$

so $\nabla f(1,0,-1) = (3,1,3)$. In general this vector is normal to the tangent plane at $(1,0,-1)$ and by observation it is also perpendicular to $\mathbf{x}'(1)$ so the curve is tangent to the surface when $t = 1$.

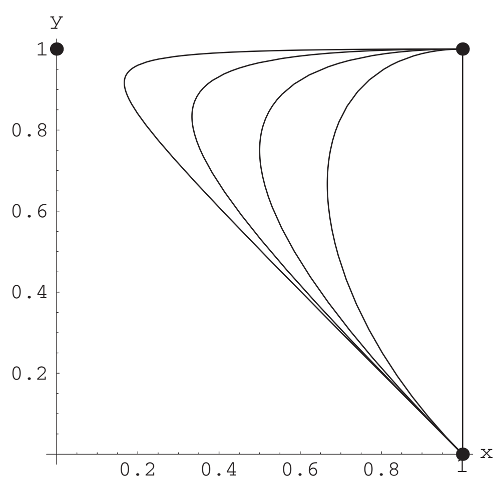
**6. (a)** We'll convert distance to inches and then $r = 240 - 3t$ while $\theta = 4\pi t$.

**(b)** $x = r\cos\theta$ and $y = r\sin\theta$ so $x(t) = (240-3t)\cos 4\pi t$ and $y(t) = (240-3t)\sin 4\pi t$.

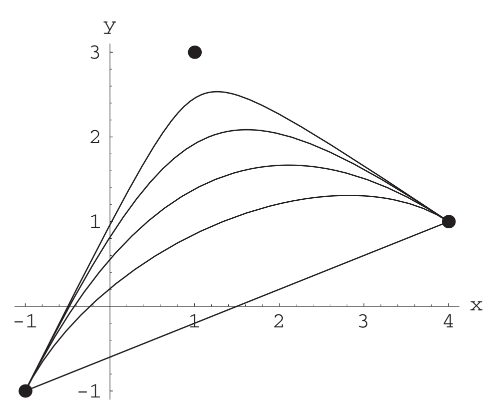
**(c)** It takes Gregor 80 minutes to reach the center so

$$\begin{aligned}\text{Distance} &= \int_0^{80} \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt \\ &= \int_0^{80} \sqrt{[-3\cos 4\pi t - 4\pi(240-3t)\sin 4\pi t]^2 + [-3\sin 4\pi t + 4\pi(240-3t)\cos 4\pi t]^2}\,dt \\ &= \int_0^{80} \sqrt{9 + 16\pi^2(240-3t)^2}\,dt = 120638 \text{ inches } \approx 1.90401 \text{ miles.}\end{aligned}$$

**7.** For $w = 0$ we just get the line segment joining the points $x_1$ and $x_2$. As $w$ increases the curve becomes more bent in the direction of the control point.




**8.** We see the same pattern as in Exercise 7.




**9. (a)** There's nothing much to show. In the equations given in (1), at $t = 0$ all but the first terms in the numerator and denominator disappear and you get $(x(0), y(0)) = (x_1, y_1)$. At $t = 1$ all but the last terms in the numerator and denominator disappear and you get $(x(1), y(1)) = (x_3, y_3)$.

**(b)** Here we get

$$\begin{aligned}\mathbf{x}(1/2) &= \left(\frac{x_1/4 + wx_2/2 + x_3/4}{(1+w)/2}, \frac{y_1/4 + wy_2/2 + y_3/4}{(1+w)/2}\right)\\ &= \frac{1}{1+w}\left(\frac{x_1+x_3}{2} + wx_2, \frac{y_1+y_3}{2} + wy_2\right)\\ &= \frac{1}{1+w}\left[\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right) + (wx_2, wy_2)\right]\\ &= \frac{1}{1+w}\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right) + \frac{w}{1+w}(x_2, y_2).\end{aligned}$$

Note that for $w \geq 0$, both $1/(1+w)$ and $w/(1+w)$ are between 0 and 1 and sum to 1. This tells us that the point **x**(1/2) lies on the line segment joining $(x_2, y_2)$ to the midpoint of the line segment joining $(x_1, y_1)$ to $(x_3, y_3)$.

**10.** If you're doing this by hand, first simplify the denominator in the expression for $x(t)$ and $y(t)$ by realizing the sum of the first and last terms is 1. In other words, $(1-t)^2 + 2wt(1-t) + t^2 = 2wt(1-t)$. Crank it through and find that $\mathbf{x}'(0) = 2\,w(x_2 - x_1, y_2 - y_1)$ and $\mathbf{x}'(1) = 2\,w(x_3 - x_2, y_3 - y_2)$.

Let $l_0$ be the tangent line to the curve at **x**(0). Then

$$l_0(t) = \mathbf{x}(0) + t\mathbf{x}'(0) = (x_1, y_1) + 2wt(x_2 - x_1, y_2 - y_1).$$

This cries out for us to check out $t = 1/(2\,w)$. We see that

$$l_0(1/(2\,w)) = (x_1, y_1) + (x_2 - x_1, y_2 - y_1) = (x_2, y_2).$$

Similarly, let $l_1$ be the tangent line to the curve at **x**(1). Then

$$l_1(t) = \mathbf{x}(1) + t\mathbf{x}'(1) = (x_3, y_3) + 2wt(x_3 - x_2, y_3 - y_2).$$

At $t = -1/(2\,w)$ we see that

$$l_1(-1/(2\,w)) = (x_3, y_3) - (x_3 - x_2, y_3 - y_2) = (x_2, y_2).$$

In other words, the point $(x_2, y_2)$ is on both of the tangent lines.

**11. (a)** Use part (b) of Exercise 9.

$$\begin{aligned}a = \|\mathbf{x}(1/2) - (x_2, y_2)\| &= \left\|\frac{1}{1+w}\left(\frac{x_1+x_3}{2} + wx_2, \frac{y_1+y_3}{2} + wy_2\right) - (x_2, y_2)\right\|\\ &= \left\|\left(\frac{1}{2(1+w)}\right)(x_1 - 2x_2 + x_3, y_1 - 2y_2 + y_3)\right\|\\ &= \left(\frac{1}{2(1+w)}\right)\sqrt{(x_1 - 2x_2 + x_3)^2 + (y_1 - 2y_2 + y_3)^2}\end{aligned}$$

**(b)** This is a similar calculation.

$$\begin{aligned}b &= \left\|\mathbf{x}(1/2) - \left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)\right\|\\ &= \left\|\frac{1}{1+w}\left(\frac{x_1+x_3}{2} + wx_2, \frac{y_1+y_3}{2} + wy_2\right) - \left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)\right\|\\ &= \left\|\left(\frac{w}{2(1+w)}\right)(-x_1 + 2x_2 - x_3, -y_1 + 2y_2 - y_3)\right\|\\ &= \left(\frac{w}{2(1+w)}\right)\sqrt{(x_1 - 2x_2 + x_3)^2 + (y_1 - 2y_2 + y_3)^2}\end{aligned}$$

**(c)** It's kind of amazing, but

$$\frac{b}{a} = \frac{\left(\frac{w}{2(1+w)}\right)\sqrt{(x_1 - 2x_2 + x_3)^2 + (y_1 - 2y_2 + y_3)^2}}{\left(\frac{1}{2(1+w)}\right)\sqrt{(x_1 - 2x_2 + x_3)^2 + (y_1 - 2y_2 + y_3)^2}} = w.$$

**12. (a)** Start with $y' = 2x$. So $y'(-2) = -4$ and $y'(2) = 4$. The two tangent lines $y - 4 = -4(x+2)$ and $y - 4 = 4(x-2)$ can be rewritten as $y = -4x - 4$ and $y = 4x - 4$. The point of intersection is $(0, -4)$ and so, by Exercise 10, this is the third control point.

**(b)** The deal here is that we are actually going to end up with $y = x^2$ between $x = -2$ and $x = 2$. Because **x**(1/2) must be on the line segment connecting the control point we found in (a) to the midpoint of the the line segment connecting the other two control points, it must be on the $y$-axis. The only point on the parabola that satisfies this is the origin. The constant $w$ is the ratio of the distance between (0, 0) and (0, 4) and the distance between (0, 0) and $(0, -4)$. In this case, $w = 1$.

**(c)** The Bézier parametrization is

$$\begin{cases} x(t) = -2(1-t)^2 + 2t^2 = 4t - 2 \\ y(t) = 4(1-t)^2 - 4(2t)(1-t) + 4t^2 = (4t-2)^2. \end{cases}$$

**13. (a)** We have $\mathbf{x}'(t) = \left(\cos t, -\sin t + \frac{1}{2}\cot\frac{t}{2}\sec^2\frac{t}{2}\right)$. Using the double angle formula, we have

$$\tfrac{1}{2}\cot\frac{t}{2}\sec^2\frac{t}{2} = \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}} = \frac{1}{\sin t}.$$

Hence $\mathbf{x}'(t) = \left(\cos t, \frac{1}{\sin t} - \sin t\right)$. Thus, $\mathbf{x}'(t) = (0, 0)$ if and only if $t = \pi/2$.

**(b)** The tangent line to the tractrix at the point $\mathbf{x}(t_0)$ is given by $\mathbf{l}(s) = \mathbf{x}(t_0) + s\mathbf{x}'(t_0)$. This line crosses the $y$-axis when $x = 0$ and if we explicitly compute the first component of $\mathbf{l}(s)$, we see that the condition for crossing is

$$\sin t_0 + s\cos t_0 = 0 \Leftrightarrow s = -\tan t_0.$$

The length of the segment we seek is given by

$$\begin{aligned} \|\mathbf{l}(-\tan t_0) - \mathbf{x}(t_0)\| &= \|\mathbf{x}(t_0) - \tan t_0 \mathbf{x}'(t_0) - \mathbf{x}(t_0)\| \\ &= |\tan t_0| \|\mathbf{x}'(t_0)\|. \end{aligned}$$

Using the work from part (a), the length of the segment is

$$\begin{aligned} |\tan t_0|\sqrt{\cos^2 t_0 + (\csc\, t_0 - \sin t_0)^2} &= |\tan t_0|\sqrt{\cos^2 t_0 + \csc^2\, t_0 - 2 + \sin^2 t_0} \\ &= |\tan t_0|\sqrt{\csc^2\, t_0 - 1} = |\tan t_0||\cot t_0| = 1. \end{aligned}$$

**14. (a)** Now we have $\mathbf{y}'(r) = (e^r, \sqrt{1 - e^{2r}})$ by the fundamental theorem of calculus. So the tangent line at the point $\mathbf{y}(r_0)$ is $\mathbf{m}(s) = \mathbf{y}(r_0) + s\mathbf{y}'(r_0)$. This line crosses the $y$-axis when $x = 0 \Leftrightarrow e^{r_0} + se^{r_0} = 0 \Leftrightarrow s = -1$. As in Exercise 13, we compute $\|\mathbf{m}(-1) - \mathbf{y}(r_0)\| = \|\mathbf{y}(r_0) - \mathbf{y}'(r_0) - \mathbf{y}(r_0)\| = \|\mathbf{y}'(r_0)\| = \sqrt{e^{2r_0} + 1 - e^{2r_0}} = 1$.

**(b)** Note that, for $\rho < 0$, the integrand $\sqrt{1 - e^{2\rho}}$ is positive. Hence for $r < 0$, the integral $\int_0^r \sqrt{1 - e^{2\rho}}\, d\rho$ is *negative*. Since the exponential $e^r$ varies between 0 and 1 as $r$ varies from $-\infty$ to 0, we see that **y** covers just the bottom half of the tractrix.

**15.** If $r = f(\theta)$, then we may write $\mathbf{x}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. Hence $\mathbf{v} = \mathbf{x}'(\theta) = (f'(\theta)\cos\theta - f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta)$ and $\|\mathbf{v}\| = \sqrt{f'(\theta)^2 + f(\theta)^2} = \sqrt{r'^2 + r^2}$. Also

$$\mathbf{a} = \mathbf{x}''(\theta) = (f''(\theta)\cos\theta - 2f'(\theta)\sin\theta - f(\theta)\cos\theta, f''(\theta)\sin\theta + 2f'(\theta)\cos\theta - f(\theta)\sin\theta).$$

If we calculate $\mathbf{v} \times \mathbf{a}$, we find (after same algebra)

$$\mathbf{v} \times \mathbf{a} = (-f(\theta)f''(\theta) + 2f'(\theta)^2 + f(\theta)^2)\mathbf{k} = (r^2 - rr'' + 2r'^2)\mathbf{k}.$$

Hence, using formula (17), we have

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{|r^2 - rr'' + 2r'^2|}{(r^2 + r'^2)^{3/2}}.$$

**16.** For the lemniscate $r^2 = \cos 2\theta$, so that, differentiating with respect to $\theta$, we have $2rr' = -2\sin 2\theta$. Hence

$$r' = -\frac{1}{r}\sin 2\theta \quad \text{so } r'^2 = \frac{1}{r^2}\sin^2 2\theta = \frac{\sin^2 2\theta}{\cos 2\theta}.$$

Thus

$$r^2 + r'^2 = \cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta} = \frac{1}{\cos 2\theta}.$$

Now, if we differentiate the equation $rr' = -\sin 2\theta$, we obtain

$$rr'' + r'^2 = -2\cos 2\theta.$$

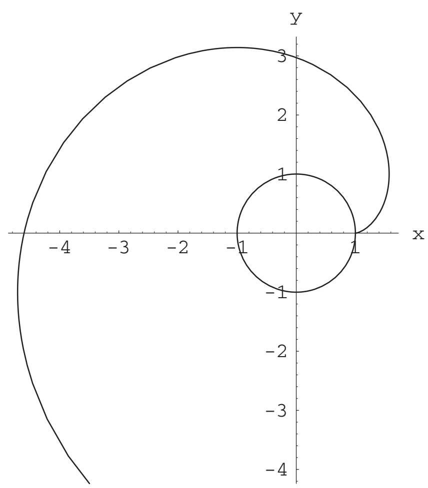
From Exercise 15, we must compute $\kappa = \dfrac{|r^2 - rr'' + 2r'^2|}{(r^2 + r'^2)^{3/2}}$. The denominator is easy; for the numerator, we have

$$\begin{aligned} r^2 - rr'' + 2r'^2 &= r^2 - (rr'' + r'^2) + 3r'^2 \\ &= \cos 2\theta - (-2\cos 2\theta) + \frac{3\sin^2 2\theta}{\cos 2\theta} \\ &= 3\cos 2\theta + \frac{3\sin^2 2\theta}{\cos 2\theta} = \frac{3}{\cos 2\theta}. \end{aligned}$$

Hence $\kappa(\theta) = \dfrac{|3/\cos 2\theta|}{(1/\cos 2\theta)^{3/2}} = 3\sqrt{\cos 2\theta}$.

**17. (a)** The involute of $\mathbf{x}(t) = (a\cos t, a\sin t)$ is $\mathbf{y}(t) = (a\cos t, a\sin t) - at(-\sin t, \cos t)$.

**(b)** The circle and involute are shown below.




**18.** We actually can afford to be a bit sloppy. Look first at $\mathbf{y}'(t) = \mathbf{x}'(t) - s'(t)\mathbf{T}(t) - s(t)\mathbf{T}'(t)$. By the fundamental theorem of calculus, $s'(t) = \|\mathbf{x}'(t)\|$ so $s'(t)\mathbf{T}(t) = \mathbf{x}'(t)$. So we can now say that $\mathbf{y}'(t) = -s(t)\mathbf{T}'(t)$. But by the Frenet–Serret formulas, $\mathbf{T}'(t) = s'(t)\kappa\mathbf{N}$. This means that $\mathbf{y}'(t) = -s(t)s'(t)\kappa\mathbf{N}$. In other words, the tangent vector to the involute is in the opposite direction to the normal vector to the curve, so the unit tangent vector to the involute at $t$ is the opposite of the unit normal vector $\mathbf{N}(t)$ to the original path $\mathbf{x}$.

**19. (a)** This first conclusion is pretty much by definition. Analytically,

$$\|\mathbf{y}(t) - \mathbf{x}(t)\| = \|\mathbf{x}(t) - s(t)\mathbf{T}(t) - \mathbf{x}(t)\| = \|s(t)\mathbf{T}(t)\| = |s(t)|\|\mathbf{T}(t)\| = |s(t)| = s(t).$$

This last fact follows because $s(t) \geq 0$. Finally we note that $s(t)$ is the distance traveled from $\mathbf{x}(t_0)$ to $\mathbf{x}(t)$ along the underlying curve of $\mathbf{x}$.

**(b)** We calculated the distance from $\mathbf{x}$ to $\mathbf{y}$ in part (a). We should also observe that this is the distance along the tangent line to $\mathbf{x}$ at time $t$ as it included the point $\mathbf{x}(t)$ and was in the direction $\mathbf{T}(t)$. The conclusion follows—it is as if you are unwinding a taut string from around $\mathbf{x}$: at each point $\mathbf{y}(t)$ is at a point in the direction of the tangent to $\mathbf{x}(t)$ of distance equal to the distance already traveled along $\mathbf{x}$. In other words, the distance is equal to the string already unraveled.

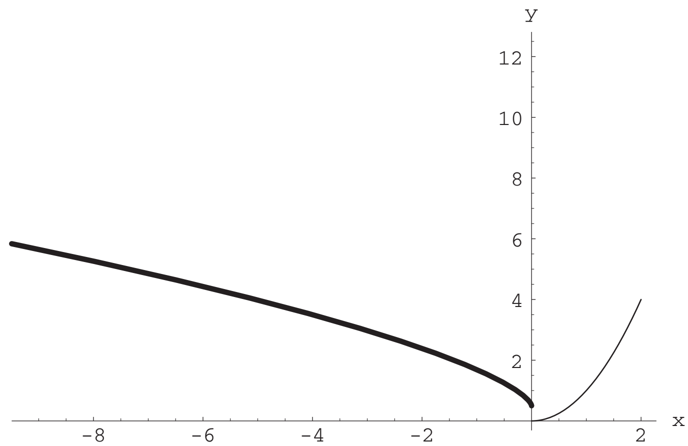
**20. (a)** The tangent vector is $\mathbf{T} = \mathbf{x}'(t)/\|\mathbf{x}'(t)\| = (1, 2t)/\sqrt{1+4t^2}$. The normal vector is in the $xy$-plane perpendicular to $\mathbf{T}$, pointing in the direction that $\mathbf{T}$ is changing: $\mathbf{N} = (-2t, 1)/\sqrt{1+4t^2}$. We'll use the formula (from Section 3.2):

$$\kappa = \frac{\|\mathbf{x}' \times \mathbf{x}''\|}{\|\mathbf{x}'\|^3} = \frac{\|(1, 2t, 0) \times (0, 2, 0)\|}{\|(1, 2t, 0)\|^3} = \frac{\|(0, 0, 2)\|}{(1+4t^2)^{3/2}} = \frac{2}{(1+4t^2)^{3/2}}.$$

**(b)** The formula for the evolute is:

$$\mathbf{y}(t) = \mathbf{x}(t) + \frac{1}{\kappa}\mathbf{N}(t) = (t + (1+4t^2)(-t), t^2 + (1+4t^2)/2) = (-4t^3, 3t^2 + 1/2).$$

**(c)** In the figure below, the evolute increases in the opposite direction as the parabola. We can see that because the parabola is "straightening out" so the curvature is decreasing so $1/\kappa$ is increasing. The evolute is made up of points distance $1/\kappa$ from the parabola in the normal direction.

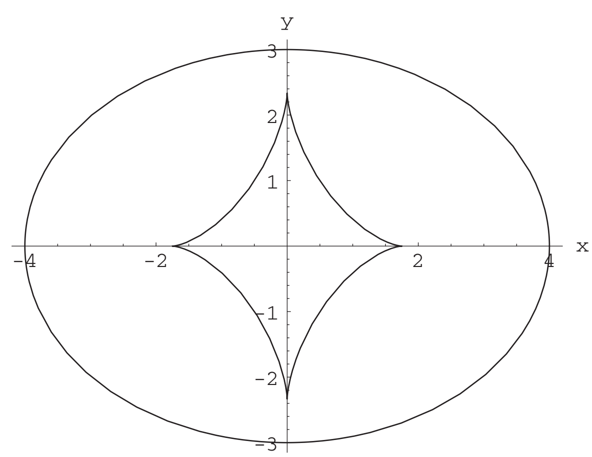



**21.** The curvature of a circle of radius $a$ is $\kappa = 1/a$. Recall from high school geometry that a tangent to a circle at a given point is perpendicular to a radial line at that point. If the normal vector is oriented inward, then the evolute consists of points of distance equal to the radius of the circle in the direction of the center of the circle. In other words, the evolute of a circle is the center of that circle. Analytically, this is $\mathbf{e}(t) = \mathbf{x}(t) + a\mathbf{N}(t)$.

**22. (a)** Using a computer we get that the evolute of the ellipse $(a \cos t, b \sin t)$ is

$$\left(\cos t\left[a - \frac{b(b^2\cos^2 t + a^2\sin^2 t)}{|ab|}\right], \cos t\left[b - \frac{a(b^2\cos^2 t + a^2\sin^2 t)}{|ab|}\right]\right).$$

**(b)** An example when $a = 3$ and $b = 4$ is shown below. As $a$ gets close to $b$ the ellipse approaches a circle and the evolute shrinks to a point.




**23.** You may initially get an ugly looking expression. After some coaxing and explicit help, your computer algebra system should be able to help you to find that $\mathbf{N}(t) = \frac{1}{\sqrt{2}}\left(\frac{\sin t}{\sqrt{1-\cos t}}, -\sqrt{1-\cos t}\right)$ and $\kappa = \frac{1}{2\sqrt{2}a\sqrt{1-\cos t}}$, and to reduce the formula for your evolute of a cycloid to $(at + a\sin t, a\cos t - a)$. This is another cycloid.

**24.** Punch this into *Mathematica* and you will get

$$\left(2a\cos t(1 + a\cos t) - \frac{2a^3(1 + a^2 + 2a\cos t)(\cos t + a\cos 2t)}{|a^2 + 2a^4 + 3a^3\cos t|},\right.$$

$$\left. 2a\sin t(1 + a\cos t) - \frac{a^2(1 + 2a\cos t)(1 + a^2 + 2a\cos t)}{|a^2 + 2a^4 + 3a^3\cos t|}\right).$$

**25.** Assume that $\mathbf{x}$ is a unit speed curve. To get the direction of the tangent, consider $\mathbf{e}'(t) = \mathbf{x}'(t) + (-1/\kappa^2)\kappa'\mathbf{N}(t) + (1/\kappa)\mathbf{N}'(t)$. By the Frenet–Serret equations, $\mathbf{N}'(t) = -\kappa\mathbf{T}$ since $\mathbf{x}$ is a planar curve. So we see what remains is $\mathbf{e}'(t) = (-1/\kappa^2)\kappa'\mathbf{N}(t)$. This tells us that the unit tangent vector to the evolute is the parallel to the unit normal vector to the original path.

**26.** First, $\kappa = \|\mathbf{v} \times \mathbf{a}\|/\|\mathbf{v}\|^3$. We know that $\mathbf{v}$ is a unit vector so $\|\mathbf{v}\|^3 = 1$. This means that $\kappa = \|\mathbf{v} \times \mathbf{a}\|$ and also that the tangential component of acceleration is $\frac{d}{dt}\|\mathbf{v}\| = 0$ so $\mathbf{v}$ is perpendicular to $\mathbf{a}$. Finally, this means that $\|\mathbf{v} \times \mathbf{a}\| = \|\mathbf{v}\|\|\mathbf{a}\| = 1$.

**27.** **(a)** $[x'(s)]^2 + [y'(s)]^2 = [\cos g(s)]^2 + [\sin g(s)]^2 = 1$.

**(b)** $\mathbf{v}(s) = (\cos g(s), \sin g(s))$ and $\mathbf{a}(s) = (-g'(s)\sin g(s), g'(s)\cos g(s))$ so

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \|(0, 0, g'(s))\| = |g'(s)|.$$

**(c)** We use the defining equations with $g'(s) = \kappa(s)$.

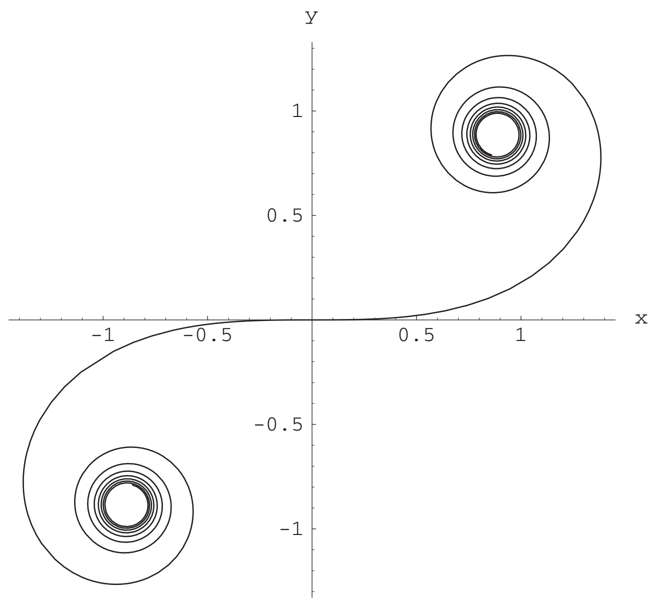
**(d)** There is more than one solution. For $s \geq 0$ we have $\kappa = s$ therefore, $g(s) = s^2/2$ so

$$x(s) = \int_0^s \cos(t^2/2)\,dt \quad \text{and} \quad y(s) = \int_0^s \sin(t^2/2)\,dt.$$

For $s < 0$, one solution corresponds to $g(s) = -s^2/2$ because for $s < 0, g'(s) = -s = |s|$. By formula (8) in Section 3.2, $\kappa$ will always be non-negative, so we can also take $g(s) = s^2/2$ for $s < 0$. Because cosine is an even function and sine is an odd function, our two solutions are

$$x(s) = \int_0^s \cos(t^2/2)\,dt \quad \text{and} \quad y(s) = \pm\int_0^s \sin(t^2/2)\,dt.$$

**(e)** The graph of the clothoid is shown below.




**28.** **(a)** $-\tau\mathbf{N} = d\mathbf{B}/ds$ so if $\tau \equiv 0$ then $\mathbf{B}$ is constant.

**(b)** The velocity $\mathbf{v}(0)$ is in the tangent direction $\mathbf{T}$, so $\mathbf{T}$ lies in the $xy$-plane. The acceleration $\mathbf{a}(0)$ has components in the direction of $\mathbf{T}$ and $\mathbf{N}$ and since $\mathbf{a}(0)$ and $\mathbf{T}$ are in the $xy$-plane, so is $\mathbf{N}$. The binormal vector $\mathbf{B}$ must be length one and perpendicular to the plane containing $\mathbf{T}$ and $\mathbf{N}$, so $\mathbf{B} = \pm\mathbf{k}$.

**(c)** Combining the results from parts (a) and (b), we know that $\mathbf{B} \equiv \mathbf{k}$ or $\mathbf{B} \equiv -\mathbf{k}$. It is always true that $\mathbf{v} \cdot \mathbf{B} = 0$ and $\mathbf{a} \cdot \mathbf{B} = 0$. In this case that is equivalent to $\mathbf{v} \cdot \mathbf{k} = 0$ and $\mathbf{a} \cdot \mathbf{k} = 0$. But $\mathbf{v}(t) \cdot \mathbf{k} = (x'(t), y'(t), z'(t)) \cdot (0, 0, 1) = z'(t)$. We conclude that $z'(t)$ is always zero so $z(t)$ is constant. Since we assumed that $z(0) = 0, z(t) \equiv 0$ and the path remains in the $xy$-plane.

**(d)** Look at the plane determined by $\mathbf{v}(0)$ and $\mathbf{a}(0)$. By part (b), $\mathbf{B}$ will be perpendicular to that plane. By part (a), $\mathbf{B}$ will be constant. Part (c) shows that motion will always be orthogonal to the direction of $\mathbf{B}$. It is harder to see in this case, but we can translate the problem so that $\mathbf{x}(0)$ is the origin and rotate so that $\mathbf{a}(0)$ and $\mathbf{v}(0)$ are in the $xy$-plane, make our conclusions then translate and rotate the solution curve back.

**29.** Note that we may write $\mathbf{x}(s) = (x(s), y(s), 0)$, where $s$ is the arclength parameter.

**(a)** $\mathbf{T} = (x'(s), y'(s), 0)$, so $\mathbf{N} = \dfrac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \dfrac{1}{\sqrt{x''^2 + y''^2}}(x''(s), y''(s), 0)$. (Hence $\kappa = \sqrt{x''^2 + y''^2}$.) Thus

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \left(0, 0, \frac{x'y'' - x''y'}{\kappa}\right) = (0, 0, \pm 1)$$

because $\mathbf{B}$ must be a *unit* vector. Now $\mathbf{B}$ must vary continuously, so either $\mathbf{B} = (0, 0, 1)$ or $(0, 0, -1)$—but in either case, it must be constant.

**(b)** $\mathbf{B}'(s) = -\tau\mathbf{N}$. $\mathbf{B}$ is constant (by part (a)), so $\mathbf{B}'(s) = \mathbf{0}$. Thus, since $\mathbf{N}$ is *never* zero, we may conclude that $\tau \equiv 0$.

**30.** We have $0 = \kappa = \left\|\frac{d\mathbf{T}}{ds}\right\|$. Thus $d\mathbf{T}/ds = \mathbf{0}$ so $\mathbf{T}$ must be a constant vector. Since $\mathbf{x}$ is parametrized by arclength, $\mathbf{x}'(s) = \mathbf{T}$ is constant. We may integrate to find:

$$\begin{aligned} \mathbf{x}(s) &= \int_{s_0}^{s} \mathbf{x}'(\sigma)\, d\sigma + \mathbf{x}(s_0) = \int_{s_0}^{s} \mathbf{T}\, d\sigma + \mathbf{x}(s_0) = (s - s_0)\mathbf{T} + \mathbf{x}(s_0) \\ &= s\mathbf{T} + (\mathbf{x}(s_0) - s_0\mathbf{T}), \end{aligned}$$

which is of the form $s\mathbf{a} + \mathbf{b} \Rightarrow$ straight line.

**31. (a)** The plan is to find the curvature of the strake by finding the curvature of the helix along the pipe. The radius $r$ will be the reciprocal of this curvature (since the curvature of a circle of radius $r$ is $1/r$). The path is $\mathbf{x}(t) = (a\cos\, t, a\sin\, t, ht/2\pi)$ so $\mathbf{x}'(t) = (-a\sin\, t, a\cos\, t, h/2\pi)$ and $\mathbf{x}''(t) = (-a\cos t, -a\sin t, 0)$.

$$\begin{aligned} \kappa &= \frac{\|(-a\sin t, a\cos t, h)/2\pi \times (-a\cos t, -a\sin t, 0)\|}{\|(-a\sin t, a\cos t, h/2\pi)\|^3} = \frac{\|((ah/2\pi)\sin t, -(ah/2\pi)\cos t, a^2)\|}{(a^2 + [h/2\pi]^2)^{3/2}} \\ &= \frac{a\sqrt{h^2/4\pi^2 + a^2}}{(a^2 + [h/2\pi]^2)^{3/2}} = \frac{a}{a^2 + h^2/4\pi^2} \quad \text{so} \quad r = \frac{(a^2 + h^2/4\pi^2)}{a}. \end{aligned}$$

**(b)** If $a = 3$ and $h = 25$ then $r = (9 + 625/4\pi^2)/3 \approx 8.2771$.

**32.** Since $\mathbf{x}(t) = (a\cos t, a\sin t, bt)$ is not parametrized by arclength, we may rewrite it as $\mathbf{x}(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\right.$ $\left. a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right)$ where $s = \sqrt{a^2 + b^2}\, t$ is arclength (see Example 3 in §3.2). Following Example 9, we have

$$\mathbf{T}(s) = \left(-\frac{a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right).$$

So the tangent spherical image is a circle of radius $a/\sqrt{a^2+b^2}$ in the plane $z = b/\sqrt{a^2+b^2}$.

$\mathbf{N}(s) = \left(-\cos\frac{s}{\sqrt{a^2+b^2}}, -\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right)$; normal spherical image is a unit circle in the $xy$-plane.

$\mathbf{B}(s) = \left(\frac{b}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}, -\frac{b}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\right)$. Thus the binormal spherical image is a circle of radius $b/\sqrt{a^2+b^2}$ in the plane $z = a/\sqrt{a^2+b^2}$.

**33.** By Example 7 of §3.2 and Exercise 30: $\mathbf{x}$ is a straight-line path $\Leftrightarrow \kappa = 0 = \left\|\frac{d\mathbf{T}}{ds}\right\| \Leftrightarrow \mathbf{T}$ is constant.

**34.** By Exercises 28 and 29, $\mathbf{x}$ is a plane curve $\Leftrightarrow \mathbf{B}$ is constant.

**35.** $\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}$ by the Frenet–Serret formula. Now for $\mathbf{N}$ to be defined $\kappa \neq 0$, so if $\tau = 0$, then $\mathbf{N}' = -\kappa\mathbf{T} \neq \mathbf{0}$ (hence $\mathbf{N}$ is not constant). If $\tau \neq 0$, then for $\mathbf{N}'$ to be $\mathbf{0}$, $\mathbf{T}$ and $\mathbf{B}$ would have to be parallel, which they aren't.

**36. (a)** $\mathbf{x}(t) = r(t)\cos\theta(t)\mathbf{i} + r(t)\sin\theta(t)\mathbf{j} + z(t)\mathbf{k} = r(t)(\cos\theta(t)\mathbf{i} + \sin\theta(t)\mathbf{j}) + z(t)\mathbf{k} = r(t)\mathbf{e}_r + z(t)\mathbf{e}_z$.

**(b)** We prepare for part (c) by calculating:

$$\begin{aligned} \frac{d\mathbf{e}_r}{dt} &= -\theta'(t)\sin\theta(t)\mathbf{i} + \theta'(t)\cos\theta(t)\mathbf{j} = \theta'(t)\mathbf{e}_\theta, \\ \frac{d\mathbf{e}_\theta}{dt} &= -\theta'(t)\cos\theta(t)\mathbf{i} - \theta'(t)\sin\theta(t)\mathbf{j} = -\theta'(t)\mathbf{e}_r, \text{ and} \\ \frac{d\mathbf{e}_z}{dt} &= \mathbf{0} \end{aligned}$$

**(c)** We use the results of parts (a) and (b) to calculate:

$$\begin{aligned}
\mathbf{v}(t) &= \frac{d}{dt}\mathbf{x}(t) = \frac{d}{dt}[r(t)\mathbf{e}_r + z(t)\mathbf{e}_z] \\
&= r'(t)\mathbf{e}_r + r(t)\frac{d\mathbf{e}_r}{dt} + z'(t)\mathbf{e}_z + z(t)\frac{d\mathbf{e}_z}{dt} \\
&= r'(t)\mathbf{e}_r + r(t)\theta'(t)\mathbf{e}_\theta + z'(t)\mathbf{e}_z, \text{ and} \\
\mathbf{a}(t) &= \frac{d}{dt}[r'(t)\mathbf{e}_r + r(t)\theta'(t)\mathbf{e}_\theta + z'(t)\mathbf{e}_z] \\
&= r''(t)\mathbf{e}_r + r'(t)\theta'(t)\mathbf{e}_\theta + r'(t)\theta'(t)\mathbf{e}_\theta + r(t)\theta''(t)\mathbf{e}_\theta - r(t)[\theta'(t)]^2\mathbf{e}_r + z''(t)\mathbf{e}_z \\
&= (r''(t) - r(t)[\theta'(t)]^2)\mathbf{e}_r + (2r'(t)\theta'(t) + r(t)\theta''(t))\mathbf{e}_\theta + z''(t)\mathbf{e}_z.
\end{aligned}$$

**37.** $\mathbf{x}(t) = (\sin 2t, \sqrt{2}\cos 2t, \sin 2t - 2)$

**(a)** The loop first closes up when $t = \pi$ so the length of the loop is

$$\text{Length} = \int_0^\pi \sqrt{[2\cos 2t]^2 + [-2\sqrt{2}\sin 2t]^2 + [2\cos 2t]^2}\, dt = \int_0^\pi 2\sqrt{2}\, dt = 2\pi\sqrt{2}.$$

**(b)** By Definition 3.2, the path is a flow line if $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$. Here

$$\mathbf{x}'(t) = (2\cos 2t, -2\sqrt{2}\sin 2t, 2\cos 2t) \quad \text{and} \quad \mathbf{F}(\mathbf{x}(t)) = (\sqrt{2}\cos 2t, -2\sin 2t, \sqrt{2}\cos 2t).$$

So $\mathbf{x}$ is a flow line of the vector field $\sqrt{2}\mathbf{F}(x, y, z) = \sqrt{2}y\mathbf{i} - 2\sqrt{2}x\mathbf{j} + \sqrt{2}y\mathbf{k}$.

**38.** Poor Livinia, she's been caught in an oven back in Chapter 2, and now here in Chapter 3 she's still looking to get warm.

**(a)** We saw in Section 2.6 that the gradient is the direction of quickest increase. Livinia should head in a direction parallel to the gradient. In other words, at each point she should travel in the direction $k\nabla T$ so $\mathbf{x}'(t) = k\nabla T$ so $\mathbf{x}$ is a path of $k\nabla T$ for $k \geq 0$.

**(b)** If $T(x, y, z) = x^2 - 2y^2 + 3z^2$ then $\nabla T = (2x, -4y, 6z)$. We also know that the initial position is $(2, 3, -1)$. This means that $x' = 2x$ and $x(0) = 2$ so $x(t) = 2e^{2kt}$. Similarly, $y(t) = 3e^{-4kt}$ and $z(t) = -e^{6kt}$. So the equation of the path is $\mathbf{x}(t) = (2e^{2kt}, 3e^{-4kt}, -e^{6kt})$.

**39.** $\mathbf{F} = u(x, y)\mathbf{i} - v(x, y)\mathbf{j}$ is an incompressible, irrotational vector field and so $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$.

**(a)** The Cauchy–Riemann equations follow immediately from the assumptions:

$$\begin{aligned}
0 = \nabla \cdot \mathbf{F} &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}, \text{ so} \\
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \text{ and} \\
0 = (\nabla \times \mathbf{F}) \cdot \mathbf{k} &= -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \text{ so} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}$$

**(b)** Take the partial derivative with respect to $x$ of both sides of the equation: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2}.$$

An analogous calculation shows the result for $v$.

**40.** $\mathbf{F}$ is a gradient field so $\mathbf{F} = \nabla f(\mathbf{x}(t))$. Also $\mathbf{F} = m\mathbf{a}$. From Section 3.3 we know that if $\mathbf{x}$ is a path on an equipotential surface of $\mathbf{F}$ then $f(\mathbf{x}(t))$ is constant so $\frac{d}{dt}f(\mathbf{x}(t)) = 0$. So

$$0 = \frac{d}{dt}f(\mathbf{x}(t)) = \nabla f \cdot \frac{d}{dt}\mathbf{x}(t) = m\mathbf{a} \cdot \mathbf{v}.$$

From Section 3.2 formulas (14) and (16), we see that

$$m\mathbf{a} \cdot \mathbf{v} = m\, \dot{s}\, \ddot{s} = \frac{1}{2}m\frac{d}{dt}(\dot{s}^2).$$

So, since the derivative of $\dot{s}^2 = 0$, we conclude that $\dot{s}^2$ is constant and hence $\dot{s}$ is constant.

**41.** Using Section 3.1, Exercise 28:

$$\begin{aligned}\frac{d\mathbf{l}}{dt} = \frac{d}{dt}(\mathbf{x} \times m\mathbf{v}) &= \frac{d\mathbf{x}}{dt} \times m\mathbf{v} + \mathbf{x} \times m\frac{d\mathbf{v}}{dt} \\ &= \mathbf{v} \times m\mathbf{v} + \mathbf{x} \times m\mathbf{a} \\ &= \mathbf{0} + \mathbf{x} \times m\mathbf{a} = \mathbf{M}.\end{aligned}$$

**42.** If $\mathbf{F}$ is a central force, then $\mathbf{F}$ is always parallel to $\mathbf{x}$. Hence $\mathbf{M} = \mathbf{x} \times \mathbf{F} = \mathbf{0}$, By Exercise 41, $\mathbf{M} = \frac{d\mathbf{l}}{dt}$ so $\mathbf{l}$ must be constant.

**43.** Notice that $\nabla \times \mathbf{F} = (0, 2e^{-x}\cos z, 0) \neq \mathbf{0}$. If $\mathbf{F}$ were a gradient field $\nabla f$ of class $C^2$, then, by Theorem 4.3, $\nabla \times \mathbf{F} = \nabla \times (\nabla f) = \mathbf{0}$.

**44.** Note that $\nabla \cdot \mathbf{F} = y^2 + 1 + e^z + x^2 e^z > 0$ for all $(x, y, z) \in \mathbf{R}^3$. But if $\mathbf{F} = \nabla \times \mathbf{G}$, then $\nabla \cdot \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{G}) \equiv 0$ for any vector field $\mathbf{G}$ of class $C^2$. Thus $\mathbf{F} \neq \nabla \times \mathbf{G}$.

## Chapter 4

# Maxima and Minima in Several Variables

### 4.1 Differentials and Taylor's Theorem

*In Exercises 1–7 we will first calculate $f(x), f'(x), \ldots, f^{(k)}(x)$ and $f(a), f'(a), \ldots, f^{(k)}(a)$. Then we'll plug into the formula for Taylor's theorem in one variable (Theorem 1.1 in the text):*

$$p_k(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

**1.** Here $a = 0$ and $k = 4$:

$$\begin{aligned} f(x) &= e^{2x} & f(0) &= 1 \\ f^{(n)}(x) &= 2^n e^{2x} & f^{(n)}(0) &= 2^n \end{aligned}$$

so

$$\begin{aligned} p_4(x) &= 1 + 2x + \frac{4}{2}x^2 + \frac{8}{6}x^3 + \frac{16}{24}x^4 \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4. \end{aligned}$$

**2.** Here $a = 0$ and $k = 3$:

$$\begin{aligned} f(x) &= \ln(1+x) & f(0) &= 0 \\ f'(x) &= \frac{1}{1+x} & f'(0) &= 1 \\ f''(x) &= -\frac{1}{(1+x)^2} & f''(0) &= -1 \\ f'''(x) &= -2\left(\frac{-1}{(1+x)^3}\right) & f'''(0) &= 2, \end{aligned}$$

so

$$\begin{aligned} p_3(x) &= 0 + x - \frac{1}{2}x^2 + \frac{2}{6}x^3 \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3. \end{aligned}$$

**3.** Here $a = 1$ and $k = 4$:

$$\begin{aligned} f(x) &= \frac{1}{x^2} & f(1) &= 1 \\ f'(x) &= -\frac{2}{x^3} & f'(1) &= -2 \\ f''(x) &= \frac{6}{x^4} & f''(1) &= 6 \\ f'''(x) &= -\frac{24}{x^5} & f'''(1) &= -24 \\ f''''(x) &= \frac{120}{x^6} & f''''(1) &= 120, \end{aligned}$$

so

$$\begin{aligned} p_4(x) &= 1 - 2(x-1) + \frac{6}{2}(x-1)^2 - \frac{24}{6}(x-1)^3 + \frac{120}{24}(x-1)^4 \\ &= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4. \end{aligned}$$

*Students sometimes forget that the Taylor polynomial depends on the choice of a. Some texts include the parameter a in the notation to stress this fact. A nice way to remind your students of this dependence on a is to either assign Exercises 4 and 5 or 6 and 7 together.*

We'll do the scratch work for both Exercises 4 and 5 together:

$$\begin{array}{lll} f(x) = \sqrt{x} & f(1) = 1 & f(9) = 3 \\ f'(x) = \dfrac{1}{2\sqrt{x}} & f'(1) = \dfrac{1}{2} & f'(9) = \dfrac{1}{6} \\ f''(x) = \dfrac{-1}{4x^{3/2}} & f''(1) = -\dfrac{1}{4} & f''(9) = -\dfrac{1}{108} \\ f'''(x) = \dfrac{3}{8x^{5/2}} & f'''(1) = \dfrac{3}{8} & f'''(9) = \dfrac{1}{648}. \end{array}$$

**4.** Here $a = 1$ and $k = 3$ so, using the work above:

$$p_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3.$$

**5.** Here $a = 9$ and $k = 3$ so, using the work above:

$$p_3(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3.$$

We'll do the scratch work for both Exercises 6 and 7 together:

$$\begin{array}{lll} f(x) = \sin x & f(0) = 0 & f(\pi/2) = 1 \\ f'(x) = \cos x & f'(0) = 1 & f'(\pi/2) = 0 \\ f''(x) = -\sin x & f''(0) = 0 & f''(\pi/2) = -1 \\ f'''(x) = -\cos x & f'''(0) = -1 & f'''(\pi/2) = 0 \\ f''''(x) = \sin x & f''''(0) = 0 & f''''(\pi/2) = 1 \\ f'''''(x) = \cos x & f'''''(0) = 1 & f'''''(\pi/2) = 0. \end{array}$$

**6.** Here $a = 0$ and $k = 5$ so, using the work above:

$$p_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}.$$

**7.** Here $a = \pi/2$ and $k = 5$ so, using the work above:

$$p_5(x) = 1 - \frac{(x-\pi/2)^2}{2} + \frac{(x-\pi/2)^4}{24}.$$

*Three notes:*

- *It makes sense to assign Exercises 8, 9, 16, and 21 together as they explore the same function. Exercise 14 is a higher-dimensional analogue.*
- *In Exercises 8–15, we again do the preliminary calculations and then substitute into the formulas given in Theorem 1.3*

$$p_1(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

*and Theorem 1.5*

$$p_2(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^{n} f_{x_i}(\mathbf{a})(x_i - a_i) + \frac{1}{2}\sum_{i,j=1}^{n} f_{x_i x_j}(\mathbf{a})(x_i - a_i)(x_j - a_j)$$

$$= p_1(\mathbf{x}) + \frac{1}{2}\sum_{i,j=1}^{n} f_{x_i x_j}(\mathbf{a})(x_i - a_i)(x_j - a_j).$$

- *Just as in the one-variable versions of Taylor's theorem, note the lower degree polynomials are contained in the expressions for the higher degree ones.*

We'll do the scratch work for both Exercises 8 and 9 together:

$$\begin{aligned}
f(x,y) &= \frac{1}{x^2+y^2+1} & f(0,0) &= 1 & f(1,-1) &= 1/3\\
f_x(x,y) &= \frac{-2x}{(x^2+y^2+1)^2} & f_x(0,0) &= 0 & f_x(1,-1) &= -2/9\\
f_y(x,y) &= \frac{-2y}{(x^2+y^2+1)^2} & f_y(0,0) &= 0 & f_y(1,-1) &= 2/9\\
f_{xx}(x,y) &= \frac{6x^2-2y^2-2}{(x^2+y^2+1)^3} & f_{xx}(0,0) &= -2 & f_{xx}(1,-1) &= 2/27\\
f_{yy}(x,y) &= \frac{6y^2-2x^2-2}{(x^2+y^2+1)^3} & f_{yy}(0,0) &= -2 & f_{yy}(1,-1) &= 2/27\\
f_{xy}(x,y) &= \frac{8xy}{(x^2+y^2+1)^3} & f_{xy}(0,0) &= 0 & f_{xy}(1,-1) &= -8/27
\end{aligned}$$

**8.** $\mathbf{a} = (0,0)$ so, using the work above:

$$p_1(\mathbf{x}) = f(0,0) + Df(0,0)\mathbf{x} = 1 \quad \text{and}$$

$$p_2(\mathbf{x}) = p_1(\mathbf{x}) + \frac{1}{2}(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2)$$

$$= 1 - x^2 - y^2.$$

**9.** $\mathbf{a} = (1,-1)$ so, using the work above:

$$p_1(\mathbf{x}) = f(1,-1) + Df(1,-1)(\mathbf{x} - (1,-1)) = \frac{1}{3} + \begin{bmatrix} -\frac{2}{9} & \frac{2}{9} \end{bmatrix} \begin{bmatrix} x-1 \\ y+1 \end{bmatrix}$$

$$= \frac{1}{3} - \frac{2(x-1)}{9} + \frac{2(y+1)}{9} \quad \text{and}$$

$$p_2(\mathbf{x}) = p_1(\mathbf{x}) + \frac{1}{2}(f_{xx}(1,-1)(x-1)^2 + 2f_{xy}(1,-1)(x-1)(y+1) + f_{yy}(1,-1)(y+1)^2)$$

$$= \frac{1}{3} - \frac{2(x-1)}{9} + \frac{2(y+1)}{9} + \frac{(x-1)^2}{27} - \frac{8(x-1)(y+1)}{27} + \frac{(y+1)^2}{27}.$$

**10.** Here $\mathbf{a} = (0,0)$ and

$$\begin{aligned}
f(x,y) &= e^{2x+y} & f(0,0) &= 1\\
f_x(x,y) &= 2e^{2x+y} & f_x(0,0) &= 2\\
f_y(x,y) &= e^{2x+y} & f_y(0,0) &= 1\\
f_{xx}(x,y) &= 4e^{2x+y} & f_{xx}(0,0) &= 4\\
f_{yy}(x,y) &= e^{2x+y} & f_{yy}(0,0) &= 1\\
f_{xy}(x,y) &= 2e^{2x+y} & f_{xy}(0,0) &= 2,
\end{aligned}$$

so

$$\begin{aligned} p_1(\mathbf{x}) &= f(0,0) + Df(0,0)\mathbf{x} = 1 + 2x + y \quad \text{and} \\ p_2(\mathbf{x}) &= 1 + 2x + y + \frac{1}{2}(4x^2 + 2(2)xy + y^2) \\ &= 1 + 2x + y + 2x^2 + 2xy + \frac{y^2}{2} \end{aligned}$$

**11.** Here $\mathbf{a} = (0, \pi)$ and

$$\begin{aligned} f(x,y) &= e^{2x}\cos 3y & f(0,\pi) &= -1 \\ f_x(x,y) &= 2e^{2x}\cos 3y & f_x(0,\pi) &= -2 \\ f_y(x,y) &= -3e^{2x}\sin 3y & f_y(0,\pi) &= 0 \\ f_{xx}(x,y) &= 4e^{2x}\cos 3y & f_{xx}(0,\pi) &= -4 \\ f_{yy}(x,y) &= -9e^{2x}\cos 3y & f_{yy}(0,\pi) &= 9 \\ f_{xy}(x,y) &= -6e^{2x}\sin 3y & f_{xy}(0,\pi) &= 0, \end{aligned}$$

so

$$\begin{aligned} p_1(\mathbf{x}) &= -1 - 2x \quad \text{and} \\ p_2(\mathbf{x}) &= -1 - 2x + \frac{1}{2}(-4x^2 + 9(y-\pi)^2) \\ &= -1 - 2x - 2x^2 + \frac{9}{2}(y-\pi)^2. \end{aligned}$$

**12.** Here $\mathbf{a} = (0, 0, 2)$ and

$$\begin{aligned} f(x,y,z) &= ye^{3x} + ze^{2y} & f(0,0,2) &= 2 \\ f_x(x,y,z) &= 3ye^{3x} & f_x(0,0,2) &= 0 \\ f_y(x,y,z) &= e^{3x} + 2ze^{2y} & f_y(0,0,2) &= 5 \\ f_z(x,y,z) &= e^{2y} & f_y(0,0,2) &= 1 \\ f_{xx}(x,y,z) &= 9ye^{3x} & f_{xx}(0,0,2) &= 0 \\ f_{xy}(x,y,z) &= 3e^{3x} & f_{xy}(0,0,2) &= 3 = f_{yx}(0,0,2) \\ f_{xz}(x,y,z) &= 0 & f_{xz}(0,0,2) &= 0 = f_{zx}(0,0,2) \\ f_{yy}(x,y,z) &= 4ze^{2y} & f_{yy}(0,0,2) &= 8 \\ f_{yz}(x,y,z) &= 2e^{2y} & f_{yz}(0,0,2) &= 2 = f_{zy}(0,0,2) \\ f_{zz}(x,y,z) &= 0 & f_{yy}(0,0,2) &= 0, \end{aligned}$$

so

$$\begin{aligned} p_1(\mathbf{x}) &= 2 + 5y + 1(z-2) = 5y + z \quad \text{and} \\ p_2(\mathbf{x}) &= 5y + z + \frac{1}{2}(6xy + 8y^2 + 4y(z-2)) \\ &= y + z + 3xy + 4y^2 + 2yz. \end{aligned}$$

**13.** Here $\mathbf{a} = (2, -1, 1)$ and

$$\begin{aligned}
f(x, y, z) &= xy - 3y^2 + 2xz & f(2, -1, 1) &= -1 \\
f_x(x, y, z) &= y + 2z & f_x(2, -1, 1) &= 1 \\
f_y(x, y, z) &= x - 6y & f_y(2, -1, 1) &= 8 \\
f_z(x, y, z) &= 2x & f_y(2, -1, 1) &= 4 \\
f_{xx}(x, y, z) &= 0 & f_{xx}(2, -1, 1) &= 0 \\
f_{xy}(x, y, z) &= 1 & f_{xy}(2, -1, 1) &= 1 = f_{yx}(2, -1, 1) \\
f_{xz}(x, y, z) &= 2 & f_{xz}(2, -1, 1) &= 2 = f_{zx}(2, -1, 1) \\
f_{yy}(x, y, z) &= -6 & f_{yy}(2, -1, 1) &= -6 \\
f_{yz}(x, y, z) &= 0 & f_{yz}(2, -1, 1) &= 0 = f_{zy}(2, -1, 1) \\
f_{zz}(x, y, z) &= 0 & f_{yy}(2, -1, 1) &= 0,
\end{aligned}$$

so

$$\begin{aligned}
p_1(\mathbf{x}) &= -1 + 1(x - 2) + 8(y + 1) + 4(z - 1) = 1 + x + 8y + 4z \quad \text{and} \\
p_2(\mathbf{x}) &= 1 + x + 8y + 4z + \frac{1}{2}(2(x - 2)(y + 1) + 4(x - 2)(z - 1) - 6(y + 1)^2) \\
&= xy - 3y^2 + 2xz.
\end{aligned}$$

Note that the second-order polynomial matches the original function exactly. This makes sense, since $f$ is itself a polynomial of degree two.

**14.** Here $\mathbf{a} = (0, 0, 0)$ and there is quite a bit of symmetry so we'll only calculate:

$$\begin{aligned}
f(x, y, z) &= \frac{1}{x^2 + y^2 + z^2 + 1} & f(0, 0, 0) &= 1 \\
f_x(x, y, z) &= \frac{-2x}{(x^2 + y^2 + z^2 + 1)^2} & f_x(0, 0, 0) &= 0 = f_y(0, 0, 0) = f_z(0, 0, 0) \\
f_{xx}(x, y, z) &= \frac{6x^2 - 2y^2 - 2z^2 - 2}{(x^2 + y^2 + z^2 + 1)^3} & f_{xx}(0, 0, 0) &= -2 = f_{yy}(0, 0, 0) = f_{zz}(0, 0, 0) \\
f_{xy}(x, y) &= \frac{8xy}{(x^2 + y^2 + z^2 + 1)^3} & f_{xy}(0, 0, 0) &= 0 = f_{xz}(0, 0, 0) = f_{yz}(0, 0, 0)
\end{aligned}$$

so

$$\begin{aligned}
p_1(\mathbf{x}) &= 1 \quad \text{and} \\
p_2(\mathbf{x}) &= 1 + \frac{1}{2}(-2x^2 - 2y^2 - 2z^2) \\
&= 1 - x^2 - y^2 - z^2.
\end{aligned}$$

**15.** Again $\mathbf{a} = (0, 0, 0)$ and there is quite a bit of symmetry so we'll only calculate:

$$\begin{aligned}
f(x, y, z) &= \sin xyz & f(0, 0, 0) &= 0 \\
f_x(x, y, z) &= yz \cos xyz & f_x(0, 0, 0) &= 0 = f_y(0, 0, 0) = f_z(0, 0, 0) \\
f_{xx}(x, y, z) &= -y^2z^2 \sin xyz & f_{xx}(0, 0, 0) &= 0 = f_{yy}(0, 0, 0) = f_{zz}(0, 0, 0) \\
f_{xy}(x, y) &= z \cos xyz - xyz^2 \sin xyz & f_{xy}(0, 0, 0) &= 0 = f_{xz}(0, 0, 0) = f_{yz}(0, 0, 0)
\end{aligned}$$

$$\text{so} \quad p_1(\mathbf{x}) = 0 \quad \text{and} \quad p_2(\mathbf{x}) = 0.$$

**16.** From Exercise 8 we can read off that the Hessian $Hf(0, 0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$.

**17.** $f(x, y) = \cos x \sin y$

$$\begin{aligned} f_x(x,y) &= -\sin x \sin y & f_y(x,y) &= \cos x \cos y \\ f_{xx}(x,y) &= -\cos x \sin y & f_{yx}(x,y) &= -\sin x \cos y \\ f_{xy}(x,y) &= -\sin x \cos y & f_{yy}(x,y) &= -\cos x \sin y \end{aligned}$$

so

$$Hf\left(\frac{\pi}{4}, \frac{\pi}{3}\right) = \begin{bmatrix} -\frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{6}}{4} \end{bmatrix}.$$

**18.** $f(x, y, z) = \dfrac{z}{\sqrt{xy}}$

$$\begin{aligned} f_x(x,y,z) &= -\frac{z}{2x^{3/2}y^{1/2}} & f_y(x,y,z) &= -\frac{z}{2x^{1/2}y^{3/2}} & f_z(x,y,z) &= \frac{1}{\sqrt{xy}} \\ f_{xx}(x,y,z) &= \frac{3z}{4x^{5/2}y^{1/2}} & f_{yx}(x,y,z) &= \frac{z}{4x^{3/2}y^{3/2}} & f_{zx}(x,y,z) &= -\frac{1}{3x^{3/2}y^{1/2}} \\ f_{xy}(x,y,z) &= \frac{z}{4x^{3/2}y^{3/2}} & f_{yy}(x,y,z) &= \frac{3z}{4x^{1/2}y^{5/2}} & f_{zy}(x,y,z) &= -\frac{1}{2x^{1/2}y^{3/2}} \\ f_{xz}(x,y,z) &= -\frac{1}{2x^{3/2}y^{1/2}} & f_{yz}(x,y,z) &= -\frac{1}{2x^{1/2}y^{3/2}} & f_{zz}(x,y,z) &= 0 \end{aligned}$$

so

$$Hf(1,2,-4) = \begin{bmatrix} -\frac{3}{\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{3}{4\sqrt{2}} & -\frac{1}{4\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{4\sqrt{2}} & 0 \end{bmatrix}.$$

**19.** $f(x, y, z) = x^3 + x^2y - yz^2 + 2z^3$

$$\begin{aligned} f_x(x,y,z) &= 3x^2 + 2xy & f_y(x,y,z) &= x^2 - z^2 & f_z(x,y,z) &= -2yz + 6z^2 \\ f_{xx}(x,y,z) &= 6x + 2y & f_{yx}(x,y,z) &= 2x & f_{zx}(x,y,z) &= 0 \\ f_{xy}(x,y,z) &= 2x & f_{yy}(x,y,z) &= 0 & f_{zy}(x,y,z) &= -2z \\ f_{xz}(x,y,z) &= 0 & f_{yz}(x,y,z) &= -2z & f_{zz}(x,y,z) &= -2y + 12z \end{aligned}$$

so

$$Hf(1,0,1) = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 12 \end{bmatrix}.$$

**20.** $f(x, y, z) = e^{2x-3y} \sin 5z$

$$\begin{aligned} f_x(x,y,z) &= 2e^{2x-3y}\sin 5z & f_y(x,y,z) &= -3e^{2x-3y}\sin 5z & f_z(x,y,z) &= 5e^{2x-3y}\cos 5z \\ f_{xx}(x,y,z) &= 4e^{2x-3y}\sin 5z & f_{yx}(x,y,z) &= -6e^{2x-3y}\sin 5z & f_{zx}(x,y,z) &= 10e^{2x-3y}\cos 5z \\ f_{xy}(x,y,z) &= -6e^{2x-3y}\sin 5z & f_{yy}(x,y,z) &= 9e^{2x-3y}\sin 5z & f_{zy}(x,y,z) &= -15e^{2x-3y}\cos 5z \\ f_{xz}(x,y,z) &= 10e^{2x-3y}\cos 5z & f_{yz}(x,y,z) &= -15e^{2x-3y}\cos 5z & f_{zz}(x,y,z) &= -25e^{2x-3y}\sin 5z \end{aligned}$$

so

$$Hf(0,0,0) = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 0 & -15 \\ 10 & -15 & 0 \end{bmatrix}.$$

*For Exercises 21–25 you'll need formula (10):* $p_2(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})\mathbf{h} + (1/2)\mathbf{h}^T Hf(\mathbf{a})\mathbf{h}$ *where* $\mathbf{h} = \mathbf{x} - \mathbf{a}$.

**21.** Use the work from Exercises 8 and 16:

$$\begin{aligned} p_2(\mathbf{x}) &= f(0,0) + Df(0,0)\mathbf{x} + \frac{1}{2}\mathbf{x}^T \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x} \\ &= 1 + \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

**22.** Use the work from Exercise 11:

$$p_2(x,y) = f(0,\pi) + Df(0,\pi)\begin{bmatrix} x \\ y-\pi \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x & y-\pi \end{bmatrix} Hf(0,\pi)\begin{bmatrix} x \\ y-\pi \end{bmatrix}$$
$$= -1 + \begin{bmatrix} -2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-\pi \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x & y-\pi \end{bmatrix}\begin{bmatrix} -4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} x \\ y-\pi \end{bmatrix}.$$

**23.** Use the work from Exercise 12:

$$p_2(x,y,z) = f(0,0,2) + Df(0,0,2)\begin{bmatrix} x \\ y \\ z-2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x & y & z-2 \end{bmatrix} Hf(0,0,2)\begin{bmatrix} x \\ y \\ z-2 \end{bmatrix}$$
$$= 2 + \begin{bmatrix} 0 & 5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z-2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x & y & z-2 \end{bmatrix}\begin{bmatrix} 0 & 3 & 0 \\ 3 & 8 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z-2 \end{bmatrix}.$$

**24.** Use the work from Exercise 19:

$$p_2(\mathbf{x}) = f(1,0,1) + Df(1,0,1)(\mathbf{x}-(1,0,1)) + \frac{1}{2}(\mathbf{x}-(1,0,1))^T\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}(\mathbf{x}-(1,0,1))$$
$$= 3 + \begin{bmatrix} 3 & 0 & 6 \end{bmatrix}\begin{bmatrix} x-1 \\ y \\ z-1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x-1 & y & z-1 \end{bmatrix}\begin{bmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 12 \end{bmatrix}\begin{bmatrix} x-1 \\ y \\ z-1 \end{bmatrix}.$$

*Exercises 25 and 26 are related and could be assigned together. To make it a cohesive single problem, you may want to tell the students to use the function from Exercise 26 in place of the function given in Exercise 25.*

**25.** The function is $f(x_1, x_2, \ldots, x_n) = e^{x_1+2x_2+\cdots+nx_n}$.

**(a)** $Df(x_1, x_2, \ldots, x_n) = e^{x_1+2x_2+\cdots+nx_n}\begin{bmatrix} 1 & 2 & \cdots & n \end{bmatrix}$, and therefore $Df(0,0,\ldots,0) = \begin{bmatrix} 1 & 2 & \cdots & n \end{bmatrix}$. Taking second derivatives and evaluating at the origin results in:

$$Hf(0,0,\ldots,0) = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 4 & 6 & \cdots & 2n \\ 3 & 6 & 9 & \cdots & 3n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 2n & 3n & \cdots & n^2 \end{bmatrix}.$$

**(c)** Since (c) follows immediately from (a) we will skip (b) for a moment.

$$p_2(\mathbf{x}) = f(0,0,\ldots,0) + Df(0,0,\ldots,0)\mathbf{x} + \frac{1}{2}\mathbf{x}^T Hf(0,0,\ldots,0)\mathbf{x}$$
$$= 1 + \begin{bmatrix} 1 & 2 & \cdots & n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 4 & 6 & \cdots & 2n \\ 3 & 6 & 9 & \cdots & 3n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 2n & 3n & \cdots & n^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

**(b)** Now we can read the answer to (b) right off of our answer to (c).

$$p_1(\mathbf{x}) = 1 + x_1 + 2x_2 + \cdots + nx_n \quad \text{and}$$
$$p_2(\mathbf{x}) = 1 + x_1 + 2x_2 + \cdots + nx_n + \frac{1}{2}\sum_{i,j=1}^{n} ijx_ix_j.$$

**26.** This is an extension of a special case of Exercise 25. Note that $f_{x_i x_j x_k}(0,0,0) = ijk$ so

$$\begin{aligned} p_3(\mathbf{x}) &= 1 + x + 2y + 3z + \frac{1}{2}(x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz) \\ &\quad + \frac{1}{6}(x^3 + 8y^3 + 27z^3 + 6x^2y + 9x^2z + 12xy^2 + 36y^2z + 27xz^2 + 54yz^2 + 36xyz). \end{aligned}$$

**27.** $Df(x,y,z) = \begin{bmatrix} 4x^3 + 3x^2y - z^2 + 2xy + 3y & x^3 + 6y^2 + x^2 + 3x & -2xz - 1 \end{bmatrix}$ and

$$Hf(x,y,z) = \begin{bmatrix} 12x^2 + 6xy + 2y & 3x^2 + 2x + 3 & -2z \\ 3x^2 + 2x + 3 & 12y & 0 \\ -2z & 0 & -2x \end{bmatrix}.$$

The only non-zero third derivatives are

$$f_{xxx}(x,y,z) = 24x + 6y \qquad f_{xxy}(x,y,z) = 6x + 2$$
$$f_{xzz}(x,y,z) = -2 \qquad f_{yyy}(x,y,z) = 12$$

and their permutations.

**(a)** Here $\mathbf{a} = (0,0,0)$ so $f(0,0,0) = 2$, $Df(0,0,0) = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$, and $Hf(0,0,0) = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$$\begin{aligned} p_3(\mathbf{x}) &= 2 - z + 3xy + \frac{1}{6}(6x^2y - 6xz^2 + 12y^3) \\ &= 2 - z + 3xy + x^2y - xz^2 + 2y^3. \end{aligned}$$

**(b)** Here $f(1,-1,0) = -4$, $Df(1,-1,0) = \begin{bmatrix} -4 & 11 & -1 \end{bmatrix}$, and $Hf(1,-1,0) = \begin{bmatrix} 4 & 8 & 0 \\ 8 & -12 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

$$\begin{aligned} p_3(\mathbf{x}) &= -4 - 4(x-1) + 11(y+1) - z \\ &\quad + \frac{1}{2}[4(x-1)^2 + 16(x-1)(y+1) - 12(y+1)^2 - 2z^2] \\ &\quad + \frac{1}{6}[18(x-1)^3 + 3(8)(x-1)^2(y+1) - 3(2)(x-1)z^2 + 12(y+1)^3] \\ &= -4 - 4(x-1) + 11(y+1) - z + 2(x-1)^2 + 8(x-1)(y+1) - 6(y+1)^2 - z^2 \\ &\quad + 3(x-1)^3 + 4(x-1)^2(y+1) - (x-1)z^2 + 2(y+1)^3. \end{aligned}$$

*Exercises 28 and 32 are used in Exercise 33 (a) and (b). From Definition 1.4, the total differential of $f$ is* $df(\mathbf{a},\mathbf{h}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{a})\, dx_i$.

**28.** $f(x,y) = x^2y^3$ so $df(x,y,\mathbf{h}) = 2xy^3\,dx + 3x^2y^2\,dy$.

**29.** $f(x,y,z) = x^2 + 3y^2 - 2z^3$ so $df(x,y,z,\mathbf{h}) = 2x\,dx + 6y\,dy - 6z^2\,dz$.

**30.** $f(x,y,z) = \cos(xyz)$ so $df(x,y,z,\mathbf{h}) = -yz\sin(xyz)\,dx - xz\sin(xyz)\,dy - xy\sin(xyz)\,dz$.

**31.** $f(x,y,z) = e^x\cos y + e^y \sin z$ so $df(x,y,z,\mathbf{h}) = e^x\cos y\,dx + (-e^x\sin y + e^y\sin z)\,dy + e^y\cos z\,dz$.

**32.** $f(x,y,z) = 1/\sqrt{xyz}$ so $df(x,y,z,\mathbf{h}) = -\frac{1}{2}(xyz)^{-3/2}(yz\,dx + xz\,dy + xy\,dz)$.

**33. (a)** Use the function from Exercise 28: $f(x,y) = x^2y^3$ with $x = 7$, $y = 2$, $dx = .07$, and $dy = -.02$. So

$$\begin{aligned} (7.07)^2(1.98)^3 &\approx 7^2 2^3 + df((7,2),(.07,-.02)) = 2(7)(2^3)(.07) + 3(7^2)(2^2)(-.02) \\ &= -3.92. \end{aligned}$$

**(b)** Use the function from Exercise 32: $f(x, y, z) = 1/\sqrt{xyz}$ with $x = 4$, $y = 2$, $z = 2$, $dx = .1$, $dy = -.04$, and $dz = .05$. So

$$\begin{aligned}\frac{1}{\sqrt{(4.1)(1.96)(2.05)}} &\approx \frac{1}{\sqrt{(4)(2)(2)}} - \frac{1}{2}(16)^{-3/2}(4(.1) + 8(-.04) + 8(.05)) \\ &= \frac{1}{4} - \frac{1}{128}(.48) = .24625.\end{aligned}$$

**(c)** Here the function is $f(x, y, z) = x\cos(yz)$ with $x = 1$, $y = \pi$, $z = 0$, $dx = .1$, $dy = -.03$, and $dz = .12$. So

$$(1.1)\cos((\pi - 0.03)(0.12)) \approx 1 + (\cos 0)(.1) - (\pi \sin 0)(.12) = 1.1.$$

**34.** $dg(x, y, z, \mathbf{h}) = (3x^2 - 2y + 2xz)\,dx + (-2x)\,dy + (x^2 + 7)\,dz$, so $dg(1, -2, 1, \mathbf{h}) = 9\,dx - 2\,dy + 8\,dz$. This means that changes in $x$ have the most effect.

**35.** Although students will probably solve this more formally, they should see that, intuitively, changes in the upper left entry are multiplied by the largest number so that is the entry for which the value of the determinant is most sensitive.

**36.** $r = 2$, $dr = .1$, $h = 3$, and $dh = .05$.

**(a)** $V = \pi r^2 h$, so $dV = 2\pi rh\,dr + \pi r^2\,dh = 2\pi(2)(3)(.1) + \pi(2^2)(.05) = 1.4\pi$.

**(b)** $S = 2\pi rh + 2\pi r^2$, so $dS = (2\pi h + 4\pi r)\,dr + 2\pi r\,dh = (2\pi(3) + 4\pi(2))(.1) + 2\pi(2)(.05) = 1.6\pi$.

**37.** Let $x$ denote the diameter of the can, $y$ the height. Then the volume $V$ is given by

$$V = \pi\left(\frac{x}{2}\right)^2 y = \frac{\pi}{4}x^2 y.$$

The change in volume, $\Delta V$, that occurs when $x$ and $y$ are changed by small amounts $dx$ and $dy$ is given approximately by the differential:

$$\Delta V \approx dV = \frac{\pi}{2}xy\,dx + \frac{\pi}{4}x^2\,dy.$$

When $x = 5$ and $y = 12$ this becomes

$$dV = \pi\left(30\,dx + \frac{25}{4}\,dy\right).$$

If $x$ is decreased by 0.5 cm, so that $dx = -0.5$, then

$$dV = \pi\left(-15 + \frac{25}{4}\,dy\right).$$

For $dV$ to be zero (which represents approximately no change in volume), we see that

$$dy = \frac{60}{25} = 2.4 \text{ cm}.$$

**38.** **(a)** The area $A$ is given by

$$A = \frac{1}{2}ab\sin\theta, \quad \text{so} \quad dA = \frac{1}{2}b\sin\theta\,da + \frac{1}{2}a\sin\theta\,db + \frac{1}{2}ab\cos\theta\,d\theta.$$

With $a = 3$, $b = 4$, and $\theta = \pi/3$, this becomes

$$dA = \sqrt{3}\,da + \frac{3\sqrt{3}}{4}\,db + 3\,d\theta.$$

Thus, at these values, the area is most sensitive to changes in the angle $\theta$.

**(b)** We use the differential appearing in part (a):

$$\Delta A \approx dA = \sqrt{3}\,da + \frac{3\sqrt{3}}{4}\,db + 3\,d\theta.$$

If the measurement of $a$ is in error by at most 5%, then

$$|da| \le 0.05(3) = 0.15.$$

Similarly,

$$|db| \le 0.05(4) = 0.2 \quad \text{and} \quad |d\theta| \le 0.02\left(\frac{\pi}{3}\right) = 0.00\overline{6}\pi.$$

Hence the maximum error that results in the calculated value of the area is

$$|dA| \le \sqrt{3}(0.15) + \frac{3\sqrt{3}}{4}(0.2) + 0.02\pi \approx 0.58245 \text{ cm}^2.$$

The percentage error that this represents is calculated as

$$\frac{|dA|}{A} \le \frac{0.58245}{3\sqrt{3}} \approx 0.112,$$

or 11.2%.

**39.** We are told that $dr = dh$ and know that $V = (1/3)\pi r^2 h$. So $dV = (2/3)\pi rh\, dr + (1/3)\pi r^2 dh = (28\pi/3)\, dr$. Now we want $|dV|$ to be at most .2 so $|dV| = (28\pi/3)|dr| \le .2$ or $|dr| \le .3/(14\pi) \approx .0068209$.

**40.** $V = xyz$ where $x = 3$, $y = 4$, $z = 2$ and we assume that $dx = dy = dz$. So $dV = (4)(2)\,dx + (3)(2)\,dy + (3)(4)\,dz = 26dx$. We want $|dV| \le .2$ so $|dx| \le .2/26 \approx .00769$. This is a percentage error of $.2/24 = .8333\%$.

**41. (a)** We do the preliminary calculations:

$$\begin{aligned}
f(x,y) &= \cos x \sin y & f(0,\pi/2) &= 1\\
f_x(x,y) &= -\sin x \sin y & f_x(0,\pi/2) &= 0\\
f_y(x,y) &= \cos x \cos y & f_y(0,\pi/2) &= 0\\
f_{xx}(x,y) &= -\cos x \sin y & f_{xx}(0,\pi/2) &= -1\\
f_{yy}(x,y) &= -\cos x \sin y & f_{yy}(0,\pi/2) &= -1\\
f_{xy}(x,y) &= -\sin x \cos y & f_{xy}(0,\pi/2) &= 0
\end{aligned}$$

So $p_2(\mathbf{x}) = 1 - x^2/2 - (y-\pi/2)^2/2$.

**(b)** We'll just follow the estimate in Example 12 in the text: "since all partial derivatives of $f$ will be the product of sines and cosines and hence no larger than 1 in magnitude" and $|h_1|$ and $|h_2|$ are each no more than .3,

$$|R_2(0,\pi/2,h_1,h_2)| \le \frac{1}{6}(|h_1|^3 + 3h_1^2|h_2| + 3|h_1|h_2^2 + |h_2|^3) \le \frac{1}{6}(8\cdot(0.3)^3) = .036.$$

**42. (a)** We do the preliminary calculations:

$$\begin{aligned}
f(x,y) &= e^{x+2y} & f(0,0) &= 1\\
f_x(x,y) &= e^{x+2y} & f_x(0,0) &= 1\\
f_y(x,y) &= 2e^{x+2y} & f_y(0,0) &= 2\\
f_{xx}(x,y) &= e^{x+2y} & f_{xx}(0,0) &= 1\\
f_{yy}(x,y) &= 4e^{x+2y} & f_{yy}(0,0) &= 4\\
f_{xy}(x,y) &= 2e^{x+2y} & f_{xy}(0,0) &= 2.
\end{aligned}$$

So $p_2(\mathbf{x}) = 1 + x + 2y + x^2/2 + 2xy + 2y^2$.

**(b)** This time each third derivative has a factor of $e^{x+2y}$ in it. Each derivative with respect to $y$ brings out an additional factor of two. Here $|h_1|$ and $|h_2|$ are no more than .1 and on our set $e^{x+2y} \le e^{.3} < 2$. So

$$|R_2(0,0,h_1,h_2)| \le (2)\frac{1}{6}(|h_1|^3 + 6h_1^2|h_2| + 12|h_1|h_2^2 + 8|h_2|^3) \le \frac{1}{3}(27\cdot(0.1)^3) = .009.$$

**43. (a)** The preliminary calculations for $f(x,y) = e^{2x}\cos y$ are

$$\begin{aligned}
f(x,y) &= e^{2x}\cos y & f(0,\pi/2) &= 0\\
f_x(x,y) &= 2e^{2x}\cos y & f_x(0,\pi/2) &= 0\\
f_y(x,y) &= -e^{2x}\sin y & f_y(0,\pi/2) &= -1\\
f_{xx}(x,y) &= 4e^{2x}\cos y & f_{xx}(0,\pi/2) &= 0\\
f_{xy}(x,y) &= -2e^{2x}\sin y & f_{xy}(0,\pi/2) &= -2 = f_{yx}(0,\pi/2)\\
f_{yy}(x,y) &= -e^{2x}\cos y & f_{yy}(0,\pi/2) &= 0.
\end{aligned}$$

Thus

$$p_2(x, y, z) = -\left(y - \frac{\pi}{2}\right) + \frac{1}{2}\left(-4x\left(y - \frac{\pi}{2}\right)\right) = \frac{\pi}{2} - y - 2x\left(y - \frac{\pi}{2}\right).$$

**(b)** The eight third-order partial derivatives are:

$$\begin{aligned}
f_{xxx}(x, y) &= 8e^{2x}\cos y \\
f_{xxy}(x, y) &= -4e^{2x}\sin y = f_{xyx}(x, y) = f_{yxx}(x, y) \\
f_{xyy}(x, y) &= -2e^{2x}\cos y = f_{yxy}(x, y) = f_{yyx}(x, y) \\
f_{yyy}(x, y) &= e^{2x}\sin y,
\end{aligned}$$

Lagrange's form of the remainder tells us that

$$\left|R_2\left(x, y, 0, \frac{\pi}{2}\right)\right| = \frac{1}{3!}\left|\sum_{i,j,k=1}^{2} f_{x_i x_j x_k}(\mathbf{z}) h_i h_j h_k\right|,$$

where $\mathbf{z}$ is a point on the line segment joining $(0, \pi/2)$ and $(x, y)$. Note that the exponential function $e^{2x}$ increases with $x$ and the sine and cosine have maximum values of 1. Thus

$$|f_{xxx}(x, y)| \le 8e^{0.4},$$

and similar results apply to the other third-order partials. Hence

$$\begin{aligned}
\left|R_2\left(x, y, 0, \frac{\pi}{2}\right)\right| &\le \frac{1}{6}\left(8e^{0.4}|h_1|^3 + 3\cdot 4e^{0.4}|h_1|^2|h_2| + 3\cdot 2e^{0.4}|h_1|\,|h_2|^2 + e^{0.4}|h_2|^3\right) \\
&= \frac{e^{0.4}}{6}\left(8|h_1|^3 + 12|h_1|^2|h_2| + 6|h_1|\,|h_2|^2 + |h_2|^3\right).
\end{aligned}$$

If $|h_1| \le 0.2$ and $|h_2| \le 0.1$, then

$$\left|R_2\left(x, y, 0, \frac{\pi}{2}\right)\right| \le \frac{e^{0.4}}{6}\left(8(0.008) + 12(0.004) + 6(0.002) + 0.001\right) \approx 0.03108.$$

## 4.2 Extrema of Functions

**1.** $f(x, y) = 4x + 6y - 12 - x^2 - y^2$ so $f_x(x, y) = 4 - 2x$, $f_y(x, y) = 6 - 2y$, $f_{xx}(x, y) = -2$, $f_{xy}(x, y) = 0$, and $f_{yy}(x, y) = -2$.

**(a)** To find the critical point we will set each of the first partial derivatives equal to 0 and solve: $f_x(x, y) = 0$ when $4 - 2x = 0$ or when $x = 2$ and $f_y(x, y) = 0$ when $6 - 2y = 0$ or when $y = 3$. So $f$ has a unique critical point at (2, 3).

**(b)** The increment

$$\begin{aligned}
\Delta f &= f(2 + \Delta x, 3 + \Delta y) - f(2, 3) \\
&= 4(2 + \Delta x) + 6(3 + \Delta y) - 12 - (2 + \Delta x)^2 - (3 + \Delta y)^2 \\
&\quad - (4(2) + 6(3) - 12 - 2^2 - 3^2) = -(\Delta x)^2 - (\Delta y)^2.
\end{aligned}$$

This tells us that little changes in $x$ and/or $y$ result in a decrease in the value of $f$. This means that $f$ must have a local maximum at (2, 3).

**(c)** The Hessian is $Hf(2, 3) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ so $d_1 = -2$ and $d_2 = 4$ so by the second derivative test, $f$ has a local maximum at (2, 3).

**2.** $g(x, y) = x^2 - 2y^2 + 2x + 3$ so $g_x(x, y) = 2x + 2$, $g_y(x, y) = -4y$, $g_{xx}(x, y) = 2$, $g_{xy}(x, y) = 0$, and $g_{yy}(x, y) = -4$.

**(a)** To find the critical point we will set each of the first partial derivatives equal to 0 and solve: $g_x(x, y) = 0$ when $2x + 2 = 0$ or when $x = -1$ and $g_y(x, y) = 0$ when $-4y = 0$ or when $y = 0$. So $g$ has a unique critical point at $(-1, 0)$.

**(b)** The increment

$$\begin{aligned}\Delta g &= g(-1+\Delta x, \Delta y) - g(-1,0)\\ &= (-1+\Delta x)^2 - 2(\Delta y)^2 + 2(-1+\Delta x) + 3 - ((-1)^2 + 2(-1) + 3)\\ &= (\Delta x)^2 - 2(\Delta y)^2.\end{aligned}$$

This tells us that any changes in $x$ result in an increase in the value of $g$ and little changes in $y$ result in a decrease in the value of $g$. This means that $f$ must have a saddle at $(-1, 0)$.

**(c)** The Hessian is $Hg(-1,0) = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$ so $d_1 = 2$ and $d_2 = -8$, so by the second derivative test, $g$ has a saddle at $(-1, 0)$.

*In Exercises 3–20, most of the mistakes will be algebra mistakes made in solving for the critical points. For Exercises 3–14, you are using the familiar rule for the second derivative test at a point* $\mathbf{a} = (a, b)$ *where* $f_x(\mathbf{a}) = 0 = f_y(\mathbf{a})$. *The determinant of the Hessian is often referred to as the discriminant:*

$$D(a,b) = |Hf(a,b)| = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

*The second derivative test (see Example 5) is then*

- *if* $D(a,b) > 0$ *and*

  *if* $f_{xx}(a,b) > 0$ *then* $f$ *has a local minimum at* $(a,b)$
  *if* $f_{xx}(a,b) < 0$ *then* $f$ *has a local maximum at* $(a,b)$

- *if* $D(a,b) < 0$ *then* $f$ *has a saddle at* $(a,b)$.
- *Otherwise the test tells us nothing.*

*In many calculus classes students never see the extension of this test to higher dimensions. In Exercises 15–20, the students will need to use the* $\mathbf{R}^3$ *version of the second derivative test.*

**3.** $f(x,y) = 2xy - 2x^2 - 5y^2 + 4y - 3$, so $f_x(x,y) = 2y - 4x$ and $f_y(x,y) = 2x - 10y + 4$. At a critical point $2y - 4x = 0$ so $y = 2x$. Also $4 = 10y - 2x = 10y - y = 9y$ so $y = 4/9$ and $x = 2/9$. So $f$ has a critical point at (2/9, 4/9).
We easily calculate the Hessian $Hf = \begin{bmatrix} -4 & 2 \\ 2 & -10 \end{bmatrix}$ so $d_1 = -4$ and $d_2 = 36$. So $f$ has a local maximum at (2/9, 4/9).

**4.** $f(x,y) = \ln(x^2+y^2+1)$, so $f_x(x,y) = \dfrac{2x}{x^2+y^2+1}$ and $f_y(x,y) = \dfrac{2y}{x^2+y^2+1}$. The only critical point of $f$ is at the origin.
The second derivatives are $f_{xx}(x,y) = \dfrac{-2x^2+2y^2+2}{(x^2+y^2+1)^2}$, $f_{yy}(x,y) = \dfrac{2x^2-2y^2+2}{(x^2+y^2+1)^2}$, and also $f_{xy}(x,y) = \dfrac{4xy}{(x^2+y^2+1)^2}$. At the origin, the Hessian $Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ so $d_1 = 2$ and $d_2 = 4$. So $f$ has a local minimum at (0, 0).

**5.** $f(x,y) = x^2 + y^3 - 6xy + 3x + 6y$, so $f_x(x,y) = 2x - 6y + 3$ and $f_y(x,y) = 3y^2 - 6x + 6$. At a critical point for $f$, $2x = 6y - 3$ and $0 = 3y^2 - 6x + 6$ so $0 = y^2 - 2x + 2$. Substituting, $0 = y^2 - 6y + 5 = (y-1)(y-5)$. We have critical points at (3/2, 1) and (27/2, 5).
The second derivatives are $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 6y$, and $f_{xy}(x,y) = -6$. $d_1 = 2$ and $d_2 = 12y - 36$. In other words, $d_1$ is always positive and $d_2$ is positive when $y = 5$ and negative when $y = 1$ so by the second derivative test $f$ has a saddle point at (3/2, 1) and $f$ has a local minimum at (27/2, 5).

**6.** $f(x,y) = y^4 - 2xy^2 + x^3 - x$, so $f_x(x,y) = -2y^2 + 3x^2 - 1$ and $f_y(x,y) = 4y^3 - 4xy = 4y(y^2 - x)$. At a critical point for $f$, $y = 0$ or $y^2 = x$. If $y = 0$ then $x = \pm 1/\sqrt{3}$. If $y^2 = x$ then $0 = 3x^2 - 2x - 1 = (3x+1)(x-1)$. This gives us that $x = 1$ or $x = -1/3$ but $x$ can't be negative. So there are four critical points for $f$: $(\pm 1/\sqrt{3}, 0)$, and $(1, \pm 1)$.
The second derivatives are $f_{xx}(x,y) = 6x$, $f_{yy}(x,y) = 12y^2 - 4x$, and $f_{xy}(x,y) = -4y$. $d_1 = 6x$ and $d_2 = 8(9xy^2 - 2y^2 - 3x^2)$. We'll calculate $d_i$ at each critical point to classify them:

| Critical Point | $d_1$ | $d_2$ | Classification |
|---|---|---|---|
| $(1/\sqrt{3}, 0)$ | $6/\sqrt{3}$ | $-8$ | saddle |
| $(-1/\sqrt{3}, 0)$ | $-6/\sqrt{3}$ | $-8$ | saddle |
| $(1, -1)$ | 6 | 32 | local minimum |
| $(1, 1)$ | 6 | 32 | local minimum |

**7.** $f(x,y) = xy + \dfrac{8}{x} + \dfrac{1}{y}$, so $f_x(x,y) = y - \dfrac{8}{x^2}$ and $f_y(x,y) = x - \dfrac{1}{y^2}$. At a critical point for $f$, $x = \dfrac{1}{y^2}$ and $y = \dfrac{8}{x^2} = 8y^4$ so $0 = y(8y^3 - 1)$ so either $y = 0$ or $y = 1/2$. Since $y = 0$ is not in the domain of $f$, the only critical point of $f$ is at (4, 1/2).
The second derivatives are $f_{xx}(x,y) = \dfrac{16}{x^3}$, $f_{yy}(x,y) = \dfrac{2}{y^3}$, and $f_{xy}(x,y) = 1$. $d_1 = \dfrac{16}{x^3}$ and $d_2 = \dfrac{32}{x^3y^3} - 1$. At our critical point both $d_1$ and $d_2$ are positive so (4, 1/2) is a local minimum.

**8.** $f(x,y) = e^x \sin y$ so $f_x(x,y) = e^x \sin y$ and $f_y(x,y) = e^x \cos y$. There are no values of $x$ and $y$ for which both first partials are 0 so there are no critical points.

**9.** $f(x,y) = e^{-y}(x^2 - y^2)$, so $f_x(x,y) = 2xe^{-y}$ and $f_y(x,y) = -e^{-y}(x^2 - y^2 + 2y)$. At a critical point for $f$, $x = 0$ and $0 = -y^2 + 2y = -y(y-2)$ so the critical points of $f$ are at (0, 0) and (0, 2).
The second derivatives are $f_{xx}(x,y) = 2e^{-y}$, $f_{yy}(x,y) = e^{-y}(x^2 - y^2 + 4y - 2)$, and $f_{xy}(x,y) = -2xe^{-y}$. $d_1 > 0$ and $d_2(0,y) = -2e^{-2y}(y^2 - 4y + 2)$. In other words, $d_1$ is always positive and $d_2$ is negative when $y = 0$ and positive when $y = 2$ so by the second derivative test $f$ has a saddle point at (0, 0) and $f$ has a local minimum at (0, 2).

**10.** $f(x,y) = x + y - x^2y - xy^2$, so $f_x(x,y) = 1 - 2xy - y^2$ and $f_y(x,y) = 1 - 2xy - x^2$. At a critical point for $f$, $x^2 = y^2$ so $x = \pm y$. If $x = y$, then $0 = 1 - 2xy - y^2 = 1 - 3x^2$ so $x = y = \pm 1/\sqrt{3}$. If $x = -y$, then $0 = 1 - 2xy - y^2 = 1 + y^2$ for which there are no real solutions. So the critical points for $f$ are $\pm(1/\sqrt{3}, 1/\sqrt{3})$.
The second derivatives are $f_{xx}(x,y) = -2y$, $f_{yy}(x,y) = -2x$, and $f_{xy}(x,y) = -2x - 2y$. $d_1 = -2y$ and $d_2 = -4x^2 - 4xy - 4y^2$. At the critical points $d_2$ is negative and $d_1$ is non-zero so $f$ has a saddle point at both $\pm(1/\sqrt{3}, 1/\sqrt{3})$.

**11.** $f(x,y) = x^2 - y^3 - x^2y + y$, so $f_x(x,y) = 2x - 2xy = 2x(1-y)$ and $f_y(x,y) = -3y^2 - x^2 + 1$. At a critical point for $f$, either $x = 0$ or $y = 1$. When $x = 0$, $y$ must be $\pm 1/\sqrt{3}$. No solution corresponds to $y = 1$, So the critical points for $f$ are $(0, \pm 1/\sqrt{3})$.
The second derivatives are $f_{xx}(x,y) = 2 - 2y$, $f_{yy}(x,y) = -6y$, and $f_{xy}(x,y) = -2x$. $d_1 = 2 - 2y$ and $d_2 = -12y + 12y^2 - 4x^2$. At $(0, -1/\sqrt{3})$, $d_1$ is positive and $d_2$ is positive so $f$ has a local minimum at $(0, -1/\sqrt{3})$. At $(0, 1/\sqrt{3})$, $d_1$ is positive and $d_2$ is negative so $f$ has a saddle point at $(0, 1/\sqrt{3})$.

**12.** $f(x,y) = e^{-x}(x^2 + 3y^2)$, so $f_x(x,y) = (2x - x^2 - 3y^2)e^{-x}$ and $f_y(x,y) = 6ye^{-x}$. From $f_y$ we see that at a critical point for $f$, we must have $y = 0$. Plugging back into $f_x$ we conclude that there are critical points at (0, 0) and at (2, 0).
The second derivatives are $f_{xx}(x,y) = (2 - 4x + x^2 + 3y^2)e^{-x}$, $f_{yy}(x,y) = -6e^{-x}$, and $f_{xy}(x,y) = -6ye^{-x}$. $d_1 = (2 - 4x + x^2 + 3y^2)e^{-x}$ and $d_2 = 6e^{-2x}(1 - 4x + x^2 - 3y^2)$. At (0, 0), $d_1$ and $d_2$ are positive so $f$ has a local minimum at (0, 0). At (2, 0), $d_1$ and $d_2$ are negative so $f$ has a saddle point at (2, 0).

**13.** $f(x,y) = 2x - 3y + \ln xy$, so $f_x(x,y) = 2 + 1/x$ and $f_y(x,y) = -3 + 1/y$. The critical point is $(-1/2, 1/3)$.
The second derivatives are $f_{xx}(x,y) = -1/x^2$, $f_{yy}(x,y) = -1/y^2$, and $f_{xy}(x,y) = 0$. $d_1 = -1/x^2$ and $d_2 = 1/x^2y^2$. At $(-1/2, 1/3)$, $d_1$ is negative and $d_2$ is positive so $f$ has a local max at $(-1/2, 1/3)$.

**14.** $f(x,y) = \cos x \sin y$, so $f_x(x,y) = -\sin x \sin y$ and $f_y(x,y) = \cos x \cos y$. The critical points are of the form $(n\pi, \pi/2 + m\pi)$ and $(\pi/2 + n\pi, m\pi)$ where $m$ and $n$ are integers.
The second derivatives are $f_{xx}(x,y) = -\cos x \sin y$, $f_{yy}(x,y) = -\cos x \sin y$, and $f_{xy}(x,y) = -\sin x \cos y$. $d_1 = -\cos x \sin y$ and $d_2 = \cos^2 x \sin^2 y - \sin^2 x \cos^2 y$. At points of the form $(n\pi, \pi/2 + m\pi)$, $d_1$ alternates between negative and positive values while $d_2$ is positive so $f$ has an alternating string of local maxs and mins at such points. At the point $(0, \pi/2)$, for example, $f$ has a local max. At points of the form $(\pi/2 + n\pi, m\pi)$, $d_1 = 0$ and $d_2$ is negative so such points are saddle points.

**15.** $f(x,y,z) = x^2 - xy + z^2 - 2xz + 6z$, so $f_x(x,y,z) = 2x - y - 2z$, $f_y(x,y,z) = -x$ and $f_z(x,y,z) = 2z - 2x + 6$. From the second equation, $x = 0$. From the third, then, $z = -3$ and from the first it follows that $y = 6$.
The second derivatives are $f_{xx}(x,y,z) = 2$, $f_{yy}(x,y,z) = 0$, $f_{zz}(x,y,z) = 2$, $f_{xy}(x,y,z) = -1$, $f_{xz}(x,y,z) = -2$ and $f_{yz}(x,y,z) = 0$. $d_1 = 2$, $d_2 = -1$ and $d_3 = -2$ so $f$ has a saddle point at $(0, 6, -3)$.

**16.** $f(x,y,z) = (x^2 + 2y^2 + 1)\cos z$, so $f_x(x,y,z) = 2x \cos z$, $f_y(x,y,z) = 4y \cos z$ and $f_z(x,y,z) = -(x^2 + 2y^2 + 1)\sin z$. From the third equation, $z = n\pi$. The other two equations imply that $x$ and $y$ both are 0. So the critical points are of the form $(0, 0, n\pi)$.
The second derivatives are $f_{xx}(x,y,z) = 2\cos z$, $f_{yy}(x,y,z) = 4\cos z$, $f_{zz}(x,y,z) = -(x^2+2y^2+1)\cos z$, $f_{xy}(x,y,z) = 0$, $f_{xz}(x,y,z) = -2x \sin z$ and $f_{yz}(x,y,z) = -4y \sin z$. $d_1 = 2\cos z$ and $d_2 = 8\cos^2 z$. It is easier to calculate $d_3$ at our critical point. In this case $d_3(0,0,n\pi) = \mp 8$ while $d_1(0,0,n\pi) = \pm 2$, $d_2 = 8$. So $f$ has saddle points at $(0, 0, n\pi)$.

**17.** $f(x,y,z) = x^2 + y^2 + 2z^2 + xz$ so $f_x(x,y,z) = 2x + z$, $f_y(x,y,z) = 2y$, and $f_z(x,y,z) = 4z + x$. It is easy to see that the only critical point is at the origin.
The Hessian is $Hf = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$ so $d_1 = 2$, $d_2 = 4$, and $d_3 = 14$. By the second derivative test, $f$ has a local minimum

at (0, 0, 0).

**18.** $f(x, y, z) = x^3 + xz^2 - 3x^2 + y^2 + 2z^2$ so $f_x(x, y, z) = 3x^2 + z^2 - 6x$, $f_y(x, y, z) = 2y$, and $f_z(x, y, z) = 2xz + 4z = 2z(x + 2)$. We see immediately that at a critical point of $f$, $y = 0$ and either $z = 0$ or $x = -2$. If $z = 0$ then $0 = 3x^2 - 6x = 3x(x - 2)$ so $x = 0$ or $x = 2$. If $x = -2$ then $z^2 = -24$ for which there are no real solutions. We conclude that $f$ has critical points at (0, 0, 0) and (2, 0, 0).

The Hessian is $Hf = \begin{bmatrix} 6x-6 & 0 & 2z \\ 0 & 2 & 0 \\ 2z & 0 & 2x+4 \end{bmatrix}$ so $Hf(x, 0, 0) = \begin{bmatrix} 6x-6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2x+4 \end{bmatrix}$. This makes it easier to calculate $d_1(x, 0, 0) = 6x - 6$, $d_2(x, 0, 0) = 2d_1(x, 0, 0)$, and $d_3(x, 0, 0) = (2x + 4)\,d_2$. At (0, 0, 0) all three $d_i$'s are negative and at (2, 0, 0) all three are positive. By the second derivative test, $f$ has a saddle point at (0, 0, 0) and a local minimum at (2, 0, 0).

**19.** $f(x, y, z) = xy + xz + 2yz + \dfrac{1}{x}$ so $f_x(x, y, z) = y + z - \dfrac{1}{x^2}$, $f_y(x, y, z) = x + 2z$, and $f_z(x, y, z) = x + 2y$. We see immediately that at a critical point of $f$, $y = z$ so both $2z = -x$ and $2z = \dfrac{1}{x^2}$ so $-x = \dfrac{1}{x^2}$ so $x = -1$. Therefore, $f$ has a critical point at $(-1, 1/2, 1/2)$.

The Hessian is $Hf = \begin{bmatrix} 2/x^3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ so $d_1(-1, 1/2, 1/2) = -2$, $d_2(-1, 1/2, 1/2) = -1$, and $d_3(-1, 1/2, 1/2) = 12$.

This is the case of the second derivative test where the conditions are valid but neither of the first two cases holds so $f$ has a saddle point at $(-1, 1/2, 1/2)$.

**20.** $f(x, y, z) = e^x(x^2 - y^2 - 2z^2)$ so $f_x(x, y, z) = e^x(x^2 + 2x - y^2 - 2z^2)$, $f_y(x, y, z) = -2ye^x$, and $f_z(x, y, z) = -4ze^x$. We see immediately that at a critical point of $f$, $y = z = 0$ and therefore $0 = x^2 + 2x = x(x + 2)$. The two critical points of $f$ are (0, 0, 0) and $(-2, 0, 0)$.

$$\text{The Hessian is } Hf = \begin{bmatrix} e^x(x^2 + 4x + 2 - y^2 - 2z^2) & -2ye^x & -4ze^x \\ -2ye^x & -2e^x & 0 \\ -4ze^x & 0 & -4e^x \end{bmatrix} \text{ so}$$

$$Hf(x, 0, 0) = \begin{bmatrix} e^x(x^2 + 4x + 2) & 0 & 0 \\ 0 & -2e^x & 0 \\ 0 & 0 & -4e^x \end{bmatrix}.$$

For (0, 0, 0), $d_1 > 0$, $d_2 < 0$, and $d_3 < 0$ so $f$ has a saddle at (0, 0, 0). For $(-2, 0, 0)$, $d_1 < 0$, $d_2 > 0$, and $d_3 < 0$ so $f$ has a local maximum at $(-2, 0, 0)$.

**21. (a)** $f(x, y) = \dfrac{2y^3 - 3y^2 - 36y + 2}{1 + 3x^2}$ so $f_x(x, y) = \dfrac{6x(2y^3 - 3y^2 - 36y + 2)}{(1 + 3x^2)}$ and $f_y(x, y) = \dfrac{6(y^2 - y - 6)}{1 + 3x^2}$ $= \dfrac{6(y - 3)(y + 2)}{1 + 3x^2}$. From $f_y$ we see that either $y = 3$ or $y = -2$. Neither of these values makes $f_x = 0$ so $x = 0$. The critical points for $f$ are $(0, -2)$ and $(0, 3)$.

**(b)**

$$Hf = \begin{bmatrix} \dfrac{6(3x-1)(3x+1)(2y^3 - 3y^2 - 36y + 2)}{(3x^2+1)^3} & -\dfrac{36x(y-3)(y+2)}{(3x^2+1)^2} \\ -\dfrac{36x(y-3)(y+2)}{(3x^2+1)^2} & \dfrac{6(2y-1)}{3x^2+1} \end{bmatrix} \text{ and}$$

$$Hf(0, y) = \begin{bmatrix} -6(2y^3 - 3y^2 - 36y + 2) & 0 \\ 0 & 6(2y - 1) \end{bmatrix}.$$

At $(0, -2)$ we find that $d_1 < 0$ and $d_2 > 0$ so $f$ has a local maximum at $(0, -2)$. At (0, 3) we find that $d_1 > 0$ and $d_2 > 0$ so $f$ has a local minimum at (0, 3).

**22. (a)** $f(x, y) = kx^2 - 2xy + ky^2$ so $f_x(x, y) = 2kx - 2y$ and $f_y(x, y) = -2x + 2ky$. We see that the origin is a critical point for any value of $k$. The Hessian is $\begin{bmatrix} 2k & -2 \\ -2 & 2k \end{bmatrix}$ so $d_1 = 2k$ and $d_2 = 4k^2 - 4$. For $f$ to have a non-degenerate local maximum or minimum $d_2 > 0$ so $k^2 - 1 > 0$ so either $k > 1$ or $k < -1$. If $k > 1$, then $d_1 > 0$ and the origin is a non-degenerate local minimum. If $k < -1$, then $d_1 < 0$ and the origin is a non-degenerate local maximum.

**(b)** $g(x, y, z) = kx^2 + kxz - 2yz - y^2 + kz^2/2$ so $g_x(x, y, z) = 2kx + kz$, $g_y(x, y, z) = -2z - 2y$, and $g_z(x, y, z) = kx - 2y + kz$. The Hessian is $\begin{bmatrix} 2k & 0 & k \\ 0 & -2 & -2 \\ k & -2 & k \end{bmatrix}$. First note that $d_1 = 2k$ and $d_2 = -4k$. These are of opposite signs so a non-degenerate local minimum is not possible. For a non-degenerate local maximum we need $d_1 < 0$ and $d_2 > 0$ so $k < 0$. We also need $d_3 = 2k(-k - 4) < 0$ so $k < -4$. So we have a non-degenerate local maximum when $k < -4$.

**23.** If you think of this problem geometrically it should be reasonably straightforward. The slices through the origin where only one variable is allowed to change are parabolas. They open up if the coefficient of the term containing that variable is positive and down if it is negative. This tells you that if all of the coefficients are positive then we have a local minimum, if all of the coefficients are negative then we have a local maximum, and if some are positive and some are negative then we have a saddle point.

**(a)** $f(x, y) = ax^2 + by^2$ so $f_x(x, y) = 2ax$ and $f_y(x, y) = 2by$. Since neither $a$ nor $b$ is 0, the critical point must be the origin. The Hessian is $Hf = \begin{bmatrix} 2a & 0 \\ 0 & 2b \end{bmatrix}$. The first condition is that $d_2 > 0$ so $4ab > 0$ so $a$ and $b$ are the same sign. Also, $d_1 = 2a$ so when $a$ and $b$ are negative the origin is a local maximum and when $a$ and $b$ are positive the origin is a local minimum.

**(b)** $f(x, y) = ax^2 + by^2 + cz^2$ so $f_x(x, y, z) = 2ax$, $f_y(x, y, z) = 2by$ and $f_z(x, y, z) = 2cz$. Since none of $a$, $b$ and $c$ is 0, the critical point must be the origin. The Hessian is $Hf = \begin{bmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{bmatrix}$. Again, in either case $d_2 > 0$ so $4ab > 0$ so $a$ and $b$ are the same sign. Also, $d_1 = 2a$ and $d_3 = 8abc$. In either case $d_1$ and $d_3$ must be the same sign. When $a$, $b$ and $c$ are negative the origin is a local maximum and when $a$, $b$ and $c$ are positive the origin is a local minimum.

**(c)** Really the analysis is no harder, it is just harder to write down. The function is now $f(x_1, x_2, \ldots, x_n) = a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2$. The first derivatives are $f_{x_i}(x_1, x_2, \ldots, x_n) = 2a_ix_i$. Because none of the $a_i$ is zero and all of the first derivatives are 0, we conclude that the only critical point is at the origin. The Hessian is an $n \times n$ matrix with zeros everywhere off of the main diagonal and the entry in position $(i, i)$ is $2a_i$. We easily calculate $d_i = 2^i a_1 a_2 \ldots a_i$. As above, $d_2$ must be positive so both $a_1$ and $a_2$ are of the same sign. We could continue to argue that $d_4 = 4a_3a_4d_2$ so $a_3$ and $a_4$ must be of the same sign. In fact, we can continue that reasoning to say for $k$ odd, $a_k$ and $a_{k+1}$ must be of the same sign. For $f$ to have a local maximum $d_1 < 0$ so $a_1$ and $a_2$ are both negative. Also, $d_k = 2a_kd_{k-1}$ and for $k$ odd $d_k < 0$ so we can move up through the entries and argue that all of the $a_i$'s must be negative. Similarly, for $f$ to have a local minimum all of the $a_i$'s must be positive.

*Note: In Exercises 24–27 we have used a computer algebra system. In fact, I've used Mathematica. In Exercise 24, I've included a list of the relevant commands. These were adapted for each of the exercises.*

**24.** We'll use the following sequence of commands:

- $f[x\_, y\_] = y^4 - 2xy^2 + x^3 - x$
- Solve $[\{D[f[x, y], x] == 0, D[f[x, y], y] == 0\}]$
- $H = \{\{\partial_{x,x} f[x, y], \partial_{x,y} f[x, y]\}, \{\partial_{y,x} f[x, y], \partial_{y,y} f[x, y]\}\}$
- MatrixForm $[H/.\{x \to 1, y \to -1\}]$ (since $(1, -1)$ is the critical point found in the second step)

This is how you define the function, solve $\nabla f = 0$, create the Hessian and display it at the critical points.
In this case we get the following solutions to the simultaneous equations: $(-1/3, \pm i/\sqrt{3})$, $(1, \pm 1)$, and $(\pm 1/\sqrt{3}, 0)$. Let's examine the real-valued solutions.

At $(1, 1)$ the Hessian is $\begin{bmatrix} 6 & -4 \\ -4 & 8 \end{bmatrix}$. This means that $d_1 > 0$ and $d_2 > 0$ so $(1, 1)$ is a local minimum.

At $(1, -1)$ the Hessian is $\begin{bmatrix} 6 & 4 \\ 4 & 8 \end{bmatrix}$. This means that $d_1 > 0$ and $d_2 > 0$ so $(1, -1)$ is a local minimum.

At $(-1/\sqrt{3}, 0)$ the Hessian is $\begin{bmatrix} -2\sqrt{3} & 0 \\ 0 & 4/\sqrt{3} \end{bmatrix}$. This means that $d_1 < 0$ and $d_2 < 0$ so $(-1/\sqrt{3}, 0)$ is a saddle point.

At $(1/\sqrt{3}, 0)$ the Hessian is $\begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & -4/\sqrt{3} \end{bmatrix}$. This means that $d_1 < 0$ and $d_2 < 0$ so $(1/\sqrt{3}, 0)$ is a saddle point.

**25.** The commands are the same as those outlined in Exercise 24. The critical points are $(0, 0)$, $(\pm\sqrt{3/2}, 0)$, and $\pm(1/\sqrt{2}, -1/\sqrt{2})$.

At $(0, 0)$ the Hessian is $\begin{bmatrix} 0 & -3 \\ -3 & -2 \end{bmatrix}$. This means that $d_1 = 0$ and $d_2 < 0$ so $(0, 0)$ is a saddle point.

At $(\pm\sqrt{3/2}, 0)$ the Hessian is $\begin{bmatrix} 0 & 6 \\ 6 & -2 \end{bmatrix}$. Again, $d_1 = 0$ and $d_2 < 0$ so both $(\sqrt{3/2}, 0)$ and $(-\sqrt{3/2}, 0)$ are saddle points.

At $\pm(1/\sqrt{2}, -1/\sqrt{2})$ the Hessian is $\begin{bmatrix} -6 & 0 \\ 0 & -2 \end{bmatrix}$. This means that $d_1 < 0$ and $d_2 > 0$ so both $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$ are local maxima.

**26.** We need to slightly alter the commands from the previous two exercises. The command to find the roots specified by the three first partials is now:

Solve $[\{D[f[x,y,z],x] == 0, D[f[x,y,z],y] == 0, D[f[x,y,z],z] == 0\}]$.

We also need to change the specification of the Hessian to:

$$H = \{\{\partial_{x,x} f[x,y,z], \partial_{x,y} f[x,y,z], \partial_{x,z} f[x,y,z]\},$$
$$\{\partial_{y,x} f[x,y,z], \partial_{y,y} f[x,y,z], \partial_{y,z} f[x,y,z]\},$$
$$\{\partial_{z,x} f[x,y,z], \partial_{z,y} f[x,y,z], \partial_{z,z} f[x,y,z]\}\}$$

Finally, it will be helpful to use the computer to calculate the determinant. For *Mathematica* you type Det[M] where $M$ is the matrix for which you wish to calculate the determinant.

The critical points are at $(1 - 2\sqrt{2}, -\sqrt{2(4-\sqrt{2})}, -\sqrt{4-\sqrt{2}})$, $(1 - 2\sqrt{2}, \sqrt{2(4-\sqrt{2})}, \sqrt{4-\sqrt{2}})$, $(1 + 2\sqrt{2}, -\sqrt{2(4+\sqrt{2})}, \sqrt{4+\sqrt{2}})$, $(1 + 2\sqrt{2}, \sqrt{2(4+\sqrt{2})}, -\sqrt{4+\sqrt{2}})$, and (0, 0, 0).

At (0, 0, 0) the Hessian is $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -4 \end{bmatrix}$. So $d_1 < 0$, $d_2 > 0$ and $d_3 < 0$ so (0, 0, 0) is a local max.

At $(1 - 2\sqrt{2}, -\sqrt{2(4-\sqrt{2})}, -\sqrt{4-\sqrt{2}})$ the Hessian is $\begin{bmatrix} -2 & \sqrt{4-\sqrt{2}} & \sqrt{2(4-\sqrt{2})} \\ \sqrt{4-\sqrt{2}} & -2 & 2\sqrt{2} \\ \sqrt{2(4-\sqrt{2})} & 2\sqrt{2} & -4 \end{bmatrix}$.

So, $d_1 = -2 < 0$ and $d_2 = \sqrt{2} > 0$ and $d_3 = 64 - 16\sqrt{2} > 0$ so $(1 - 2\sqrt{2}, -\sqrt{2(4-\sqrt{2})}, -\sqrt{4-\sqrt{2}})$ is a saddle point.

At $(1 - 2\sqrt{2}, \sqrt{2(4-\sqrt{2})}, \sqrt{4-\sqrt{2}})$ the Hessian is $\begin{bmatrix} -2 & -\sqrt{4-\sqrt{2}} & -\sqrt{2(4-\sqrt{2})} \\ -\sqrt{4-\sqrt{2}} & -2 & 2\sqrt{2} \\ -\sqrt{2(4-\sqrt{2})} & 2\sqrt{2} & -4 \end{bmatrix}$.

So, $d_1 = -2 < 0$ and $d_2 = \sqrt{2} > 0$ and $d_3 = 64 - 16\sqrt{2} > 0$ so $(1 - 2\sqrt{2}, \sqrt{2(4-\sqrt{2})}, \sqrt{4-\sqrt{2}})$ is a saddle point.

At $(1 + 2\sqrt{2}, -\sqrt{2(4+\sqrt{2})}, \sqrt{4+\sqrt{2}})$ the Hessian is $\begin{bmatrix} -2 & -\sqrt{4+\sqrt{2}} & \sqrt{2(4+\sqrt{2})} \\ -\sqrt{4+\sqrt{2}} & -2 & -2\sqrt{2} \\ \sqrt{2(4+\sqrt{2})} & -2\sqrt{2} & -4 \end{bmatrix}$.

So, $d_1 = -2 < 0$ and $d_2 = -\sqrt{2} < 0$ and $d_3 = 64 + 16\sqrt{2} > 0$ so $(1 + 2\sqrt{2}, -\sqrt{2(4+\sqrt{2})}, \sqrt{4+\sqrt{2}})$ is a saddle point.

At $(1 + 2\sqrt{2}, \sqrt{2(4+\sqrt{2})}, -\sqrt{4+\sqrt{2}})$ the Hessian is $\begin{bmatrix} -2 & \sqrt{4+\sqrt{2}} & -\sqrt{2(4+\sqrt{2})} \\ \sqrt{4+\sqrt{2}} & -2 & -2\sqrt{2} \\ -\sqrt{2(4+\sqrt{2})} & -2\sqrt{2} & -4 \end{bmatrix}$.

So, $d_1 = -2 < 0$ and $d_2 = -\sqrt{2} < 0$ and $d_3 = 64 + 16\sqrt{2} > 0$ so $(1 + 2\sqrt{2}, \sqrt{2(4+\sqrt{2})}, -\sqrt{4+\sqrt{2}})$ is a saddle point.

**27.** The commands are extended as they were in Exercise 26. The critical points are (0, 0, 0, 0), $(-\sqrt{2}, 2\sqrt{2}, 1, -\sqrt{2})$, $(\sqrt{2}, 2\sqrt{2}, -1, -\sqrt{2})$, $(-\sqrt{2}, -2\sqrt{2}, -1, \sqrt{2})$, and $(\sqrt{2}, -2\sqrt{2}, 1, \sqrt{2})$.

At (0, 0, 0, 0) the Hessian is $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$. So $d_1 = -2 < 0$, $d_2 = 0$, $d_3 = 0$, and $d_4 = -8 < 0$, so (0, 0, 0, 0) is a saddle point.

At $(-\sqrt{2}, 2\sqrt{2}, 1, -\sqrt{2})$ the Hessian is $\begin{bmatrix} -2 & -1 & -2\sqrt{2} & 0 \\ -1 & 0 & \sqrt{2} & 1 \\ -2\sqrt{2} & \sqrt{2} & -4 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$. So $d_1 = -2 < 0$, $d_2 = -1 < 0$, $d_3 = 16 > 0$, and $d_4 = 32 > 0$, so $(-\sqrt{2}, 2\sqrt{2}, 1, -\sqrt{2})$ is a saddle point.

At $(\sqrt{2}, 2\sqrt{2}, -1, -\sqrt{2})$ the Hessian is $\begin{bmatrix} -2 & 1 & -2\sqrt{2} & 0 \\ 1 & 0 & -\sqrt{2} & 1 \\ -2\sqrt{2} & -\sqrt{2} & -4 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$. So $d_1 = -2 < 0$, $d_2 = -1 < 0$, $d_3 = 16 > 0$, and $d_4 = 32 > 0$, so $(\sqrt{2}, 2\sqrt{2}, -1, -\sqrt{2})$ is a saddle point.

At $(-\sqrt{2}, -2\sqrt{2}, -1, \sqrt{2})$ the Hessian is $\begin{bmatrix} -2 & 1 & 2\sqrt{2} & 0 \\ 1 & 0 & \sqrt{2} & 1 \\ 2\sqrt{2} & \sqrt{2} & -4 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$. So $d_1 = -2 < 0$, $d_2 = -1 < 0$, $d_3 = 16 > 0$, and $d_4 = 32 > 0$, so $(-\sqrt{2}, -2\sqrt{2}, -1, \sqrt{2})$ is a saddle point.

At $(\sqrt{2}, -2\sqrt{2}, 1, \sqrt{2})$ the Hessian is $\begin{bmatrix} -2 & -1 & 2\sqrt{2} & 0 \\ -1 & 0 & -\sqrt{2} & 1 \\ 2\sqrt{2} & -\sqrt{2} & -4 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$. So $d_1 = -2 < 0$, $d_2 = -1 < 0$, $d_3 = 16 > 0$, and $d_4 = 32 > 0$, so $(\sqrt{2}, -2\sqrt{2}, 1, \sqrt{2})$ is a saddle point.

**28.** We want to maximize $V = xyz$ subject to the constraint $2xy + 2xz + 2yz = c$. Solve the second equation for $z = \dfrac{c - 2xy}{2x + 2y}$ and substitute to get

$$V(x, y) = \frac{cxy - 2x^2y^2}{2x + 2y}.$$

The derivatives are $V_x = -\dfrac{y^2(2x^2 + 4xy - c)}{2(x + y)^2}$ and $V_y = -\dfrac{x^2(2y^2 + 4xy - c)}{2(x + y)^2}$. Since neither $x$ nor $y$ could be zero (we wouldn't have a box), a critical point of $f$ occurs when both $2x^2+4xy-c = 0$ and $2y^2+4xy-c = 0$. Solving these together we find that $x^2 = y^2$ and since $x$ and $y$ are positive we conclude that $x = y$. Substituting back in, $0 = 2x^2 + 4xy - c = 2x^2 + 4x^2 - c = 6x^2 - c$ so $x = y = \sqrt{c/6}$. $z = \dfrac{c - 2xy}{2x + 2y} = \dfrac{c - (c/3)}{4\sqrt{c/6}} = \sqrt{c/6}$. So our only critical point is when the box is a cube. To conclude that this is a local maximum we see that $d_1 = -\dfrac{y^2(c + 2y^2)}{(x + y)^3} < 0$ and at our critical point $d_2 = -\dfrac{2x^2y^2(2x^2 + 8xy + 2y^2 - 3c)}{(x + y)^4} = -\dfrac{2x^4(12x^2 - 3c)}{(2x)^4} = -\dfrac{2c - 3c}{8} > 0$. So the largest rectangular box with fixed surface area is a cube.

**29.** We will actually minimize the square of the distance (i.e., the sum of the squares of the differences in each direction): $D(x, y) = x^2 + y^2 + (3x - 4y - 24)^2$ so $D_x(x, y) = 20x - 24y - 144$ and $D_y(x, y) = 34y - 24x + 192$. Set these equal to 0 and solve to get that the point on the plane closest to the origin is $(36/13, -48/13, -12/13)$.

**30.** Again we will minimize the square of the distance. For points $(x, y, z)$ on the surface we have $z^2 = 4 - xy$, so that the square of the distance $x^2 + y^2 + z^2 = x^2 + y^2 + 4 - xy$; thus we consider the function $D(x, y) = x^2 - xy + y^2 + 4$. We have $D_x(x, y) = 2x - y$ and $D_y(x, y) = 2y - x$. Set the partial derivatives equal to 0 and solve the system

$$\begin{cases} 2x - y = 0 \\ -x + 2y = 0 \end{cases}.$$

The only solution is $(0, 0)$. This solution corresponds to the points $(0, 0, 2)$ and $(0, 0, -2)$ on the surface $xy + z^2 = 4$. To see that these points really do give the minimum distance, we rewrite $D$ as

$$D(x, y) = x^2 - xy + y^2 + 4 = \left(x - \frac{y}{2}\right)^2 + \frac{3y^2}{4} + 4.$$

Thus we see that $D(x, y) \geq 4$ for all $(x, y)$ and $D = 4$ exactly when $x = y = 0$.

**31.** We solve

$$\begin{cases} R_x(x, y) = 8 - 2x + 2y = 0 \\ R_y(x, y) = 6 - 4y + 2x = 0 \end{cases}.$$

Adding the two equations gives $14 - 2y = 0$ which implies that $y = 7$. Using this in the first equation gives $22 - 2x = 0$ so that $x = 11$. Hence $(11, 7)$ is the unique critical point. A quick check with the Hessian

$$HR(11,7) = \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$$

reveals that $d_1 = -2$, $d_2 = 8 - 4 = 4$, so this critical point yields a maximum value of $R$. (Note: we may rewrite the revenue function as $R(x, y) = 8x + 6y - (x - y)^2 - y^2$. From this it is clear that this critical point must be a global maximum.) Thus you should manufacture 1100 units of model X and 700 units of model Y.

*Exercises 32–39 force us to check values on the border of our region.*

**32.** $f(x, y) = x^2 + xy + y^2 - 6y$ so $f_x(x, y) = 2x + y$ and $f_y(x, y) = x + 2y - 6$. At a critical point for $f$, $y = -2x$ so $6 = x + 2y = -3x$. Our only critical point is $(-2, 4)$. We need to check the value of $f$ at the critical point and along the boundary of the region $-3 \le x \le 3$, $0 \le y \le 5$.

- $f(-2, 4) = -12$,
- $f(-3, y) = 9 - 9y + y^2$ has a minimum of $-11.25$ at $y = 4.5$ and a maximum of 9 at $y = 0$,
- $f(3, y) = 9 - 3y + y^2$ has a minimum of 27/4 at $y = 3/2$ and a maximum of 19 at $y = 5$,
- $f(x, 0) = x^2$ which has a minimum of 0 at $x = 0$ and a maximum of 9 at $x = \pm 3$,
- $f(x, 5) = x^2 + 5x - 5$ has a minimum of $-11.25$ at $x = -5/2$ and a maximum of 19 at $x = 3$.

The absolute maximum is, therefore, 19 at (3, 5) and the absolute minimum is $-12$ at $(-2, 4)$.

**33.** $f(x, y, z) = x^2 + xz - y^2 + 2z^2 + xy + 5x$ so $f_x(x, y, z) = 2x + y + z + 5$, $f_y(x, y, z) = x - 2y$, and $f_z(x, y, z) = x + 4z$. At a critical point for $f$, $x = 2y = -4z$ so $-5 = 2x + y + z = -8z - 2z + z = -9z$. Our only critical point is $(-20/9, -10/9, 5/9)$ which is not within our region. We need to check the value of $f$ along the boundary of the region $-5 \le x \le 0, 0 \le y \le 3, 0 \le z \le 2$. This consists of six two-dimensional faces, twelve one-dimensional edges and eight vertices.

- $f(x, 0, 0) = x^2 + 5x$ has a minimum of $-6.25$ at $x = -5/2$ and a maximum of 0 at $x = -5$ or 0,
- $f(x, 0, 2) = x^2 + 7x + 8$ has a minimum of $-4.25$ at $x = -7/2$ and a maximum of 8 at $x = 0$,
- $f(x, 3, 0) = x^2 + 8x - 9$ has a minimum of 25 at $x = -4$ and a maximum of $-9$ at $x = 0$,
- $f(x, 3, 2) = x^2 + 10x - 1$ has a minimum of $-26$ at $x = -5$ and a maximum of $-1$ at $x = 0$,
- $f(-5, y, 0) = -y^2 - 5y$ has a minimum of $-24$ at $y = 3$ and a maximum of 0 at $y = 0$,
- $f(0, y, 0) = -y^2$ has a minimum of $-9$ at $y = 3$ and a maximum of 0 at $y = 0$,
- $f(-5, y, 2) = -y^2 - 5y - 2$ has a minimum of $-26$ at $y = 3$ and a maximum of $-2$ at $y = 0$,
- $f(0, y, 2) = 8 - y^2$ has a minimum of $-1$ at $y = 3$ and a maximum of 8 at $y = 0$,
- $f(-5, 0, z) = 2z^2 - 5z$ has a minimum of $-25/8$ at $z = 5/4$ and a maximum of 0 at $z = 0$,
- $f(0, 0, z) = 2z^2$ has a minimum of 0 at $z = 0$ and a maximum of 8 at $z = 2$,
- $f(-5, 3, z) = 2z^2 - 5z - 24$ has a minimum of $-217/8$ at $z = 5/4$ and a maximum of $-24$ at $z = 0$,
- $f(0, 3, z) = 2z^2 - 9$ has a minimum of $-9$ at $z = 0$ and a maximum of $-1$ at $z = 2$.

You also must check for extrema on each face and at each vertex. When you do you find: The absolute maximum is 8 at (0, 0, 2) and the absolute minimum is $-191/7$ at $(-32/7, 3, 8/7)$.

**34.** In a fit of compassion, the author of the text has not forced Livinia the housefly to walk around the metal plate in search of the hottest and coldest points. The temperature is $T(x, y) = 2x^2 + y^2 - y - 3$ so $T_x(x, y) = 4x$ and $T_y(x, y) = 2y - 1$. We have a critical point for $T$ at (0, 1/2) and $T(0, 1/2) = 2.75$. To check the temperature of the boundary we note that it is a unit disk and so $x = \cos\theta$ and $y = \sin\theta$. We can rewrite $T(\theta) = 2\cos^2\theta + \sin^2\theta - \sin\theta + 3 = \cos^2\theta - \sin\theta + 4$. Then $T_\theta(\theta) = -2\cos\theta\sin\theta - \cos\theta = -\cos\theta(2\sin\theta + 1)$. We, therefore, have critical points on the boundary when $\cos\theta = 0$ (so $\theta = \pi/2$ or $3\pi/2$) and when $\sin\theta = -1/2$ (so $\theta = 7\pi/6$ or $11\pi/6$). Checking the values we see that $T(\pi/2) = 3$, $T(3\pi/2) = 5$ and $T(7\pi/6) = T(11\pi/6) = 21/4$. We conclude that the coldest spot on the plate is at (0, 1/2) where the temperature is 11/4 and the two hottest spots are at $(\pm\sqrt{3}/2, -1/2)$ where the temperature is 21/4.

**35.** Because the function is "separable", we can analyze it without calculus. The maximum value for $f$ is 1 and the minimum value for $f$ is $-1$. The absolute maximum is achieved at $(\pi/2, 0)$, $(\pi/2, 2\pi)$, and $(3\pi/2, \pi)$. The absolute minimum is achieved at $(3\pi/2, 0)$, $(3\pi/2, 2\pi)$, and $(\pi/2, \pi)$.

**36.**

$$\frac{\partial f}{\partial x} = -2\sin x$$
$$\frac{\partial f}{\partial y} = 3\cos y$$

So "ordinary" critical points on $\{(x, y)|0 \le x \le 4,\ 0 \le y \le 3\}$ are at $\left(0, \frac{\pi}{2}\right), \left(\pi, \frac{\pi}{2}\right)$. (In fact, $\left(\pi, \frac{\pi}{2}\right)$ is the only critical point that's actually in the interior of the rectangle.)
Now we look at the boundary of the rectangle:

$$\begin{aligned}
&f_1(x) = f(x, 0) = 2\cos x && f_1'(x) = -2\sin x \text{ so critical points at } (0, 0), (\pi, 0);\\
&f_2(x) = f(x, 3) = 2\cos x + 3\sin 3 && f_2'(x) = -2\sin x \text{ so critical points at } (0, 3), (\pi, 3);\\
&f_3(y) = f(0, y) = 2 + 3\sin y && f_3'(y) = 3\cos y \text{ so critical point at } \left(0, \tfrac{\pi}{2}\right);\\
&f_4(y) = f(4, y) = 2\cos 4 + 3\sin y && f_4'(y) = 3\cos y \text{ so critical point at } \left(4, \tfrac{\pi}{2}\right).
\end{aligned}$$

Now we compare values:

| $(x, y)$ | $f(x, y) = 2\cos x + 3\sin y$ |
|---|---|
| $\left(0, \frac{\pi}{2}\right)$ | 5 |
| $\left(\pi, \frac{\pi}{2}\right)$ | 1 |
| (0, 0) | 2 |
| $(\pi, 0)$ | $-2$ |
| (0, 3) | $2 + 3\sin 3 \approx 2.423$ |
| $(\pi, 3)$ | $-2 + 3\sin 3 \approx -1.577$ |
| $\left(4, \frac{\pi}{2}\right)$ | $2\cos 4 + 3 \approx 1.693$ |
| (4, 0) | $2\cos 4 \approx -1.307$ |
| (4, 3) | $2\cos 4 + 3\sin 3 \approx -0.884$ |

Thus the absolute minimum occurs at $(\pi, 0)$ and is $-2$. The absolute maximum occurs at $\left(0, \frac{\pi}{2}\right)$ and is 5.

**37.** $f(x, y) = 2x^2 - 2xy + y^2 - y + 3$, so $f_x(x, y) = 4x - 2y$ and $f_y(x, y) = -2x + 2y - 1$. At a critical point for $f$ we have $y = 2x$, so $-2x + 4x - 1 = 0$. Thus the only critical point is $(\frac{1}{2}, 1)$.

Now we need to consider the boundary of the region. It consists of three parts: (1) the horizontal line $y = 0$, where $0 \le x \le 2$; (2) the vertical line $x = 0$, where $0 \le y \le 2$; (3) the line $x + y = 2$ (or $y = 2 - x$), where $0 \le x \le 2$. Thus we compare

- $f(\frac{1}{2}, 1) = \frac{5}{2}$,
- $f(x, 0) = 2x^2 + 3$ has a minimum of 3 at $x = 0$ and a maximum of 11 at $x = 2$,
- $f(0, y) = y^2 - y + 3$ has a minimum of $\frac{11}{4}$ at $y = \frac{1}{2}$ and a maximum of 5 at $y = 2$,
- $f(x, 2 - x) = 5x^2 - 7x + 5$ has a minimum of $\frac{51}{20}$ at $x = \frac{7}{10}$ and a maximum of 11 at $x = 2$

Thus the absolute minimum is $\frac{5}{2}$ occurring at $(\frac{1}{2}, 1)$ and the absolute maximum is 11 occurring at $(2, 0)$.

**38.** $f(x, y) = x^2 y$ so $f_x(x, y) = 2xy$ and $f_y(x, y) = x^2$. Therefore the only ordinary critical point is $(0, 0)$. The boundary of $D$ may be parametrized by $x = 2\cos t$, $y = \sqrt{3}\sin t$ for $0 \le t < 2\pi$. Thus

$$F(t) = f(2\cos t, \sqrt{3}\sin t) = 4\sqrt{3}\cos^2 t \sin t$$

and

$$\begin{aligned}
F'(t) &= 4\sqrt{3}\left(-2\cos t \sin^2 t + \cos^3 t\right)\\
&= 4\sqrt{3}\cos t\left(-2(1 - \cos^2 t) + \cos^2 t\right) = 4\sqrt{3}\cos t(3\cos^2 t - 2).
\end{aligned}$$

We see that $F'(t) = 0$ when either $\cos t = 0$ (in which case $\sin t = \pm 1$) or $\cos t = \pm\sqrt{2/3}$ (in which case $\sin t = \pm 1/\sqrt{3}$). Thus, in addition to $(0, 0)$, we need to consider six more points: $(0, \pm\sqrt{3})$, $(\pm 2\sqrt{2/3}, \pm 1)$. From the following table

| $(x, y)$ | $f(x, y) = x^2y$ |
|---|---|
| $(0, 0)$ | $0$ |
| $(0, \pm\sqrt{3})$ | $0$ |
| $\left(\pm\frac{2\sqrt{2}}{\sqrt{3}}, 1\right)$ | $\frac{8}{3}$ |
| $\left(\pm\frac{2\sqrt{2}}{\sqrt{3}}, -1\right)$ | $-\frac{8}{3}$ |

we see that absolute minima occur at $(2\sqrt{2/3}, -1)$ and $(-2\sqrt{2/3}, -1)$ and absolute maxima at $(2\sqrt{2/3}, 1)$ and $(-2\sqrt{2/3}, 1)$.

**39.** The boundary of the closed ball is given by $x^2 + y^2 - 2y + z^2 + 4z = 0$. Completing the square, we find $x^2 + y^2 - 2y + 1 + z^2 + 4z + 4 = 5$ or $x^2 + (y-1)^2 + (z+2)^2 = 5$. (Note also that $x^2 + y^2 - 2y + z^2 + 4z = x^2 + (y-1)^2 + (z+2)^2 - 5$.)

The function $f(x, y, z) = e^{1-x^2-y^2+2y-z^2-4z}$ has

$$\begin{aligned} f_x(x, y, z) &= -2xe^{1-x^2-y^2+2y-z^2-4z} = 0 && \text{when } x = 0 \\ f_y(x, y, z) &= (-2y+2)e^{1-x^2-y^2+2y-z^2-4z} = 0 && \text{when } y = 1 \\ f_z(x, y, z) &= (-2z-4)e^{1-x^2-y^2+2y-z^2-4z} = 0 && \text{when } z = -2 \end{aligned}$$

So $(0, 1, -2)$ is an interior critical point (the only one). Note that on the boundary $x^2 + y^2 - 2y + z^2 + 4z = 0$, we have

$$f(x, y, z) = e^{1-0} = e$$

$$f(0, 1, -2) = e^{1-(-5)} = e^6 \leftarrow \text{so absolute max is at } (0, 1, -2).$$

The absolute minimum of $e$ occurs at *all* points of the boundary. If we set $w = x^2+y^2-2y+z^2+4z$, then $f(x, y, z) = e^{1-w}$, so that it's clear that the minimum must occur when $w = 0$ (since $w \le 0$ defines the domain we are to consider). Likewise, the maximum must occur at the center of the ball.

*It's good to take a step back and see that sometimes we can tell what type of critical point we have without using the tools we've developed. In single-variable calculus, when the second derivative test failed to tell us anything we returned either to the first derivative test or to an analysis of the function.*

*In Exercises 40–45, the exponents are all at least two so (see, for example, Section 2.4, Exercise 27) when the Hessian is evaluated at the origin, all of the entries will be 0. The fact that $Hf(\mathbf{0}) = \mathbf{0}$ means that the Hessian doesn't provide us with any information about the nature of the critical point at the origin. This is part (a) for Exercises 40–45. By a deleted neighborhood of the origin, we will mean a neighborhood of the origin with the origin removed.*

**40.** $f(x, y) = x^2y^2$: in every deleted neighborhood of the origin $f(x, y) > 0$ so $f(0, 0) < f(x, y)$ for every point $(x, y)$ near but not equal to (0, 0) so $f$ has a local minimum at the origin.

**41.** $f(x, y) = 4 - 3x^2y^2$: in every deleted neighborhood of the origin $x^2y^2 > 0$ so $-3x^2y^2 < 0$ so $f(x, y) < 4$ so $f(0, 0) > f(x, y)$ for every point $(x, y)$ near but not equal to (0, 0) so $f$ has a local maximum at the origin.

**42.** $f(x, y) = x^3y^3$: in every deleted neighborhood of the origin in quadrants I and III $f(x, y) > 0$ and in quadrants II and IV $f(x, y) < 0$ so $f$ has neither a minimum nor a maximum at the origin.

**43.** $f(x, y, z) = x^2y^3z^4$: in every deleted neighborhood of the origin where $y > 0$, $f(x, y, z) > 0$; when $y < 0$, $f(x, y, z) < 0$ so $f$ has neither a minimum nor a maximum at the origin.

**44.** $f(x, y, z) = x^2y^2z^4$: in every deleted neighborhood of the origin $f(x, y, z) > 0$ so $f(0, 0, 0) < f(x, y, z)$ for every point $(x, y, z)$ near but not equal to (0, 0, 0) so $f$ has a local minimum at the origin.

**45.** $f(x, y, z) = 2 - x^4y^4 - z^4$: in every deleted neighborhood of the origin $x^4y^4 + z^4 > 0$ so $f(x, y, z) < 2$ so $f(0, 0, 0) > f(x, y, z)$ for every point $(x, y, z)$ near but not equal to (0, 0, 0) so $f$ has a local maximum at the origin.

**46.** $f(x, y) = e^{x^2+5y^2}$. Notice that $e^u$ is a monotone increasing function of $u$ and $x^2 + 5y^2$ has a unique minimum at (0, 0). So $f$ has a local minimum at (0, 0) so $f(0, 0) = 1$ is a global minimum.

**47.** $f(x, y, z) = e^{2-x^2-2y^2-3x^4}$. Notice that $e^u$ is a monotone increasing function of $u$ and $2 - x^2 - 2y^2 - 3x^4$ has a unique maximum of 2 at (0, 0, 0). So $f$ has a local maximum at (0, 0, 0), so $f(0, 0, 0) = e^2$ is a global maximum.

**48.** $f(x, y) = x^3 + y^3 - 3xy + 7$.

**(a)** The first partial derivatives are $f_x(x, y) = 3x^2 - 3y$ and $f_y(x, y) = 3y^2 - 3x$ so we have critical points at (0, 0) and (1, 1). At the origin we have a saddle point. For the behavior at (1, 1), $d_1(1, 1) = 6$ and $d_2(1, 1) = 36 - 9 = 27$. By the second derivative test we have a local minimum.

**(b)** We know there are no global extrema. Look along the $x$-axis. The function is $f(x, 0) = x^3 + 7$. As $x \to \infty$ $f$ increases without bound and as $x \to -\infty$ $f$ decreases without bound.

**49.** There can't be a global maximum because, for example, for fixed $y$, as $x \to 0+$ the function grows without bound. $f_x(x, y) = y - 1/x$ and $f_y(x, y) = x + 2 - 2/y$ so $f$ has a critical point at (2, 1/2). From the Hessian $\begin{bmatrix} 1/4 & 1 \\ 1 & 8 \end{bmatrix}$ we see that there is a local minimum at (2, 1/2) of $2 + \ln 2$. Note that $f_x(2, y) = y - 1/2$.

We would like to now conclude that $f$ has a unique critical point at (2, 1/2) which is a local minimum and hence it is a global minimum—such a conclusion seems reasonable, but, as Exercise 52 will demonstrate, is not correct. Consider $f_x(x, 1/2) = 1/2 - 1/x$. For $x > 2$ this is positive and so $f$ is increasing along this line. Now look at $f_y(x, y) = x + 2 - 2/y$ for $x \geq 2$. When $y > 1/2$ this is positive and when $0 < y < 1/2$ this is negative. So as we move vertically away from the line $y = 1/2$ for $x \geq 2$ we see that $f$ is increasing. A similar analysis for the remaining regions shows that $f$ has a global minimum at (2, 1/2).

**50.** First we'll determine the local extrema. We have $f_x(x, y, z) = 3x^2 + 6x - 3z$, $f_y(x, y, z) = 2ye^{y^2+1}$, and $f_z(x, y, z) = 2z - 3x$. Thus the critical points are $(0, 0, 0)$ and $(-1/2, 0, -3/4)$. The Hessian is

$$Hf(x, y, z) = \begin{bmatrix} 6x + 6 & 0 & -3 \\ 0 & (2 + 4y^2)e^{y^2+1} & 0 \\ -3 & 0 & 2 \end{bmatrix}.$$

Thus

$$Hf(0, 0, 0) = \begin{bmatrix} 6 & 0 & -3 \\ 0 & 2e & 0 \\ -3 & 0 & 2 \end{bmatrix}$$

whose sequence of principal minors is $d_1 = 6$, $d_2 = 12e$, $d_3 = 6e$. Thus $(0, 0, 0)$ yields a local minimum. In addition,

$$Hf\left(-\tfrac{1}{2}, 0, -\tfrac{3}{4}\right) = \begin{bmatrix} 3 & 0 & -3 \\ 0 & 2e & 0 \\ -3 & 0 & 2 \end{bmatrix}$$

whose sequence of principal minors is $d_1 = 3$, $d_2 = 6e$, $d_3 = -6e$. Hence this critical point is a saddle point.

There are no global extrema. If we fix $y$ and $z$ both equal to zero, then $f(x, 0, 0) = x^3 + 3x^2 + e$. As $x \to +\infty$, the expression $x^3 + 3x^2 + e$ grows without bound and as $x \to -\infty$, it decreases without bound.

**51. (a)** We have

$$\frac{\partial f}{\partial x} = -\frac{2}{3}\left[(x-1)(y-2)\right]^{-1/3}(y-2) = -\frac{2(y-2)^{2/3}}{3(x-1)^{1/3}}$$

$$\frac{\partial f}{\partial y} = -\frac{2}{3}\left[(x-1)(y-2)\right]^{-1/3}(x-1) = -\frac{2(x-1)^{2/3}}{3(y-2)^{1/3}}.$$

Note that $\partial f/\partial x$ is undefined when $x = 1$ and zero when $y = 2$ (and $x \neq 1$). Similarly, $\partial f/\partial y$ is undefined when $y = 2$ and zero when $x = 1$ (and $y \neq 2$). Hence the set of critical points consists of all points on the lines $x = 1$ and $y = 2$. Note that these critical points are not isolated.

**(b)** The domain of $f$ is all of $\mathbf{R}^2$; the expression $[(x-1)(y-2)]^{2/3}$ is always nonnegative and is zero only when either $x = 1$ or $y = 2$. Thus $f(x, y) \leq 3$ for all $(x, y) \in \mathbf{R}^2$ and $f(x, y) = 3$ precisely when either $x = 1$ or $y = 2$. Hence there are (global) maxima of 3 along these lines.

**52. (a)** Say that $f$ has a local maximum at $x_0$ and no other critical points. Assume that $f(x_0)$ is not the global maximum. Then there exists a point $x_1$ such that $f(x_1) > f(x_0)$. By the extreme value theorem, on the closed interval with endpoints $x_0$ and $x_1$ there must be a global maximum and a global minimum somewhere on that closed interval. The global minimum could not be at $x_0$ since it is a local maximum. It could not be at $x_1$, since $f(x_1) > f(x_0)$. The global minimum must be somewhere on the open interval and it must be at a critical point. This contradicts the assumption that there were no other critical points. If instead the unique critical point of $f$ were a local minimum, then just modify the argument appropriately.

**(b)** $f(x, y) = 3ye^x - e^{3x} - y^3$ so $f_x(x, y) = 3ye^x - 3e^{3x}$ and $f_y(x, y) = 3e^x - 3y^2$. Solving, $y = 0$ or $y = 1$, but $y$ can't be 0 since $e^x = y^2$. The only critical point for $f$ is at (0, 1) and $f(0, 1) = 1$. Also, $d_1(0, 1) = f_{xx}(0, 1) = -6$ and $d_2(0, 1) = 27$ so at (0, 1) $f$ has a local maximum. Along the $y$-axis, $f(0, y) = 3y - 1 - y^3$, so as $y \to -\infty$ we see that $f$ increases without bound.

**53. (a)** Let the local maxima occur at $a < b$. Consider $f$ on [*a, b*]. By the extreme value theorem, $f$ must attain both a maximum and minimum somewhere on [*a, b*]. The minimum cannot occur at $a$ or $b$ since local maxima occur there. Hence there must be some $c$ is the *open* interval (*a, b*) that gives an absolute minimum on [*a, b*]—hence it must be at least a local minimum on **R**.

**(b)**

$$f_x(x, y) = -2(xy^2 - y - 1)y^2$$
$$f_y(x, y) = -2(xy^2 - y - 1)(2xy - 1) - 4(y^2 - 1)y$$

For $f_x = 0$, either $xy^2 - y - 1 = 0$ or $y^2 = 0$ (so $y = 0$). If $y = 0$, then the $f_y = 0$ equation becomes $-2(-1)(-1) = 0$, which is false. Thus $xy^2 - y - 1 = 0$ and the $f_y = 0$ equation becomes $-4y(y^2 - 1) = 0$. Since $y \neq 0$, we must have $y^2 - 1 = 0$ or $y = \pm 1$. With $y = 1$ in the $f_x = 0$ equation, we have $-2(x - 2) = 0 \Rightarrow x = 2$. With $y = -1$ in the $f_x = 0$ equation, we have $-2(x + 1 - 1) = 0$ so $x = 0$. So we have two critical points: (2, 1) and $(0, -1)$. The Hessian matrix is

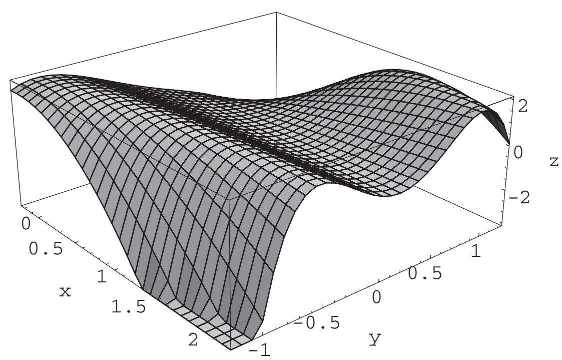
$$Hf(x, y) = \begin{bmatrix} -2y^4 & -2(4xy^3 - 3y^2 - 2y) \\ -2(4xy^3 - 3y^2 - 2y) & -2(6x^2y^2 - 6xy - 2x + 1) - 4(3y^2 - 1) \end{bmatrix}$$

so

$$Hf(2, 1) = \begin{bmatrix} -2 & -6 \\ -6 & -26 \end{bmatrix} \text{ sequence of minors is } -2, 16 \Rightarrow \text{local max;}$$

$$Hf(0, -1) = \begin{bmatrix} -2 & 2 \\ 2 & -10 \end{bmatrix} \text{ sequence of minors is } -2, 16 \Rightarrow \text{local max}.$$

**(c)** Best left to a computer. Stay close to the critical points to see the surface details well.




## 4.3 Lagrange Multipliers

**1.** The plane is given by $2x - 3y - z = 4$. There will be only one critical point in each case. Geometrically, it cannot be a local maximum because there will always be points nearby which are farther away. There is at least one point on the plane closest to the origin so the single critical point will be at this point. You can also perform the second derivative test.

**(a)** We'll minimize the square of the distance: $D(x, y) = x^2 + y^2 + (2x - 3y - 4)^2$. The partials are $D_x(x, y) = 10x - 12y - 16$ and $D_y(x, y) = 20y - 12x + 24$. Set these equal to zero and solve simultaneously to find the critical point $(4/7, -6/7, -2/7)$.

**(b)** Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = 2x - 3y - z = 4$. We solve the system

$$\begin{cases} 2x = 2\lambda \\ 2y = -3\lambda \\ 2z = -\lambda \\ 2x - 3y - z = 4. \end{cases}$$

We see that $x = \lambda$ so $y = -(3/2)x$ and $z = -x/2$. Substituting into the last equation: $2x + 9x/2 + x/2 = 4$ so $x = 4/7$ and our critical point is $(4/7, -6/7, -2/7)$.

**2.** The function is $f(x, y) = y$ subject to the constraint $g(x, y) = 2x^2 + y^2 = 4$. We solve the system

$$\begin{cases} 0 = 4\lambda x \\ 1 = 2\lambda y \\ 2x^2 + y^2 = 4. \end{cases}$$

From the first equation, $\lambda x = 0$, but $\lambda \neq 0$ since $2\lambda y \neq 0$. Hence we must have $x = 0$, so $y^2 = 4$; therefore the critical points are $(0, \pm 2)$.

**3.** The function is $f(x, y) = 5x + 2y$ subject to the constraint $g(x, y) = 5x^2 + 2y^2 = 14$. We solve the system

$$\begin{cases} 5 = 10\lambda x \\ 2 = 4\lambda y \\ 5x^2 + 2y^2 = 14. \end{cases}$$

By either of the first two equations we see that $\lambda \neq 0$. Together, the first two equations imply that $x = y$ so $7x^2 = 14$ so the critical points are $\pm(\sqrt{2}, \sqrt{2})$.

**4.** The function is $f(x, y) = xy$ subject to the constraint $g(x, y) = 2x - 3y = 6$. We solve the system

$$\begin{cases} y = 2\lambda \\ x = -3\lambda \\ 2x - 3y = 6. \end{cases}$$

If $\lambda$ were 0, then both $x$ and $y$ would be 0 which would contradict the third equation. In short, $\lambda \neq 0$. In that case, the first two equations imply that $x = -(3/2)y$ so $-3y - 3y = 6$ or $y = -1$. The critical point is at $(3/2, -1)$.

**5.** The function is $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = 2x + 3y + z = 6$. We solve the system

$$\begin{cases} yz = 2\lambda \\ xz = 3\lambda \\ xy = \lambda \\ 2x + 3y + z = 6. \end{cases}$$

One possibility is that two of $x$, $y$, and $z$ are zero. In this case the three possible critical points are (3, 0, 0), (0, 2, 0), and (0, 0, 6). If none of $x$, $y$, and $z$ is zero then the first two equations imply that $x = (3/2)y$, and the second and third equations together imply that $3y = z$. Hence, $3y + 3y + 3y = 6$, so the final critical point is (1, 2/3, 2).

**6.** The function is $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = x + y - z = 1$. We solve the system

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = -\lambda \\ x + y - z = 1. \end{cases}$$

We see immediately that $x = y = -z$, which implies that $x + x + x = 1$. Therefore, the critical point is $(1/3, 1/3, -1/3)$.

**7.** The function is $f(x, y, z) = 3 - x^2 - 2y^2 - z^2$ subject to the constraint $g(x, y, z) = 2x + y + z = 2$. We solve the system

$$\begin{cases} -2x = 2\lambda \\ -4y = \lambda \\ -2z = \lambda \\ 2x + y + z = 2. \end{cases}$$

Immediately we have $\lambda = -x = -4y = -2z \iff x = 4y = 2z$. Thus $x = 2z$ and $y = z/2$ so that the last equation of the system becomes $4z + z/2 + z = 2 \iff z = 4/11$. Therefore, there is a unique critical point of $\left(\frac{8}{11}, \frac{2}{11}, \frac{4}{11}\right)$.

**8.** The function is $f(x, y, z) = x^6 + y^6 + z^6$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 6$. We solve the system

$$\begin{cases} 6x^5 = 2\lambda x \\ 6y^5 = 2\lambda y \\ 6z^5 = 2\lambda z \\ x^2 + y^2 + z^2 = 6. \end{cases}$$

The first equation of the system implies either $x = 0$ or $\lambda = 3x^4$. Similarly, the second equation implies either $y = 0$ or $\lambda = 3y^4$ and the third equation implies either $z = 0$ or $\lambda = 3z^4$. No more than two of $x$, $y$, or $z$ can be zero, or else the constraint $x^2 + y^2 + z^2 = 6$ cannot be satisfied. Let us suppose that $y = z = 0$. Then $x = \pm\sqrt{6}$ from the constraint. Hence $(\pm\sqrt{6}, 0, 0)$ are two of the critical points. Similarly, if $x = z = 0$, then we obtain $(0, \pm\sqrt{6}, 0)$ as additional critical points, and if $x = y = 0$ we obtain $(0, 0, \pm\sqrt{6})$. If just $z = 0$, then $\lambda = 3x^4 = 3y^4$, so $x = \pm y$ and the constraint $x^2 + y^2 + z^2 = 6$ implies $2x^2 = 6$ or $x = \pm\sqrt{3}$ and there are thus four more critical points $(\pm\sqrt{3}, \pm\sqrt{3}, 0)$. In a similar manner $(\pm\sqrt{3}, 0, \pm\sqrt{3})$ and $(0, \pm\sqrt{3}, \pm\sqrt{3})$ are critical points. Finally, if none of $x$, $y$, or $z$ is zero, then $\lambda = 3x^4 = 3y^4 = 3z^4$, which implies $x = \pm y = \pm z$. Hence the last equation of the system implies that $3x^2 = 6$, so $x = \pm\sqrt{2}$. Therefore, there are eight more critical points, namely $(\pm\sqrt{2}, \pm\sqrt{2}, \pm\sqrt{2})$. Thus there are 26 critical points in all.

**9.** The function is $f(x, y, z) = 2x+y^2-z^2$ subject to the two constraints $g_1(x, y, z) = x-2y = 0$ and $g_2(x, y, z) = x+z = 0$. We solve the system

$$\begin{cases} 2 = \lambda + \mu \\ 2y = -2\lambda \\ -2z = \mu \\ x = 2y \\ x = -z. \end{cases}$$

Solving, we see that $2 = \lambda + \mu = -y - 2z = -x/2 + 2x = 3x/2$. So the critical point is $(4/3, 2/3, -4/3)$.

**10.** The function is $f(x, y, z) = 2x + y^2 + 2z$ subject to the two constraints $g_1(x, y, z) = x^2 - y^2 = 1$ and $g_2(x, y, z) = x + y + z = 2$. We solve the system

$$\begin{cases} 2 = 2\lambda x + \mu \\ 2y = -2\lambda y + \mu \\ 2 = \mu \\ x^2 - y^2 = 1 \\ x + y + z = 2. \end{cases}$$

The third equation of the system implies that the first equation becomes $2\lambda x = 0$. Thus either $\lambda = 0$ or $x = 0$. If $x = 0$, the fourth equation becomes $-y^2 = 1$, which has no solution. If $\lambda = 0$, then the second equation becomes $2y = 2 \iff y = 1$. Hence $x^2 = 2$ in the fourth equation. Using the last equation, we see that $(\sqrt{2}, 1, 1 - \sqrt{2})$ and $(-\sqrt{2}, 1, 1 + \sqrt{2})$ are the only critical points.

**11.** The function is $f(x, y, z) = xy + yz$ subject to the two constraints $g_1(x, y, z) = x^2 + y^2 = 1$ and $g_2(x, y, z) = yz = 1$. We solve the system

$$\begin{cases} y = 2\lambda x \\ x + z = 2\lambda y + \mu z \\ y = \mu y \\ x^2 + y^2 = 1 \\ yz = 1. \end{cases}$$

The third equation of the system implies that either $\mu = 1$ or $y = 0$. However, $y$ cannot be zero from the last equation. Thus $\mu = 1$ and the second equation reduces to $x = 2\lambda y$, and the first equation becomes $y = 4\lambda^2 y$. Thus either $y = 0$ (which we reject) or $\lambda = \pm 1/2$. This in turn implies that $x = \pm y$, and the fourth equation thus becomes $2x^2 = 1$, so that $x = \pm 1/\sqrt{2}$ and $y = \pm 1/\sqrt{2}$. Now $z = 1/y$ from the last equation, so there are four critical points:

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right).$$

**12.** The function is $f(x, y, z) = x+y+z$ subject to the two constraints $g_1(x, y, z) = y^2-x^2 = 1$ and $g_2(x, y, z) = x+2z = 1$. We solve the system

$$\begin{cases} 1 = -2\lambda x + \mu \\ 1 = 2\lambda y \\ 1 = 2\mu \\ y^2 - x^2 = 1 \\ x + 2z = 1. \end{cases}$$

Solving, we see that $\mu = 1/2$ and $2\lambda = 1/y$ so $1/2 = -2\lambda x = -x/y$ or $y = -2x$. This means that $1 = y^2 - x^2 = 3x^2$, so the critical points are $(-1/\sqrt{3}, 2/\sqrt{3}, (3 + \sqrt{3})/6)$ and $(1/\sqrt{3}, -2/\sqrt{3}, (3 - \sqrt{3})/6)$.

**13.** **(a)** The function is $f(x, y) = x^2 + y$ subject to the constraint $g(x, y) = x^2 + 2y^2 = 1$. We solve the system

$$\begin{cases} 2x = 2x\lambda \\ 1 = 4y\lambda \\ x^2 + 2y^2 = 1. \end{cases}$$

From the first equation, we see that either $x = 0$ or $\lambda = 1$. If $\lambda = 1$, then $y = 1/4$, so $x = \pm\sqrt{7/8}$. If $x = 0$, then $y = \pm\sqrt{1/2}$. In short, the critical points are $(\pm\sqrt{7/8}, 1/4)$ and $(0, \pm\sqrt{1/2})$.

**(b)** $L(\lambda; x, y) = x^2 + y - \lambda(x^2 + 2y^2 - 1)$ so

$$H(\lambda; x, y) = \begin{bmatrix} 0 & -2x & -4y \\ -2x & 2 - 2\lambda & 0 \\ -4y & 0 & -4\lambda \end{bmatrix}.$$

So $-d_3 = -16y[x^2 + 1/2 - 2y]$. Substitute the critical points to find that there are local maxima at $(\pm\sqrt{7/8}, 1/4)$ and local minima at $(0, \pm\sqrt{1/2})$.

**14. (a)** The function is $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$ subject to the three constraints $g_1(x, y, z, w) = 2x + y + z = 1$, $g_2(x, y, z, w) = x - 2z - w = -2$ and $g_3(x, y, z, w) = 3x + y + 2w = -1$. We solve the system

$$\begin{cases} 2x = 2\lambda + \mu + 3\nu \\ 2y = \lambda + \nu \\ 2z = \lambda - 2\mu \\ 2w = -\mu + 2\nu \\ 2x + y + z = 1 \\ x - 2z - w = -2 \\ 3x + y + 2w = -1. \end{cases}$$

After a great deal of fussing we find that there is a critical point at $\frac{1}{68}(-11, 15, 75, -25)$.

**(b)**

$$HL(\lambda, \mu, \nu, x, y, z, w) = \begin{bmatrix} 0 & 0 & 0 & -2 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & -3 & -1 & 0 & -2 \\ -2 & -1 & -3 & 2 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 2 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

We calculate $-d_7 = 628$ and conclude that $f$ has a local minimum at the critical point.

*Note: For Exercises 15–19 the Mathematica code would be similar to that in Exercise 15.*

**15.** Input the following three lines into *Mathematica* (or the equivalent into your favorite computer algebra system)

$f = 3xy - 4z$

$g = 3x + y - 2xz$

Solve $[\{D[f, x] == \lambda D[g, x], D[f, y] == \lambda D[g, y], D[f, z] == \lambda D[g, z], 3x + y - 2x\,z == 1\}]$
The solutions are

- $\lambda = \sqrt{6}, (x, y, z) = (\sqrt{2/3}, 1/2, (12 - \sqrt{6})/8)$ and
- $\lambda = -\sqrt{6}, (x, y, z) = (-\sqrt{2/3}, 1/2, (12 + \sqrt{6})/8)$.

**16.** Use the same basic code you used in Exercise 15, allowing for two Lagrange multipliers. The solution is $\lambda_1 = 482/121$, $\lambda_2 = -107/121$, $(x, y, z) = (31, 29, 5)/11$.

**17.** Many solutions are returned by *Mathematica*. They are

- $(0, -1, 0)$ for $\lambda = -3/2$
- $(0, 1, 0)$ for $\lambda = 3/2$
- $(-2/3, -2/3, -1/3)$ and $(2/3, -2/3, 1/3)$ for $\lambda = -4/3$
- $(-1, 0, 0)$ and $(1, 0, 0)$ for $\lambda = -1$
- $(0, 0, -1)$ and $(0, 0, 1)$ for $\lambda = 0$ and
- $(\sqrt{11/2}/8, -3/8, -3\sqrt{11/2}/8)$ and $(-\sqrt{11/2}/8, -3/8, 3\sqrt{11/2}/8)$ for $\lambda = 1/8$.

**18.** The solutions given are

- $(1, -1/2, \pm\sqrt{3/2})$ for $\lambda = -1$
- $((-1 - \sqrt{5})/2, (-3 - \sqrt{5})/4, \pm i5^{1/4}/\sqrt{2})$ for $\lambda = (1 + \sqrt{5})/2$.
- $((1 - \sqrt{5})/2, (-3 + \sqrt{5})/4, \pm 5^{1/4}/\sqrt{2})$ for $\lambda = (1 - \sqrt{5})/2$.
- $(-i, i, 0)$ and $(i, -i, 0)$ for $\lambda = -2$, and
- $(-1, -1, 0)$ and $(1, 1, 0)$ for $\lambda = 2$.

Note that several of the solutions are complex and, for the purposes of this discussion, can be discarded.

**19.** Here there are two solutions:

- $(w, x, y, z) = ((1 - \sqrt{2})/2, 1/\sqrt{2}, 1/\sqrt{2}, (1 - \sqrt{2})/2)$ for $\lambda_1 = 2 - 1/\sqrt{2}, \lambda_2 = 1 - \sqrt{2}$, and $\lambda_3 = 0$, and
- $(w, x, y, z) = ((1 + \sqrt{2})/2, -1/\sqrt{2}, -1/\sqrt{2}, (1 + \sqrt{2})/2)$ for $\lambda_1 = 2 + 1/\sqrt{2}, \lambda_2 = 1 + \sqrt{2}$, and $\lambda_3 = 0$.

**20.** **(a)** We need to solve

$$\begin{cases} 3x^2 = \lambda y \\ 6y = \lambda x \\ xy = -4 \end{cases}$$

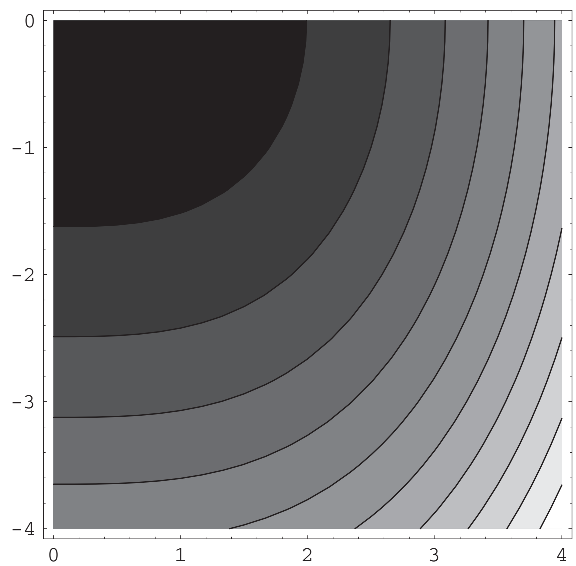
Substitute $y = -4/x$ into the second equation to get $\lambda = -24/x^2$. Substitute both of these into the right side of the first equation to get $x^5 = -32$ or $x = 2$. So $y = -2$ and $\lambda = -6$.

**(b)** The Hessian in this case is $\begin{bmatrix} 0 & 2 & -2 \\ 2 & 12 & 6 \\ -2 & 6 & 6 \end{bmatrix}$. Following the rule for the second derivative test for constrained local extrema, note that $n = 2$ and $k = 1$ so the only relevant term in sequence (1) is

$$(-1)^1 d_3 = (-1)[(-2)(24) - 2(36)] = 120 > 0.$$

We conclude that there is a constrained local minimum at the point $(2, -2)$.

**(c)** You can see from the figure below that there is a constrained local minimum at $(2, -2)$ on the curve. This will be the point at which the constraint curve is tangent to one of the level curves.




**21.** The symmetry of the problem suggests the answer, but we are maximizing $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = x + y + z = 18$. We solve the system

$$\begin{cases} yz = \lambda \\ xz = \lambda \\ xy = \lambda \\ x + y + z = 18. \end{cases}$$

None of the solutions that corresponds to one of $x$, $y$, and $z$ being zero is a maximum. The solution we get is $x = y = z$, so $3x = 18$, so the maximum product occurs at the point (6, 6, 6).

**22.** First, a sphere is a compact surface and the function $f$ is continuous so, by the extreme value theorem, we know that both a minimum and a maximum must be attained. We find the extrema of $f(x, y, z) = x + y - z$ subject to the constraint $x^2 + y^2 + z^2 = 81$. We solve the system

$$\begin{cases} 1 = 2\lambda x \\ 1 = 2\lambda y \\ -1 = 2\lambda z \\ x^2 + y^2 + z^2 = 81. \end{cases}$$

We see that $x = y = -z$, so the critical points are $(3\sqrt{3}, 3\sqrt{3}, -3\sqrt{3})$ and $(-3\sqrt{3}, -3\sqrt{3}, 3\sqrt{3})$. By evaluating at $f(x, y, z) = x + y - z$, we see that the first must yield a maximum of $9\sqrt{3}$, and the second a minimum of $-9\sqrt{3}$.

**23.** This is a nice problem to assign because by this point some students are only checking boundary values. We are looking for the maximum and minimum values of $f(x, y) = x^2 + xy + y^2$ constrained to be inside the closed disk $g(x, y) = x^2 + y^2 \le 4$. First we find the critical points without paying attention to the constraint. The partial derivatives are $f_x(x, y) = 2x + y$ and $f_y(x, y) = x + 2y$ so we have a critical point at the origin, and $f(0, 0) = 0$. Next we look for extrema of the function on the boundary of the disk by solving the system

$$\begin{cases} 2x + y = 2\lambda x \\ x + 2y = 2\lambda y \\ x^2 + y^2 = 4. \end{cases}$$

From the first two equations we see that $x^2 = y^2$ so $x = \pm y$ and $x = \pm\sqrt{2}$. Substituting, we find that the minimum is 0 at the origin and the maximum is 6 at $(\sqrt{2}, \sqrt{2})$.

**24.** We are maximizing $V(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = 2x + 2y + z \le 108$. In this case, the maximum must occur on the boundary because the only unconstrained critical point requires two of the coordinates to be zero—these points are on the boundary and give the (degenerate) minimum solution of 0. We solve the system

$$\begin{cases} yz = 2\lambda \\ xz = 2\lambda \\ xy = \lambda \\ 2x + 2y + z = 108. \end{cases}$$

Since none of $x$, $y$, or $z$ can be zero, we find that $x = y = z/2$, so $3z = 108$ and the critical point is (18, 18, 36). So the dimensions are 18" by 18" by 36".

**25.** We are maximizing $f(r, h) = \pi r^2 h$ subject to the constraint that $g(r, h) = 2\pi rh + 2\pi r^2 = c$. We solve the system

$$\begin{cases} 2\pi rh = \lambda(2\pi h + 4\pi r) \\ \pi r^2 = 2\lambda\pi r \\ 2\pi rh + 2\pi r^2 = c. \end{cases}$$

Since $r \ne 0$ the second equation implies that $r = 2\lambda$, so, substituting this into the first equation, we see that $h = 2r$. Hence, the height should equal the diameter.

**26.** We are minimizing the cost which is $C(r, h) = \pi r^2 + 2(2\pi rh) + 5(2\pi r^2) = 11\pi r^2 + 4\pi rh$ subject to the constraint $g(r, h) = \pi r^2 h + (2/3)\pi r^3 = 900\pi$. We solve the system

$$\begin{cases} 22\pi r + 4\pi h = \pi\lambda(2rh + 2r^2) \\ 4\pi r = \lambda\pi r^2 \\ \pi r^2 h + (2/3)\pi r^3 = 900\pi. \end{cases}$$

As above, we see that $4 = \lambda r$ so $22\pi r + 4\pi h = (4\pi/r)(2rh + 2r^2)$ or $14r = 4h$. Substituting, $900 = (7/2)r^3 + (2/3)r^3 = (25/6)r^3$ so the radius is 6 feet and the height is 21 feet.

**27.** We wish to minimize $M(x, y, z) = xz - y^2 + 3x + 3$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 9$. We solve the system

$$\begin{cases} z + 3 = 2\lambda x \\ -2y = 2\lambda y \\ x = 2\lambda z \\ x^2 + y^2 + z^2 = 9. \end{cases}$$

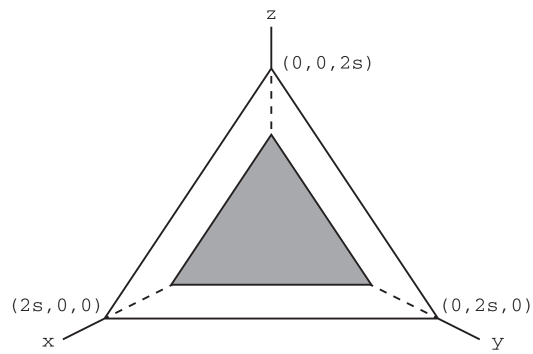
Either $y = 0$ or $\lambda = -1$. If $y = 0$, then $z = -3$ or 3/2 so we get $(0, 0, -3)$ and $(\pm 3\sqrt{3}/2, 0, 3/2)$ as critical points. If $\lambda = -1$, we find the critical points are $(-2, 2, 1)$ and $(-2, -2, 1)$. Comparing values of $M$, the minimum of $-9$ is attained at either $(-2, 2, 1)$ or $(-2, -2, 1)$.

**28.** It's easier to maximize the *square* of the area $f(x, y, z) = s(s - x)(s - y)(s - z)$ subject to $x + y + z = 2s$ $(= P)$, a constant.

Thus $\nabla f = \lambda \nabla g$ (where $g(x, y, z) = x + y + z$) gives us the system:

$$\begin{cases} -s(s - y)(s - z) = \lambda \\ -s(s - x)(s - z) = \lambda \\ -s(s - x)(s - y) = \lambda \\ \quad x + y + z = 2s \qquad (0 < x, y, z \le s) \end{cases}$$

Hence $-s(s-y)(s-z) = -s(s-x)(s-z) = -s(s-x)(s-y)$. The first equality implies $z = s$ or $x = y$. Note that $z = s$ means $f = 0$—so there's zero area which cannot possibly be maximum. Thus $x = y$. From $-s(s-x)(s-z) = -s(s-x)(s-y)$ we similarly conclude that $y = z$. Hence $x = y = z$ $\left(= \frac{2}{3}s\right)$ gives us our critical point and corresponds to having an equilateral triangle. Our constraint looks like a portion of a plane. The dark triangle in the figure below is the part to be considered—it's where $f$ is $\geq 0$. Therefore, the point $\left(\frac{2}{3}s, \frac{2}{3}s, \frac{2}{3}s\right)$ yields the maximum.



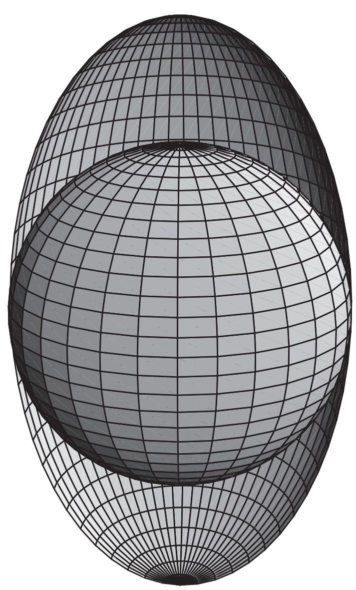

**29.** A sphere centered at the origin has equation $x^2+y^2+z^2 = r^2$. Thus we want to maximize $f(x,y,z) = x^2+y^2+z^2$ subject to the constraint $g(x,y,z) = 3x^2+2y^2+z^2 = 6$. We can solve this using Lagrange multipliers, but we must make sure we find an *inscribed* sphere. We consider the system

$$\begin{cases} 2x = 6\lambda x & 1^{\text{st}} \text{ equation gives } x = 0 \text{ or } \lambda = 1/3 \\ 2y = 4\lambda y & 2^{\text{nd}} \text{ equation gives } y = 0 \text{ or } \lambda = 1/2 \\ 2z = 2\lambda z & 3^{\text{rd}} \text{ equation gives } z = 0 \text{ or } \lambda = 1 \\ 3x^2+2y^2+z^2 = 6 & \text{(Note that we can't have } x = y = z = 0 \\ & \quad \text{and still satisfy the constraint.)} \end{cases}$$

Thus if $\lambda = 1/3$, $y = z = 0$ and the constraint implies $x = \pm\sqrt{2}$. If $\lambda = 1/2$, $x = z = 0$ and $y = \pm\sqrt{3}$. Finally, if $\lambda = 1$, then $x = y = 0$ and $z = \pm\sqrt{6}$. Comparing values, we have

$$f(\pm\sqrt{2},0,0) = 2, \quad f(0,\pm\sqrt{3},0) = 3, \quad f(0,0,\pm\sqrt{6}) = 6,$$

so that it's tempting to say that the largest sphere has a radius of $\sqrt{6}$. However, such a sphere is not actually inscribed in the ellipsoid. The largest sphere that actually remains inscribed in the ellipsoid has a radius of $\sqrt{2}$.



**30.** This is just Exercise 1 with two constraints. We are minimizing $f(x, y, z) = x^2 + y^2 + z^2$ with the constraints $g_1(x, y, z) = 2x + y + 3z = 9$ and $g_2(x, y, z) = 3x + 2y + z = 6$. We solve the system

$$\begin{cases} 2x = 2\lambda + 3\mu \\ 2y = \lambda + 2\mu \\ 2z = 3\lambda + \mu \\ 2x + y + 3z = 9 \\ 3x + 2y + z = 6. \end{cases}$$

Eliminate $\lambda$ and $\mu$ and then solve to get a critical point at (1, 2/5, 11/5).

**31.** This is just Exercise 22 translated by $(2, 5, -1)$. We are minimizing $f(x, y, z) = (x-2)^2 + (y-5)^2 + (z+1)^2$ with the constraints $g_1(x, y, z) = x - 2y + 3z = 8$ and $g_2(x, y, z) = 2z - y = 3$. We solve the system

$$\begin{cases} 2(x-2) = \lambda \\ 2(y-5) = -2\lambda - \mu \\ 2(z+1) = 3\lambda + 2\mu \\ x - 2y + 3z = 8 \\ 2z - y = 3. \end{cases}$$

Eliminate $\mu$ by combining the second and third equations and then substitute $2(x-2)$ for $\lambda$. Solve to get a critical point at (9/2, 2, 5/2).

**32.** We want to maximize and minimize the distance function $\sqrt{x^2 + y^2 + z^2}$, but the task is equivalent to finding the extrema of the *square* of the distance. Hence we find the extrema of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the two constraints that $g_1(x, y, z) = x + y + z = 4$ and $g_2(x, y, z) = x^2 + y^2 - z = 0$. Note that $f$ is continuous and the ellipse defined by the constraints is compact, so the extreme value theorem guarantees that $f$ has a global maximum and a global minimum on the ellipse. From the Lagrange multiplier equation $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$, plus the constraints, we see that we must solve the system

$$\begin{cases} 2x = 2\lambda_1 x + \lambda_2 \\ 2y = 2\lambda_1 y + \lambda_2 \\ 2z = -\lambda_1 + \lambda_2 \\ x + y + z = 4 \\ x^2 + y^2 - z = 0. \end{cases}$$

The first two equations imply $\lambda_2 = 2x - 2\lambda_1 x = 2y - 2\lambda_1 y$, so that $2x(1 - \lambda_1) = 2y(1 - \lambda_1)$. Hence either $\lambda_1 = 1$ or $x = y$. If $\lambda_1 = 1$, then $\lambda_2 = 0$ and the third equation becomes $2z = -1$, so $z = -1/2$. The last two equations are thus $x + y - 1/2 = 4$ and $x^2 + y^2 + 1/2 = 0$. However, there can be no real solutions to $x^2 + y^2 = -1/2$. Therefore, the case that $\lambda_1 = 1$ leads to no critical points.

If $x = y$, then the last two equations become $2x + z = 4$ and $2x^2 - z = 0$. Hence $z = 4 - 2x$, so that $2x^2 - z = 0$ is equivalent to $2x^2 + 2x - 4 = 0$, which has solutions $x = -2, 1$. Therefore our critical points are $(-2, -2, 8)$ and $(1, 1, 2)$.

Finally, note that $f(-2, -2, 8) = 72 > f(1, 1, 2) = 6$. Hence, in view of the initial observations above, $(1, 1, 2)$ is the point on the ellipse nearest the origin and $(-2, -2, 8)$ the point farthest from the origin.

**33.** This is the same as Exercise 32 except that we are trying to find extrema for $f(x, y, z) = z$ and the plane has the equation $g_1(x, y, z) = x + y + 2z = 2$. Again, using a computer algebra system we find that the lowest point is at (1/2, 1/2, 1/2) and the highest is at $(-1, -1, 2)$.

**34.** Minimize $f(x, y, u, v) = (x-u)^2 + (y-v)^2$ subject to the two constraints: $g_1(x, y, u, v) = x^2 + 2y^2 = 1$ and $g_2(x, y, u, v) = u + v = 4$. We solve the system

$$\begin{cases} 2(x-u) = 2\lambda x \\ -2(x-u) = \mu \\ 2(y-v) = 2\lambda y \\ -2(y-v) = \mu \\ x^2 + 2y^2 = 1 \\ u + v = 4. \end{cases}$$

Solving you get two critical points $(x, y, u, v) = (\sqrt{2/3}, \sqrt{1/6}, 2+\sqrt{6}/12, 2-\sqrt{6}/12)$ for which the square of the distance is $35/4 - 2\sqrt{6} \approx 3.85$ and $(x, y, u, v) = (-\sqrt{2/3}, -\sqrt{1/6}, 2-\sqrt{6}/12, 2+\sqrt{6}/12)$ for which the square of the distance is $35/4 + 2\sqrt{6} \approx 13.65$. The minimum distance is $\sqrt{35/4 - 2\sqrt{6}} \approx 1.96$.

**35.** **(a)** $f(x, y) = x + y$ with the constraint $xy = 6$ so we solve the system

$$\begin{cases} 1 = 2\lambda y \\ 1 = 2\lambda x \\ xy = 6. \end{cases}$$

So $x = y$ and the critical points are at $\pm(\sqrt{6}, \sqrt{6})$.

**(b)** The constraint curve is not connected. There are two distinct components. Although $(-\sqrt{6}, -\sqrt{6})$ produces a local maximum of $-2\sqrt{6}$ on its component, the value of the function at any point on the other component is greater. Similarly, $(\sqrt{6}, \sqrt{6})$ produces a local minimum of $2\sqrt{6}$ on its component, but the value of the function at any point on the other component is less.

**36.** We use a Lagrange multiplier to find the maximum value of $f(\alpha, \beta, \gamma) = \sin\alpha \sin\beta \sin\gamma$ subject to the constraint that $\alpha + \beta + \gamma = \pi$. (Note that we also assume that each of $\alpha$, $\beta$, $\gamma$ must be strictly between 0 and $\pi$.) The system of equations to consider is

$$\begin{cases} \cos\alpha \sin\beta \sin\gamma = \lambda \\ \sin\alpha \cos\beta \sin\gamma = \lambda \\ \sin\alpha \sin\beta \cos\gamma = \lambda \\ \alpha + \beta + \gamma = \pi. \end{cases}$$

The first two equations imply that $\cos\alpha \sin\beta \sin\gamma = \sin\alpha \cos\beta \sin\gamma$ This holds if either $\cos\alpha \sin\beta = \sin\alpha \cos\beta$ or $\sin\gamma = 0$. However, if $\sin\gamma = 0$, then $\gamma$ is 0 or $\pi$ which we have already ruled out. (Also, $f$ would necessarily be zero and clearly not maximized since any acute triangle will yield a positive value of $f$.) Hence

$$\cos\alpha \sin\beta = \sin\alpha \cos\beta \iff \sin\alpha \cos\beta - \cos\alpha \sin\beta = 0 \iff \sin(\alpha - \beta) = 0.$$

It follows that $\alpha = \beta$. Similarly, the second and third equations together imply that $\sin\alpha \cos\beta \sin\gamma = \sin\alpha \sin\beta \cos\gamma$ Thus either $\sin\alpha = 0$ (which we reject) or

$$\cos\beta \sin\gamma = \sin\beta \cos\gamma \iff \sin\beta \cos\gamma - \cos\beta \sin\gamma = 0 \iff \sin(\beta - \gamma) = 0.$$

Hence $\beta = \gamma$ and so $\alpha = \beta = \gamma = \pi/3$ using the last equation. Therefore, the maximum value of $f$ is $3\sqrt{3}/8$.

**37.** Let $P$ have coordinates $(x, y, z)$. The square of the distance from $P$ to the origin is given by the function $f(x, y, z) = x^2 + y^2 + z^2$ and the coordinates of $P$ must satisfy $g(x, y, z) = c$. Thus if $f$ is maximized at $P$, then, since $\nabla g(x, y, z)$ is given never to vanish, $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ for some $\lambda$. If we write this out, we find

$$(2x, 2y, 2z) = \lambda \nabla g(x, y, z).$$

But

$$(2x, 2y, 2z) = 2(x, y, z) = 2\overrightarrow{OP},$$

where $\overrightarrow{OP}$ denotes the displacement vector from the origin to $P$. Therefore,

$$\overrightarrow{OP} = \frac{\lambda}{2} \nabla g(x, y, z);$$

that is, $\overrightarrow{OP}$ is parallel to $\nabla g$. (Note that $\overrightarrow{OP}$ must be nonzero if the distance from the origin to $P$ is to be *maximized*.) Since the gradient vector $\nabla g$ at $P$ is known to perpendicular to the level set of $g$ through $P$, the result follows.

**38.** This is a non-linear version of Exercise 30. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $g_1(x, y, z) = x^2 + y^2 = 4$ and $g_2(x, y, z) = 2x + 2y + z = 2$. We solve the system

$$\begin{cases} 2x = 2\lambda x + 2\mu \\ 2y = 2\lambda y + 2\mu \\ 2z = \mu \\ x^2 + y^2 = 4 \\ 2x + 2y + z = 2. \end{cases}$$

Solving we see that either $x = y$ or $\lambda = 1$. If $x = y$ then $x = y = \pm\sqrt{2}$ and $z = 2 \mp 4\sqrt{2}$. The farthest point is $(-\sqrt{2}, -\sqrt{2}, 2 + 4\sqrt{2})$. If $\lambda = 1$ then $x = (1 \pm \sqrt{7})/2, y = (1 \mp \sqrt{7})/2$, and $z = 0$—these last two are the closest points.

**39.** We want to find the extreme values of the function $f(x, y) = x^2 + y^2$ (the square of the distance from the point $(x, y)$ to the origin) subject to the constraint $g(x, y) = 3x^2 - 4xy + 3y^2 = 50$. (Note that there will be a global maximum and a global minimum by the extreme value theorem since the ellipse is a compact set in $\mathbf{R}^2$.) We solve the system

$$\begin{cases} 2x = \lambda(6x - 4y) \\ 2y = \lambda(-4x + 6y) \\ 3x^2 - 4xy + 3y^2 = 50. \end{cases}$$

The first two equations together imply

$$\frac{1}{\lambda} = \frac{6x - 4y}{2x} = \frac{-4x + 6y}{2y} \iff 3 - \frac{2y}{x} = 3 - \frac{2x}{y} \iff y^2 = x^2.$$

Thus $y = \pm x$. If $y = x$, then the last equation becomes

$$3x^2 - 4x^2 + 3x^2 = 50 \iff x^2 = 25 \iff x = \pm 5.$$

Thus there are two critical points $(5, 5)$ and $(-5, -5)$. If $y = -x$, then the last equation becomes

$$3x^2 + 4x^2 + 3x^2 = 50 \iff x^2 = 5 \iff x = \pm\sqrt{5}.$$

Hence there are two more critical points $(\sqrt{5}, -\sqrt{5})$ and $(-\sqrt{5}, \sqrt{5})$. Finally, we have

$$f(5, 5) = f(-5, -5) = 50 \quad \text{and} \quad f(\sqrt{5}, -\sqrt{5}) = f(-\sqrt{5}, \sqrt{5}) = 10,$$

so that $(5, 5)$ and $(-5, -5)$ are the points on the ellipse farthest from the origin and $(\sqrt{5}, -\sqrt{5})$ and $(-\sqrt{5}, \sqrt{5})$ are the points nearest the origin.

**40. (a)** This follows immediately from the extreme value theorem. The constraint defines a quarter circle, including the endpoints, which is a compact set in $\mathbf{R}^2$ and the function $f(x, y) = \sqrt{x} + 8\sqrt{y}$ is continuous whenever $x$ and $y$ are both nonnegative.

**(b)** The system we consider is

$$\begin{cases} \left(\dfrac{1}{2\sqrt{x}}, \dfrac{8}{2\sqrt{y}}\right) = \lambda(2x, 2y) \\ x^2 + y^2 = 17 \end{cases} \quad \text{or} \quad \begin{cases} \dfrac{1}{2\sqrt{x}} = 2\lambda x \\ \dfrac{4}{\sqrt{y}} = 2\lambda y \\ x^2 + y^2 = 17. \end{cases}$$

The first two equations of the system together imply that

$$2\lambda = \frac{1}{2x^{3/2}} = \frac{4}{y^{3/2}} \implies y^{3/2} = 8x^{3/2} \implies y = 4x.$$

Using this result in the last equation gives $x^2 + 16x^2 = 17$. Thus $x = 1$ since we only want $x$ (and $y$) nonnegative. Thus the only critical point we identify in this manner is $(1, 4)$.

**(c)** Note that $\nabla f(x, y)$ is undefined if either $x$ or $y$ is zero. Given the constraint, this means that we should also consider the points $(\sqrt{17}, 0)$ and $(0, \sqrt{17})$. Comparing values, we have

- $f(1, 4) = 17$,
- $f(\sqrt{17}, 0) = \sqrt[4]{17}$,
- $f(0, \sqrt{17}) = 8\sqrt[4]{17} \approx 16.24$.

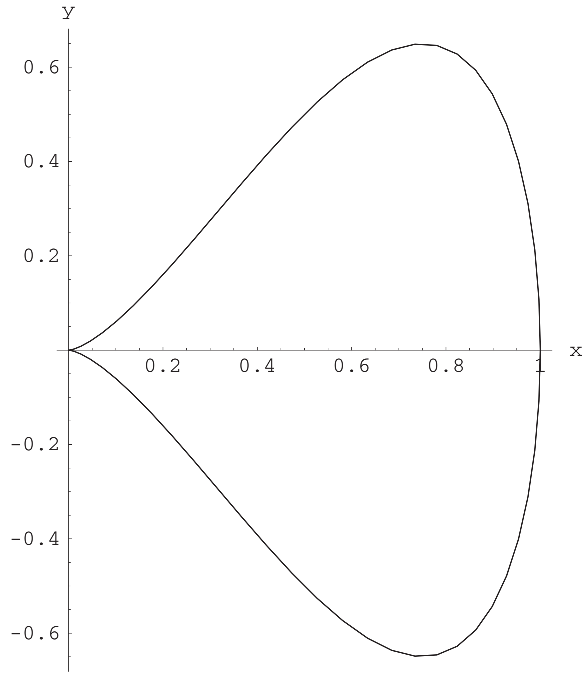
Hence $(1, 4)$ yields the global maximum and $(\sqrt{17}, 0)$ the global minimum on the quarter circle.

**41. (a)** The system is

$$\begin{cases} 1 = \lambda(16x^3 - 12x^2) \\ 0 = 2\lambda y \\ y^2 - 4x^3 + 4x^4 = 0. \end{cases}$$

The second equation implies that either $y = 0$ or $\lambda = 0$. But $\lambda = 0$ cannot satisfy the first equation, so $y = 0$. The last equation implies $4x^3(1 - x) = 0$; thus $x = 0$ or 1. But $x = 0$ cannot satisfy the first equation. Thus the only solution to the system is $(1, 0)$.

**(b)** The graph of the curve (known as the **piriform**) is shown in the figure below. From it, it's clear that the maximum value of $f(x, y) = x$ occurs at (1, 0) and the minimum value at (0, 0).




**(c)** Note that $\nabla g(x, y) = (16x^3 - 12x^2, 2y) = (0, 0)$ at (0, 0) (and at (3/4, 0)). (0, 0) is a point on the curve ((3/4, 0) is not). It's the singular point of the piriform and, although not a solution to the Lagrange multiplier system in part (a), it must be considered as a possible site for extrema.

**42. (a)** The relevant Lagrange multiplier system to solve is

$$\begin{cases} 2x = 0 \\ 2y = 0 \\ 0 = \lambda \\ z = c \end{cases} \qquad \text{The obvious unique solution is } (0, 0, c) \text{ with } \lambda = 0.$$

**(b)** $L(l; x, y, z) = x^2 + y^2 - l(z - c)$. With $c$ as a constant and $x_1 = x, x_2 = y, x_3 = z$, we have

$$HL(l; x, y, z) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = HL(0; 0, 0, c).$$

The second derivative test asks us to calculate $(-1)^1 d_3$ and $(-1)^1 d_4$ or

$$-d_3 = -\det \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 0; \quad -d_4 = -\det \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = -(-4) = 4.$$

Thus the second derivative test seems to suggest that we've found a saddle point.

**(c)** Now we let $x_1 = z, x_2 = y, x_3 = x$ and look at

$$HL(l; z, y, x) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

In this case we find

$$-d_3 = -\det \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = -(-2) = 2 \quad \text{and}$$

$$-d_4 = -\det \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = -(-4) = 4.$$

This time the sound derivative test suggests a local minimum.

**(d)** Indeed, inspection tells us that the expression $x^2 + y^2$ attains a *global* minimum at $x = y = 0$. So to satisfy the constraint $z = c$, we see that (0, 0, $c$) yields a global minimum. The difference between the results of (b) and (c) can be explained by looking at $\partial g/\partial x$ vs. $\partial g/\partial z$: $\partial g/\partial x = 0$, but $\partial g/\partial z = 1 \neq 0$.

In part (b), we did not satisfy the hypothesis of the second derivative test that the variables be ordered so that

$$\det \begin{bmatrix} \dfrac{\partial g_1}{\partial x_1}(\mathbf{a}) & \cdots & \dfrac{\partial g_1}{\partial x_k}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_k}{\partial x_1}(\mathbf{a}) & \cdots & \dfrac{\partial g_k}{\partial x_k}(\mathbf{a}) \end{bmatrix} \neq 0.$$

(The determinant in this situation is just $\partial g/\partial x$.) In part (c), we did satisfy the hypothesis, since $\partial g/\partial z \neq 0$.

**43. (a)** In order for $(\boldsymbol{\lambda}, \mathbf{a})$ to be a solution of the constrained problem, $(\boldsymbol{\lambda}, \mathbf{a})$ must solve the system

$$\begin{cases} f_{x_i}(\mathbf{a}) = \displaystyle\sum_{j=1}^{k} \lambda_j (g_j)_{x_i}(\mathbf{a}) & \text{for } 1 \leq i \leq n \\ g_j(\mathbf{a}) = c_j & \text{for } 1 \leq j \leq k. \end{cases}$$

On the other hand, an unconstrained critical point for $L$ must be where all first partials are zero. In other words, we must have

$$L_{l_j} = 0, \quad 1 \leq j \leq k \quad \text{and} \quad L_{x_j} = 0, \quad 1 \leq j \leq n.$$

Upon explicit calculation of the partials these equations are:

$$\begin{cases} f_{l_j}(\mathbf{a}) - (g_j(\mathbf{a}) - c_j) = 0 & \text{for } 1 \leq j \leq k, \text{ and} \\ f_{x_j}(\mathbf{a}) - \sum_{i=1}^{k} \lambda_i (g_i)_{x_j}(\mathbf{a}) = 0 & \text{for } 1 \leq j \leq n. \end{cases}$$

This is the same system as that for the constrained case.

**(b)** Calculate the Hessian in four blocks. All of the entries in the upper left $k \times k$ block are 0. This is because the entry in position ($i$, $j$) is $L_{l_i l_j}$ and the highest power of any $l_i$ appearing in $L$ is 1. The top right block with $k$ rows and $n$ columns gives back the negative first partials of the constraint conditions because the entry in position $(k + i, j)$ is $L_{x_i l_j} = -(g_j - c_j)_{x_i} = -(g_j)_{x_i}$. The lower left block of $n$ rows and $k$ columns is just the transpose of this last block. The lower right $n \times n$ block is such that the entry in position $(k + i, k + j) = L_{x_i x_j} = (f - \sum_{q=1}^{k} l_q g_q)_{x_i x_j}$. When $\boldsymbol{\lambda}$ and **a** are substituted for **l** and **x**, the desired matrix is obtained.

**44.** We find extreme values of $f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^{n} x_i y_i$ subject to the two constraints $g_1(x_1, \ldots, x_n,\ y_1, \ldots,$ $y_n) = {x_1}^2 + \cdots + {x_n}^2 = 1$ and $g_2(x_1, \ldots, x_n, y_1, \ldots, y_n) = {y_1}^2 + \cdots + {y_n}^2 = 1$. Thus we look at $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$

together with the constraints to solve

$$\begin{cases} \left.\begin{array}{l} y_1 = 2\lambda_1 x_1 \\ \vdots \\ y_n = 2\lambda_1 x_n \end{array}\right\} & \text{The first } n \text{ equations (and the last) imply} \quad \begin{aligned} 1 &= \sum {y_i}^2 = 4{\lambda_1}^2 \sum {x_i}^2 \\ &= 4{\lambda_1}^2 \cdot 1 \\ &\text{so } \lambda_1 = \pm\frac{1}{2}. \end{aligned} \\ \left.\begin{array}{l} x_1 = 2\lambda_2 y_1 \\ \vdots \\ x_n = 2\lambda_2 y_n \end{array}\right\} & \text{The next } n \text{ equations (and the next-to-last) imply} \quad \begin{aligned} 1 &= \sum {x_i}^2 = 4{\lambda_2}^2 \sum {y_i}^2 = 4{\lambda_2}^2 \\ &\text{so } \lambda_2 = \pm\frac{1}{2}. \end{aligned} \\ \sum {x_i}^2 = 1 & \\ \sum {y_i}^2 = 1 & \end{cases}$$

Putting all the information together, we find that $\mathbf{x} = \mathbf{y}$ (when $\lambda_1 = \lambda_2 = \frac{1}{2}$) and $\mathbf{x} = -\mathbf{y}$ (when $\lambda_1 = \lambda_2 = -\frac{1}{2}$). When $\mathbf{x} = \mathbf{y}$, $f(\mathbf{x}, \mathbf{y}) = \sum {x_i}^2 = 1$. When $\mathbf{x} = -\mathbf{y}$, $f(\mathbf{x}, -\mathbf{x}) = \sum(-{x_i}^2) = -1$. Though it takes a little bit of argumentation, the hypersphere in $\mathbf{R}^n$ is compact—hence so is the *product* of hyperspheres in $\mathbf{R}^{2n}(= \mathbf{R}^n \times \mathbf{R}^n)$. Thus we find maximum and minimum values of $+1$ and $-1$, respectively.

**45. (a)**

$$\sum_{i=1}^{n} {u_i}^2 = {u_1}^2 + \cdots + {u_n}^2 = \frac{{x_1}^2}{(\sqrt{{x_i}^2})^2} + \frac{{x_2}^2}{(\sqrt{{x_i}^2})^2} + \cdots + \frac{{x_n}^2}{(\sqrt{{x_i}^2})^2} = \frac{\sum {x_i}^2}{\sum {x_i}^2} = 1.$$

So **u** is an the unit hypersphere. The case for **v** is identical.

**(b)** By Exercise 44, we have $-1 \le \sum_{i=1}^{n} u_i v_i \le 1$. Hence

$$\begin{aligned} -1 &\le \sum_i \left( \frac{x_i}{\sqrt{\sum_j x_j^2}} \right) \left( \frac{y_i}{\sqrt{\sum_j y_j^2}} \right) \le 1 \\ \Leftrightarrow -\sqrt{\sum_j x_j^2}\sqrt{\sum_j y_j^2} &\le \sum_i x_i y_i \le \sqrt{\sum_j x_j^2}\sqrt{\sum_j y_i^2} \\ \Leftrightarrow -\|\mathbf{x}\|\|\mathbf{y}\| &\le \mathbf{x} \cdot \mathbf{y} \le \|\mathbf{x}\|\|\mathbf{y}\| \\ \Leftrightarrow \qquad |\mathbf{x} \cdot \mathbf{y}| &\le \|\mathbf{x}\|\|\mathbf{y}\|. \end{aligned}$$

## 4.4 Some Applications of Extrema

**1.** This problem can be done using calculators or the following table to help with Proposition 4.1:

| $x_i$ | $y_i$ | ${x_i}^2$ | $x_i y_i$ |
|---|---|---|---|
| 0 | 2 | 0 | 0 |
| 1 | 3 | 1 | 3 |
| 2 | 5 | 4 | 10 |
| 3 | 3 | 9 | 9 |
| 4 | 2 | 16 | 8 |
| 5 | 7 | 25 | 35 |
| 6 | 7 | 36 | 42 |
| 21 | 29 | 91 | 107 |

$$\text{So} \quad m_0 = \frac{7(107) - (21)(29)}{7(91) - (21)^2} = \frac{140}{196} = \frac{35}{49} \approx .71428$$

$$\text{and} \quad b_0 = \frac{(91)(29) - (21)(107)}{7(91) - (21)^2} = \frac{392}{196} = \frac{98}{49} = 2.$$

The equation of the least squares line is $y = (35/49)x + 2$.

**2.** Again, using Proposition 4.1,

$$m = \frac{2(x_1y_1 + x_2y_2) - (x_1 + x_2)(y_1 + y_2)}{2(x_1^2 + x_2^2) - (x_1 + x_2)^2} = \frac{(x_1 - x_2)(y_1 - y_2)}{(x_1 - x_2)^2} = \frac{y_1 - y_2}{x_1 - x_2}.$$

$$b = \frac{(x_1^2 + x_2^2)(y_1 + y_2) - (x_1 + x_2)(x_1y_1 + x_2y_2)}{2(x_1^2 + x_2^2) - (x_1 + x_2)^2} = \frac{(x_1y_2 - x_2y_1)(x_1 - x_2)}{(x_1 - x_2)^2} = \frac{x_1y_2 - x_2y_1}{x_1 - x_2}.$$

You can check that $(x_1, y_1)$ and $(x_2, y_2)$ are both on the line

$$y = \left(\frac{y_1 - y_2}{x_1 - x_2}\right)x + \frac{x_1y_2 - x_2y_1}{x_1 - x_2}.$$

**3. (a)** As in the text, the function $D(a, b)$ will be the sum of the squares of the differences between the observed $y$ values and the $y$ values on the curve $y = a/x + b$. This means that

$$D(a, b) = \sum_{i=1}^{n}(y_i - (a/x_i + b))^2.$$

**(b)** Make the substitution $X_i = 1/x_i$ and then fit the line $y = aX + b$ to this transformed data using Proposition 4.1. We get

$$a = \frac{n\sum X_iy_i - \left(\sum X_i\right)\left(\sum y_i\right)}{n\sum X_i^2 - \left(\sum X_i\right)^2} \quad \text{and} \quad b = \frac{\left(\sum X_i^2\right)\left(\sum y_i\right) - \left(\sum X_i\right)\left(\sum X_iy_i\right)}{n\sum X_i^2 - \left(\sum X_i\right)^2}.$$

Transform the data back, replacing $X_i$ with $1/x_i$, then the curve of the form $y = a/x + b$ that best fits the data has

$$a = \frac{n\sum y_i/x_i - \left(\sum 1/x_i\right)\left(\sum y_i\right)}{n\sum 1/x_i^2 - \left(\sum 1/x_i\right)^2} \quad \text{and} \quad b = \frac{\left(\sum 1/x_i^2\right)\left(\sum y_i\right) - \left(\sum 1/x_i\right)\left(\sum y_i/x_i\right)}{n\sum 1/x_i^2 - \left(\sum 1/x_i\right)^2}.$$

**4.** We'll use the results of Exercise 3 and organize our sums with the following table:

| $1/x_i$ | $y_i$ | $1/x_i^2$ | $y_i/x_i$ |
|---|---|---|---|
| 1 | 0 | 1 | 0 |
| 1/2 | −1 | 1/4 | −1/2 |
| 2 | 1 | 4 | 2 |
| 1/3 | −1/2 | 1/9 | −1/6 |
| 23/6 | −1/2 | 193/36 | 8/6 |

So $$a = \frac{4(8/6) - (23/6)(-1/2)}{4(193/36) - (23/6)^2} = \frac{261}{243} = \frac{29}{27}$$

and $$b = \frac{(193/36)(-1/2) - (23/6)(8/6)}{4(193/36) - (23/6)^2} = -\frac{561}{486} = -\frac{187}{162}.$$

The equation of the least squares curve of the desired form is $y = 29/(27x) - 187/162$.

**5.** Again the function $D(a, b, c)$ will be the sum of the squares of the differences between the observed $y$ values and the $y$ values on the curve $y = ax^2 + bx + c$. This means that

$$D(a,b,c) = \sum_{i=1}^{n}(y_i - (ax_i^2 + bx_i + c))^2$$
$$= \sum_{i=1}^{n} y_i^2 + a^2\sum_{i=1}^{n} x_i^4 + b^2\sum_{i=1}^{n} x_i^2 + nc^2 - 2a\sum_{i=1}^{n} x_i^2 y_i - 2b\sum_{i=1}^{n} x_i y_i - 2c\sum_{i=1}^{n} y_i$$
$$+ 2ab\sum_{i=1}^{n} x_i^3 + 2ac\sum_{i=1}^{n} x_i^2 + 2bc\sum_{i=1}^{n} x_i \quad \text{so}$$
$$D_a(a,b,c) = 2a\sum_{i=1}^{n} x_i^4 - 2\sum_{i=1}^{n} x_i^2 y_i + 2b\sum_{i=1}^{n} x_i^3 + 2c\sum_{i=1}^{n} x_i^2,$$
$$D_b(a,b,c) = 2b\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} x_i y_i + 2a\sum_{i=1}^{n} x_i^3 + 2c\sum_{i=1}^{n} x_i, \quad \text{and}$$
$$D_c(a,b,c) = 2cn - 2\sum_{i=1}^{n} y_i + 2a\sum_{i=1}^{n} x_i^2 + 2b\sum_{i=1}^{n} x_i.$$

Set each of the partial derivatives equal to zero, move the term with coefficient $-2$ to the other side, and divide by 2 to get the desired equations.

**6.** You may want to point out to the students that the independent variable $x$ corresponds to hours of sleep because that is what (in theory) Egbert can control.

**(a)** To get a line $y = ax + b$ we'll need

| $x_i$ | $y_i$ | $x_i^2$ | $x_i y_i$ |
|---|---|---|---|
| 8 | 85 | 64 | 680 |
| 8.5 | 72 | 72.25 | 612 |
| 9 | 95 | 81 | 855 |
| 7 | 68 | 49 | 476 |
| 4 | 52 | 16 | 208 |
| 8.5 | 75 | 72.25 | 637.5 |
| 7.5 | 90 | 56.25 | 675 |
| 6 | 65 | 36 | 390 |
| 58.5 | 602 | 446.75 | 4533.5 |

Using the formulas in Proposition 4.1 you'll find that the least squares line is

$$y = (4204/607)x + (14935/607) \approx 6.93x + 24.6.$$

**(b)** We will need some additional data:

| $x_i$ | $x_i^3$ | $x_i^4$ | $x_i^2 y_i$ |
|---|---|---|---|
| 8 | 512 | 4096 | 5440 |
| 8.5 | 614.125 | 5220.0625 | 5202 |
| 9 | 729 | 6561 | 7695 |
| 7 | 343 | 2401 | 3332 |
| 4 | 64 | 256 | 832 |
| 8.5 | 614.125 | 5220.0625 | 5418.75 |
| 7.5 | 421.875 | 3164.0625 | 5062.5 |
| 6 | 216 | 1296 | 2340 |
| 58.5 | 3514.125 | 28214.1875 | 35322.25 |

Use the formulas given in Exercise 5 to obtain the system

$$\begin{cases} 28214.1875a + 3514.125b + 446.75c = 35322.25 \\ 3514.125a + 446.75b + 58.5c = 4533.5 \\ 446.75a + 58.5b + 8c = 602. \end{cases}$$

Solve this system to get the following (approximate) quadratic:

$$y = -.192044054x^2 + 9.42923983x + 17.02314387.$$

**(c)** Plugging 6.8 into the linear model predicts that Egbert will get 71.7, plugging 6.8 into the quadratic model predicts that Egbert will get 72.26.

**7. (a)** We are required to show that **F** is a gradient (conservative) vector field. Clearly if $V(x, y) = x^2 + 2xy + 3y^2 + x + 2y$ then $-\nabla V = (-2x - 2y - 1)\mathbf{i} + (-2x - 6y - 2)\mathbf{j} = \mathbf{F}$.

**(b)** We find equilibrium points of **F** when $\mathbf{F} = \mathbf{0}$. Solve the system of equations

$$\begin{cases} -2x - 2y = 1 \\ -2x - 6y = 2 \end{cases}$$

and find one solution at $(-1/4, -1/4)$. The Hessian is

$$HV = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}$$

so both $d_1$ and $d_2$ are positive so the equilibrium is stable.

**8.** Here $V(x, y) = 2x^2 - 8xy - y^2 + 12x - 8y + 12$ so $\nabla V = -\mathbf{F} = (4x - 8y + 12, -8x - 2y - 8)$. This is **0** at $(-11/9, 8/9)$. The Hessian is

$$HV = \begin{bmatrix} 4 & -8 \\ -8 & -2 \end{bmatrix}.$$

Note that $d_1 > 0$ and $d_2 < 0$ so the equilibrium at $(-11/9, 8/9)$ is not stable.

**9.** Here $V(x, y, z) = 3x^2 + 2xy + z^2 - 2yz + 3x + 5y - 10$ so $\nabla V = -\mathbf{F} = (6x + 2y + 3, 2x - 2z + 5, -2y + 2z)$. This is **0** at $(-1, 3/2, 3/2)$. The Hessian is

$$HV = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{bmatrix}.$$

Note that $d_1 > 0$, $d_2 < 0$, and $d_3 > 0$ so the equilibrium at $(-1, 3/2, 3/2)$ is not stable.

**10. (a)** Here we are looking for constrained equilibria (as in Example 3 in the text). Our equation is $\mathbf{F} - \nabla V = \lambda \nabla g$ where $g(x, y, z) = 2x^2 + 3y^2 + z^2 = 1$, $\mathbf{F} = -mg\mathbf{k}$, and $V(x, y, z) = 2x$. So our system of equations is

$$\begin{cases} -2 = 4\lambda x \\ 0 = 6\lambda y \\ -mg = 2\lambda z \\ 2x^2 + 3y^2 + z^2 = 1. \end{cases}$$

Note from the first equation that $\lambda \neq 0$ so by the second equation $y = 0$. From the third equation $2\lambda = -mg/z$ so $z = mgx$. Substituting into the equation of the ellipsoid, $2x^2 + m^2g^2x^2 = 1$ so $x = \pm 1/\sqrt{2 + m^2g^2}$. So our two equilibria are at $\pm(1/\sqrt{2 + m^2g^2}, 0, mg/\sqrt{2 + m^2g^2})$.

**(b)** Note the direction of the force is $(-2, 0, -mg)$ so $-(1/\sqrt{2 + m^2g^2}, 0, mg/\sqrt{2 + m^2g^2})$ is a stable equilibrium.

**11.** Maximize $R(x, y, z) = xyz^2 - 25000x - 25000y - 25000z$ subject to the constraint $x + y + z = 200000$. Our system of equations is

$$\begin{cases} yz^2 - 25000 = \lambda \\ xz^2 - 25000 = \lambda \\ 2xyz - 25000 = \lambda \\ x + y + z = 200000. \end{cases}$$

The hidden condition is that all of the variables are non-negative. This means that we are finding a maximum on the triangular portion of the plane that lies in the first octant. The maximum revenue will occur at a boundary point or at a critical point. Along the boundary at least one of the variables is 0 and the revenue is at most 0 when at least one of $x$, $y$ and $z$ is 0. We will see the value of $R$ at the critical point is greater and therefore that it is our global maximum. Assume none of the variables is zero. Then, from the first two equations, since $z \neq 0$ then $x = y$. From the third equation paired with either of the first two we see that $z = 2x = 2y$. Finally, since their sum is 200000 we find the solution (50000, 50000, 100000) is where the maximum revenue occurs.

**12.** This is similar to Example 4 from the text. We are maximizing $U(x_1, x_2, x_3) = x_1x_2 + 2x_1x_3 + x_1x_2x_3$ subject to the constraint $g(x_1, x_2, x_3) = x_1 + 4x_2 + 2x_3 = 90$. Our system of equations is

$$\begin{cases} x_2 + 2x_3 + x_2x_3 = \lambda \\ x_1 + x_1x_3 = 4\lambda \\ 2x_1 + x_1x_2 = 2\lambda \\ x_1 + 4x_2 + 2x_3 = 90. \end{cases}$$

The only solution of this system with all three of the $x_i$'s non-negative is (33.0149, 6.37314, 15.7463). You can only order integer amounts, so experiment with the different ways of rounding to obtain a maximum at (34, 6, 16).

**13.** We maximize the function $B$ subject to the constraint $15x + 10y = 500$. Using a Lagrange multiplier, we solve the system

$$\begin{cases} 8x = 15\lambda \\ 2y = 10\lambda \\ 15x + 10y = 500. \end{cases}$$

The first two equations imply that $5\lambda = \frac{8}{3}x = y$. Using this in the constraint equation yields

$$15x + \frac{80}{3}x = 500 \iff x = 12.$$

Thus $(x, y) = (12, 32)$ is our only critical point. We should compare the yield $B$ at this point with that at the boundary values of $\left(\frac{100}{3}, 0\right)$ (all irrigation) and $(0, 50)$ (all fertilizer). We have

$$B(12, 32) = 2200,\ B\left(\tfrac{100}{3}, 0\right) = 5044.\overline{4},\ B(0, 50) = 3100.$$

Thus she should forgo the fertilizer entirely and simply irrigate the field.

**14.** **(a)** We maximize the given production function $f$ subject to the constraint $8x + 2y = 1000$. Using a Lagrange multiplier, the system we must consider is

$$\begin{cases} 4y - 2 = 8\lambda \\ 4x - 8 = 2\lambda \\ 8x + 2y = 1000. \end{cases}$$

The first two equations of the system imply that

$$4\lambda = 8x - 16 = 2y - 1 \quad \Longrightarrow \quad 8x = 2y + 15.$$

Using this in the last equation we have $4y + 15 = 1000 \iff y = 985/4$. Hence $x = 1015/16$. (Note that in the constraint $8x + 2y = 1000$, we must have $0 \le x \le 125$ and $0 \le y \le 500$. The endpoints $(125, 0)$ and $(0, 500)$ give negative values for $f$ and so $(1015/16, 985/4)$ must yield the maximum value of $f$ on the line segment described by the constraints.) Hence the manufacturer should purchase 63.4375 lb of cashmere and 246.2516 lb of cotton. The ratio of cotton to cashmere is $4\left(\frac{985}{1015}\right) \approx 3.88$.

**(b)** Most of the essential features of the situation remain unchanged. The constraint equation becomes $8x + 2y = B$, so that the relevant system to solve is

$$\begin{cases} 4y - 2 = 8\lambda \\ 4x - 8 = 2\lambda \\ 8x + 2y = B. \end{cases}$$

As before, $8x = 2y + 15$ and, using this we find that

$$(x, y) = \left(\frac{B + 15}{16}, \frac{B - 15}{4}\right)$$

is the critical point that maximizes $f$. Thus the ratio of cotton to cashmere should be

$$\frac{(B - 15)/4}{(B + 15)/16} = 4\left(\frac{B - 15}{B + 15}\right).$$

As $B$ becomes very large, we have

$$\lim_{B \to +\infty} 4\left(\frac{B - 15}{B + 15}\right) = \lim_{B \to +\infty} \frac{4\,(1 - 15/B)}{1 + 15/B} = 4,$$

which is the ratio of the cost of cashmere to that of cotton.

**15.** **(a)** This is an example of the Cobb-Douglas production function with $p = w = 1$ (see Example 5 from the text). The only critical point will be $(K, L) = ((1/3)360000, (2/3)360000) = (120000, 240000)$.

**(b)** $\partial Q/\partial K = 20(L/K)^{2/3}$ and so at (120000, 240000), $\partial Q/\partial K = 20(2)^{2/3}$. On the other hand, $\partial Q/\partial L = 40(K/L)^{1/3}$ and so at (120000, 240000), $\partial Q/\partial L = 40(1/2)^{1/3}$. These quantities are equal at the critical point.

**16.** This time we are minimizing $pK + wL = M$ subject to the constraint $Q(K, L) = c$. Our system of equations is

$$\begin{cases} p = \lambda \dfrac{\partial Q}{\partial K} \\ w = \lambda \dfrac{\partial Q}{\partial L}. \end{cases}$$

Since none of $p$, $q$, and $\lambda$ is 0, we can divide the top equation by $p\lambda$, divide the bottom equation by $q\lambda$ and the result is immediate.

## True/False Exercises for Chapter 4

**1.** True.
**2.** False. (The increment measures the change in the function.)
**3.** True.
**4.** True.
**5.** True.
**6.** False. ($p_2(x, y) = 1 - 3x + y + 3x^2 + 2xy$.)
**7.** False. ($f$ is most sensitive to changes in $y$.)
**8.** False. (The result is true if $f$ is of class $C^2$.)
**9.** False.
**10.** True.
**11.** True.
**12.** False. (The set is not bounded.)
**13.** False. (Consider the function $f(x, y) = x^2 + y^2$.)
**14.** True. (This ball is compact.)
**15.** True.
**16.** False. (The point **a** might not be a critical point.)
**17.** False. (The point is not a critical point of the function.)
**18.** False. (The point (0, 0, 0) gives a local minimum.)
**19.** True.
**20.** True.
**21.** False. (The critical point is a saddle point.)
**22.** False. (A local extremum can occur where a partial derivative fails to exist.)
**23.** False. (Extrema may also occur at points where $g = c$ and $\nabla g = \mathbf{0}$.)
**24.** False. (Solutions to the system only give critical points.)
**25.** False. (You will have to solve a system of 7 equations in 7 unknowns.)
**26.** True.
**27.** True.
**28.** True.
**29.** False. (The equilibrium points are the critical points of the potential function.)
**30.** False. (This is only true at values of labor and capital that maximize the output.)

## Miscellaneous Exercises for Chapter 4

**1.** If $V = \pi r^2 h$ then $dV = 2\pi rh\, dr + \pi r^2 dh$, so in order for $V$ to be equally sensitive to small changes in $r$ and $h$, we must be at a point $(r_0, h_0)$ where $2\pi r_0 h_0 \approx \pi r_0^2$ so $r_0 = 2h_0$.

**2.** **(a)** If $f(x_1, x_2, \ldots, x_n) = e^{-x_1^2 - x_2^2 - \cdots - x_n^2}$, then $f_{x_i}(x_1, x_2, \ldots, x_n) = -2x_i e^{-x_1^2 - x_2^2 - \cdots - x_n^2}$ and is 0 only when $x_i = 0$. So the only critical point is at the origin.

**(b)** If $i \neq j$, then $f_{x_i x_j}(x_1, x_2, \ldots, x_n) = 4x_i x_j e^{-x_1^2 - x_2^2 - \cdots - x_n^2}$, so $f_{x_i x_j}(0, 0, \ldots, 0) = 0$. Also $f_{x_i x_i}(x_1, x_2, \ldots, x_n) = (-2 + 4x_i^2)e^{-x_1^2 - x_2^2 - \cdots - x_n^2}$, so $f_{x_i x_i}(0, 0, \ldots, 0) = -2$. The Hessian is an $n \times n$ diagonal matrix with $-2$'s on the main diagonal and 0's everywhere else. It is easy to calculate $d_i(0, 0, \ldots, 0) = (-2)^i$ and so by the second derivative test, $f$ has a local maximum at the origin.

**3.** We are asked to maximize the profit $P(x, y) = (x - 2)(80 - 100x + 40y) + (y - 4)(20 + 60x - 35y) = -100x^2 + 40x - 35y^2 + 80y + 100xy - 240$. The partial derivatives are $P_x(x, y) = -200x + 100y + 40$ and $P_y(x, y) = 100x - 70y + 80$. These are both zero at (27/10, 5). You can read the Hessian right off the first derivatives and you see that $d_1 = -200 < 0$ and $d_2 = 4000 > 0$ so profit is maximized when you charge \$2.70 for Mocha and \$5 for Kona.

**4. (a)** Revenue is $R(x, y, z) = 1000x(4 - 0.02x) + 1000y(4.5 - 0.05y) + 1000z(5 - 0.1z) = -20x^2 + 4000x - 50y^2 + 4500y - 100z^2 + 5000z$.

**(b)** When $(x, y, z) = (6, 5, 4)$, the prices of brands X, Y and Z are, respectively, \$3.88, \$4.25, and \$4.60, and when $(x, y, z) = (1, 3, 3)$, the prices are \$3.98, \$4.35, and \$4.70. The difference is $R(1, 3, 3) - R(6, 5, 4) = 31,130 - 62,930 = -31,800$. The revenue will decline by \$31,800 if the prices are raised.

**(c)** The partial derivatives are $R_x(x, y, z) = 4000 - 40x$, $R_y(x, y, z) = 4500 - 100y$, and $R_z(x, y, z) = 5000 - 200z$. Thus the critical point is $(100, 45, 25)$ and hence the selling prices should be \$2 for brand $X$, \$2.25 for brand $Y$ and \$2.50 for brand $Z$.

**5.** We note that there must be both a (global) maximum and a minimum value of $f$ because the constraint equation defines the surface of a sphere, which is compact, and $f$ is continuous, so that the extreme value theorem applies.

**(a)** We find the extrema of $f(x, y, z) = x - \sqrt{3}y$ subject to $g(x, y, z) = x^2 + y^2 + z^2 = 4$. Using the Lagrange multiplier method, we solve

$$\begin{cases} 1 = 2\lambda x \\ -\sqrt{3} = 2\lambda y \\ 0 = 2\lambda z \\ x^2 + y^2 + z^2 = 4. \end{cases}$$

From the first equation, we must have $\lambda \neq 0$, so from the third equation $z = 0$. Then the first two equations imply that $y = -\sqrt{3}x$. Thus, since $x^2 + 3x^2 = 4$, our critical points are $\pm(1, -\sqrt{3}, 0)$. We evaluate $f$ at these points to find that we have a maximum of 4 at $(1, -\sqrt{3}, 0)$ and a minimum of $-4$ at $(-1, \sqrt{3}, 0)$.

**(b)** Now we are looking at the function $g(\varphi, \theta) = f(2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi) = 2\sin\varphi\cos\theta - 2\sqrt{3}\sin\varphi\sin\theta$. Thus $g_\varphi(\varphi, \theta) = 2\cos\varphi\cos\theta - 2\sqrt{3}\cos\varphi\sin\theta$ and $g_\theta(\varphi, \theta) = -2\sin\varphi\sin\theta - 2\sqrt{3}\sin\varphi\cos\theta$ so that we should solve

$$\begin{cases} -2\sin\varphi(\sin\theta + \sqrt{3}\cos\theta) = 0 \\ 2\cos\varphi(\cos\theta - \sqrt{3}\sin\theta) = 0. \end{cases}$$

Either $\varphi = 0, \pi$ and $\cos\theta = \sqrt{3}\sin\theta$ (hence $\tan\theta = 1/\sqrt{3}$ so $\theta = \pi/6, 7\pi/6$), or $\varphi = \pi/2, 3\pi/2$ and $\sin\theta = -\sqrt{3}\cos\theta$ (hence $\tan\theta = -\sqrt{3}$ so $\theta = 2\pi/3, 5\pi/3$). Note that these points are (using $(x, y, z) = (2\sin\varphi\cos\theta,$ $2\sin\varphi\sin\theta, 2\cos\varphi)$):

$$\begin{aligned}
(\varphi, \theta) &= \left(0, \tfrac{\pi}{6}\right) &&\Longleftrightarrow && (x, y, z) = (0, 0, 2) \\
(\varphi, \theta) &= \left(0, \tfrac{7\pi}{6}\right) &&\Longleftrightarrow && (x, y, z) = (0, 0, 2) \\
(\varphi, \theta) &= \left(\pi, \tfrac{\pi}{6}\right) &&\Longleftrightarrow && (x, y, z) = (0, 0, -2) \\
(\varphi, \theta) &= \left(\pi, \tfrac{7\pi}{6}\right) &&\Longleftrightarrow && (x, y, z) = (0, 0, -2) \\
(\varphi, \theta) &= \left(\tfrac{\pi}{2}, \tfrac{2\pi}{3}\right) &&\Longleftrightarrow && (x, y, z) = (-1, \sqrt{3}, 0) \\
(\varphi, \theta) &= \left(\tfrac{\pi}{2}, \tfrac{5\pi}{3}\right) &&\Longleftrightarrow && (x, y, z) = (1, -\sqrt{3}, 0) \\
(\varphi, \theta) &= \left(\tfrac{3\pi}{2}, \tfrac{2\pi}{3}\right) &&\Longleftrightarrow && (x, y, z) = (-1, \sqrt{3}, 0) \\
(\varphi, \theta) &= \left(\tfrac{3\pi}{2}, \tfrac{5\pi}{3}\right) &&\Longleftrightarrow && (x, y, z) = (1, -\sqrt{3}, 0).
\end{aligned}$$

We obtain the same points as in part (a), plus the additional critical points $(0, 0, 2)$ and $(0, 0, -2)$, which are not global extrema, since $f(0, 0, \pm 2) = 0$.

**6. (a)** Here we are maximizing $T(x, y, z) = 200xyz^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 1$. Using the

Lagrange multiplier method, we solve

$$\begin{cases} 200yz^2 = 2\lambda x \\ 200xz^2 = 2\lambda y \\ 400xyz = 2\lambda z \\ x^2 + y^2 + z^2 = 1. \end{cases}$$

From the third equation, $\lambda \neq 0$ so $2\lambda = 400xy$ so $2x^2 = 2y^2 = z^2$. From the last equation we see that $4x^2 = 1$ so our critical points are the eight possible combinations of $x = \pm 1/2, y = \pm 1/2$ and $z = \pm 1/\sqrt{2}$. The temperature is a maximum of 25 when the sign of $x$ and $y$ are the same. This is at the four points $\pm(1/2, 1/2, \pm 1/\sqrt{2})$.

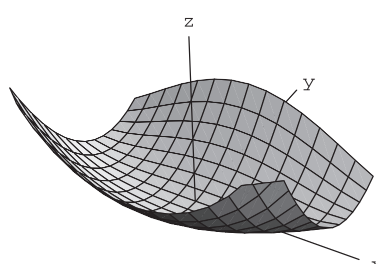
**7.** **(a)** $f_x(x, y) = 2x(-3y + 4x^2)$ while $f_y(x, y) = 2y - 3x^2$. From $f_x$ we see that either $x = 0$ or $y = (4/3)x^2$. But from the second equation $y = (3/2)x^2$. So we conclude that the only solution is at (0, 0).

**(b)** $f_{xx}(x, y) = 6(-y + 4x^2), f_{xy}(x, y) = -6x$, and $f_{yy}(x, y) = 2$. At the origin, the Hessian is $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ and so the determinant is 0 and the critical point is degenerate.

**(c)** If $y = mx$ then the original equation becomes $F(x) = m^2x^2 - 3mx^3 + 2x^4$. We calculate $F'(x) = 2m^2x - 9mx^2 + 8x^3 = 2x(m^2 - 9mx/2 + 4x^2)$. From the second derivative we see that $F''(x) = 2m^2 - 18mx + 24x^2$. This is positive at $x = 0$ for all $m \neq 0$ so there is a minimum for $x = 0$ along any line other than the two axes. When $m = 0, F'(x) = 8x^3$ and so the first derivative test implies that there is a minimum at $x = 0$ when $m = 0$. Finally, consider $G(y) = f(0, y) = y^2$. This clearly has a minimum at $y = 0$. We've shown that along any line through the origin, $f$ has a minimum at (0, 0).

**(d)** Consider $g(x) = f(x, 3x^2/2) = (-x^2/2)(x^2/2) = -x^4/4$. From the derivative $g'(x) = -x^3$ we see that $g$ has a maximum at $x = 0$ and hence $f$ has a maximum at the origin when constrained to the given parabola. This means that the origin is actually a saddle point for $f$.

**(e)** A portion of the surface is shown below.

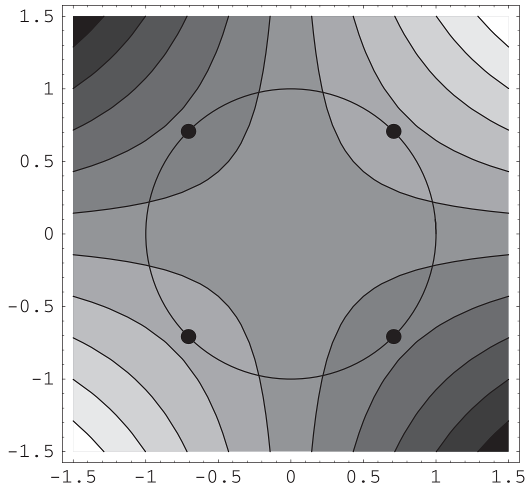



**8.** **(a)** Here we are finding the critical points of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 1 = 0$. So taking the partials of $f(x, y) = \lambda g(x, y)$ along with the constraint we get the following system of equations.

$$\begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ 1 = x^2 + y^2. \end{cases}$$

The solutions correspond to $\lambda = \pm 1/2$ and are the four critical points $(1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2}), (1/\sqrt{2}, -1/\sqrt{2})$, and $(-1/\sqrt{2}, -1/\sqrt{2})$.

**(b)** Here is a contour plot of $f(x, y) = xy$ along with the constraint curve $x^2 + y^2 = 1$ and the four critical points.



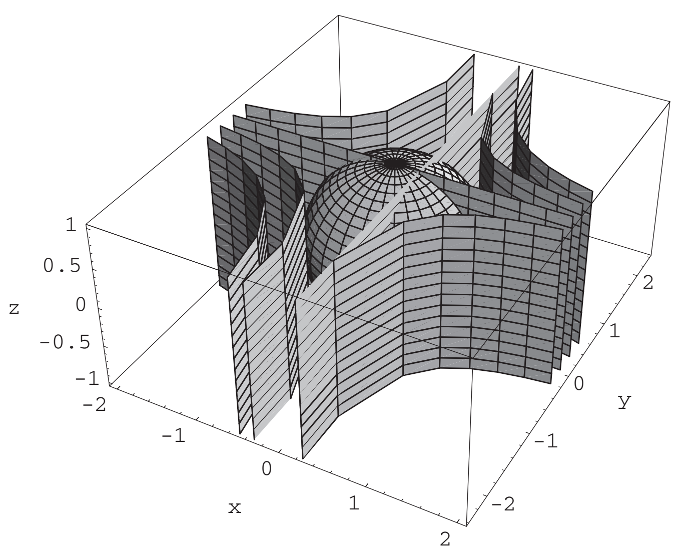

**(c)** You can see from the figure that $f$ is at its highest value along the constraint curve at two of the critical points and at its lowest at two of the others. In particular, $f$ has a constrained max at $\pm(1/\sqrt{2}, 1/\sqrt{2})$ and has a constrained min at $\pm(1/\sqrt{2}, -1/\sqrt{2})$.

**9. (a)** Here we are finding the critical points of $f(x, y, z) = xy$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. So taking the partials of $f(x, y, z) = \lambda g(x, y, z)$ along with the constraint we get the following system of equations.

$$\begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ 0 = 2\lambda z \\ 1 = x^2 + y^2 + z^2 \end{cases}$$

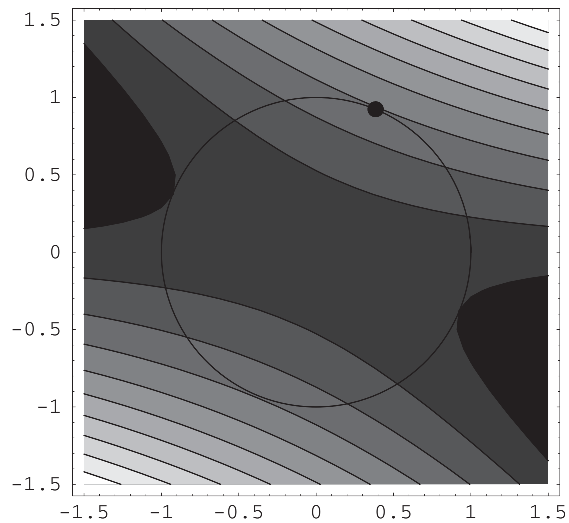
This problem is very similar to Exercise 8 and so it is no surprise that we again get four critical points corresponding to $\lambda = \pm 1/2$. They are $(1/\sqrt{2}, 1/\sqrt{2}, 0)$, $(-1/\sqrt{2}, 1/\sqrt{2}, 0)$, $(1/\sqrt{2}, -1/\sqrt{2}, 0)$, and $(-1/\sqrt{2}, -1/\sqrt{2}, 0)$. We also get critical points at the two poles corresponding to $\lambda = 0$. These are at $(0, 0, \pm 1)$.

**(b)** Of course, it is harder to represent this situation than its lower-dimensional counterpart. Here are some level sets, the unit sphere and the critical points.



**(c)** The arguments that $f$ has a constrained max at $\pm(1/\sqrt{2}, 1/\sqrt{2}, 0)$ and has a constrained min at $\pm(1/\sqrt{2}, -1/\sqrt{2}, 0)$ are the same as in Exercise 8. The two poles must be saddle points. If you travel in a direction where $y = x$, $f(x, y)$ is increasing while if you travel in a direction where $y = -x$, $f(x, y)$ is decreasing. So there are saddle points at $(0, 0, \pm 1)$.

**10.** From the diagram you can see that we are minimizing $f(x, y) = (x + y)y$ subject to the constraint that $x^2 + y^2 = 1$. Because this is a physical problem, we can assume that $x > 0$ and $y > 0$. A look at the contour plot for $f$ along with the constraint curve lets us see that this solution will be a max.




Our system of equations is

$$\begin{cases} y &= 2\lambda x \\ x + 2y &= 2\lambda y \\ 1 &= x^2 + y^2 \end{cases}$$

Solving gives us one solution for which $x$ and $y$ are positive, namely $x = (\sqrt{2 + \sqrt{2}}/2)(\sqrt{2} - 1)$ and $y = (\sqrt{2 + \sqrt{2}}/2)$. The area of the rectangle is, therefore, $(\sqrt{2} + 1)/2$.

**11.** Minimize $f(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2$ subject to the constraint $g(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n = 1$ where not all the $a_i$'s are zero. We solve

$$\begin{cases} 2x_i = a_i\lambda & \text{for } 1 \le i \le n \\ a_1x_1 + a_2x_2 + \cdots + a_nx_n = 1. \end{cases}$$

This means that our constrained critical point is at $x_i = a_i\lambda/2$ and $2/(a_1^2 + a_2^2 + \cdots + a_n^2) = \lambda$ so $x_i = a_i/(a_1^2 + a_2^2 + \cdots + a_n^2)$. So our minimum is

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= f\left(\frac{a_1}{a_1^2 + a_2^2 + \cdots + a_n^2}, \frac{a_2}{a_1^2 + a_2^2 + \cdots + a_n^2}, \ldots, \frac{a_n}{a_1^2 + a_2^2 + \cdots + a_n^2}\right) \\ &= \left(\frac{a_1}{a_1^2 + a_2^2 + \cdots + a_n^2}\right)^2 + \left(\frac{a_2}{a_1^2 + a_2^2 + \cdots + a_n^2}\right)^2 + \cdots + \left(\frac{a_n}{a_1^2 + a_2^2 + \cdots + a_n^2}\right)^2 \\ &= \frac{1}{a_1^2 + a_2^2 + \cdots + a_n^2}. \end{aligned}$$

**12.** Minimize the function $f(x_1, x_2, \ldots, x_n) = (a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2$ subject to the constraint $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ where not all the $a_i$'s are zero. We solve

$$\begin{cases} 2a_i(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = 2\lambda x_i \;\; \text{for } 1 \le i \le n, \text{ and} \\ x_1^2 + x_2^2 + \cdots + x_n^2 = 1. \end{cases}$$

From the first equation, $x_i^2 = (a_i x_i/\lambda)(a_1x_1 + a_2x_2 + \cdots + a_nx_n)$, so

$$\begin{aligned}\lambda &= a_1x_1(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + a_2x_2(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + \cdots \\ &\quad + a_nx_n(a_1x_1 + a_2x_2 + \cdots + a_nx_n) \\ &= (a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2 \quad \text{so} \\ x_i &= \frac{a_i}{a_1x_1 + a_2x_2 + \cdots + a_nx_n} \quad \text{and finally,} \\ \sum_{i=1}^n x_i^2 &= \frac{\sum_{i=1}^n a_i^2}{(a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2} = 1.\end{aligned}$$

Now we can substitute back into the original equation:

$$\begin{aligned}f(x_1, x_2, \ldots, x_n) &= f\left(\frac{a_1}{a_1x_1 + a_2x_2 + \cdots + a_nx_n}, \ldots, \frac{a_n}{a_1x_1 + a_2x_2 + \cdots + a_nx_n}\right) \\ &= \left(\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{a_1x_1 + a_2x_2 + \cdots + a_nx_n}\right)^2 \\ &= \left(\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{(a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2}\right)(a_1^2 + a_2^2 + \cdots + a_n^2) \\ &= a_1^2 + a_2^2 + \cdots + a_n^2.\end{aligned}$$

**13.** Since the faces are parallel to the coordinate planes, we can reduce the problem to maximizing $M(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = x^2 + 2y^2 + 4z^2 = 12$, where $x$, $y$, and $z$ are all positive. Here, by the symmetry of the problem, we are maximizing the volume of one eighth of the box and therefore we will have the dimensions of the box itself by doubling $x$, $y$, and $z$. We solve

$$\begin{cases} yz = 2\lambda x \\ xz = 4\lambda y \\ xy = 8\lambda z \\ x^2 + 2y^2 + 4z^2 = 12. \end{cases}$$

So $x^2 = 2y^2 = 4z^2$ and $12z^2 = 12$ so a critical point is $(2, \sqrt{2}, 1)$. The dimensions of the box are twice these values so the largest box is $4 \times 2\sqrt{2} \times 2$.

**14.** We are minimizing the cost of producing a sphere and a cylinder of equal radii with the given constraints. We also need to convert 8000 gallons to $8000/7.480520 \approx 1069.444$ cubic feet. So minimize $V(r, h) = 2\pi rh + 8\pi r^2$ subject to $g(r, h) = \pi r^2 h + (4/3)\pi r^3 = 1069.444$. We solve

$$\begin{cases} 2\pi h + 16\pi r = \lambda(2\pi rh + 4\pi r^2) \\ 2\pi r = \lambda(\pi r^2) \\ \pi r^2 h + (4/3)\pi r^3 = 1069.444. \end{cases}$$

Physically, $r$ cannot be zero, so by the second equation $\lambda = 2/r$ and then by the first $h = 4r$ and so by the third $1069.444 = 4\pi r^3 + (4/3)\pi r^3 = (16\pi/3)r^3$. Therefore, the best dimensions are $r \approx \sqrt[3]{63.8277} \approx 3.9964$ feet and $h \approx 15.9856$ feet.

**15.** Minimize $M(x, y, z) = x^2 + y^2 + z^2$ subject to $x^2 - (y - z)^2 = 1$. We solve

$$\begin{cases} 2x = 2\lambda x \\ 2y = -2\lambda(y - z) \\ 2z = 2\lambda(y - z) \\ x^2 - (y - z)^2 = 1. \end{cases}$$

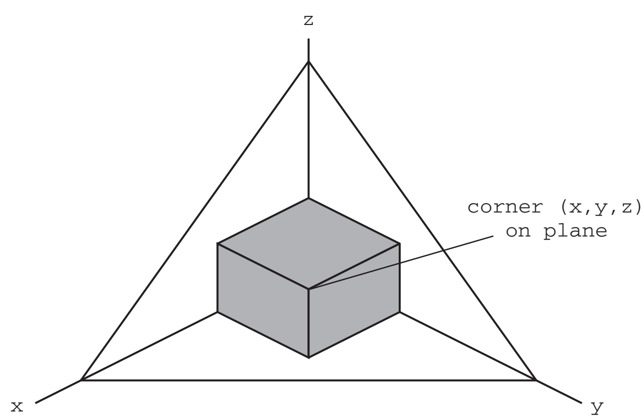
Since the last equation implies that $x \neq 0$, the first equation gives us that $\lambda = 1$, so $y = z = 0$ and thus $x = \pm 1$. The minimum distance is, therefore, 1.

**16.** Place the vertex of the cone at the North Pole $(0, 0, a)$, with the axis of symmetry of the cone coinciding with the $z$-axis. The height of the cone is $h$ and the radius is $r$. We are maximizing $V(r, h) = (1/3)\pi r^2 h$ subject to the constraint $(h-a)^2 + r^2 = a^2$ or $g(h, a) = h^2 - 2ha + r^2 = 0$. We solve

$$\begin{cases} (2/3)\pi r h = 2\lambda r \\ (1/3)\pi r^2 = 2\lambda(h-a) \\ h^2 - 2ha + r^2 = 0. \end{cases}$$

From the first equation we know $\lambda \neq 0$ and $\pi h/3 = \lambda$. So substitute this into the second equation to find that $r^2 = 2h^2 - 2ah$. Solve this with the third equation to find that $h = 4a/3$ and $r = 2\sqrt{2}a/3$.

**17.** We want to maximize $V = xyz$ subject to $bcx + acy + abz = abc$.




Thus we solve

$$\begin{cases} \nabla V = \lambda \nabla(bcx + acy + abz) \\ bcx + acy + abz = abc \end{cases} \quad \text{or} \quad \begin{cases} yz = \lambda bc \\ xz = \lambda ac \\ xy = \lambda ab \\ \dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1. \end{cases}$$

Hence $\lambda = \dfrac{yz}{bc} = \dfrac{xz}{ac} = \dfrac{xy}{ab}$.

$$\frac{yz}{bc} = \frac{xz}{ac} \Leftrightarrow z = 0 \quad \text{or} \quad y = \frac{bc}{ac}x = \frac{b}{a}x.$$

Now $z = 0$ makes $V = 0$, so this cannot possibly maximize. Thus $y = (b/a)x$. Now $\dfrac{yz}{bc} = \dfrac{xy}{ab} \Leftrightarrow y = 0$ (reject) or $z = \dfrac{bc}{ab}x$ or $z = \dfrac{c}{a}x$. Hence the constraint becomes

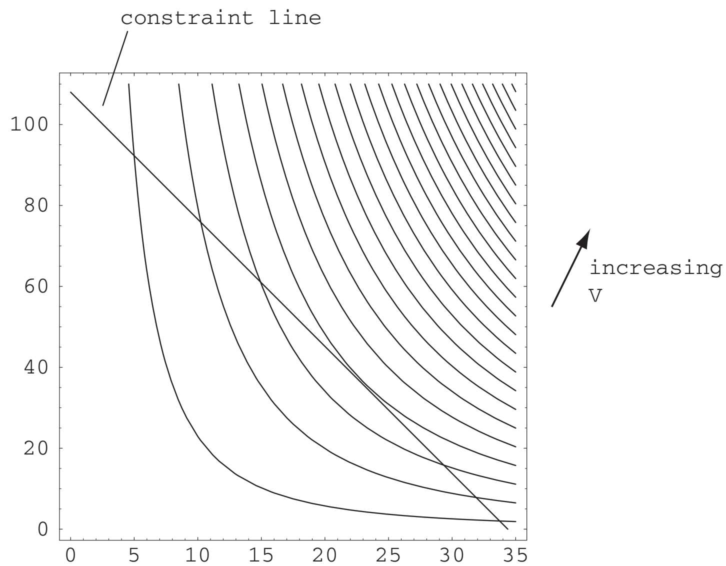
$$\frac{x}{a} + \frac{x}{a} + \frac{x}{a} = 1 \quad \text{so} \quad x = a/3 \Rightarrow y = b/3 \quad z = c/3.$$

**18.** We have $V(x, y) = \pi\left(\dfrac{x}{2}\right)^2 y = \dfrac{\pi}{4}x^2 y$ with $\pi x + y \leq 108$.

**(a)** We maximize $V$ subject to $g(x, y) = \pi x + y = 108$. Thus, with a Lagrange multiplier we solve

$$\begin{cases} \dfrac{\pi x y}{2} = \pi\lambda \\ \dfrac{\pi x^2}{4} = \lambda \\ \pi x + y = 108 \end{cases}.$$

The first two equations imply that $\lambda = \dfrac{xy}{2} = \dfrac{\pi x^2}{4}$ so that either $x = 0$ (which we reject) or $\dfrac{y}{2} = \dfrac{\pi x}{4}$, so $y = \dfrac{\pi x}{2}$. Thus, in the constraint we must have $\pi x + \dfrac{\pi x}{2} = 108$ so $x = \dfrac{2 \cdot 108}{3\pi} = \dfrac{72}{\pi}$. Hence the maximizing dimensions are $\dfrac{72}{\pi}''$ diameter, $36''$ length. (That these dimensions really do maximize volume may be seen from the following picture.)




**(b)** Perhaps this is an easier method: $\pi x + y = 108 \Leftrightarrow y = 108 - \pi x$ so $v(x) = V(x, 108 - \pi x) = \dfrac{\pi x^2}{4}(108 - \pi x)$ defined on $\left[0, \dfrac{108}{\pi}\right]$. Thus $v'(x) = \dfrac{\pi}{4}(216x - 3\pi x^2)$ so critical points are $x = 0, \dfrac{72}{\pi}$.

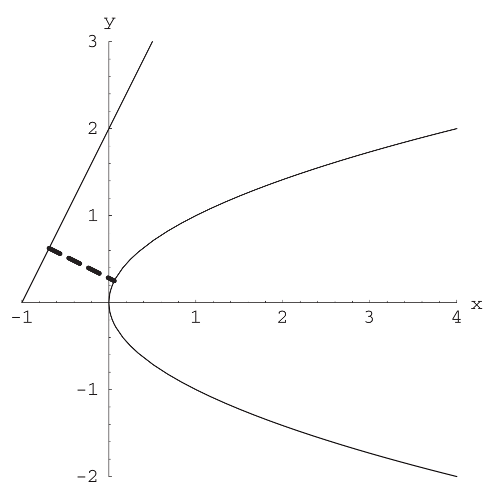
Compare values: $v(0) = 0$, $v\left(\dfrac{72}{\pi}\right) > 0$, $v\left(\dfrac{108}{\pi}\right) = 0$, so $x = 72/\pi$ must give the *absolute* maximum.

**19.** The two equations are $x = y/2 - 1$ and $x = y^2$. We will minimize the square of the distance between a point $(x_1, y_1)$ on the line and a point $(x_2, y_2)$ on the parabola. Maximize $f(y_1, y_2) = (y_1/2 - 1 - y_2^2)^2 + (y_2 - y_1)^2$. Take the first partials:

$$f_{y_1}(y_1, y_2) = \frac{5}{2}y_1 - 1 - y_2^2 - 2y_2 \quad \text{and}$$

$$f_{y_2}(y_1, y_2) = 4y_2^2 - 2y_1y_2 + 6y_2 - 2y_1.$$

Set these equal to zero and solve to find the critical point at $(y_1, y_2) = (5/8, 1/4)$. The minimal distance is therefore $3\sqrt{5}/8$.




**20. (a)** For each section the time is the distance divided by the rate and the hypotenuse is the altitude divided by the cosine of the angle that is formed by the altitude and the hypotenuse. So

$$T(\theta_1, \theta_2) = \frac{a}{v_1 \cos\theta_1} + \frac{b}{v_2 \cos\theta_2}.$$

**(b)** We are minimizing time subject to the constraint that the horizontal separation is constant: $a\tan\theta_1 + b\tan\theta_2 = c$. We solve

$$\begin{cases} \dfrac{a\sin\theta_1}{v_1\cos^2\theta_1} = \dfrac{\lambda a}{\cos^2\theta_1} \\ \dfrac{b\sin\theta_2}{v_2\cos^2\theta_2} = \dfrac{\lambda b}{\cos^2\theta_2} \\ a\tan\theta_1 + b\tan\theta_2 = c. \end{cases}$$

The first two equations immediately give the result: $\dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{v_1}{v_2}$.

**21.** We are minimizing the square of the distance $f(x, y) = (x - x_0)^2 + (y - y_0)^2$ subject to the constraint $ax + by = d$. We solve

$$\begin{cases} 2(x - x_0) = a\lambda \\ 2(y - y_0) = b\lambda \\ ax + by = d. \end{cases}$$

Solving, we see that $x = (a\lambda + 2x_0)/2$ and $y = (b\lambda + 2y_0)/2$ so substituting for $x$ and $y$ in the third equation $(a^2 + b^2)\lambda = 2(d - ax_0 - by_0)$. Also substituting for $x$ and $y$ in $f$ we see that

$$\begin{aligned} f(x, y) &= \left(\frac{a\lambda}{2}\right)^2 + \left(\frac{b\lambda}{2}\right)^2 = \left(\frac{a^2 + b^2}{4}\right)\lambda^2 = \frac{a^2 + b^2}{4}\left(\frac{2(d - ax_0 - by_0)}{a^2 + b^2}\right)^2 \\ &= \frac{(d - ax_0 - by_0)^2}{a^2 + b^2} \end{aligned}$$

so the distance $D$ is the square root of this: $D = \dfrac{|ax_0 + by_0 - d|}{\sqrt{a^2 + b^2}}$.

**22.** This is very similar to Exercise 21. Minimize the square of the distance $f(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$ subject to the constraint $ax + by + cz = d$. We solve

$$\begin{cases} 2(x - x_0) = a\lambda \\ 2(y - y_0) = b\lambda \\ 2(z - z_0) = c\lambda \\ ax + by + cz = d. \end{cases}$$

Solving, we see that $x = (a\lambda + 2x_0)/2, y = (b\lambda + 2y_0)/2$ and $z = (c\lambda + 2z_0)/2$ so substituting for $x$, $y$ and $z$ in the fourth equation $(a^2 + b^2 + c^2)\lambda = 2(d - ax_0 - by_0 - cz_0)$. Also substituting for $x$, $y$ and $z$ in $f$ we see that

$$f(x, y, z) = \left(\frac{a^2 + b^2 + c^2}{4}\right)\lambda^2 = \frac{(d - ax_0 - by_0 - cz_0)^2}{a^2 + b^2}$$

so the distance $D$ is the square root of this: $D = \dfrac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$.

**23. (a)** We solve

$$\begin{cases} 2xy^2z^2 = 2\lambda x \\ 2x^2yz^2 = 2\lambda y \\ 2x^2y^2z = 2\lambda z \\ x^2 + y^2 + z^2 = a^2. \end{cases}$$

If $\lambda = 0$, then at least one of $x$, $y$ and $z$ is 0 and this corresponds to a minimum. If $\lambda \neq 0$, we see that, at a critical point, $x^2 = y^2 = z^2$, so $3x^2 = a^2$ or $x^2 = a^2/3$. Therefore, at a critical point,

$$f(x, y, z) = \left(\frac{a^2}{3}\right)^3 = \frac{a^6}{27}.$$

**(b)** In part (a) we showed $x^2y^2z^2 \leq (a^2/3)^3 = [(x^2 + y^2 + z^2)/3]^3$ and so this result follows immediately.

**(c)** We make the appropriate adjustments to parts (a) and (b) and maximize $f(x_1, x_2, \ldots, x_n) = x_1^2 x_2^2 \cdots x_n^2$ subject to $x_1^2 + x_2^2 + \cdots + x_n^2 = a^2$. Because, as in part (a), the case $\lambda = 0$ corresponds to a minimum, we see that at a maximum no $x_i$ is 0. So we solve

$$\begin{cases} 2x_1^2 x_2^2 \cdots x_n^2 / x_i = 2\lambda x_i & \text{for } 1 \leq i \leq n \\ x_1^2 + x_2^2 + \cdots + x_n^2 = a^2. \end{cases}$$

At a maximum, $x_1^2 = x_2^2 = \cdots = x_n^2$, so $x_i^2 = a^2/n$. Therefore, the maximum of $f$ is $(a^2/n)^n$. So we conclude that $x_1^2 x_2^2 \cdots x_n^2 \leq (a^2/n)^n = [(x_1^2 + x_2^2 + \cdots + x_n^2)/n]^n$. The result follows immediately.

**(d)** We found that $f$ was maximized when $x_1^2 = x_2^2 = \cdots = x_n^2$ so, since here we are assuming $x_i > 0$ for all $i$, the equality holds when $x_1 = x_2 = \cdots = x_n$.

**24. (a)**

$$\frac{\partial f}{\partial x_k} = \sum_{j=1}^{n} a_{k_j} x_j + \sum_{i=1}^{n} a_{ik} x_i = \sum_{j=1}^{n} (a_{jk} + a_{kj}) x_j$$

$$\frac{\partial g}{\partial x_k} = 2x_k.$$

Thus the Lagrange multiplier system is

$$\begin{cases} \sum_j (a_{j1} + a_{1j}) x_j = 2\lambda x_1 \\ \vdots \\ \sum_j (a_{jn} + a_{nj}) x_j = 2\lambda x_n \\ x_1^2 + \cdots + x_n^2 = 1. \end{cases}$$

**(b)** Because $A$ is symmetric, $a_{jk} = a_{kj}$ so the system becomes

$$\begin{cases} \sum_j 2a_{1j} x_j = 2\lambda x_1 \\ \vdots \\ \sum_j 2a_{nj} x_j = 2\lambda x_n \\ x_1^2 + \cdots + x_n^2 = 1. \end{cases}$$

The first $n$ equations come from $\nabla f = \lambda \nabla g$ and simplify to

$$\begin{cases} \sum_j a_{1j} x_j = \lambda x_1 \\ \vdots \\ \sum_j a_{nj} x_j = \lambda x_n. \end{cases}$$

Note that $\sum_j a_{kj} x_j$ is the dot product of the $k$th row of $A$ with $\mathbf{x}$. So the $n$ equations, taken together, express

$$A\mathbf{x} = \lambda \mathbf{x}.$$

**(c)**

$$\begin{aligned} f(x_1, \ldots, x_n) &= \mathbf{x}^T A \mathbf{x} = \mathbf{x}^T (\lambda \mathbf{x}) \quad (\mathbf{x} \text{ is an eigenvector}) \\ &= \lambda(\mathbf{x}^T \mathbf{x}) = \lambda \mathbf{x} \cdot \mathbf{x} \\ &= \lambda \|\mathbf{x}\|^2 = \lambda \cdot 1, \end{aligned}$$

since $\mathbf{x}$ is assumed to be a unit vector.

**25. (a)** To set things up using Lagrange multipliers, we solve

$$\begin{cases} 2ax + 2by = 2\lambda x \\ 2bx + 2cy = 2\lambda y \\ x^2 + y^2 = 1 \end{cases} \Leftrightarrow \begin{cases} (a-\lambda)x + by = 0 \\ bx + (c-\lambda)y = 0 \\ x^2 + y^2 = 1. \end{cases}$$

In the last system, multiply the first equation by $\lambda - c$ and the second by $b$, then add to obtain:

$$((a-\lambda)(\lambda - c) + b^2)x = 0.$$

Now multiply the first equation by $b$ and the second by $\lambda - a$, then add to get:

$$(b^2 + (\lambda - a)(c - \lambda))y = 0.$$

Since $x^2 + y^2 = 1$, we cannot have both $x$ and $y$ equal to 0. Thus

$$b^2 + (\lambda - a)(c - \lambda) = 0 \Leftrightarrow \lambda^2 - (a+c)\lambda + ac - b^2 = 0.$$

Hence

$$\lambda_1, \lambda_2 = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac - b^2)}}{2}.$$

**(b)** Rewriting, $\lambda_1, \lambda_2 = \dfrac{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2}$. $(a-c)^2 + 4b^2 \geq 0$ so the eigenvalues are always real.

**26. (a)** $\lambda_1 = \lambda_2 \Leftrightarrow (a-c)^2 + 4b^2 = 0 \Leftrightarrow a = c, b = 0$ so $f(x, y) = a(x^2 + y^2)$.

**(b)** The eigenvalues are the max and min values of $f$ on the circle. If both are positive, then $f$ has a positive minimum on the circle; hence $f$ must be positive on the entire circle.

**(c)** If both eigenvalues are negative, then $f$ has a negative maximum on the circle—so $f$ must be negative on the entire circle.

**27. (a)**

$$f(kx_1, \ldots, kx_n) = \sum_{i,j=1}^{n} a_{ij}(kx_i)(kx_j) = k^2 \sum_{i,j=1}^{n} a_{ij}x_i x_j$$

**(b)** Let $\mathbf{u} = \mathbf{x}/\|\mathbf{x}\|$ when $\mathbf{x} \neq \mathbf{0}$. Then $\mathbf{u}$ is a point on the unit hypersphere. If $f$ has a positive minimum on the hypersphere, then $f$ must be positive on the entire hypersphere. Hence, for $\mathbf{x} \neq \mathbf{0}$:

$$f(\mathbf{x}) = f(k\mathbf{u}) = k^2 f(\mathbf{u}) > 0 \quad (k = \|\mathbf{x}\|).$$

The case where $f$ has a negative maximum on the hypersphere is similar.

**(c)** Clearly the converses of the results of part (b) hold (i.e., if $f(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$, then $f$ is positive on the hypersphere ...). From Exercise 24, the minimum value of $f$ is the smallest eigenvalue of $A$. Thus the quadratic form is positive definite $\Leftrightarrow$ $f(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$ $\Leftrightarrow$ $f$ is positive on the hypersphere $\Leftrightarrow$ the smallest eigenvalue of $A$ is positive $\Leftrightarrow$ all eigenvalues are positive. (The negative definite result is similar.)

## Chapter 5

# Multiple Integration

### 5.1 Introduction: Areas and Volumes

**1.**

$$\int_0^2 \int_1^3 (x^2 + y)\, dy\, dx = \int_0^2 (x^2 y + y^2/2)\Big|_{y=1}^{y=3} dx = \int_0^2 ((3x^2 + 9/2) - (x^2 + 1/2))\, dx$$
$$= \int_0^2 (2x^2 + 4)\, dx = (2x^3/3 + 4x)\Big|_0^2 = 40/3.$$

**2.**

$$\int_0^\pi \int_1^2 (y \sin x)\, dy\, dx = \int_0^\pi \left(\frac{y^2}{2} \sin x\right)\Big|_{y=1}^{y=2} dx = \int_0^\pi \left((2 \sin x) - \left(\frac{1}{2} \sin x\right)\right) dx$$
$$= \frac{3}{2} \int_0^\pi (\sin x)\, dx = -\frac{3}{2}(\cos x)\Big|_0^\pi = \frac{3}{2} + \frac{3}{2} = 3.$$

**3.**

$$\int_{-2}^4 \int_0^1 (xe^y)\, dy\, dx = \int_{-2}^4 (xe^y)\Big|_{y=0}^{y=1} dx = \int_{-2}^4 (x(e-1))\, dx$$
$$= \frac{x^2}{2}(e-1)\Big|_{-2}^4 = (8-2)(e-1) = 6(e-1).$$

**4.**

$$\int_0^{\pi/2} \int_0^1 (e^x \cos y)\, dx\, dy = \int_0^{\pi/2} (e^x \cos y)\Big|_{x=0}^{x=1} dy = \int_0^{\pi/2} ((e-1) \cos y)\, dy$$
$$= (e-1) \sin y\Big|_0^{\pi/2} = e - 1.$$

**5.**

$$\int_1^2 \int_0^1 (e^{x+y} + x^2 + \ln y)\, dx\, dy = \int_1^2 \int_0^1 (e^x e^y + x^2 + \ln y)\, dx\, dy = \int_1^2 \left(e^x e^y + \frac{x^3}{3} + x \ln y\right)\Big|_{x=0}^{x=1} dy$$
$$= \int_1^2 \left((e-1)e^y + \frac{1}{3} + \ln y\right) dy = \left((e-1)e^y + \frac{y}{3} + y \ln y - y\right)\Big|_1^2$$
$$= (e-1)(e^2 - e) + \frac{1}{3} + 2 \ln 2 - 1 = e^3 - 2e^2 + e - \frac{2}{3} + 2 \ln 2.$$

**6.**

$$\int_1^9 \int_1^e \left(\frac{\ln \sqrt{x}}{xy}\right) dx\, dy = \frac{1}{2} \int_1^9 \int_1^e \left(\frac{\ln x}{xy}\right) dx\, dy \quad \text{(treat } y \text{ as a constant—use substitution)}$$
$$= \frac{1}{2} \int_1^9 \frac{(\ln x)^2}{2y}\Big|_{x=1}^{x=e} dy = \frac{1}{2} \int_1^9 \left(\frac{1}{2y}\right) dy = \frac{1}{4} \ln y\Big|_1^9 = \frac{\ln 9}{4}.$$

**7. (a)** Here we are fixing $x$ and finding the area of the slices:

$$A(x) = \int_0^2 (x^2 + y^2 + 2)\,dy = \left( x^2 y + \frac{y^3}{3} + 2y \right)\Big|_0^2 = 2x^2 + 20/3.$$

Now we "add up the areas of these slices":

$$V = \int_{-1}^2 A(x)\,dx = \int_{-1}^2 (2x^2 + 20/3)\,dx = \left( \frac{2}{3}x^3 + \frac{20}{3}x \right)\Big|_{-1}^2 = \left( \frac{16}{3} + \frac{40}{3} \right) - \left( -\frac{2}{3} - \frac{20}{3} \right) = 26.$$

**(b)** Now we fix $y$ and find the area of the slices:

$$\begin{aligned} A(y) &= \int_{-1}^2 (x^2 + y^2 + 2)\,dx = \left( \frac{x^3}{3} + y^2 x + 2x \right)\Big|_{-1}^2 \\ &= \left( \frac{8}{3} + 2y^2 + 4 \right) - \left( -\frac{1}{3} - y^2 - 2 \right) = 9 + 3y^2. \end{aligned}$$

Adding up the area of these slices:

$$V = \int_0^2 A(y)\,dy = \int_0^2 (9 + 3y^2)\,dy = (9y + y^3)\Big|_0^2 = 26.$$

**8.** Here we are calculating:

$$\begin{aligned} \int_1^2 \int_0^3 (x + 3y + 1)\,dx\,dy &= \int_1^2 \left( \frac{x^2}{2} + 3yx + x \right)\Big|_0^3 dy = \int_1^2 \left( \frac{9}{2} + 9y + 3 \right)\,dy \\ &= \int_1^2 \left( \frac{15}{2} + 9y \right)\,dy = \left( \frac{15}{2}y + \frac{9}{2}y^2 \right)\Big|_1^2 \\ &= (15 + 18) - (15/2 + 9/2) = 21. \end{aligned}$$

**9.** Here we are calculating

$$\begin{aligned} \int_{-1}^2 \int_0^1 (2x^2 + y^4 \sin \pi x)\,dx\,dy &= \int_{-1}^2 \left( \frac{2}{3}x^3 - \frac{y^4}{\pi}\cos \pi x \right)\Big|_0^1 dy = \int_{-1}^2 \left( \frac{2}{3} + \frac{2y^4}{\pi} \right)\,dy \\ &= \left( \frac{2}{3}y + \frac{2y^5}{5\pi} \right)\Big|_{-1}^2 = \left( \frac{4}{3} + \frac{64}{5\pi} \right) - \left( -\frac{2}{3} - \frac{2}{5\pi} \right) = 2 + \frac{66}{5\pi}. \end{aligned}$$

**10.** This is the volume of the "rectangular box" bounded by the plane $z = 2$, the *xy*-plane, and the planes $x = 1$, $x = 3$, $y = 0$, and $y = 2$. Here we could just calculate the volume of this $2 \times 2 \times 2$ box as 8 without integrating—or

$$V = \int_0^2 \int_1^3 2\,dx\,dy = \int_0^2 2x\Big|_1^3 dy = \int_0^2 4\,dy = 4y\Big|_0^2 = 8.$$

**11.** This is the volume of the region bounded by the paraboloid $z = 16 - x^2 - z^2$, the *xy*-plane, and the planes $x = 1$, $x = 3$, $y = -2$, and $y = 2$. The volume is

$$\begin{aligned} V &= \int_1^3 \int_{-2}^2 (16 - x^2 - y^2)\,dy\,dx = \int_1^3 \left( 16y - x^2 y - \frac{y^3}{3} \right)\Big|_{-2}^2 dx = \int_1^3 \left( 64 - 4x^2 - \frac{16}{3} \right)\,dx \\ &= \left( 64x - \frac{4}{3}x^3 - \frac{16}{3}x \right)\Big|_1^3 = (192 - 36 - 16) - (64 - 4/3 - 16/3) = 248/3. \end{aligned}$$

**12.** This is the volume of the region bounded by $z = \sin x \cos y$, the *xy*-plane, and the planes $x = 0$, $x = \pi$, $y = -\pi/2$, and $y = \pi/2$. The volume is

$$\begin{aligned} V &= \int_{-\pi/2}^{\pi/2} \int_0^\pi (\sin x \cos y)\,dx\,dy = \int_{-\pi/2}^{\pi/2} (-\cos x \cos y)\Big|_0^\pi dy \\ &= 2\int_{-\pi/2}^{\pi/2} \cos y\,dy = 2\sin y\Big|_{-\pi/2}^{\pi/2} = 4. \end{aligned}$$

**13.** This is the volume of the region bounded by $z = 4 - x^2$, the *xy*-plane, and the planes $x = -2,\ x = 2,\ y = 0$, and $y = 5$. The volume is

$$V = \int_0^5 \int_{-2}^2 (4 - x^2)\,dx\,dy = \int_0^5 (4x - x^3/3)\Big|_{-2}^2\,dy = \int_0^5 ((8 - 8/3) - (-8 + 8/3))\,dy$$
$$= \int_0^5 \frac{32}{3}\,dy = \frac{32}{3}y\Big|_0^5 = 160/3.$$

**14.** This is the volume of the region bounded by $z = |x|\sin \pi y$, the *xy*-plane, and the planes $x = -2,\ x = 3,\ y = 0$, and $y = 1$. The volume is

$$V = \int_{-2}^3 \int_0^1 |x| \sin \pi y\,dy\,dx = \int_{-2}^3 -\frac{|x|}{\pi}\cos \pi y\Big|_0^1\,dx = \int_{-2}^3 \frac{2|x|}{\pi}\,dx.$$

At this point we use the definition of absolute value to split this into two quantities:

$$V = \int_{-2}^0 -\frac{2}{\pi}x\,dx + \int_0^3 \frac{2}{\pi}x\,dx = -\frac{x^2}{\pi}\Big|_{-2}^0 + \frac{x^2}{\pi}\Big|_0^3 = \frac{4}{\pi} + \frac{9}{\pi} = \frac{13}{\pi}.$$

**15.**

$$\int_{-5}^5 \int_{-1}^2 (5 - |y|)\,dx\,dy = \int_{-5}^5 (5 - |y|)x\Big|_{x=-1}^2\,dy$$
$$= \int_{-5}^5 (5 - |y|) \cdot 3\,dy = 150 - 3\int_{-5}^5 |y|\,dy$$
$$= 150 - 3\int_{-5}^0 (-y)\,dy - 3\int_0^5 y\,dy$$
$$= 150 + \frac{3}{2}y^2\Big|_{-5}^0 - \frac{3}{2}y^2\Big|_0^5 = 150 - \frac{75}{2} - \frac{75}{2} = 75.$$

The iterated integral gives the volume of the region bounded by the graph of $z = 5 - |y|$, the *xy*-plane, and the planes $x = -1$, $x = 2$, $y = -5$, $y = 5$. (The solid so described is a rectangular prism.)

**16.** We have $V = \displaystyle\int_a^b \int_c^d f(x, y)\,dy\,dx$. Since $0 \le f(x, y) \le M$, the solid bounded by $y = f(x, y)$, the *xy*-plane, and the planes $x = a$, $x = b$, $y = c$, $y = d$ sits inside the rectangular block of height $M$ and base bounded by $x = a$, $x = b$, $y = c$, $y = d$. Hence $V \le M(b - a)(d - c)$

## 5.2 Double Integrals

**1.** Since the integrand $f(x, y) = y^3 + \sin 2y$ is continuous, the double integral $\iint_R (y^3 + \sin 2y)\,dA$ exists by Theorem 2.4. Now consider a Riemann sum corresponding to the double integral that we obtain by partitioning the rectangle $[0, 3] \times [-1, 1]$ symmetrically with respect to the $x$-axis and by choosing test points $\mathbf{c}_{ij}$ in each subrectangle that are also symmetric with respect to the $x$-axis. Then

$$S = \sum_{i,j} f(\mathbf{c}_{ij})\,\Delta A_{ij} = \sum_{i,j} (y_{ij}^3 + \sin 2y_{ij})\,\Delta A_{ij}$$

(where $y_{ij}$ denotes the $y$-coordinate of $\mathbf{c}_{ij}$) must be zero since the terms cancel in pairs because $f(x, -y) = -f(x, y)$. When we shrink the rectangles in the limit, we can arrange to preserve all the symmetry. Hence the limit under such restrictions must be zero and thus the overall limit (which must exist in view of Theorem 2.4) must also be zero.

**2.** The integrand $f(x, y) = x^5 + 2y$ is continuous, so the double integral exists by Theorem 2.4. Consider a Riemann sum corresponding to the double integral that we obtain by partitioning the rectangle $[-3, 3] \times [-2, 2]$ symmetrically with respect to *both* coordinate axes and by choosing test points $\mathbf{c}_{ij}$ in each subrectangle that are also symmetric with respect to both axes. Then

$$S = \sum_{i,j} f(\mathbf{c}_{ij})\,\Delta A_{ij} = \sum_{i,j} (x_{ij}^5 + 2y_{ij})\,\Delta A_{ij} = \sum_{i,j} x_{ij}^5\,\Delta A_{ij} + \sum_{i,j} 2y_{ij}\,\Delta A_{ij}$$

must be zero since the terms in each sum will cancel in pairs (because $(-x)^5 = -x^5$ and $2(-y) = -2y$). When we shrink the rectangles in the limit, we can arrange to preserve all the symmetry. Hence the limit under such restrictions must be zero and thus the overall limit (which must exist in view of Theorem 2.4) must also be zero.

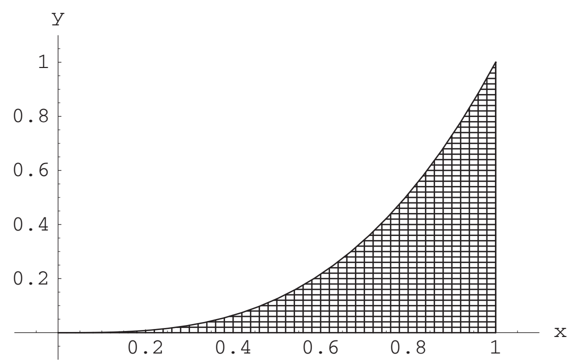
*Note: you may want to discuss Exercise 3 (b) before assigning it, to get your students in the habit of looking critically at problems before working on them.*

**3. (a)** We are computing

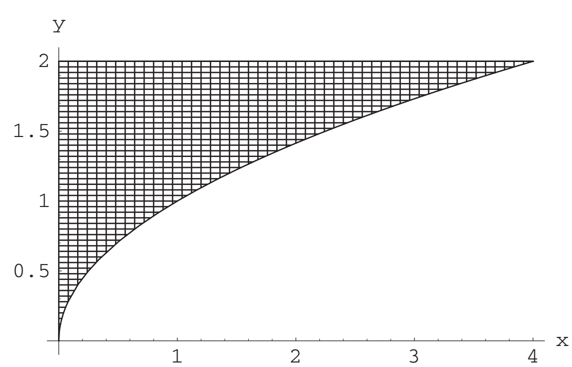
$$\int_{-2}^{2}\int_{0}^{4-x^2} x^3\,dy\,dx = \int_{-2}^{2} x^3 y\Big|_{0}^{4-x^2} dx = \int_{-2}^{2}(4x^3 - x^5)\,dx = (x^4 - x^6/6)\Big|_{-2}^{2} = 0.$$

**(b)** The integrand is an odd function depending only on $x$ and the region is symmetric about the $y$-axis. The students encounter this situation when they looked at $\int_{-a}^{a} x^3\,dx$ in first year calculus.

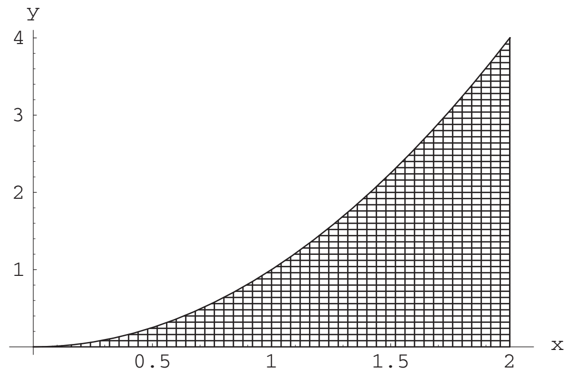
**4.** $\displaystyle\int_0^1\int_0^{x^3} 3\,dy\,dx = \int_0^1 3y\Big|_0^{x^3} dx = \int_0^1 3x^3\,dx = \frac{3}{4}x^4\Big|_0^1 = \frac{3}{4}$. The region over which we are integrating is:



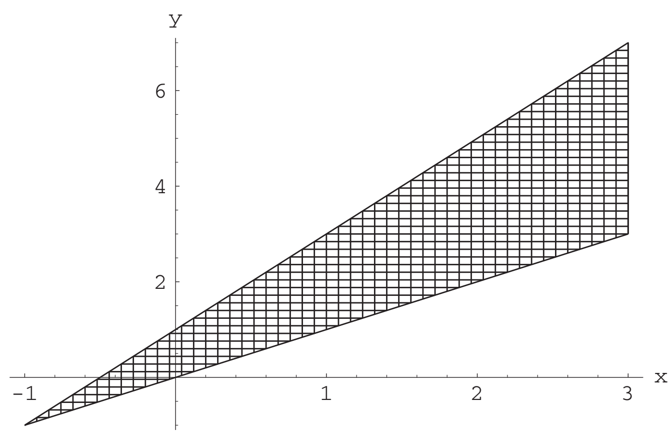

**5.** $\displaystyle\int_0^2\int_0^{y^2} y\,dx\,dy = \int_0^2 xy\Big|_0^{y^2} dy = \int_0^2 y^3\,dy = \frac{y^4}{4}\Big|_0^2 = 4$. The region over which we are integrating is:



**6.** $\int_0^2 \int_0^{x^2} y\,dy\,dx = \int_0^2 \left.\frac{y^2}{2}\right|_0^{x^2} dx = \int_0^2 \frac{x^4}{2}\,dx = \left.\frac{x^5}{10}\right|_0^2 = \frac{32}{10} = \frac{16}{5}$. The region over which we are integrating is:

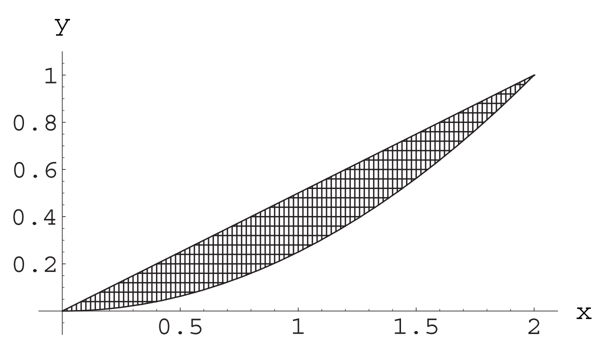



**7.** $\int_{-1}^3 \int_x^{2x+1} xy\,dy\,dx = \int_{-1}^3 \left.\frac{xy^2}{2}\right|_x^{2x+1} dx = \frac{1}{2}\int_{-1}^3 (3x^3 + 4x^2 + x)\,dx = \frac{1}{2}\left[\frac{3}{4}x^4 + \frac{4}{3}x^3 + \frac{x^2}{2}\right]\Big|_{-1}^3 = \frac{152}{3}$. The region over which we are integrating is:




**8.** $\int_0^2 \int_{x^2/4}^{x/2} (x^2 + y^2)\,dy\,dx = \int_0^2 \left(x^2 y + \frac{y^3}{3}\right)\Big|_{y=x^2/4}^{y=x/2} dx$

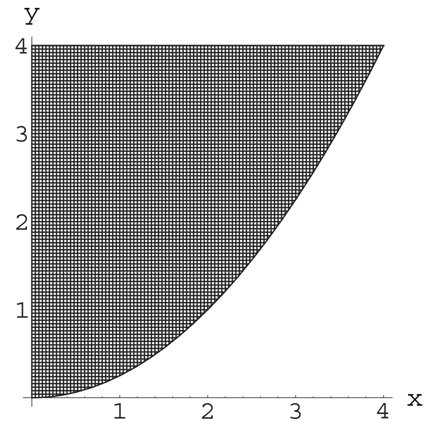
$= \int_0^2 \left(\left(\frac{x^3}{2} + \frac{x^2}{24}\right) - \left(\frac{x^4}{4} + \frac{x^6}{192}\right)\right) dx = \left[\frac{13}{96}x^4 - \frac{1}{20}x^5 - \frac{1}{1344}x^7\right]\Big|_0^2 = \frac{33}{70}$. The region over which we are integrating is:



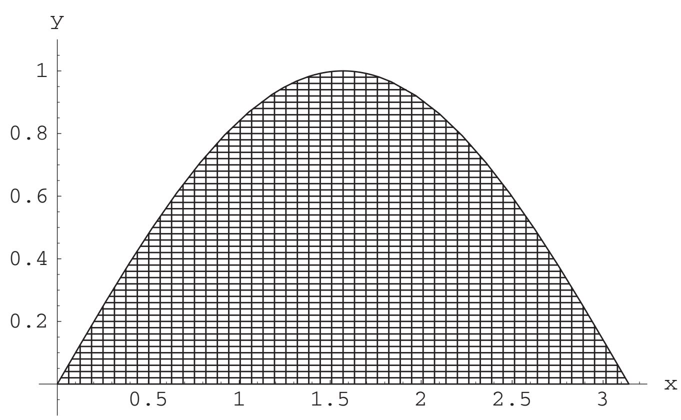

**9.** $\int_0^4 \int_0^{2\sqrt{y}} x\sin(y^2)\,dx\,dy = \int_0^4 \frac{x^2}{2}\sin(y^2)\Big|_{x=0}^{x=2\sqrt{y}} dy = \int_0^4 2y\sin(y^2)\,dy$. Now let $u = y^2$, so $du = 2y\,dy$. Then this integral becomes

$$\int_0^{16} \sin u\, du = -\cos u \Big|_0^{16} = 1 - \cos 16.$$

The region over which we are integrating is:

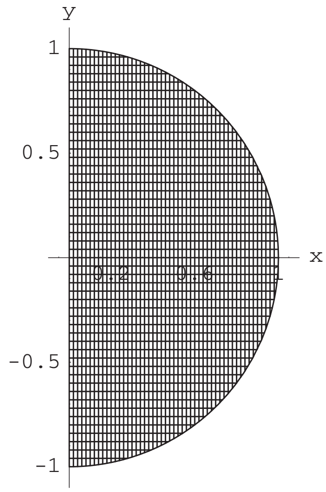



**10.** $\displaystyle\int_0^{\pi}\int_0^{\sin x} y\cos x\, dy\, dx = \int_0^{\pi} \frac{y^2}{2}\cos x\Big|_0^{\sin x} dx = \frac{1}{2}\int_0^{\pi}(\sin^2 x\cos x)\, dx =$ (using the substitution $u = \sin x$) $\displaystyle\frac{1}{2}\int_{x=0}^{x=\pi} u^2\, du = \frac{\sin^3 x}{6}\Big|_0^{\pi} = 0$. The region over which we are integrating is:



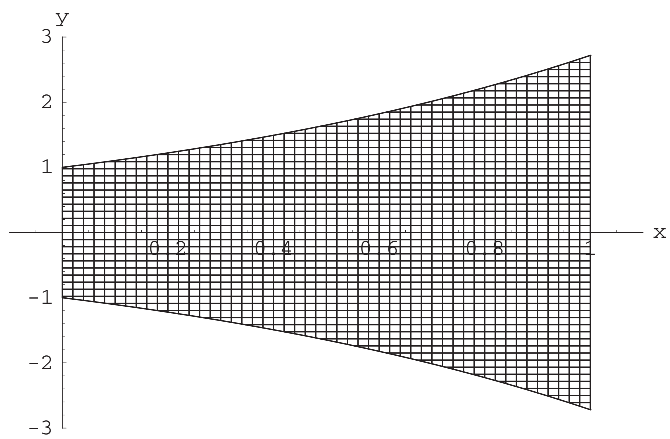

*Note: After you assign Exercises 11 and 12, together you can probe to see whether students see that they are the same. This is a nice set-up for Section 5.3 where they will learn about interchanging the order of integration.*

**11.** $\displaystyle\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3\, dy\, dx = \int_0^1 3y\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_0^1 6\sqrt{1-x^2}\, dx =$ (using the substitution $x = \sin t$) $= 3\pi/2$. You can also see that the region over which we are integrating is a half-circle of radius 1 so we have found the volume of the cylinder over this region of height 3. This figure is:




**12.** This is the same as Exercise 11 with the limits of integration reversed. The solution is again $3\pi/2$.

**13.** $\int_0^1 \int_{-e^x}^{e^x} y^3\, dy\, dx = \int_0^1 \left. \frac{y^4}{4} \right|_{-e^x}^{e^x} dx = \int_0^1 0\, dx = 0$. The region over which we are integrating is:

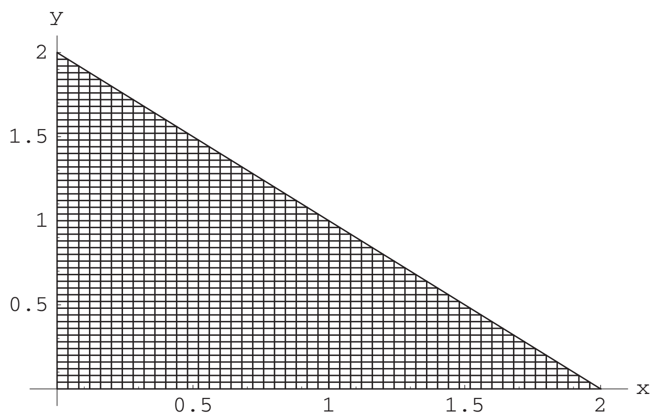



**14.** For each square in the domain we need to estimate the height of the square and multiply it by the length times the width. For our estimate we will choose the value of the height $f(\mathbf{c}_{ij})$ in the lower right corner of the square in row $i$ column $j$ as our height for the square. The heights are then:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 5 | 6 | 7 | 8 | 9 | 9 | 10 | 9 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 10 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 | 10 | 10 | 9 |
| 4 | 5 | 6 | 7 | 8 | 8 | 9 | 9 | 9 | 9 |
| 4 | 5 | 6 | 7 | 7 | 8 | 8 | 8 | 8 | 8 |

Each box has a base of area 25 so the sum of the products of 25 times the heights is 92500. Of course, this answer depends on what point in each box we chose for our estimate—your mileage may vary.

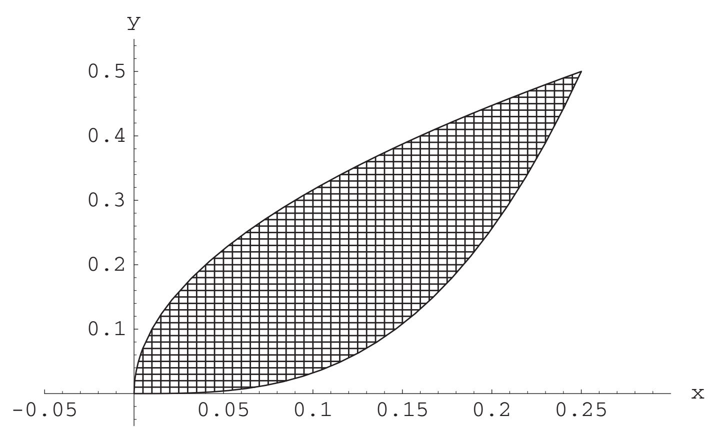
**15.** A quick sketch of the region over which we are integrating helps us set up our double integral.




$$\int_0^2 \int_0^{2-x} (1 - xy)\, dy\, dx = \int_0^2 \left( y - \frac{xy^2}{2} \right)\Big|_0^{2-x} dx = \int_0^2 (2 - 3x + 2x^2 - x^3/2)\, dx$$

$$= \left( 2x - \frac{3}{2}x^2 + \frac{2}{3}x^3 - \frac{x^4}{8} \right)\Big|_0^2 = 4 - 6 + 16/3 - 2 = 4/3.$$

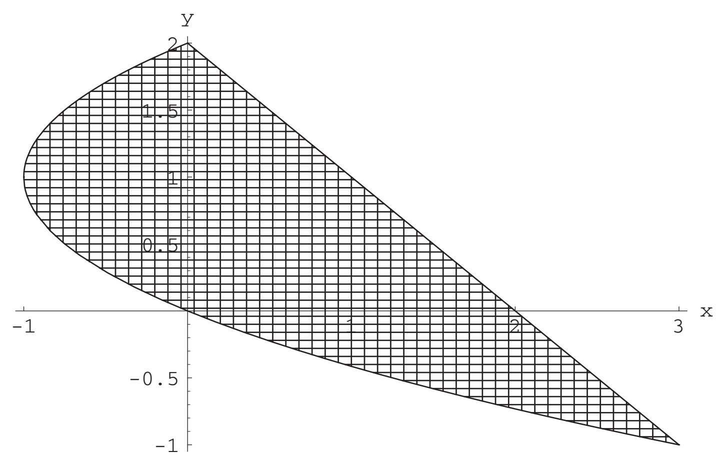
**16.** Again a sketch of the region over which we are integrating helps us set up our double integral. The top bounding curve is $y = \sqrt{x}$ and the bottom curve is $y = 32x^3$.




$$\int_0^{1/4} \int_{32x^3}^{\sqrt{x}} 3xy\, dy\, dx = \int_0^{1/4} \left( \frac{3xy^2}{2} \right)\Big|_{32x^3}^{\sqrt{x}} dx = \int_0^{1/4} \left( \frac{3}{2}x^2 - 1536x^7 \right) dx$$

$$= \left( \frac{3}{2}x^2 - 192x^8 \right)\Big|_0^{1/4} = \frac{1}{128} - \frac{3}{1024} = \frac{5}{1024}.$$

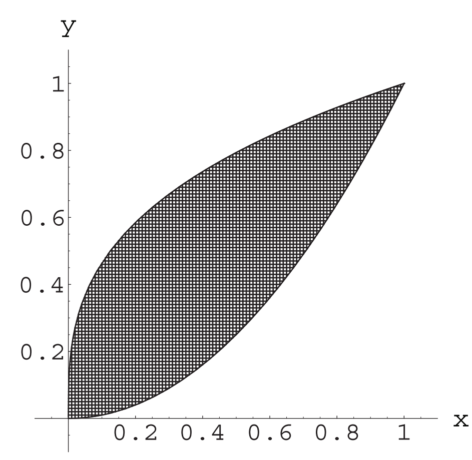
**17.** We can easily determine the limits of integration from the sketch and/or by solving for where $x+y=2$ intersects the parabola $y^2-2y-x=0$.




$$\int_{-1}^{2}\int_{y^2-2y}^{2-y}(x+y)\,dx\,dy=\int_{-1}^{2}\left(\frac{x^2}{2}+xy\right)\Big|_{y^2-2y}^{2-y}dy=\int_{-1}^{2}\left(-\frac{y^4}{2}+y^3-\frac{y^2}{2}+2\right)dy$$

$$=\left(-\frac{y^5}{10}+\frac{y^4}{4}-\frac{y^3}{6}+2y\right)\Big|_{-1}^{2}=\frac{99}{20}.$$

**18.** The region $D$ of integration has top boundary curve $x=y^3$ and bottom boundary curve $y=x^2$ and looks like:



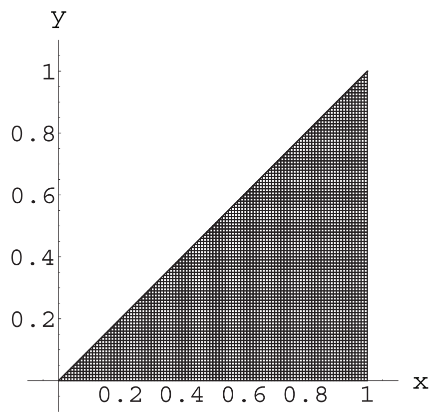

Note that $y=x^2$ may be expressed as $x=\sqrt{y}$ since the region of interest lies in the first quadrant. Hence we have a type 2 elementary region and

$$\iint_D xy\,dA=\int_0^1\int_{y^3}^{\sqrt{y}}xy\,dx\,dy=\int_0^1\frac{x^2}{2}\,y\,\Big|_{x=y^3}^{x=\sqrt{y}}dy=\int_0^1\left(\frac{y^2}{2}-\frac{y^7}{2}\right)dy$$

$$=\frac{1}{2}\left(\frac{1}{3}y^3-\frac{1}{8}y^8\right)\Big|_0^1=\frac{5}{48}.$$

Note that we may also set up this integral as $\displaystyle\int_0^1\int_{x^2}^{\sqrt[3]{x}}xy\,dy\,dx$.

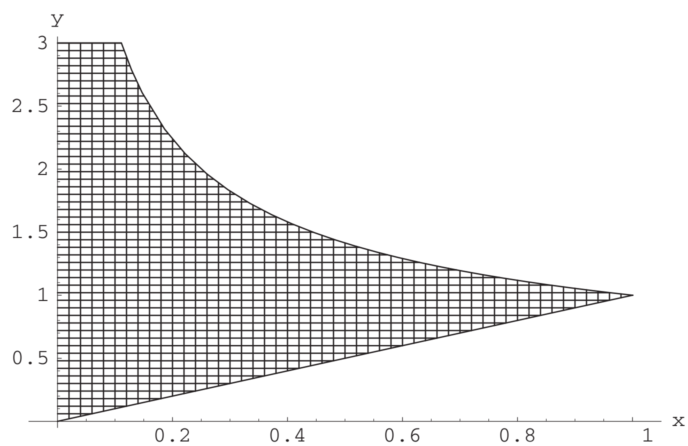
**19.** The region $D$ is triangular, with top boundary the line $y = x$ and looks like:




Viewing $D$ as a type 1 region we have

$$\begin{aligned}\iint_D e^{x^2}\,dA &= \int_0^1 \int_0^x e^{x^2}\,dy\,dx \\ &= \int_0^1 xe^{x^2}\,dx = \int_0^1 \frac{1}{2}e^u\,du \qquad \text{where } u = x^2 \\ &= \frac{1}{2}e^u\Big|_0^1 = \frac{1}{2}(e-1).\end{aligned}$$

**20.** We see from the sketch that we need to divide the integral into two pieces. For $0 \le x \le 1/9$ we see that $x \le y \le 3$ and for $1/9 \le x \le 1$ we see that $x \le y \le 1/\sqrt{x}$.

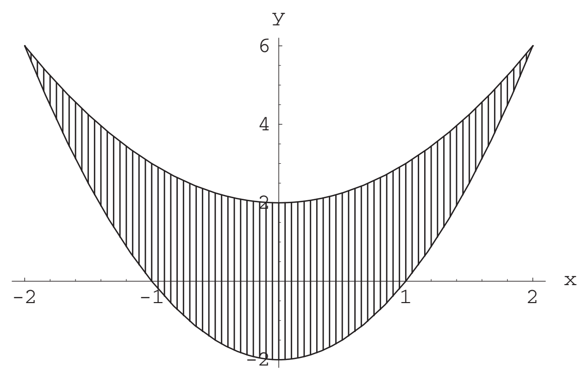



$$\begin{aligned}\iint_D 3y\,dA &= \int_0^{1/9} \int_x^3 3y\,dy\,dx + \int_{1/9}^1 \int_x^{1/\sqrt{x}} 3y\,dy\,dx \\ &= \int_0^{1/9} \frac{3}{2}y^2\Big|_x^3\,dx + \int_{1/9}^1 \frac{3}{2}y^2\Big|_x^{1/\sqrt{x}}\,dx \\ &= \int_0^{1/9}\left(\frac{27}{2} - \frac{3}{2}x^2\right)dx + \int_{1/9}^1\left(\frac{3}{2x} - \frac{3}{2}x^2\right)dx\end{aligned}$$

$$= \left(\frac{27}{2}x - \frac{1}{2}x^3\right)\Big|_0^{1/9} + \left(\frac{3}{2}\ln x - \frac{1}{2}x^3\right)\Big|_{1/9}^{1}$$

$$= \frac{3}{2} - \frac{1}{2} - \frac{3}{2}\ln(1/9) = 1 + \ln 27.$$

**21.** From the sketch below we see that this is a fairly straightforward integral.

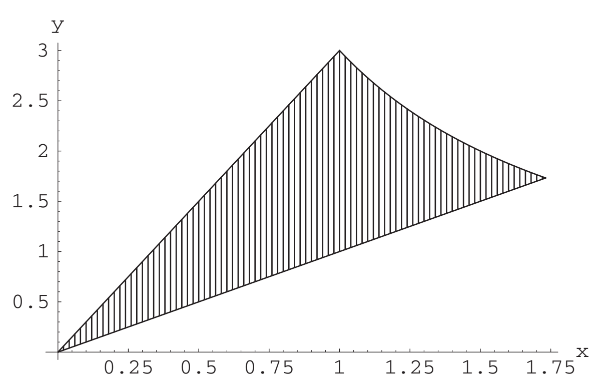



$$\iint_D (x-2y)\,dA = \int_{-2}^{2}\int_{2x^2-2}^{x^2+2}(x-2y)\,dy\,dx$$

$$= \int_{-2}^{2}(xy - y^2)\Big|_{2x^2-2}^{x^2+2}dx = \int_{-2}^{2}(3x^4 - x^3 - 12x^2 + 4x)\,dx$$

$$= (3x^5/5 - x^4/4 - 4x^3 + 2x^2)\Big|_{-2}^{2} = 192/5 - 64 = -128/5$$

**22.** From the sketch below we see that this integral needs to be done in two pieces.




$$\iint_D (x^2+y^2)\,dA = \int_0^1\int_x^{3x}(x^2+y^2)\,dy\,dx + \int_1^{\sqrt{3}}\int_x^{3/x}(x^2+y^2)\,dy\,dx$$

$$= \int_0^1 (x^2y + y^3/3)\Big|_x^{3x}dx + \int_1^{\sqrt{3}}(x^2y + y^3/3)\Big|_x^{3/x}dx$$

$$= \int_0^1 (32/3)x^3\,dx + \int_1^{\sqrt{3}}(9/x^3 + 3x - 4x^3/3)\,dx = 8/3 + 10/3 = 6$$

**23.** As in the proof of property 1 in the text, we note that the Riemann sum whose limit is

$$\iint_R cf\,dA \text{ is } \sum_{i,j=1}^{n} cf(\mathbf{c}_{ij})\Delta A_{ij} = c\sum_{i,j=1}^{n} f(\mathbf{c}_{ij})\Delta A_{ij} \to c\iint_R f\,dA.$$

**24.** $\iint_R g\,dA = \iint_R (f+[g-f])\,dA$ which, by property 1, equals $\iint_R f\,dA + \iint_R [g-f]\,dA$. But $g-f \geq 0$ so $\iint_R [g-f]\,dA \geq 0$ and so $\iint_R g\,dA \geq \iint_R f\,dA$.

**25.** Define $f^+ = \max(f,0)$ and $f^- = \max(-f,0)$. Note that both $f^+$ and $f^-$ have only non-negative values. Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Since $f^\pm \leq |f| = f^+ + f^-$ we can see that $|f|$ is Riemann integrable. Also we can use property 2 to conclude that

$$\begin{aligned}\left|\iint_R f\,dA\right| &= \left|\iint_R (f^+ - f^-)\,dA\right| = \left|\iint_R f^+\,dA - \iint_R f^-\,dA\right| \\ &\leq \iint_R f^+\,dA + \iint_R f^-\,dA = \iint_R |f|\,dA.\end{aligned}$$

**26. (a)** Intuitively, the volume of a figure with constant height should be the area of the base times the height. In this case that is just the area of the base. More formally, by Definition 2.3,

$$\iint_D 1\,dA = \lim_{\text{all } \Delta x_i, \Delta y_j \to 0} \sum_{i,j=1}^{n} \Delta x_i \Delta y_j.$$

We are assuming that $D$ is an elementary region; let's consider the case of a type 1 region, then we can rewrite the above sum as

$$\lim_{\text{all } \Delta x_i \to 0} \sum_{i=1}^{n} (\delta(c_i) - \gamma(c_i))\Delta x_i = \int_a^b (\delta(x) - \gamma(x))\,dx = \text{the area of } D.$$

The proof is not much different for the other elementary regions.

**(b)** We integrate $\iint_D 1\,dA = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx = 2\int_{-a}^{a}\sqrt{a^2-x^2}\,dx$. We've seen this above in Exercises 11 and 12. Let $x = a\sin t$ and integrate to get the desired result.

**27.** Using Exercise 26, the area is

$$\begin{aligned}\iint_A 1\,dA &= \int_0^1\int_{x^3}^{x^2} 1\,dy\,dx = \int_0^1 (x^2 - x^3)\,dx \\ &= (x^3/3 - x^4/4)\Big|_0^1 = 1/3 - 1/4 = 1/12.\end{aligned}$$

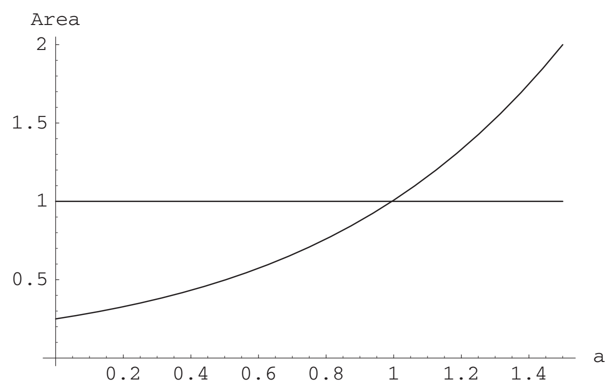
**28.** Again using Exercise 26, the area is

$$\begin{aligned}\iint_A 1\,dA &= \int_0^{\sqrt{5}-2}\int_{2x}^{1-2x-x^2} 1\,dy\,dx = \int_0^{\sqrt{5}-2} (1 - 4x - x^2)\,dx \\ &= (x - 2x^2 - x^3/3)\Big|_0^{\sqrt{5}-2} = (1/3)(10\sqrt{5} - 22).\end{aligned}$$

**29.** We integrate $\int_{-a}^{a}\int_{-\sqrt{b^2-b^2x^2/a^2}}^{\sqrt{b^2-b^2x^2/a^2}} dy\,dx = 2\int_{-a}^{a}\sqrt{b^2 - \frac{b^2x^2}{a^2}}\,dx = \frac{2b}{a}\left(\int_{-a}^{a}\sqrt{a^2-x^2}\,dx\right) = \frac{b}{a}(\pi a^2) = \pi ab.$

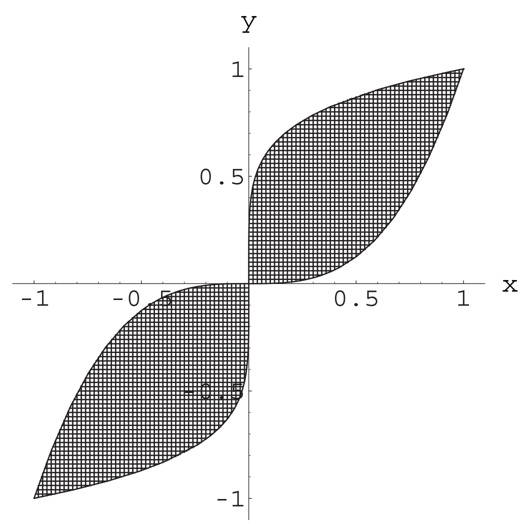
**30. (a)** For $x \geq 0$ the curve $x^3 - x$ lies below the curve $y = ax^2$ between 0 and their positive point of intersection $x = \dfrac{a+\sqrt{a^2+4}}{2}$. So the area is given by $\int_0^{(a+\sqrt{a^2+4})/2}\int_{x^3-x}^{ax^3} dy\,dx.$

**(b)** The graph of area against $a$ is:




The area is 1 at $a \approx .995$.

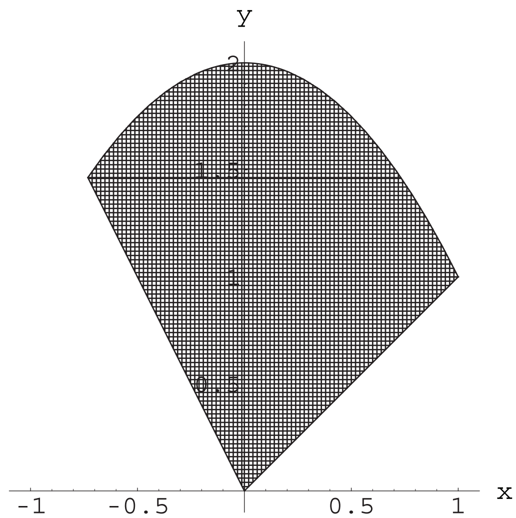
**31.** The region looks like:




By symmetry, it's enough to calculate the first quadrant area and double it. Thus

$$\text{Total area} = 2\int_0^1 \int_{x^3}^{x^{1/5}} 1\,dy\,dx = 2\int_0^1 \left(x^{1/5} - x^3\right)\,dx$$
$$= 2\left(\frac{5}{6}x^{6/5} - \frac{1}{4}x^4\right)\Big|_0^1 = 2\left(\frac{5}{6} - \frac{1}{4}\right) = \frac{7}{6}.$$

**32.** The region in question looks like:



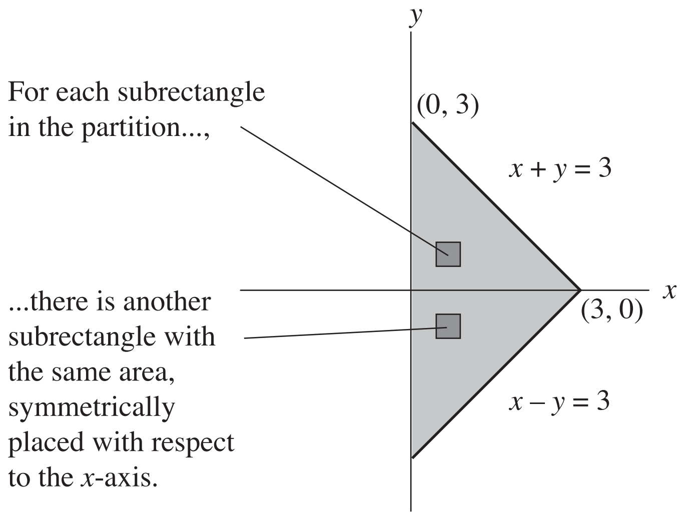

Note that the intersection point of $y = -2x$ and $y = 2 - x^2$ is $(1 - \sqrt{3}, -2 + 2\sqrt{3})$. We use the $y$-axis to divide the region into two type 1 subregions. Then

$$\begin{aligned}
\text{Area} &= \int_{1-\sqrt{3}}^{0} \int_{-2x}^{2-x^2} 1\, dy\, dx + \int_0^1 \int_x^{2-x^2} 1\, dy\, dx \\
&= \int_{1-\sqrt{3}}^{0} (2 + 2x - x^2)\, dx + \int_0^1 (2 - x - x^2)\, dx \\
&= \left(2x + x^2 - \frac{x^3}{3}\right)\Big|_{1-\sqrt{3}}^{0} + \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 \\
&= \frac{6\sqrt{3} - 8}{3} + \frac{7}{6} = \frac{4\sqrt{3} - 3}{2}.
\end{aligned}$$

**33.** First, note that the integrand is continuous; hence the integral as the limit of Riemann sums must exist. Second, note that the region $D$ is symmetric with respect to the $x$-axis. Next, note that we can break up the integral as

$$\iint_D (y^3 + e^{x^2} \sin y + 2)\, dA = \iint_D y^3\, dA + \iint_D e^{x^2} \sin y\, dA + \iint_D 2\, dA.$$

Consider first $\iint_D y^3\, dA$ and note that the integrand, $y^3$, is an odd function. Hence, in a Riemann sum, we can arrange to partition any rectangle that contains $D$ in such a way that for every subrectangle above the $x$-axis (i.e., where $y > 0$), there is a corresponding "mirror image" subrectangle—with the same area—*below* the $x$-axis (where $y < 0$). Then the "test points" in each pair of subrectangles may be chosen to have *opposite* $y$-coordinates. (See the figure below.)




The Riemann sum corresponding to this partition will be

$$\sum_{i,j} y_{ij}^3 \,\Delta A_{ij} = 0,$$

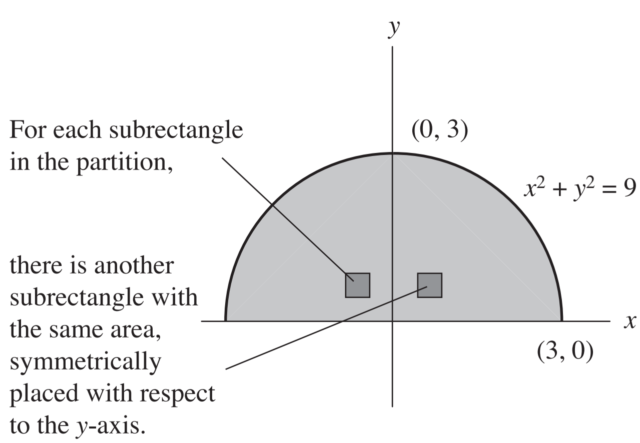
since the terms of the sum will cancel in pairs. Thus, even when we take the limit of this sum as $\Delta A_{ij} \to 0$, we still obtain zero. Therefore, we conclude that $\iint_D y^3\,dA = 0$. Using a similar argument, we find that $\iint_D e^{x^2} \sin y\,dA = 0$ as well. Hence

$$\begin{aligned}\iint_D (y^3 + e^{x^2} \sin y + 2)\,dA &= \iint_D y^3\,dA + \iint_D e^{x^2} \sin y\,dA + \iint_D 2\,dA \\ &= 0 + 0 + 2\iint_D dA = 2(\text{area of } D) \\ &= 2(9) = 18.\end{aligned}$$

**34.** First, note that the integrand is continuous; hence the integral as the limit of Riemann sums must exist. Second, note that the region $D$ is symmetric with respect to the $y$-axis. Next, note that we can break up the integral as

$$\iint_D (2x^3 - y^4 \sin x + 2)\,dA = \iint_D 2x^3\,dA - \iint_D y^4 \sin x\,dA + \iint_D 2\,dA.$$

Consider first $\iint_D 2x^3\,dA$ and note that the integrand, $2x^3$, is an odd function. Hence, in a Riemann sum, we can arrange to partition any rectangle that contains $D$ in such a way that for every subrectangle to the right of the $y$-axis (i.e., where $x > 0$), there is a corresponding "mirror image" subrectangle—with the same area—to the *left* of the $y$-axis (where $x < 0$). Then the "test points" in each pair of subrectangles may be chosen to have *opposite* $x$-coordinates. (See the figure below.)



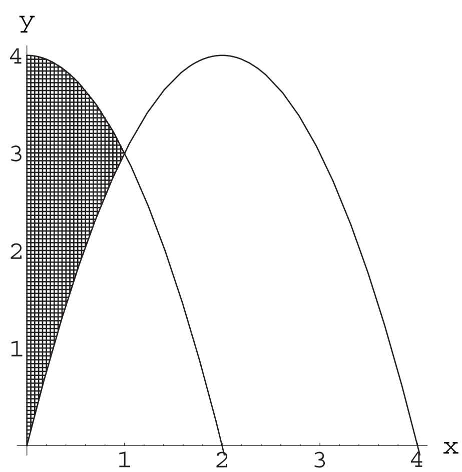
The Riemann sum corresponding to this partition will be

$$\sum_{i,j} 2x_{ij}^3 \, \Delta A_{ij} = 0,$$

since the terms of the sum will cancel in pairs. Thus, even when we take the limit of this sum as $\Delta A_{ij} \to 0$, we still obtain zero. Therefore, we conclude that $\iint_D 2x^3 \, dA = 0$. Using a similar argument, we find that $\iint_D y^4 \sin x \, dA = 0$ as well. Hence

$$\begin{aligned}\iint_D (2x^3 - y^4 \sin x + 2) \, dA &= \iint_D 2x^3 \, dA - \iint_D y^4 \sin x \, dA + \iint_D 2 \, dA \\ &= 0 + 0 + 2 \iint_D dA = 2(\text{area of } D) \\ &= 2\left(\frac{9\pi}{2}\right) = 9\pi.\end{aligned}$$

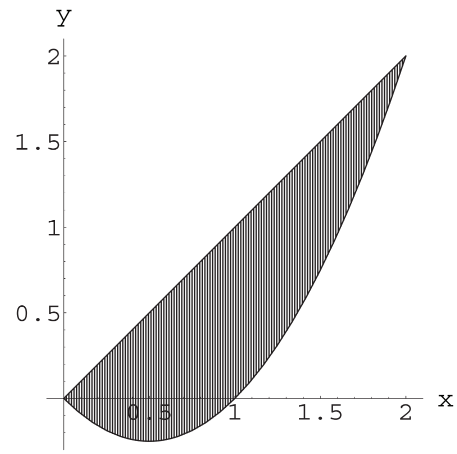
**35.** The volume is given by $\iint_D (24 - 2x - 6y) \, dA$, where $D$ is the region in the $xy$-plane bounded by $y = 4 - x^2$, $y = 4x - x^2$, and the $y$-axis. Now $D$ is a type 1 region that looks like:




Thus the volume is

$$\begin{aligned}\int_0^1 \int_{4x-x^2}^{4-x^2} (24 - 2x - 6y) \, dy \, dx &= \int_0^1 \left[(24 - 2x)y - 3y^2\right]\Big|_{y=4x-x^2}^{y=4-x^2} dx \\ &= \int_0^1 \left[(24 - 2x)(4 - 4x) - 3(4 - x^2)^2 + 3(4x - x^2)^2\right] dx \\ &= \int_0^1 \left[8(x^2 - 13x + 12) - 24x^3 + 72x^2 - 48\right] dx \\ &= \int_0^1 \left[80x^2 - 24x^3 - 104x + 48\right] dx = \frac{50}{3}.\end{aligned}$$

**36.** The volume is given by $\iint_D (x^2 + 6y^2) \, dA$, where $D$ is the region in the $xy$-plane bounded by $y = x$ and $y = x^2 - x$. This region $D$ looks like:




Therefore, the volume is

$$\int_0^1 \int_{x^2-x}^{x} (x^2+6y^2)\,dy\,dx = \int_0^2 \left[x^2(2x-x^2)+2\left(x^3-(x^2-x)^3\right)\right]\,dx$$
$$= \int_0^2 \left[-2x^6+6x^5-7x^4+6x^3\right]\,dx$$
$$= \left(-\frac{2}{7}x^7+x^6-\frac{7}{5}x^5+\frac{3}{2}x^4\right)\Bigg|_0^2 = \frac{232}{35}.$$

**37.** The graphs of $y=x^2-10$ and $y=31-(x-1)^2$ intersect at $x=-4$ and $x=5$ with the graph of $y=x^2-10$ lying below the graph of $y=31-(x-1)^2$ on this interval.

$$\int_{-4}^{5}\int_{x^2-10}^{31-(x-1)^2} (4x+2y+25)\,dy\,dx = \int_{-4}^{5} (4xy+y^2+25y)\Big|_{x^2-10}^{31-(x-1)^2}\,dx$$
$$= \int_{-4}^{5} (-12x^3-78x^2+330x+1800)\,dx$$
$$= (-3x^4-26x^3+165x^2+1800x)\Big|_{-4}^{5} = 11664.$$

**38.** **(a)** This is a special case of the region over which we integrated in Exercise 26 (b). The integral is $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2-y^2+5)\,dy\,dx.$

**(b)** You can use your favorite computer algebra system. Using *Mathematica*, enter the command:

Integrate[Integrate[$x^2-y^2+5, \{y, -\text{Sqrt}[4-x^2], \text{Sqrt}[4-x^2]\}], \{x, -2, 2\}$] or

Integrate[$x^2-y^2+5, \{x,-2,2\}, \{y, -\text{Sqrt}[4-x^2], \text{Sqrt}[4-x^2]\}$] and get the answer $20\pi$.

**39.** By symmetry we see that the volume is four times the volume of the piece over the first quadrant $(x, y \geq 0)$. In this region $|x| = x$ and $|y| = y$ so the volume is

$$4\int_0^2\int_0^{2-x} (2-x-y)\,dy\,dx = 4\int_0^2 (2y-xy-y^2/2)\Big|_0^{2-x}\,dx = 4\int_0^2 (2-2x+x^2/2)\,dx$$
$$= 4(2x-x^2+x^3/6)\Big|_0^2 = 16/3.$$

*The results demonstrated in Exercises 40 and 41 are arrived at easily but worth seeing. In Exercise 40 we have the dream situation where the double integral of a product can be split into the product of integrals. We quickly see that this only works in a very special case. In Exercise 41 we examine a function where $\iint f\,dy\,dx$ exists but $\iint f\,dA$ does not.*

**40.** **(a)** The function $h(x, y) = f(x)g(y)$ satisfies the conditions of Theorem 2.6 (Fubini's theorem) on $[a, b] \times [c, d]$. So:

$$\iint_R f(x)g(y)\,dA = \int_a^b \int_c^d f(x)g(y)\,dy\,dx.$$

For emphasis, we rewrite this last integral with parentheses and, since $f(x)$ does not depend on $y$, we have:

$$\int_a^b \left(\int_c^d f(x)g(y)\,dy\right)\,dx = \int_a^b f(x)\left(\int_c^d g(y)\,dy\right)\,dx.$$

But $\int_c^d g(y)\,dy$ is constant so we can pull it out of this last integral to get the result:

$$\int_a^b f(x)\left(\int_c^d g(y)\,dy\right)\,dx = \left(\int_c^d g(y)\,dy\right)\left(\int_a^b f(x)\,dx\right).$$

**(b)** If $D$ is an elementary region we can perform the first step above, if $D$ is not an elementary region, there's not much we can do. For example, if $D$ is a type 1 region, $D = \{(x, y)|\gamma(x) \le y \le \delta(x), a \le x \le b\}$ then

$$\iint_R f(x)g(y)\,dA = \int_a^b \left(\int_{\gamma(x)}^{\delta(x)} f(x)g(y)\,dy\right)\,dx = \int_a^b f(x)\left(\int_{\gamma(x)}^{\delta(x)} g(y)\,dy\right)\,dx.$$

**41.** **(a)** If $x$ is rational, then $\int_0^2 f(x, y)\,dy = \int_0^2 1\,dy = 2$. If $x$ is irrational, then $\int_0^2 f(x, y)\,dy = \int_0^1 0\,dy + \int_1^2 2\,dy = 2$.

**(b)** Using our answer from part (a), $\int_0^1 \int_0^2 f(x, y)\,dy\,dx = \int_0^1 2\,dx = 2$.

**(c)** If $\mathbf{c}_{ij}$ has a rational $x$ coordinate, then $f(\mathbf{c}_{ij}) = 1$ and so the Riemann sum will converge to the area of the region, which is 2.

**(d)** In this case $f(\mathbf{c}_{ij}) = 1$ for our points in the region $[0, 1] \times [0, 1]$ and $f(\mathbf{c}_{ij}) = 2$ for our points in the region $[0, 1] \times [1, 2]$. In short, the Riemann sums will converge to $(1)(1) + (2)(1) = 3$.

**(e)** As we saw in parts (c) and (d), the Riemann sum does not have a well defined limit and so $f$ fails to be integrable on $R$, even though in part (b) we actually computed the iterated integral.

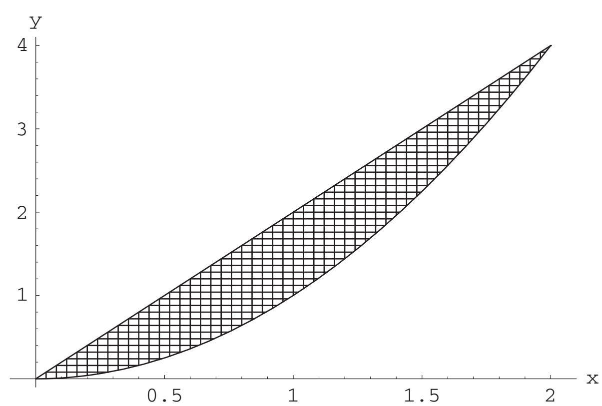
## 5.3 Changing The Order of Integration

*This is a good section in which to encourage students to explore with a computer system.*

**1.** **(a)**

$$\int_0^2 \int_{x^2}^{2x} (2x + 1)\,dy\,dx = \int_0^2 (2x + 1)(2x - x^2)\,dx = \int_0^2 (-2x^3 + 3x^2 + 2x)\,dx$$
$$= \left(-\frac{x^4}{2} + x^3 + x^2\right)\Big|_0^2 = 4.$$

**(b)** The region of integration is bounded above by $y = 2x$ and below by $y = x^2$:

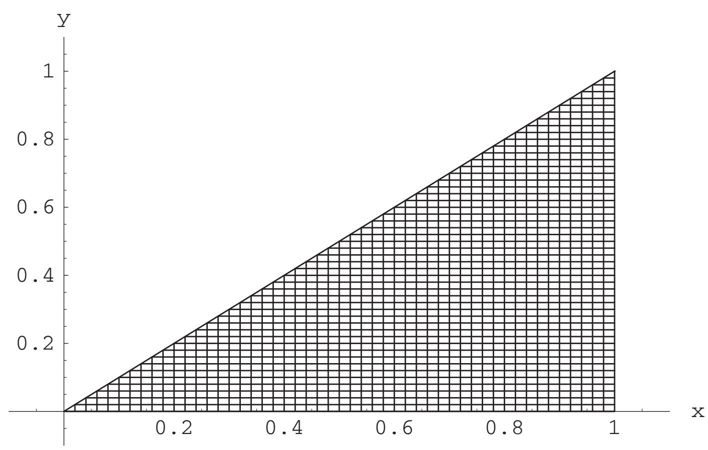


**(c)**

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (2x+1)\,dx\,dy = \int_0^4 (x^2+x)\Big|_{y/2}^{\sqrt{y}}\,dy = \int_0^4 \left(-\frac{y^2}{4} + \frac{y}{2} + \sqrt{y}\right) dy$$
$$= \left(-\frac{y^3}{12} + \frac{y^2}{4} + \frac{2y^{3/2}}{3}\right)\Big|_0^4 = 4.$$

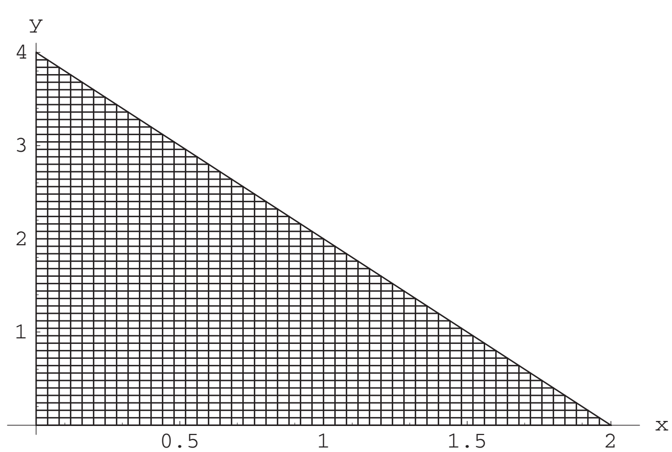
*Note: In Exercises 2–9, most students will find the biggest challenge in reversing the order of integration (the topic of this section). You may want to suggest that they reverse the order of integration in all of the exercises, but that they evaluate both iterated integrals only in Exercises 7–9.*

**2.** The region of integration is:



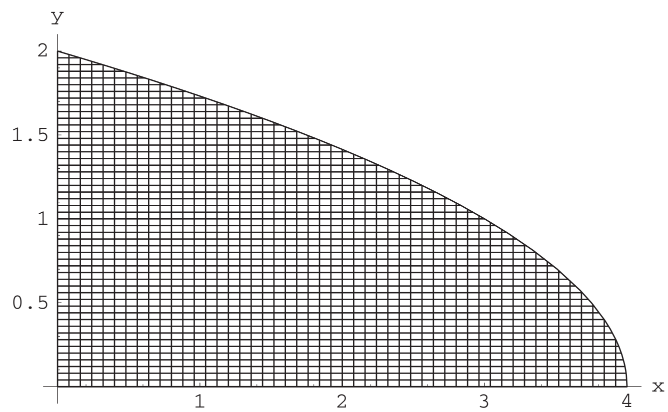

$$\int_0^1 \int_0^x (2-x-y)\,dy\,dx = \int_0^1 (2x - 3x^2/2)\,dx = 1/2 \quad \text{and}$$
$$\int_0^1 \int_y^1 (2-x-y)\,dx\,dy = \int_0^1 \frac{3}{2}(y^2-2y+1)\,dy = 1/2.$$

**3.** The region of integration is:



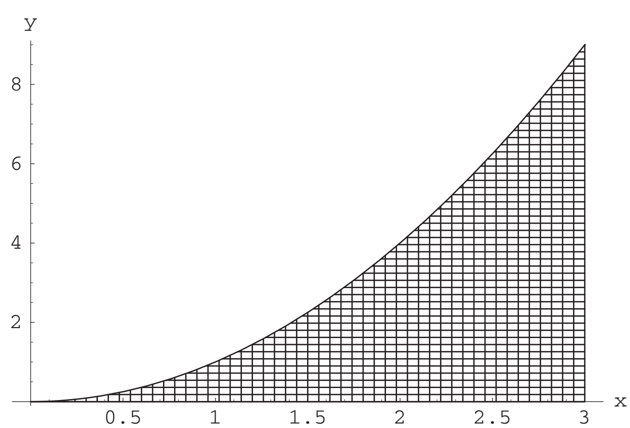
$$\int_0^2 \int_0^{4-2x} y\,dy\,dx = \int_0^2 (2x^2 - 8x + 8)\,dx = 16/3 \quad \text{and}$$
$$\int_0^4 \int_0^{2-y/2} y\,dx\,dy = \int_0^4 (-y^2/2 + 2y)\,dy = 16/3.$$

**4.** The region of integration is:



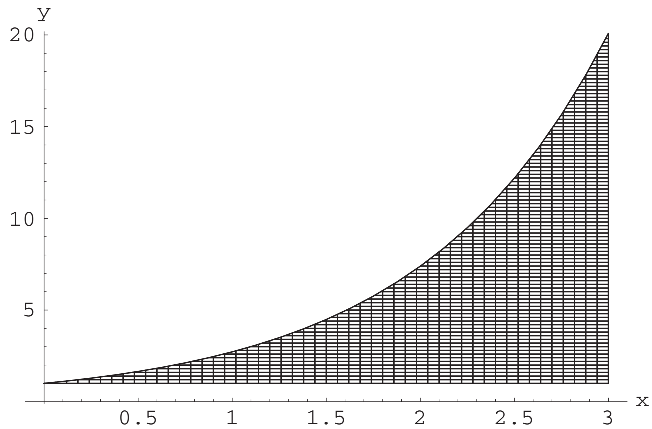

$$\int_0^2 \int_0^{4-y^2} x\,dx\,dy = \int_0^2 \left(\frac{(4-y^2)^2}{2}\right) dy = 128/15 \quad \text{and}$$
$$\int_0^4 \int_0^{\sqrt{4-x}} x\,dy\,dx = \int_0^4 (x\sqrt{4-x})\,dx = 128/15.$$

**5.** The region of integration is:




$$\int_0^9 \int_{\sqrt{y}}^3 (x+y)\,dx\,dy = \int_0^9 \frac{1}{2}(-2y^{3/2} + 5y + 9)\,dy = 891/20 \quad \text{and}$$
$$\int_0^3 \int_0^{x^2} (x+y)\,dy\,dx = \int_0^9 (x^4/2 + x^3)\,dx = 891/20.$$

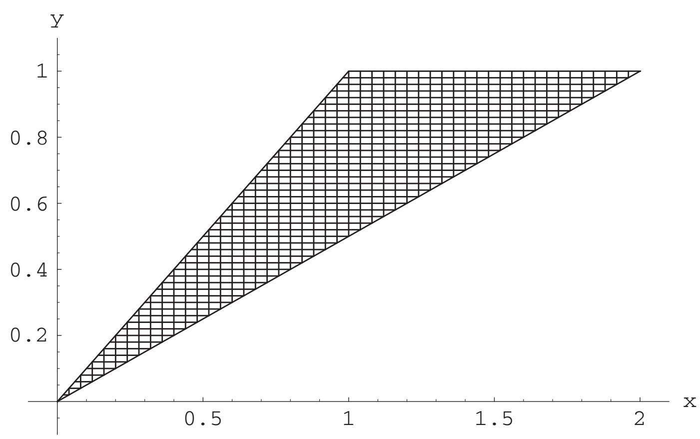
**6.** The region of integration is:




$$\int_0^3 \int_1^{e^x} 2\,dy\,dx = \int_0^3 (2e^x - 2)\,dx = 2e^3 - 8 \quad \text{and}$$

$$\int_1^{e^3} \int_{\ln y}^3 2\,dx\,dy = \int_0^3 (6 - 2\ln y)\,dy = 2e^3 - 8.$$

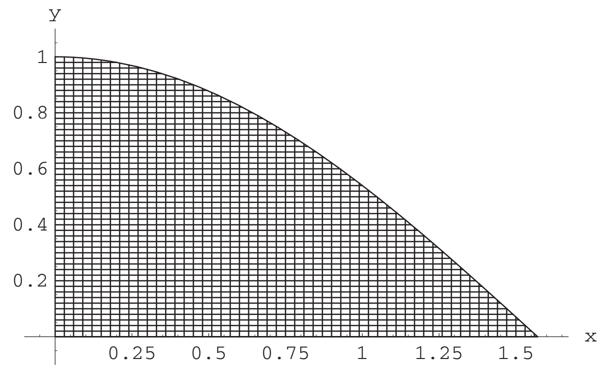
**7.** The region of integration is:




$$\int_0^1 \int_y^{2y} e^x\,dx\,dy = \int_0^1 (e^{2y} - e^y)\,dy = \frac{1}{2}(e^2 - 2e + 1) \quad \text{and}$$

$$\int_0^1 \int_{x/2}^x e^x\,dy\,dx + \int_1^2 \int_{x/2}^1 e^x\,dy\,dx = \int_0^1 (xe^x/2)\,dy + \int_1^2 (e^x - xe^x/2)\,dy = \frac{1}{2} + \frac{e^2}{2} - e.$$

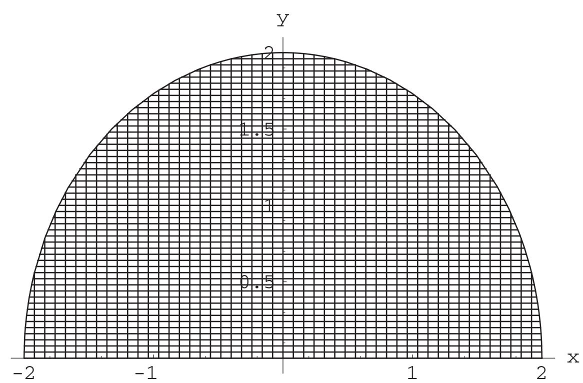
**8.** The region of integration is:




$$\int_0^{\pi/2}\int_0^{\cos x}\sin x\,dy\,dx = \int_0^{\pi/2}(\cos x\sin x)\,dx = 1/2 \quad \text{and}$$

$$\int_0^1\int_0^{\cos^{-1}y}\sin x\,dx\,dy = \int_0^1(1-y)\,dy = 1/2.$$

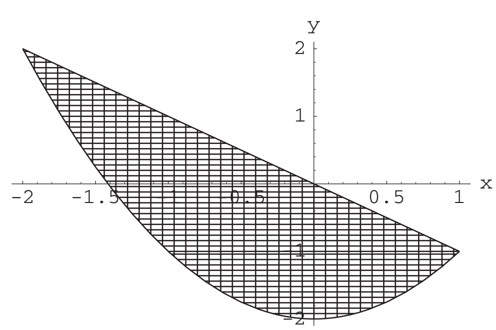
**9.** The region of integration is:




$$\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} y\,dx\,dy = \int_0^2(2y\sqrt{4-y^2})\,dy = 16/3 \quad \text{and}$$

$$\int_{-2}^2\int_0^{\sqrt{4-x^2}} y\,dy\,dx = \int_{-2}^2(-x^2/2+2)\,dx = 16/3.$$

**10.** The limits of integration describe a region $D$ bounded on the top by the line $y=-x$ and on the bottom by the parabola $y=x^2-2$, as shown in the figure.



To reverse the order of integration we must divide $D$ into two regions by the line $y = -1$. Then the original integral is equivalent to the sum

$$\int_{-2}^{-1}\int_{-\sqrt{y+2}}^{\sqrt{y+2}}(x-y)\,dx\,dy + \int_{-1}^{2}\int_{-\sqrt{y+2}}^{-y}(x-y)\,dx\,dy$$

The first of these integrals is

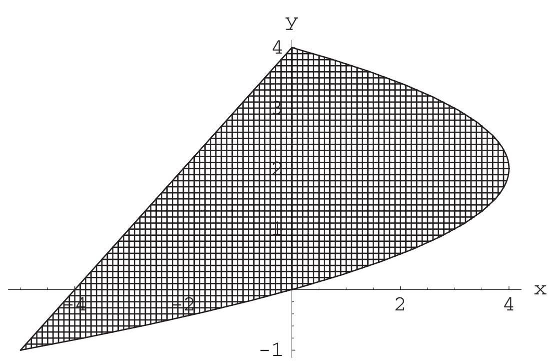
$$\int_{-2}^{-1}\int_{-\sqrt{y+2}}^{\sqrt{y+2}}(x-y)\,dx\,dy = \int_{-2}^{-1} -2y\sqrt{y+2}\,dy$$
$$= \int_{0}^{1} -2(u-2)\sqrt{u}\,du = -2\int_0^1 (u^{3/2}-2u^{1/2})\,du$$
$$= -2\left(\frac{2}{5}u^{5/2}-\frac{4}{3}u^{3/2}\right)\Big|_0^1 = -2\left(\frac{2}{5}-\frac{4}{3}\right) = \frac{28}{15}.$$

The second integral is

$$\int_{-1}^{2}\int_{-\sqrt{y+2}}^{-y}(x-y)\,dx\,dy = \int_{-1}^{2}\left(\frac{1}{2}y^2-\frac{1}{2}(y+2)+y^2-y\sqrt{y+2}\right)dy$$
$$= \left(\frac{1}{2}y^3-\frac{1}{4}y^2-y\right)\Big|_{-1}^{2} - \int_{-1}^{2} y\sqrt{y+2}\,dy = \frac{3}{4} - \int_1^4 (u-2)\sqrt{u}\,du$$
$$= \frac{3}{4} - \left(\frac{2}{5}u^{5/2}-\frac{4}{3}u^{3/2}\right)\Big|_1^4 = \frac{3}{4}-\frac{64}{5}+\frac{32}{3}+\frac{2}{5}-\frac{4}{3}$$
$$= -\frac{139}{60}.$$

Thus the final answer is $\frac{28}{15}-\frac{139}{60} = -\frac{9}{20}$.

**11.** The limits of integration describe a region $D$ bounded on the left by $x = y-4$ and on the right by the parabola $x = 4y - y^2$.



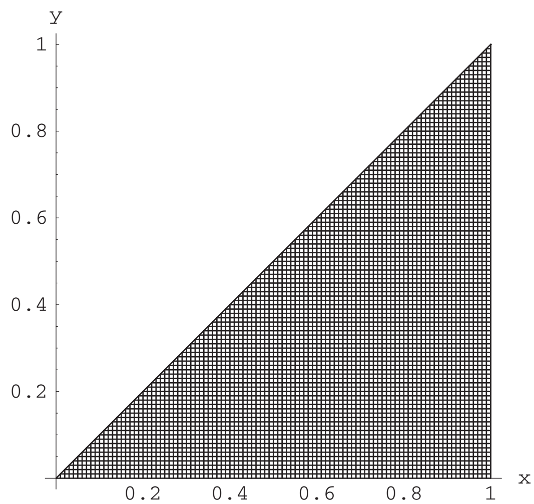

To reverse the order of integration, divide $D$ into two regions by the line $x = 0$ (the $y$-axis). The original integral is equivalent to

$$\int_{-5}^{0}\int_{2-\sqrt{4-x}}^{x+4}(y+1)\,dy\,dx + \int_0^4\int_{2-\sqrt{4-x}}^{2+\sqrt{4-x}}(y+1)\,dy\,dx$$
$$= \int_{-5}^{0}\left(\frac{1}{2}(x+4)^2+(x+4)-\frac{1}{2}(2-\sqrt{4-x})^2-2+\sqrt{4-x}\right)dx$$
$$+ \int_0^4\left(\frac{1}{2}(2+\sqrt{4-x})^2+(2+\sqrt{4-x})-\frac{1}{2}(2-\sqrt{4-x})^2-(2-\sqrt{4-x})\right)dx$$

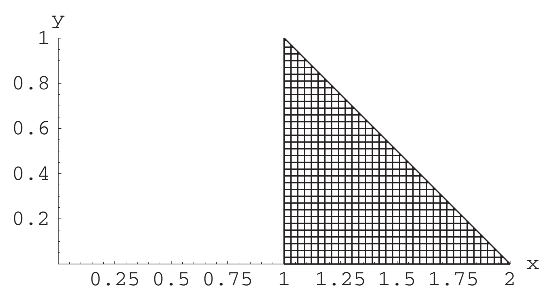
$$= \int_{-5}^{0} \left(6 + 3\sqrt{4-x} + \frac{11}{2}x^2 + \frac{1}{2}x^2\right) dx + \int_{0}^{4} 6\sqrt{4-x}\, dx$$

$$= \frac{241}{12} + 32 = \frac{625}{12}.$$

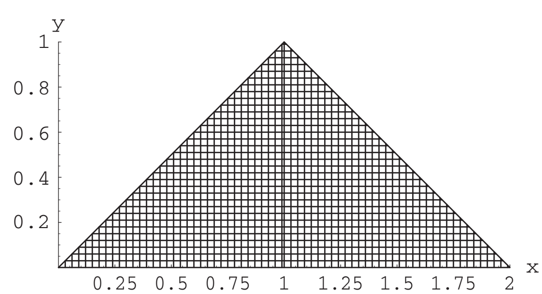
**12.** The limits of integration of the first integral describe the triangular region $D_1$ bounded on top by $y = x$:




The limits of integration of the second integral describe the triangular region $D_2$ bounded by $y = 2 - x$:




Taken together, we obtain the triangular region $D$ below



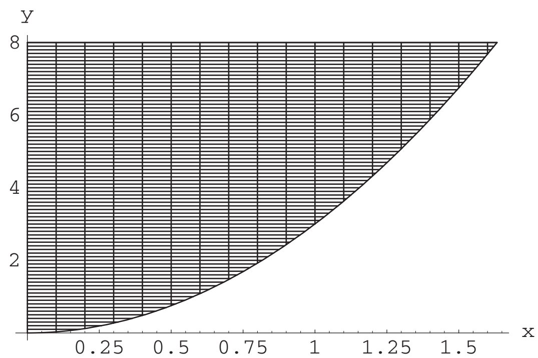

Reversing the order of integration, we find that the sum of the integrals equals

$$\int_0^1 \int_y^{2-y} \sin x\, dx\, dy = \int_0^1 (-\cos(2-y) + \cos y)\, dy$$

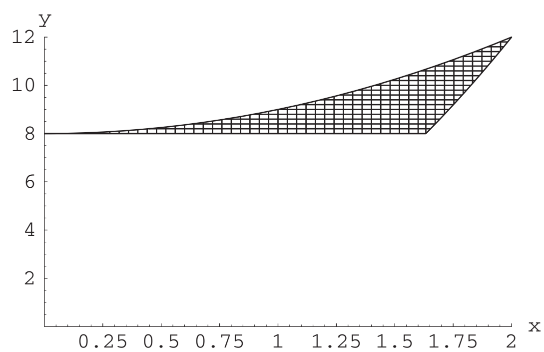
$$= (\sin(2-y) + \sin y)\Big|_0^1 = \sin 1 + \sin 1 - \sin 2$$

$$= 2\sin 1 - \sin 2.$$

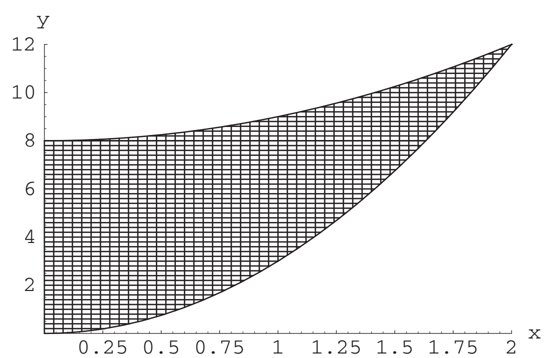
**13.** The limits of integration of the first integral describe the region $D_1$ bounded on the left by the $x$-axis, on the right by $x = \sqrt{y/3}$ (or, equivalently, by $y = 3x^2$) and on top by $x = 8$.




The limits of integration of the second integral describe the region $D_2$ bounded on the bottom by $y = 8$, on the left by $x = \sqrt{y-8}$ (which is equivalent to $y = x^2 + 8$), and on the right by $x = \sqrt{-y/3}$.




Together, $D_1$ and $D_2$ give the full region $D$ of integration.




When we reverse the order of integration, the sum of integrals is equal to

$$\begin{aligned}\int_0^2 \int_{3x^2}^{x^2+8} y\,dy\,dx &= \int_0^2 \frac{1}{2}((x^2+8)^2 - 9x^4)\,dx \\ &= \frac{1}{2}\int_0^2 (-8x^4 + 16x^2 + 64)\,dx \\ &= \frac{1}{2}\left(-\frac{256}{5} + \frac{128}{3} + 128\right) = \frac{896}{15}.\end{aligned}$$

**14.** We reverse the order of integration:

$$\int_0^1 \int_{3y}^3 \cos x^2 \, dx \, dy = \int_0^3 \int_0^{x/3} \cos x^2 \, dy \, dx = \int_0^3 (y \cos x^2) \Big|_0^{x/3} dx$$
$$= \frac{1}{3} \int_0^3 x \cos x^2 \, dx = \frac{\sin x^2}{6} \Big|_0^3 = \frac{\sin 9}{6}.$$

**15.** We reverse the order of integration:

$$\int_0^1 \int_y^1 x^2 \sin xy \, dx \, dy = \int_0^1 \int_0^x x^2 \sin xy \, dy \, dx = \int_0^1 (-x \cos xy) \Big|_0^x dx$$
$$= \int_0^1 (x - x \cos x^2) \, dx = \frac{1}{2}(x^2 - \sin x^2) \Big|_0^1 = \frac{1}{2}(1 - \sin 1).$$

**16.** We reverse the order of integration:

$$\int_0^\pi \int_y^\pi \frac{\sin x}{x} \, dx \, dy = \int_0^\pi \int_0^x \frac{\sin x}{x} \, dy \, dx = \int_0^\pi \frac{y \sin x}{x} \Big|_0^x dx = \int_0^\pi (\sin x) \, dx = -\cos x \Big|_0^\pi = 2.$$

**17.** We reverse the order of integration:

$$\int_0^3 \int_0^{9-x^2} \frac{xe^{3y}}{9-y} \, dy \, dx = \int_0^9 \int_0^{\sqrt{9-y}} \frac{xe^{3y}}{9-y} \, dx \, dy = \int_0^9 \frac{x^2 e^{3y}}{2(9-y)} \Big|_0^{\sqrt{9-y}} dy$$
$$= \int_0^9 (e^{3y}/2) \, dx = (e^{3y}/6) \Big|_0^9 = \frac{e^{27} - 1}{6}.$$

**18.** We reverse the order of integration:

$$\int_0^2 \int_{y/2}^1 e^{-x^2} \, dy \, dx = \int_0^1 \int_0^{2x} e^{-x^2} \, dy \, dx = \int_0^1 e^{-x^2} y \Big|_0^{2x} dx$$
$$= \int_0^1 (2xe^{-x^2}) \, dx = (-e^{-x^2}) \Big|_0^1 = 1 - \frac{1}{e}.$$

*Note: It's kind of interesting to see, in Exercises 19–21, that order of integration matters to us and to computer algebra systems.*

**19. (a)** After churning for a while the program returned a sum of terms that included Bessel functions, Gamma functions and other non-trivial and non-elightening results.

**(b)** You would use integration by parts twice and then substitute back in to eliminate the integral.

**(c)** In a blink of an eye you get $\int_0^1 \int_0^{2y} y^2 \cos xy \, dx \, dy = (1/4)(1 - \cos 2)$.

**20. (a)** Again, the program thought for a while and warned that inverse functions were being used and that values could be lost for multivalued inverses. This time, however, it did come up with the correct answer of $(1/4)(1 - \cos 81)$.

**(b)** The calculation $\int_0^9 \int_0^{\sqrt{y}} x \sin y^2 \, dx \, dy$ resulted in the same answer, but the solution came much more quickly.

**21. (a)** The software did nothing more than typeset the integral and leave it unevaluated.

**(b)** This time *Mathematica* quickly calculated the integral $\int_0^{\pi/2} \int_0^{\sin x} e^{\cos x} dy \, dx = e - 1$.

## 5.4 Triple Integrals

*In Exercises 1–3, use Theorem 4.5, Fubini's Theorem, to integrate in the most convenient order. Exercise 4 asks the students to reconsider what happened in Exercise 1. Exercise 3 is a nice opportunity to look back at a result from Section 5.2.*

**1.** If we integrate with respect to $x$ first, the integral simplifies:

$$\iiint_{[-1,1]\times[0,2]\times[1,3]} xyz\,dV = \int_1^3\int_0^2\int_{-1}^1 xyz\,dx\,dy\,dz = \int_1^3\int_0^2 \left.\frac{x^2yz}{2}\right|_{-1}^1 dy\,dz$$
$$= \int_1^3\int_0^2 0\,dy\,dz = 0.$$

**2.** Here order doesn't matter.

$$\iiint_{[0,1]\times[0,2]\times[0,3]} (x^2+y^2+z^2)\,dV = \int_0^1\int_0^2\int_0^3 (x^2+y^2+z^2)\,dz\,dy\,dx$$
$$= \int_0^1\int_0^2 \left.\left(x^2z+y^2z+\frac{z^3}{3}\right)\right|_0^3 dy\,dx$$
$$= \int_0^1\int_0^2 (3x^2+3y^2+9)\,dy\,dx$$
$$= \int_0^1 \left.(3x^2y+y^3+9y)\right|_0^2 dx$$
$$= \int_0^1 (6x^2+26)\,dx$$
$$= \left.(2x^3+26x)\right|_0^1 = 28.$$

**3.** You could work this out as in Exercise 2, or suggest to your students that they could extend the result they established in Exercise 40 of Section 5.2:

$$\iiint_{[1,e]\times[1,e]\times[1,e]} \left(\frac{1}{xyz}\right) dV = \left(\int_1^e \frac{1}{x}\,dx\right)\left(\int_1^e \frac{1}{y}\,dy\right)\left(\int_1^e \frac{1}{z}\,dz\right)$$
$$= \left(\int_1^e \frac{1}{x}\,dx\right)^3 = \left(\ln x\Big|_1^e\right)^3 = 1^3 = 1.$$

**4.** This works for the same reason that Exercise 1 simplified. We are integrating an odd function of $z$ on an interval that is symmetric in the $z$ coordinate and so, since $\int_{-3}^3 z\,dz = 0$, the triple integral will also be 0.

**5.**

$$\int_{-1}^2\int_1^{z^2}\int_0^{y+z} 3yz^2\,dx\,dy\,dz = \int_{-1}^2\int_1^{z^2} \left.3xyz^2\right|_0^{y+z} dy\,dz = 3\int_{-1}^2\int_1^{z^2} (y^2z^2+yz^3)\,dy\,dz$$
$$= 3\int_{-1}^2 \left.\left(\frac{y^3z^2}{3}+\frac{y^2z^3}{2}\right)\right|_1^{z^2} dz = 3\int_{-1}^2 \left(\frac{z^8}{3}+\frac{z^7}{2}-\frac{z^3}{2}-\frac{z^2}{3}\right) dz$$
$$= 3\left.\left(\frac{z^9}{27}+\frac{z^8}{16}-\frac{z^4}{8}-\frac{z^3}{9}\right)\right|_{-1}^2 = \frac{1539}{16}.$$

**6.**

$$\int_1^3\int_0^z\int_1^{xz} (x+2y+z)\,dy\,dx\,dz = \int_1^3\int_0^z \left.(xy+y^2+zy)\right|_1^{xz} dx\,dz$$
$$= \int_1^3\int_0^z (x^2z+x^2z^2+xz^2-x-z-1)\,dx\,dz$$
$$= \int_1^3 \left.\left(\frac{x^3z}{3}+\frac{x^3z^2}{3}+\frac{x^2z^2}{2}-\frac{x^2}{2}-xz-x\right)\right|_0^z dz$$
$$= \int_1^3 \left(\frac{z^5}{3}+\frac{5z^4}{6}-\frac{3z^2}{2}-z\right) dz = \frac{574}{9}.$$

**7.**

$$\begin{aligned}\int_0^1\int_{1+y}^{2y}\int_z^{y+z} z\,dx\,dz\,dy &= \int_0^1\int_{1+y}^{2y} xz\Big|_z^{y+z}\,dz\,dy\\ &= \int_0^1\int_{1+y}^{2y} yz\,dz\,dy = \int_0^1 (yz^2/2)\Big|_{1+y}^{2y}\,dy\\ &= \int_0^1\left(\frac{3y^3}{2}-y^2-\frac{y}{2}\right)dy = -\frac{5}{24}.\end{aligned}$$

**8. (a)** This is a higher-dimensional analogue of Exercise 26 from Section 5.2. Again the idea would be that if we were in four-dimensional space that a figure of constant height would have volume equal to the volume of the base multiplied by the height. In this case that would be just the volume of the base. Somehow this is a lot less physically appealing or intuitive. By Definition 4.3,

$$\iiint_W 1\,dA = \lim_{\text{all }\Delta x_i,\Delta y_j,\Delta z_k\to 0}\sum_{i,j,k=1}^{n}\Delta x_i\Delta y_j\Delta z_k.$$

The intuition follows from examining the formula above on the right. This converges to the volume of $W$. More formally, we are assuming that $W$ is an elementary region; let's consider the case of a type 1 region, then we can rewrite the sum above as

$$\begin{aligned}\lim_{\text{all }\Delta x_i,\Delta y_j\to 0}\sum_{i,j=1}^{n}\Delta x_i\Delta y_j(\psi(\mathbf{c}_{ij})-\varphi(\mathbf{c}_{ij})) &= \lim_{\text{all }\Delta x_i\to 0}\sum_{i=1}^{n}\Delta x_i\left(\int_{\gamma(c_i)}^{\delta(c_i)}(\psi(y)-\varphi(y))\,dy\right)\\ &= \int_a^b\int_{\gamma(c_i)}^{\delta(c_i)}(\psi(y)-\varphi(y))\,dy\,dx = \text{volume of } W.\end{aligned}$$

The proof is not much different for the other elementary regions.

**(b)** Work out that the equation of the circle where the two paraboloids intersect is $x^2+y^2=9/2$ so

$$\begin{aligned}\text{Volume} &= \int_{-3/\sqrt{2}}^{3/\sqrt{2}}\int_{-\sqrt{9/2-x^2}}^{\sqrt{9/2-x^2}}\int_{x^2+y^2}^{9-x^2-y^2} 1\,dz\,dy\,dx\\ &= \int_{-3/\sqrt{2}}^{3/\sqrt{2}}\int_{-\sqrt{9/2-x^2}}^{\sqrt{9/2-x^2}}(9-2x^2-2y^2)\,dy\,dx\\ &= \int_{-3/\sqrt{2}}^{3/\sqrt{2}}\left(\left[12-\frac{8}{3}x^2\right]\sqrt{\frac{9}{2}-x^2}\right)dx\\ &= \left(\sqrt{\frac{9}{2}-x^2}\left[\frac{15x}{2}-\frac{2x^3}{3}\right]+\frac{81}{4}\arcsin\left[\frac{\sqrt{2}x}{3}\right]\right)\Bigg|_{-3/\sqrt{2}}^{3/\sqrt{2}}\\ &= \frac{81\pi}{4}.\end{aligned}$$

**9.** Of course there are other ways to calculate the volume of the sphere.

$$\begin{aligned}\text{Volume} &= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} 1\,dz\,dy\,dx = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2-y^2}\,dy\,dx\\ &= \int_{-a}^{a}\left(y\sqrt{a^2-x^2-y^2}-(a^2-x^2)\arcsin\left[\frac{y}{\sqrt{a^2-x^2}}\right]\right)\Bigg|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}dx\\ &= \pi\int_{-a}^{a}(a^2-x^2)\,dx = \pi(a^2x-x^3/3)\Big|_{-a}^{a} = \frac{4\pi a^3}{3}.\end{aligned}$$

**10.** The students have seen this as the volume of a solid of revolution. We'll orient the cone so that the vertex is down at the origin and the axis is along the $z$-axis. Then the horizontal cross sections are circles of radius $rz/h$. This simplifies the following computation:

$$\text{Volume} = \int_0^h \int_{-rz/h}^{rz/h} \int_{-\sqrt{(rz/h)^2-x^2}}^{\sqrt{(rz/h)^2-x^2}} dy\,dx\,dz = \int_0^h \pi \frac{r^2}{h^2} z^2\,dz = \frac{1}{3}\pi r^2 h.$$

**11.**

$$\begin{aligned}\int_0^1 \int_{-2}^2 \int_0^{y^2} (2x - y + z)\,dz\,dy\,dx &= \int_0^1 \int_{-2}^2 \left(2xz - yz + \frac{z^2}{2}\right)\Big|_0^{y^2} dy\,dx \\ &= \int_0^1 \int_{-2}^2 (2xy^2 - y^3 + y^4/2)\,dy\,dx \\ &= \int_0^1 \left(\frac{2xy^3}{3} - \frac{y^4}{4} + \frac{y^5}{10}\right)\Big|_{-2}^2 dx \\ &= \int_0^1 \left(\frac{32x}{3} + \frac{64}{10}\right) dx \\ &= \left(\frac{16x^2}{3} + \frac{32}{5}x\right)\Big|_0^1 = \frac{176}{15}.\end{aligned}$$

**12.**

$$\begin{aligned}\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{2-x-z} y\,dy\,dz\,dx &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\frac{y^2}{2}\right)\Big|_0^{2-x-z} dz\,dx \\ &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} ((2-x-z)^2/2)\,dz\,dx \\ &= \int_{-1}^1 \left(\frac{1}{3}\sqrt{1-x^2}(2x^2 - 12x + 13)\right) \\ &= \left(\sqrt{1-x^2}\left(\frac{2x^3 - 16x^2 + 25x + 16}{12}\right) + \frac{9}{4}\arcsin x\right)\Big|_{-1}^1 \\ &= \frac{9\pi}{4}.\end{aligned}$$

**13.** Here $\int_{-3}^3 \int_{x^2}^9 \int_0^{9-y} 8\,xyz\,dz\,dy\,dx = 0$, because we are integrating an odd function in $x$ over an interval that is symmetric in $x$ (see Exercises 1 and 4).

**14.**

$$\begin{aligned}\int_0^3 \int_x^3 \int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx &= \int_0^3 \int_x^3 \frac{z^2}{2}\Big|_0^{\sqrt{9-y^2}} dy\,dx \\ &= \int_0^3 \int_x^3 ((9-y^2)/2)\,dy\,dx \\ &= \int_0^3 \left(\frac{-y^3 + 27y}{6}\right)\Big|_x^3 dx \\ &= \int_0^3 \left(\frac{x^3 - 27x + 54}{6}\right) dx = \frac{81}{8}.\end{aligned}$$

**15.** Here we are again integrating a polynomial. The only difficulty is in the set up:

$$\int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} (1 - z^2)\,dz\,dy\,dx = \frac{1}{10}.$$

**16.** Again, the set up and solution are:

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^4 3x\,dz\,dy\,dx = \frac{64}{5}.$$

**17.**

$$\int_0^3 \int_0^{3-x} \int_{-\sqrt{3-x^2/3}}^{\sqrt{3-x^2/3}} (x+y)\,dz\,dy\,dx = \int_0^3 \int_0^{3-x} (2(x+y)\sqrt{3-x^2/3})\,dy\,dx$$

$$= \int_0^3 ((9-x^2)\sqrt{3-x^2/3})\,dx$$

$$= \left(\sqrt{3-x^2/3}\left(\frac{45x-2x^3}{8}\right) + \frac{81\sqrt{3}}{8}\arcsin(x/3)\right)\Bigg|_0^3$$

$$= \frac{81\sqrt{3}\pi}{16}.$$

**18.**

$$\int_{-2}^2 \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} \int_0^{x+2} z\,dz\,dy\,dx = \int_{-2}^2 \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} ((x+2)^2/2)\,dy\,dx$$

$$= \int_{-2}^2 ((x+2)^2\sqrt{1-x^2/4})\,dx$$

$$= \left(\frac{3x^3+16x^2+18x-64}{12}\sqrt{1-x^2/4} + 5\arcsin(x/2)\right)\Bigg|_{-2}^2$$

$$= 5\pi.$$

**19.** $\int_0^1 \int_{y^2}^y \int_0^y (4x+y)\,dz\,dx\,dy = \int_0^1 \int_{y^2}^y (4x+y)y\,dx\,dy = \int_0^1 (3y^3 - 2y^5 - y^4)\,dy$
$= \left[\frac{3}{4}y^4 - \frac{1}{3}y^6 - \frac{1}{5}y^5\right]\Big|_0^1 = \frac{13}{60}.$

**20.** The surfaces $z = x^2 + 2y^2$ and $z = 6 - x^2 - y^2$ intersect where

$$x^2 + 2y^2 = 6 - x^2 - y^2 \iff 2x^2 + 3y^2 = 6.$$

Since we are only interested in the first octant part of the solid, the shadow of the solid in the $xy$-plane is the region bounded by the ellipse $2x^2 + 3y^2 = 6$ and the coordinate axes in the first quadrant. Thus we calculate:

$$\int_0^{\sqrt{3}} \int_0^{\sqrt{2-2x^2/3}} \int_{x^2+2y^2}^{6-x^2-y^2} x\,dz\,dy\,dx = \int_0^{\sqrt{3}} \int_0^{\sqrt{2-2x^2/3}} x(6-2x^2-3y^2)\,dy\,dx$$

$$= \int_0^{\sqrt{3}} \left[(6x-2x^3)\sqrt{2-\frac{2}{3}x^2} - x\left(2-\frac{2}{3}x^2\right)^{3/2}\right] dx$$

$$= \int_0^{\sqrt{3}} 2x\left(2-\frac{2}{3}x^2\right)^{3/2} dx = \int_2^0 -\frac{3}{2}u^{3/2}\,du,$$

where $u = 2 - \frac{2}{3}x^2$,

$$= \int_0^2 \frac{3}{2}u^{3/2}\,du = \frac{3}{5}u^{5/2}\Big|_0^2 = \frac{12\sqrt{2}}{5}.$$

**21.** The volume is given by

$$\iiint_W 1\,dV = \int_0^2 \int_0^{2-x} \int_0^{4-x^2} 1\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^{2-x} (4-x^2)\,dy\,dx = \int_0^2 (4-x^2)(2-x)\,dx$$

$$= \int_0^2 (x^3 - 2x^2 - 4x + 8)\,dx = \left(\frac{x^4}{4} - \frac{2x^3}{3} - 2x^2 + 8x\right)\Bigg|_0^2 = \frac{20}{3}.$$

**22.** The volume is

$$\begin{aligned}\iiint_W 1\,dV &= \int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{6-2y} 1\,dz\,dy\,dx = \int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} (6-2y)\,dy\,dx \\ &= \int_{-3}^{3}\left[6\sqrt{9-x^2}-(9-x^2)\right]\,dx = \int_{-3}^{3} 6\sqrt{9-x^2}\,dx + \int_{-3}^{3}(x^2-9)\,dx \\ &= \int_{-3}^{3} 6\sqrt{9-x^2}\,dx + \left(\frac{x^3}{3}-9x\right)\Big|_{-3}^{3} \\ &= \int_{-3}^{3} 6\sqrt{9-x^2}\,dx - 36.\end{aligned}$$

For the remaining integral, let $x = 3\sin\theta$ so that $dx = 3\cos\theta\,d\theta$. Then

$$\begin{aligned}\int_{-3}^{3} 6\sqrt{9-x^2}\,dx &= \int_{-\pi/2}^{\pi/2} 6(3\cos\theta)3\cos\theta\,d\theta = 27\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta \\ &= 27\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_{-\pi/2}^{\pi/2} = 27\pi.\end{aligned}$$

(Alternatively, we could have recognized this integral as six times the area of a semicircle of radius 3, or $6(\pi\cdot 3^2/2) = 27\pi$.) Hence the total volume is $27\pi - 36$.

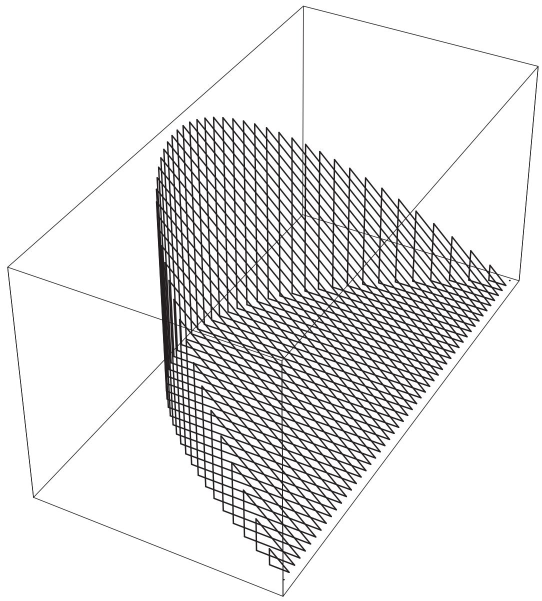
**23.**

$$\begin{aligned}\int_{-1}^{1}\int_{-\sqrt{(1-y^2)/2}}^{\sqrt{(1-y^2)/2}}\int_{4x^2+y^2}^{2-y^2} dz\,dx\,dy &= \int_{-1}^{1}\int_{-\sqrt{(1-y^2)/2}}^{\sqrt{(1-y^2)/2}}(2-4x^2-2y^2)\,dx\,dy \\ &= \int_{-1}^{1}\left(\frac{4\sqrt{2}}{3}(y^2-1)\sqrt{1-y^2}\right)\,dy \\ &= \frac{\pi}{\sqrt{2}}.\end{aligned}$$

**24.**

$$\begin{aligned}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx &= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}(2\sqrt{a^2-x^2})\,dy\,dx \\ &= \int_{-a}^{a} 4(a^2-x^2)\,dx \\ &= \frac{16a^3}{3}.\end{aligned}$$

**25.** The region looks like a wedge of cheese:



The five other forms are:

$$\begin{aligned}\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1-x} f(x,y,z)\,dz\,dx\,dy &= \int_{0}^{1}\int_{-\sqrt{x}}^{\sqrt{x}}\int_{0}^{1-x} f(x,y,z)\,dz\,dy\,dx\\ &= \int_{0}^{1}\int_{0}^{1-x}\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y,z)\,dy\,dz\,dx\\ &= \int_{0}^{1}\int_{0}^{1-z}\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y,z)\,dy\,dx\,dz\\ &= \int_{-1}^{1}\int_{0}^{1-y^2}\int_{y^2}^{1-z} f(x,y,z)\,dx\,dz\,dy\\ &= \int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{y^2}^{1-z} f(x,y,z)\,dx\,dy\,dz.\end{aligned}$$

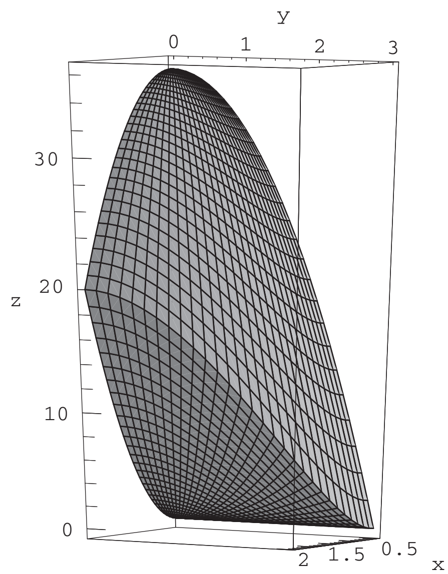
**26.** The five other forms are:

$$\begin{aligned}\int_{0}^{1}\int_{0}^{1}\int_{0}^{x^2} f(x,y,z)\,dz\,dx\,dy &= \int_{0}^{1}\int_{0}^{1}\int_{0}^{x^2} f(x,y,z)\,dz\,dy\,dx\\ &= \int_{0}^{1}\int_{0}^{1}\int_{\sqrt{z}}^{1} f(x,y,z)\,dx\,dz\,dy\\ &= \int_{0}^{1}\int_{0}^{1}\int_{\sqrt{z}}^{1} f(x,y,z)\,dx\,dy\,dz\\ &= \int_{0}^{1}\int_{0}^{x^2}\int_{0}^{1} f(x,y,z)\,dy\,dz\,dx\\ &= \int_{0}^{1}\int_{\sqrt{z}}^{1}\int_{0}^{1} f(x,y,z)\,dy\,dx\,dz.\end{aligned}$$

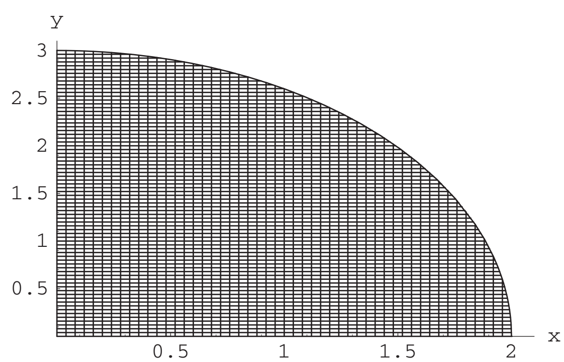
**27.** The five other forms are:

$$\int_0^2 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^2 \int_y^2 \int_0^y f(x,y,z)\,dz\,dx\,dy$$
$$= \int_0^2 \int_0^x \int_z^x f(x,y,z)\,dy\,dz\,dx$$
$$= \int_0^2 \int_z^2 \int_z^x f(x,y,z)\,dy\,dx\,dz$$
$$= \int_0^2 \int_0^y \int_y^2 f(x,y,z)\,dx\,dz\,dy$$
$$= \int_0^2 \int_z^2 \int_y^2 f(x,y,z)\,dx\,dy\,dz.$$

**28.** **(a)** The solid $W$ is bounded below by the surface $z = 5x^2$, above by the paraboloid $z = 36 - 4x^2 - 4y^2$, on the left by the *xz*-plane (i.e., $y = 0$), and in back by the *yz*-plane (i.e., $x = 0$). The solid is shown below.



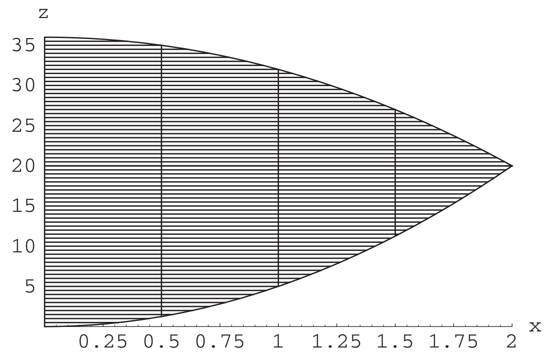

**(b)** The shadow of the solid in the *xy*-plane is a quarter of the ellipse $9x^2 + 4y^2 = 36$ (obtained by finding the intersection curve of $z = 5x^2$ and $z = 36 - 4x^2 - 4y^2$.) The shadow looks like:




Using the shadow region to reverse the order of integration between $x$ and $y$, we find that the original integral is equivalent to

$$\int_0^3 \int_0^{\frac{1}{3}\sqrt{36-4y^2}} \int_{5x^2}^{36-4x^2-4y^2} 2\,dz\,dx\,dy.$$

**(c)** In this case, we need to consider the shadow of $W$ in the *xz*-plane.



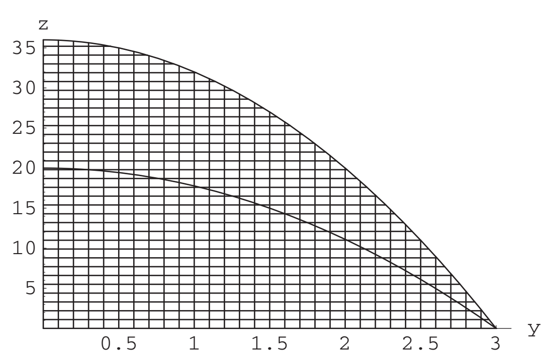

This region is bounded on the left by $x = 0$, on the bottom by $z = 5x^2$, and on top by $z = 36 - 4x^2$ (the section by $y = 0$). Now the full solid $W$ is bounded in the $y$-direction by $y = 0$ and $y = \frac{1}{2}\sqrt{36 - 4x^2 - z}$ (the latter is just the paraboloid surface). Hence the desired iterated integral is

$$\int_0^2 \int_{5x^2}^{36-4x^2} \int_0^{\frac{1}{2}\sqrt{36-4x^2-z}} 2\,dy\,dz\,dx.$$

**(d)** Here we use the same shadow in the *xz*-plane as in part (c), only to integrate with respect to $x$ before integrating with respect to $z$ will require dividing the shadow into two regions by the line $z = 20$. (Equivalently, we are dividing the solid $W$ into two solids by the plane $z = 20$.) This is why we need a sum of integrals. They are

$$\int_0^{20} \int_0^{\sqrt{z/5}} \int_0^{\frac{1}{2}\sqrt{36-4x^2-z}} 2\,dy\,dx\,dz + \int_{20}^{36} \int_0^{\frac{1}{2}\sqrt{36-z}} \int_0^{\frac{1}{2}\sqrt{36-4x^2-z}} 2\,dy\,dx\,dz.$$

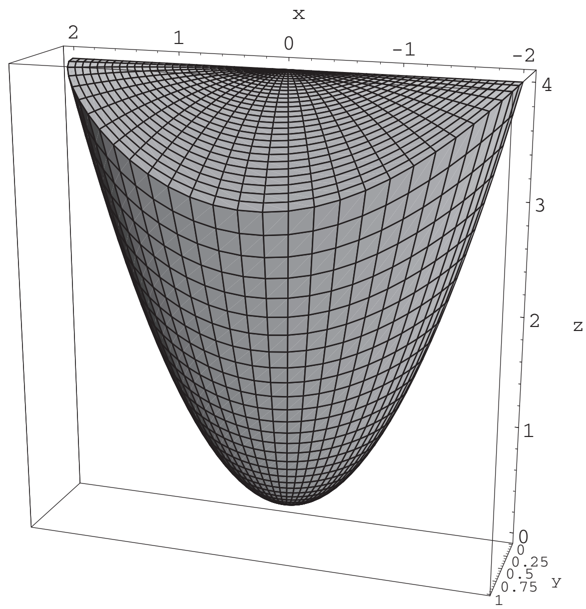
**(e)** To integrate with respect to $x$ first, we need to divide $W$ in a very different manner. The shadow in the *yz*-plane shows a region bounded on the left by $y = 0$ and above by $z = 36 - 4y^2$ (the section of the paraboloid by $x = 0$). However, the curve of intersection of the surfaces $z = 5x^2$ and $z = 36 - 4x^2 - 4y^2$ with $x$ eliminated yields the equation $z = 20 - \dfrac{20y^2}{9}$. It is along this curve that we must divide the *yz*-shadow and thus the integrals. (Note: This curve is just the shadow of the intersection curve of the two surfaces projected into the *yz*-plane.)



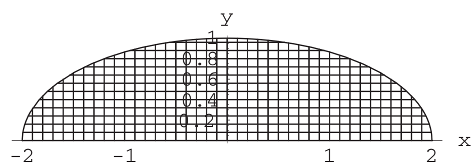

Thus the desired sum of integrals is

$$\int_0^3 \int_0^{20-20y^2/9} \int_0^{\sqrt{z/5}} 2\,dx\,dz\,dy + \int_0^3 \int_{20-20y^2/9}^{36-4y^2} \int_0^{\frac{1}{2}\sqrt{36-4y^2-z}} 2\,dx\,dz\,dy.$$

**29. (a)** The solid $W$ is bounded below by the paraboloid $z = x^2 + 3y^2$, above by the surface $z = 4 - y^2$ and in back by the plane $y = 0$. The solid is shown below.



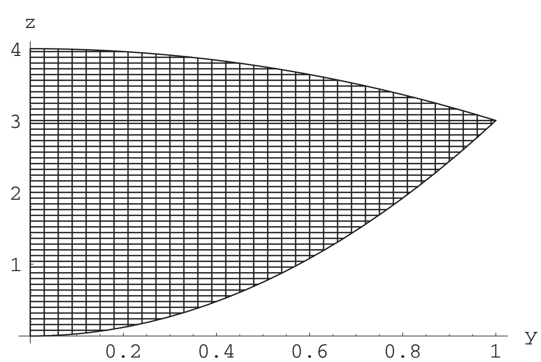

**(b)** The shadow of $W$ in the *xy*-plane is half of the region inside the ellipse $x^2 + 4y^2 = 4$ (the half with $y \geq 0$). It may be obtained by finding the intersection curve of $z = x^2 + 3y^2$ and $z = 4 - y^2$ and eliminating $z$. The shadow looks like




Using the shadow to reverse the order of integration between $x$ and $y$, we find that the original integral is equivalent to

$$\int_0^1 \int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}} \int_{x^2+3y^2}^{4-y^2} (x^3 + y^3)\, dz\, dx\, dy.$$

**(c)** We need to consider the shadow of $W$ in the *yz*-plane.



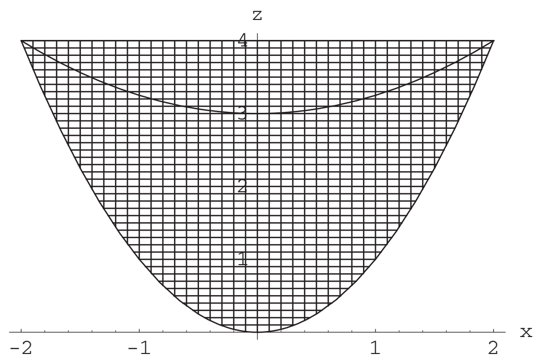

The region is bounded on the left by $y = 0$, on the bottom by $z = 3y^2$ (the section by $x = 0$) and on the top by $z = 4 - y^2$. The full solid $W$ is bounded in the $x$-direction by the paraboloid $z = x^2 + 3y^2$, which must be expressed in terms of $x$ as $x = \pm\sqrt{z - 3y^2}$. Putting all this information together, we find the desired iterated integral is

$$\int_0^1 \int_{3y^2}^{4-y^2} \int_{-\sqrt{z-3y^2}}^{\sqrt{z-3y^2}} (x^3 + y^3)\, dx\, dz\, dy.$$

**(d)** Here we use the same shadow in the *yz*-plane as in part (c), only to integrate with respect to $y$ before integrating with respect to $z$ requires dividing the shadow into two regions by the line $z = 3$. (Equivalently, we are dividing the solid $W$ by the plane $z = 3$.) This is why we need a sum of integrals. They are

$$\int_0^3 \int_0^{\sqrt{z/3}} \int_{-\sqrt{z-3y^2}}^{\sqrt{z-3y^2}} (x^3 + y^3)\, dx\, dy\, dz + \int_3^4 \int_0^{\sqrt{4-z}} \int_{-\sqrt{z-3y^2}}^{\sqrt{z-3y^2}} (x^3 + y^3)\, dx\, dy\, dz.$$

**(e)** To integrate with respect to $y$ first, we need to divide $W$ in a different manner. The shadow in the *xz*-plane shows a region bounded by $z = x^2$ (the section of the paraboloid by $y = 0$) and $z = 4$. However, the curve of intersection of the surfaces $z = x^2 + 3y^2$ and $z = 4 - y^2$ with $y$ eliminated yields the equation $z = \frac{x^2}{4} + 3$. It is along this curve that we must divide the *xz*-shadow and thus the integrals. (Note: This curve is just the shadow of the intersection curve of the two surfaces projected into the *xz*-plane.)




Thus the desired sum of integrals is

$$\int_{-2}^{2} \int_{x^2}^{(x^2/4)+3} \int_0^{\sqrt{(z-x^2)/3}} (x^3 + y^3)\, dy\, dz\, dx + \int_{-2}^{2} \int_{(x^2/4)+3}^{4} \int_0^{\sqrt{4-z}} (x^3 + y^3)\, dy\, dz\, dx.$$

## 5.5 Change of Variables

**1. (a)**

$$\mathbf{T}(u, v) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

**(b)** In this case we can see by inspection that the transformation stretches by 3 in the horizontal direction and reflects without a stretch in the vertical direction. Therefore the image $D = \mathbf{T}(D^*)$ where $D^*$ is the unit square is the rectangle $[0, 3] \times [-1, 0]$.

**2. (a)** This is similar to the map in Example 4 with a scaling factor of $1/\sqrt{2}$. We can also rewrite

$$\mathbf{T}(u, v) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

This is a rotation matrix (the determinant is 1 so there is no stretching) which rotates the unit square counterclockwise by $45^\circ$ leaving the vertex at the origin in place.

**(b)** We rewrite

$$\mathbf{T}(u, v) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

This is a rotation followed by a reflection. You can apply Proposition 5.1 and see where $\mathbf{T}$ maps each of the vertices to completely determine the image of the unit square. You will see that the vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(1, 0)$ are mapped to $(0, 0)$, $(1/\sqrt{2}, 1/\sqrt{2})$, $(\sqrt{2}, 0)$, and $(1/\sqrt{2}, -1/\sqrt{2})$.

**3.** Again, since **T** has non-zero determinant, we can apply Proposition 5.1 and see where **T** maps each of the vertices. We conclude that **T** maps $D^*$ to the parallelogram whose vertices are: (0, 0), (11, 2), (4, 3), and (15, 5).

**4.** We are trying to determine the entries $a$, $b$, $c$, and $d$ in the expression:

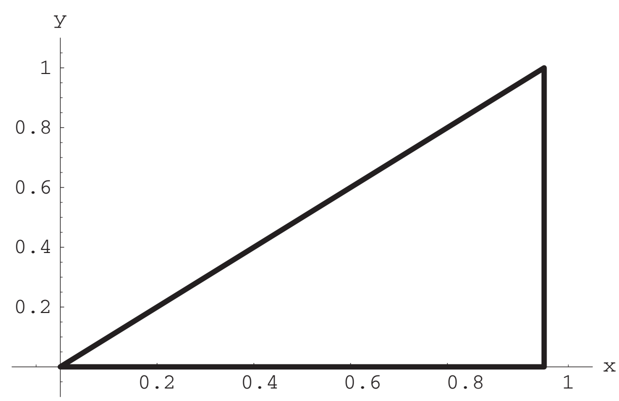
$$\mathbf{T}(u,v) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Since $\mathbf{T}(0,0) = (0,0)$ we know that the motion is not a translation. Also $\mathbf{T}(0,5) = (4,1)$ so $b = 4/5$ and $d = 1/5$. Now $\mathbf{T}(1,2) = (1,-1)$ so $a = -3/5$ and $c = -7/5$. We check with the remaining vertex: $\mathbf{T}(-1,3) = (3,2)$.

$$\mathbf{T}(u,v) = \begin{bmatrix} -3/5 & 4/5 \\ -7/5 & 1/5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

**5.** As noted in the text, we have a result for $\mathbf{R}^3$ that is analogous to Proposition 5.1, so as in Exercises 3 and 2 (b) we can just compute the images of the vertices of $W^*$. We conclude that $W^*$ maps to the parallelepiped with vertices: (0, 0, 0), (3, 1, 5), $(-1,-1,3)$, $(0,2,-1)$, (2, 0, 8), (3, 3, 4), $(-1,1,2)$, and (2, 2, 7).

**6.** You can see that $\mathbf{T}(u,v) = (u, uv)$ is not one-one on $D^*$ by observing that all points of the form (0, $v$) get mapped to the origin under **T**. In fact, you can imagine the map by picturing the left vertical side of the unit square being shrunk down to a point at the origin. The image is the triangle:




**7.** This map should be a happy memory for the students:

$$(x,y,z) = \mathbf{T}(\rho,\varphi,\theta) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$$

is familiar from their work with spherical coordinates.

**(a)** This is the unit ball: $D = \{(x,y,z) | x^2+y^2+z^2 \le 1\}$.

**(b)** This is the portion of the unit ball in the first octant: $D = \{(x,y,z) | x^2+y^2+z^2 \le 1, x, y, z \ge 0\}$.

**(c)** You can think of this as the region from part (b) with the portion corresponding to $0 \le \rho < 1/2$ removed. It is the portion in the first octant of the shell 1/2 unit thick around a sphere of radius 1/2: $D = \{(x,y,z) | 1/4 \le x^2+y^2+z^2 \le 1, x, y, z \ge 0\}$.

**8.** **(a)** $\displaystyle\int_0^1 \int_{y/2}^{(y/2)+2} (2x-y)\,dx\,dy = \int_0^1 (x^2 - xy)\Big|_{y/2}^{(y/2)+2} dy = \int_0^1 4\,dy = 4$. A sketch of $D$ is shown below.

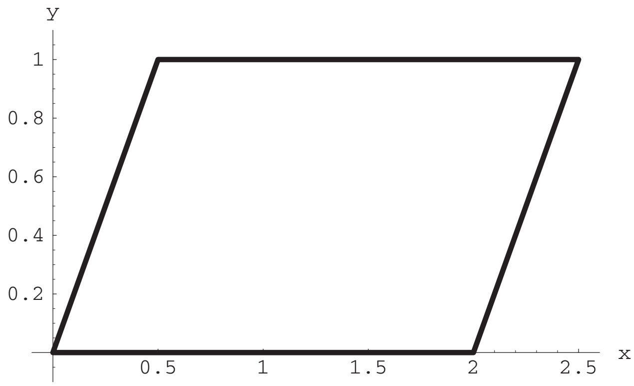
**(b)** We again can apply Proposition 5.1 and see that the vertices are mapped: $(0,0) \to (0,0)$, $(2,0) \to (4,0)$, $(1/2,1) \to (0,1)$, and $(5/2,1) \to (4,1)$ so $D^*$ is $[0,4] \times [0,1]$.

**(c)** First note that

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = 2 \quad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}.$$

Then, using the change of variables theorem,

$$\int_0^1 \int_{y/2}^{(y/2)+2} (2x-y)\,dx\,dy = \int_0^1 \int_0^4 u(1/2)\,du\,dv = \int_0^1 \frac{u^2}{4}\Big|_0^4 du = \int_0^1 4\,dv = 4.$$



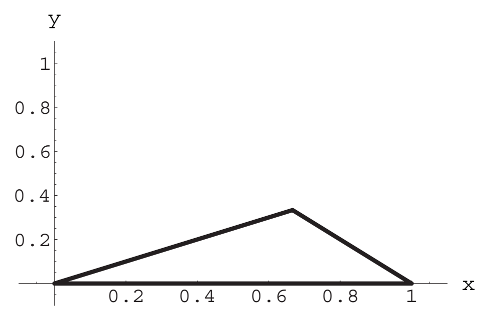

**9.** First,

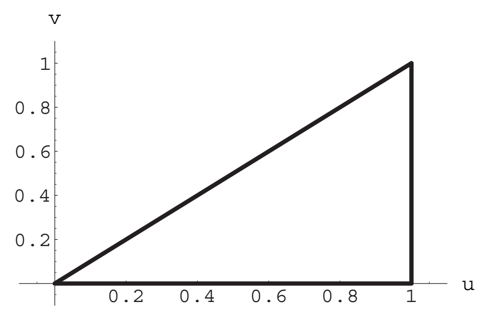
$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = 2 \quad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}.$$

Also we can rewrite $x^5(2y-x)e^{(2y-x)^2} = u^5 v e^{v^2}$, and the transformed region is $[0,2] \times [0,2]$ so

$$\int_0^2 \int_{x/2}^{(x/2)+1} x^5(2x-y)e^{(2x-y)^2}\,dy\,dx = \int_0^2 \int_0^2 u^5 v e^{v^2}(1/2)\,du\,dv = \frac{16}{3}\int_0^2 v e^{v^2}\,dv = \frac{8}{3}(e^4-1).$$

**10.** The original region $D$ is sketched below left. The transformation $u = x + y$ and $v = x - 2y$ maps $D$ to the region $D^*$ sketched below right.




We may find $\partial(x,y)/\partial(u,v)$ in two ways. First, solving for $x$ and $y$ in terms of $u$ and $v$, we have

$$x = \frac{2u+v}{3}, \qquad y = \frac{u-v}{3}.$$

Thus

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} = -\frac{2}{9} - \frac{1}{9} = -\frac{1}{3}.$$

Alternatively, we may calculate

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = -2 - 1 = -3.$$

Therefore, $\partial(x,y)/\partial(u,v) = (\partial(u,v)/\partial(x,y))^{-1} = -1/3$.

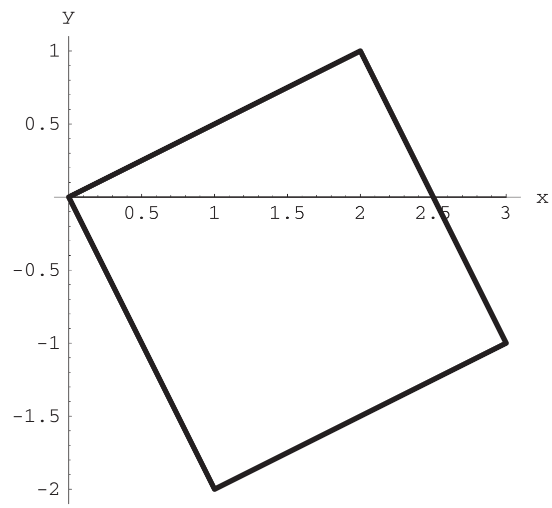
Using the change of variables theorem, our integral becomes

$$\int_0^1 \int_0^u \frac{1}{3}\left(\frac{u^{1/2}}{v^{1/2}}\right) dv\,du = \int_0^1 \frac{2}{3}u^{1/2}v^{1/2}\Big|_0^u du = \int_0^1 \frac{2}{3}u\,du = \frac{u^2}{3}\Big|_0^1 = \frac{1}{3}.$$

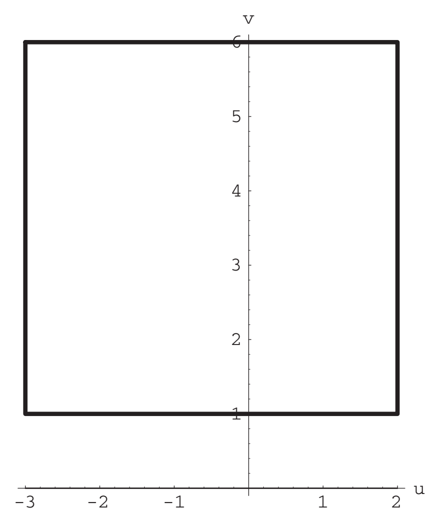
**11.** Here the problem cries out to you to let $u = 2x + y$ and $v = x - y$. Once you've made that move you can easily figure that $\partial(x,y)/\partial(u,v) = -1/3$ and that the new region is $[1,4] \times [-1,1]$. So the integral is

$$\int_1^4 \int_{-1}^1 u^2 e^v (1/3)\,dv\,du = \frac{1}{3}\int_1^4 u^2 e^v\Big|_{-1}^1 du = (e - e^{-1})\frac{u^3}{9}\Big|_1^4 = 7(e - e^{-1}).$$

**12.** If we sketch the region we get the square:




The transformation we use is $u = 2x + y - 3$ and $v = 2y - x + 6$ so $\partial(x, y)/\partial(u, v) = 1/5$ and the transformed region is the square:




Our integral is

$$\int_1^6 \int_{-3}^2 \frac{u^2}{5v^2}\, du\, dv = \frac{1}{15}\int_1^6 \frac{u^3}{v^2}\bigg|_{-3}^2 dv = \frac{7}{3}\int_1^6 v^{-2}\, dv = \frac{7}{3}\left(-\frac{1}{v}\right)\bigg|_1^6 = \frac{35}{18}.$$

*Note: In Exercises 13–17 the Jacobian for the change of variables is $r$. Assign Exercise 16 so that your students appreciate the role of the extra $r$.*

**13.** $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3\, dy\, dx = \int_0^{2\pi}\int_0^1 3r\, dr\, d\theta = \int_0^{2\pi} \frac{3}{2} d\theta = 3\pi.$

**14.** $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} dy\, dx = \int_0^{\pi/2}\int_0^2 r\, dr\, d\theta = \int_0^{\pi/2} 2\, d\theta = \pi.$

**15.** $\displaystyle\int_0^{2\pi}\int_0^3 r^4\, dr\, d\theta = \int_0^{2\pi} \frac{r^5}{5}\bigg|_0^3 d\theta = \int_0^{2\pi}\frac{243}{5}\, d\theta = \frac{486\pi}{5}.$

**16.** $\displaystyle\int_{-a}^a \int_0^{\sqrt{a^2-y^2}} e^{x^2+y^2}\, dx\, dy = \int_{-\pi/2}^{\pi/2}\int_0^a re^{r^2}\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2}e^{r^2}\bigg|_0^a d\theta = \int_{-\pi/2}^{\pi/2}\frac{1}{2}(e^{a^2}-1)d\theta = \pi(e^{a^2}-1)/2.$

**17.** $\displaystyle\int_0^3 \int_0^x \frac{dy\,dx}{\sqrt{x^2+y^2}} = \int_0^{\pi/4} \int_0^{3\sec\theta} dr\,d\theta = \int_0^{\pi/4} 3\sec\theta\,d\theta = 3\ln(\sec\theta + \tan\theta)\big|_0^{\pi/4} = \ln(1+\sqrt{2}) - \ln 1 = \ln(1+\sqrt{2}).$

**18.** This is a job for polar coordinates. The given disk has boundary circle with equation $x^2 + (y-1)^2 = 1$. In polar coordinates this equation becomes

$$r^2\cos^2\theta + (r\sin\theta - 1)^2 = 1 \Leftrightarrow r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + 1 = 1$$
$$\Leftrightarrow r^2 = 2r\sin\theta.$$

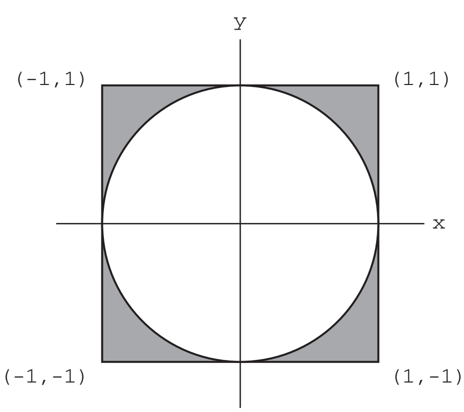
Factoring out $r$, the boundary circle has equation $r = 2\sin\theta$. In fact, this circle is completely traced by letting $\theta$ vary from 0 to $\pi$. Thus the region $D$ inside the disk is given by

$$D = \{(r,\theta) | r \le 2\sin\theta, \quad 0 \le \theta \le \pi\}.$$

Hence

$$\begin{aligned}
\iint_D \frac{1}{\sqrt{4-x^2-y^2}}\,dA &= \int_0^\pi \int_0^{2\sin\theta} \frac{1}{\sqrt{4-r^2}}\,r\,dr\,d\theta \\
&= \int_0^\pi -\frac{1}{2}(2\sqrt{4-r^2})\Big|_0^{2\sin\theta} d\theta = -\int_0^\pi (\sqrt{4-4\sin^2\theta} - 2)\,d\theta \\
&= -2\int_0^\pi \sqrt{\cos^2\theta}\,d\theta + \int_0^\pi 2\,d\theta \\
&= -2\left(\int_0^{\pi/2}\cos\theta\,d\theta + \int_{\pi/2}^{\pi}(-\cos\theta)\,d\theta\right) + 2\pi \\
&= -2\left[\sin\frac{\pi}{2} - \sin 0 - \sin\pi + \sin\frac{\pi}{2}\right] + 2\pi \\
&= -4 + 2\pi = 2\pi - 4.
\end{aligned}$$

**19.** The region in question looks like




We find $\displaystyle\iint_D y^2 dA = \iint_{\text{square}} y^2 dA - \iint_{\text{disk}} y^2 dA$

$$\iint_{\text{square}} y^2\,dA = \int_{-1}^1 \int_{-1}^1 y^2\,dx\,dy = \int_{-1}^1 2y^2\,dy = \frac{2}{3}y^3\Big|_{-1}^1 = \frac{4}{3}$$

$$\begin{aligned}
\iint_{\text{disk}} y^2\,dA &= \int_0^{2\pi} \int_0^1 r^2\sin^2\theta \cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{4}\sin^2\theta\,d\theta \\
&= \frac{1}{8}\int_0^{2\pi}(1-\cos 2\theta)\,d\theta = \frac{1}{8}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{\pi}{4}.
\end{aligned}$$

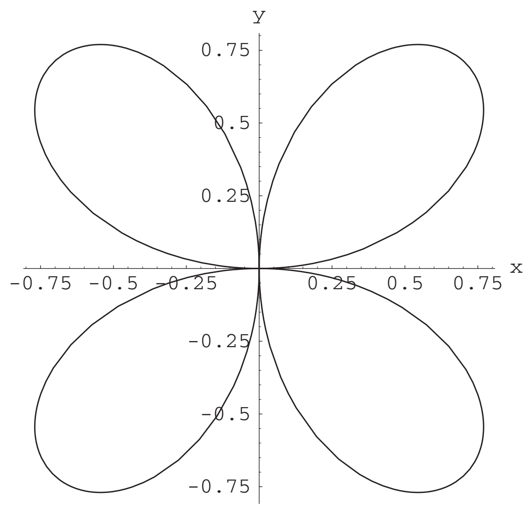
Thus

$$\iint_D y^2\,dA = \frac{4}{3} - \frac{\pi}{4} = \frac{16-3\pi}{12}.$$

**20.** A sketch of the rose is shown below. One leaf means that $0 \le \theta \le \pi/2$. The area of one leaf is

$$\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta\,d\theta = \frac{1}{16}(4\theta - \sin 4\theta)\Big|_0^{\pi/2} = \frac{\pi}{8}.$$

The total area is then four times this, or $\pi/2$.




**21.** If $n$ is *odd*, the polar equation $r = a\cos n\theta$ determines an $n$-leafed rose. Half of one of the $n$ leaves is traced as $\theta$ varies from 0 to $\pi/(2n)$. Hence the total area enclosed is

$$\begin{aligned}\iint_D 1\,dA &= 2n\int_0^{\pi/(2n)}\int_0^{a\cos n\theta} r\,dr\,d\theta = 2n\int_0^{\pi/(2n)} \frac{1}{2}(a\cos n\theta)^2\,d\theta \\ &= na^2\int_0^{\pi/(2n)} \cos^2 n\theta\,d\theta = \frac{na^2}{2}\int_0^{\pi/(2n)}(1+\cos 2n\theta)\,d\theta \\ &= \frac{na^2}{2}\left(\theta + \frac{1}{2n}\sin 2n\theta\right)\Big|_0^{\pi/(2n)} = \frac{\pi a^2}{4}.\end{aligned}$$

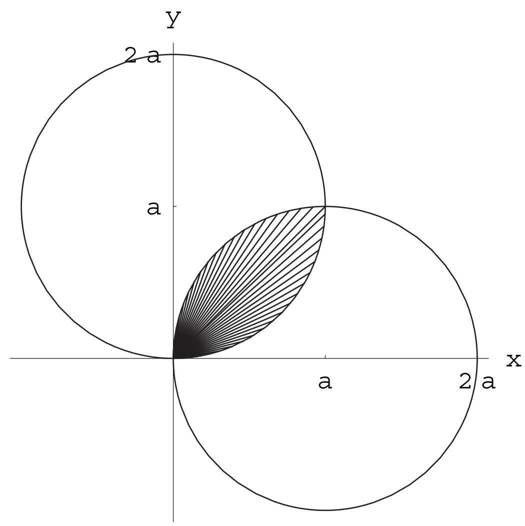
If $n$ is *even*, then the equation $r = a\cos n\theta$ determines a rose with $2n$ leaves. Half of one of these $2n$ leaves is again traced as $\theta$ varies from 0 to $\pi/(2n)$. The total area enclosed is

$$\iint_D 1\,dA = 2(2n)\int_0^{\pi/(2n)}\int_0^{a\cos n\theta} r\,dr\,d\theta.$$

Since this is just twice the previous iterated integral, there is no reason to recompute; the result is $\pi a^2/2$.

In each case the answer depends only on $a$, not the specific value of $n$ other than its parity.

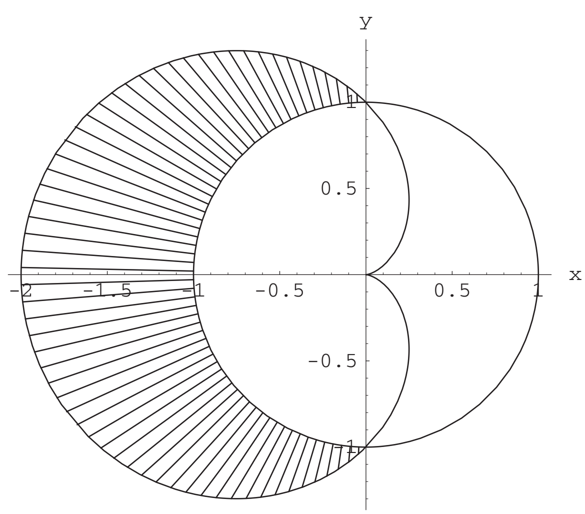
**22.** The circles $r = 2a\cos\theta$ and $r = 2a\sin\theta$ are both of radius $a$ with respective centers at $(a, 0)$ and $(0, a)$ (in Cartesian coordinates). The region in question looks like:




The circles intersect where $2a\cos\theta = 2a\sin\theta \iff \theta = \pi/4$ (also at the origin where $r = 0$). By symmetry, we have

$$\begin{aligned}\text{Area} &= \iint_D 1\,dA = 2\int_0^{\pi/4}\int_0^{2a\sin\theta} r\,dr\,d\theta \\ &= \int_0^{\pi/4}(2a\sin\theta)^2\,d\theta = \int_0^{\pi/4} 2a^2(1-\cos 2\theta)\,d\theta \\ &= 2a^2\left(\theta - \frac{1}{2}\sin 2\theta\right)\bigg|_0^{\pi/4} = \frac{\pi a^2}{2} - a^2 = \frac{(\pi-2)a^2}{2}.\end{aligned}$$

**23.** We sketch the graphs of the cardioid $r = 1 - \cos\theta$ and the circle $r = 1$:

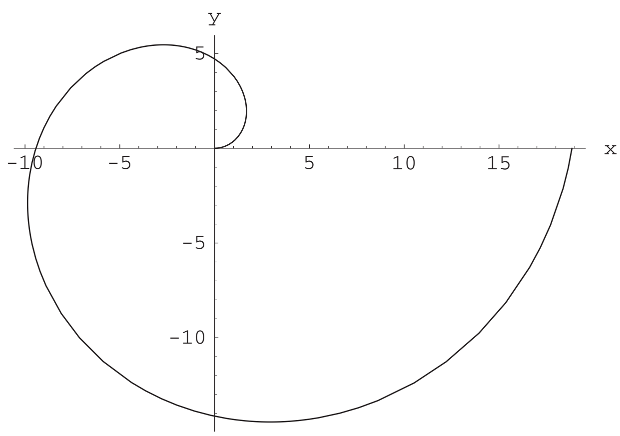



The two curves intersect when $1 - \cos\theta = 1$ which is when $\cos\theta = 0$ so the two points of intersection are $(r, \theta) = (1, \pi/2)$ and $(1, 3\pi/2)$. The region between the two graphs is where $\pi/2 \le \theta \le 3\pi/2$. The area is

$$\begin{aligned}\int_{\pi/2}^{3\pi/2}\int_1^{1-\cos\theta} r\,dr\,d\theta &= \int_{\pi/2}^{3\pi/2}\left(\frac{\cos^2\theta}{2} - \cos\theta\right)d\theta \\ &= \frac{1}{8}(2\theta - 8\sin\theta + \sin 2\theta)\bigg|_{\pi/2}^{3\pi/2} = 2 + \frac{\pi}{4}.\end{aligned}$$

**24.** We want the area "inside" the spiral shown below. The area is

$$\int_0^{2\pi}\int_0^{3\theta} r\,dr\,d\theta = \int_0^{2\pi}\frac{9}{2}\theta^2\,d\theta = \frac{3}{2}\theta^3\Big|_0^{2\pi} = 12\pi^3.$$



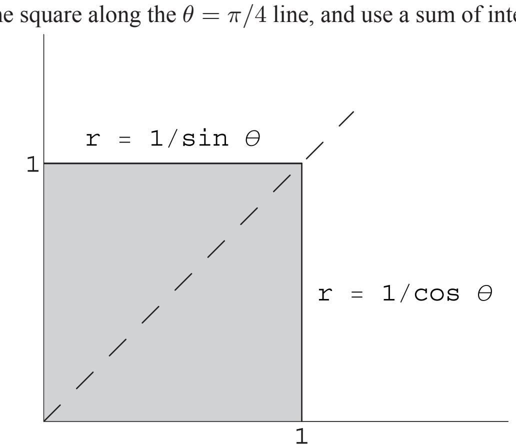

**25.** The integral is

$$\int_{\pi/3}^{\pi}\int_0^1 r\cos r^2\,dr\,d\theta = \frac{1}{2}\int_{\pi/3}^{\pi}\sin r^2\Big|_0^1\,d\theta = \frac{1}{2}\sin 1\Big|_{\pi/3}^{\pi} = \frac{\pi}{3}(\sin 1).$$

**26.** $\displaystyle\iint_D \sin(x^2+y^2)\,dA = \int_0^{\pi/2}\int_1^3 \sin(r^2)\cdot r\,dr\,d\theta$

$\displaystyle = \int_0^{\pi/2}\left(-\tfrac{1}{2}\cos(r^2)\right)\Big|_{r=1}^{r=3}\,d\theta = \int_0^{\pi/2}\tfrac{1}{2}(\cos 1-\cos 9)\,d\theta = \frac{\pi}{4}(\cos 1-\cos 9).$

**27.** Two of the edges of the unit square are given by $x = 1$ (or $r = 1/\cos\theta$ in polar coordinates) and by $y = 1$ (i.e., by $r = 1/\sin\theta$). We need to divide the square along the $\theta = \pi/4$ line, and use a sum of integrals:



Thus

$$\iint_D \frac{x}{\sqrt{x^2+y^2}}\,dA = \int_0^{\pi/4}\int_0^{1/\cos\theta} \frac{r\cos\theta}{r}\cdot r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{1/\sin\theta} r\cos\theta\,dr\,d\theta$$
$$= \int_0^{\pi/4} \frac{1}{2\cos^2\theta}\cdot\cos\theta\,d\theta + \int_{\pi/4}^{\pi/2}\frac{1}{2\sin^2\theta}\cdot\cos\theta\,d\theta$$
$$= \int_0^{\pi/4}\frac{1}{2}\sec\theta\,d\theta + \int_{\theta=\pi/4}^{\theta=\pi/2}\frac{1}{2(\sin\theta)^2}\,d(\sin\theta)$$
$$= \frac{1}{2}\ln|\sec\theta+\tan\theta|\Big|_0^{\pi/4} - \frac{1}{2\sin\theta}\Big|_{\pi/4}^{\pi/2}$$
$$= \frac{1}{2}\ln(\sqrt{2}+1) - \frac{1}{2}\ln 1 - \frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{1}{2}(\ln(\sqrt{2}+1)+\sqrt{2}-1).$$

**28.** $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{\sqrt{x^2+y^2}}^{3}\frac{e^z}{\sqrt{x^2+y^2}}\,dz\,dy\,dx = \int_0^{2\pi}\int_0^3\int_r^3 \frac{e^z}{r}\cdot r\,dz\,dr\,d\theta$

$\displaystyle= \int_0^{2\pi}\int_0^3 (e^3-e^r)\,dr\,d\theta = \int_0^{2\pi}(3e^3-e^3+1)\,d\theta = 2\pi(2e^3+1).$

**29.** $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{4-x^2-y^2} e^{x^2+y^2+z}\,dz\,dx\,dy = \int_0^{2\pi}\int_0^1\int_0^{4-r^2} e^{r^2+z}\,r\,dz\,dr\,d\theta$

$\displaystyle= \int_0^{2\pi}\int_0^1 \left(re^{r^2}\cdot e^z\right)\Big|_{z=0}^{z=4-r^2} dr\,d\theta = \int_0^{2\pi}\int_0^1 re^{r^2}\left(e^{4-r^2}-1\right)dr\,d\theta$

$\displaystyle= \int_0^{2\pi}\int_0^1\left(e^4r - re^{r^2}\right)dr\,d\theta = \int_0^{2\pi}\left(\frac{e^4}{2}-\frac{e}{2}+\frac{1}{2}\right)d\theta = \pi(e^4-e+1).$

**30.** Since $B$ is a ball we will use spherical coordinates:

$$\iiint_B \frac{dV}{\sqrt{x^2+y^2+z^2+3}} = \int_0^{2\pi}\int_0^{\pi}\int_0^2 \frac{\rho^2\sin\varphi}{\sqrt{\rho^2+3}}\,d\rho\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}\left(\left[\sqrt{7}-\frac{3}{2}\operatorname{arcsinh}(2/\sqrt{3})\right]\sin\varphi\right)d\varphi\,d\theta$$
$$= \int_0^{2\pi}(2\sqrt{7}-3\operatorname{arcsinh}(2/\sqrt{3}))\,d\theta$$
$$= 4\sqrt{7}\pi - 6\pi\operatorname{arcsinh}(2/\sqrt{3}) \quad \text{which is the same as the text's solution}$$
$$= (4\sqrt{7}-6\ln(2+\sqrt{7})+3\ln 3)\pi.$$

**31.** Here we will use cylindrical coordinates:

$$\iiint_W (x^2+y^2+2z^2)\,dV = \int_{-1}^{2}\int_0^{2\pi}\int_0^2 r(r^2+2z^2)\,dr\,d\theta\,dz$$
$$= \int_{-1}^{2}\int_0^{2\pi}(4z^2+4)\,d\theta\,dz$$
$$= \int_{-1}^{2}(8\pi z^2+8\pi)\,dz = 48\pi.$$

**32.** We use cylindrical coordinates:

$$\begin{aligned}\iiint_W \frac{z}{\sqrt{x^2+y^2}}\,dV &= \int_0^{2\pi}\int_0^3\int_{2r^2-6}^{12} \frac{z}{r}\cdot r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^3 \frac{1}{2}\left(144-(2r^2-6)^2\right)\,dr\,d\theta = \int_0^{2\pi}\int_0^3 (54+12r^2-2r^4)\,dr\,d\theta \\ &= \int_0^{2\pi}\left(54r+4r^3-\frac{2}{5}r^5\right)\Big|_0^3\,d\theta = 2\pi\left(\frac{864}{5}\right) = \frac{1728\pi}{5}.\end{aligned}$$

**33.** Again we use cylindrical coordinates:

$$\begin{aligned}\text{Volume} = \iiint_W 1\,dV &= \int_0^{2\pi}\int_0^b\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^b r\sqrt{a^2-r^2}\,dr\,d\theta = \int_0^{2\pi}\int_{a^2}^{a^2-b^2} -\frac{1}{2}\sqrt{u}\,du\,d\theta\end{aligned}$$

where $u = a^2 - r^2$,

$$\begin{aligned}&= \int_0^{2\pi}\int_{a^2-b^2}^{a^2} \frac{1}{2}\sqrt{u}\,du\,d\theta = \int_0^{2\pi}\frac{1}{3}\left(a^3-(a^2-b^2)^{3/2}\right)\,d\theta \\ &= \frac{2\pi}{3}\left[a^3-(a^2-b^2)^{3/2}\right].\end{aligned}$$

**34.** It is natural to use spherical coordinates.

$$\begin{aligned}\iiint_W \frac{dV}{\sqrt{x^2+y^2+z^2}} &= \int_0^{2\pi}\int_0^{\pi}\int_a^b (\rho\sin\varphi)\,d\rho\,d\varphi\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}((b^2-a^2)\sin\varphi)\,d\varphi\,d\theta \\ &= \int_0^{2\pi}(b^2-a^2)\,d\theta = 2\pi(b^2-a^2).\end{aligned}$$

**35.** Once again we use spherical coordinates.

$$\begin{aligned}\iiint_W \sqrt{x^2+y^2+z^2}\,e^{x^2+y^2+z^2}\,dV &= \int_0^{2\pi}\int_0^{\pi}\int_a^b (\rho^3 e^{\rho^2}\sin\varphi)\,d\rho\,d\varphi\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}([(1-a^2)e^{a^2}+(b^2-1)e^{b^2}]\sin\varphi)\,d\varphi\,d\theta \\ &= \int_0^{2\pi}((1-a^2)e^{a^2}+(b^2-1)e^{b^2})\,d\theta \\ &= 2\pi((1-a^2)e^{a^2}+(b^2-1)e^{b^2}).\end{aligned}$$

**36.** We use spherical coordinates:

$$\begin{aligned}\iiint_W (x+y+z)\,dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_a^b (\rho\sin\varphi\cos\theta+\rho\sin\varphi\sin\theta+\rho\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta\\ &= \int_0^{\pi/2}\int_0^{\pi/2}\frac{b^4-a^4}{4}\left(\sin^2\varphi(\cos\theta+\sin\theta)+\sin\varphi\cos\varphi\right)\,d\varphi\,d\theta\\ &= \frac{b^4-a^4}{4}\int_0^{\pi/2}\left[(\cos\theta+\sin\theta)\left(\frac{1}{2}\varphi-\frac{1}{4}\sin 2\varphi\right)+\frac{1}{2}\sin^2\varphi\right]\Bigg|_{\varphi=0}^{\varphi=\pi/2} d\theta\\ &= \frac{b^4-a^4}{4}\int_0^{\pi/2}\left(\frac{\pi}{4}(\cos\theta+\sin\theta)+\frac{1}{2}\right)\,d\theta\\ &= \frac{b^4-a^4}{4}\left[\frac{\pi}{4}(\sin\theta-\cos\theta)+\frac{1}{2}\theta\right]\Bigg|_0^{\pi/2}\\ &= \frac{b^4-a^4}{8}\left[\frac{\pi}{2}(1+1)+\frac{\pi}{2}\right]=\frac{3\pi(b^4-a^4)}{16}.\end{aligned}$$

**37.** We use spherical coordinates, in which case the cone $z=\sqrt{3x^2+3y^2}$ has equation

$$\rho\cos\varphi=\sqrt{3}\rho\sin\varphi \iff \tan\varphi=\frac{1}{\sqrt{3}} \iff \varphi=\frac{\pi}{6}.$$

The sphere $x^2+y^2+z^2=6z$ has spherical equation

$$\rho^2=6\rho\cos\varphi \iff \rho=6\cos\varphi.$$

Thus

$$\begin{aligned}\iiint_W z^2\,dV &= \int_0^{2\pi}\int_0^{\pi/6}\int_0^{6\cos\varphi}(\rho^2\cos^2\varphi)\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta\\ &= \int_0^{2\pi}\int_0^{\pi/6}\frac{6^5}{5}\cos^7\varphi\sin\varphi\,d\varphi\,d\theta = \int_0^{2\pi}\left(-\frac{7776}{40}\cos^8\varphi\right)\Bigg|_{\varphi=0}^{\varphi=\pi/6} d\theta\\ &= \int_0^{2\pi}\frac{972}{5}\left(1-\frac{81}{256}\right)\,d\theta=\frac{972}{5}\left(\frac{175}{128}\right)\pi=\frac{8505\pi}{32}.\end{aligned}$$

**38.** We are integrating over a cone with vertex at the origin and base the disk at height 6 with radius 3. We will use cylindrical coordinates.

$$\begin{aligned}\iiint_W (2+\sqrt{x^2+y^2})\,dV &= \int_0^3\int_0^{2\pi}\int_{2r}^6 r(2+r)\,dz\,d\theta\,dr\\ &= \int_0^3\int_0^{2\pi}(-2r^3+2r^2+12r)\,d\theta\,dr\\ &= \int_0^3(2\pi(-2r^3+2r^2+12r))\,dr=63\pi.\end{aligned}$$

*You should assign one of Exercises 39 or 40 so that your students see the benefits of using another coordinate system even when it is not explicitly called for. You might want to stress that the symmetries of the problem are what lead you, in this case, to choose cylindrical coordinates. Exercise 41 is fun because students will be tempted to use spherical coordinates—life is much easier if they use cylindrical coordinates.*

**39.** We will use cylindrical coordinates.

$$\begin{aligned}\iiint_W dV &= \int_0^1 \int_0^{2\pi} \int_{-\sqrt{10-2r^2}}^{\sqrt{10-2r^2}} r\,dz\,d\theta\,dr \\ &= \int_0^1 \int_0^{2\pi} (2r\sqrt{10-2r^2})\,d\theta\,dr \\ &= \int_0^1 (4\pi r\sqrt{10-2r^2})\,dr \\ &= \frac{4\pi}{3}(5\sqrt{10}-8\sqrt{2}).\end{aligned}$$

**40.** We will again use cylindrical coordinates.

$$\begin{aligned}\iiint_W dV &= \int_0^2 \int_0^{2\pi} \int_0^{9-r^2} r\,dz\,d\theta\,dr \\ &= \int_0^2 \int_0^{2\pi} (9r - r^3)\,d\theta\,dr \\ &= \int_0^2 (18\pi r - 2\pi r^3)\,dr \\ &= 28\pi.\end{aligned}$$

**41.** We will again use cylindrical coordinates.

$$\begin{aligned}\iiint_W (2+x^2+y^2)\,dV &= \int_3^5 \int_0^{2\pi} \int_0^{\sqrt{25-z^2}} (2+r^2)r\,dr\,d\theta\,dz \\ &= \int_3^5 \int_0^{2\pi} \left(\frac{z^4}{4} - \frac{27z^2}{2} + \frac{725}{4}\right) d\theta\,dz \\ &= \int_3^5 \left(2\pi\left[\frac{z^4}{4} - \frac{27z^2}{2} + \frac{725}{4}\right]\right) dz = \frac{656\pi}{5}.\end{aligned}$$

**42.** You can draw a million pictures, but the easiest way to visualize this is by taking an apple corer and a potato and cutting in the three orthogonal directions. This will provide you with a model that the students can hold and pass around to aid their discussion. They can easily identify symmetries and cut the model along the coordinate planes to set up the integral.
If you do this and look in the first octant, you will see a seam along the line $y = x$. If we split the integral along this line we will have 1/16 of the desired volume. Using cylindrical coordinates this means that $0 \le \theta \le \pi/4$ and the cylinder with axis of symmetry the $z$-axis gives us that $0 \le r \le a$. The hard one to see is $z$, but because we are only looking at the wedge on one side of $\theta = \pi/4$ we need only worry about one other cylinder so $0 \le z \le \sqrt{a^2 - r^2\cos^2(\theta)}$.
So the volume is

$$V = 16\int_0^{\pi/4} \int_0^a \int_0^{\sqrt{a^2-r^2\cos^2(\theta)}} r\,dz\,dr\,d\theta = 8a^3(2-\sqrt{2}).$$

## 5.6 Applications of Integration

*Exercises 1–9 concern average value.*

**1. (a)** Let's assume a 30-day month.

$$\begin{aligned}[f]_{\text{avg}} &= \frac{1}{30}\int_0^{30} I(x)\,dx = \frac{1}{30}\int_0^{30} \left(75\cos\frac{\pi x}{15} + 80\right) dx \\ &= \frac{1}{30}\left(\frac{1125}{\pi}\sin\frac{\pi x}{15} + 80x\right)\Bigg|_0^{30} = 2400/30 = 80 \text{ cases.}\end{aligned}$$

**(b)** Here the 2 cents will be a constant that pulls through the integral so the average holding cost is just 2 cents times the average daily inventory, or \$1.60.

**2.** We will divide the integral by the area:

$$[f]_{\text{avg}} = \frac{1}{(2\pi)(4\pi)}\int_0^{2\pi}\int_0^{4\pi}\sin^2 x\cos^2 y\,dy\,dx = \frac{1}{8\pi^2}\int_0^{2\pi}\left(\frac{\sin^2 x}{4}(\sin 2y + 2y)\right)\Bigg|_0^{4\pi} dx$$
$$= \frac{1}{8\pi^2}\int_0^{2\pi}(2\pi\sin^2 x)\,dx = \frac{1}{8\pi^2}\left(\frac{\pi}{2}(2x - \sin 2x)\right)\Bigg|_0^{2\pi} = \frac{2\pi^2}{8\pi^2} = \frac{1}{4}.$$

**3.** Again we will divide the integral by the area:

$$[f]_{\text{avg}} = \frac{1}{1/2}\int_0^1\int_0^{1-x} e^{2x+y}\,dy\,dx = 2\int_0^1 (e^{2x+y})\Big|_0^{1-x} dx$$
$$= 2\int_0^1 (e^{x+1} - e^{2x})\,dx = (2e^{1+x} - e^{2x})\Big|_0^1 = e^2 - 2e + 1.$$

**4.** Here we are finding the average over a ball of volume $4\pi/3$. We'll integrate using cylindrical coordinates because $z$ appears explicitly in the integrand.

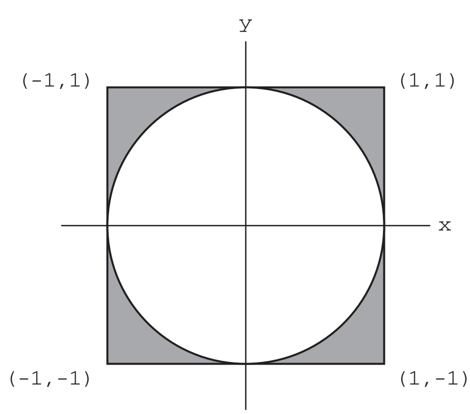
$$[g]_{\text{avg}} = \frac{1}{4\pi/3}\int_{-1}^1\int_0^{2\pi}\int_0^{\sqrt{1-z^2}} re^z\,dr\,d\theta\,dz = \frac{3}{4\pi}\int_{-1}^1\int_0^{2\pi}\left(\frac{e^z}{2}(1-z^2)\right)d\theta\,dz$$
$$= \frac{3}{4\pi}\int_{-1}^1(\pi e^z(1-z^2))\,dz = \frac{3}{4\pi}\frac{4\pi}{e} = \frac{3}{e}.$$

**5. (a)** We are told that in the $2\times 2\times 2$ cube centered at the origin, $T(x,y,z) = c(x^2+y^2+z^2)$. The average temperature of the cube is

$$[T]_{\text{avg}} = \frac{c}{8}\int_{-1}^1\int_{-1}^1\int_{-1}^1(x^2+y^2+z^2)\,dx\,dy\,dz = \frac{c}{8}\int_{-1}^1\int_{-1}^1(2z^2+2y^2+2/3)\,dy\,dz$$
$$= \frac{c}{8}\int_{-1}^1(4z^2+8/3)\,dz = \frac{c}{8}(8) = c.$$

**(b)** $T(x,y,z) = c$ when $x^2+y^2+z^2 = 1$ so the temperature is equal to the average temperature on the surface of the unit sphere.

**6.** The region looks like




and the area of it is $2^2 - \pi = 4 - \pi$. Hence the average value is

$$\frac{1}{4-\pi}\iint_D (x^2+y^2)\,dA = \frac{1}{4-\pi}\left[\iint_{D_1}(x^2+y^2)\,dA - \iint_{D_2}(x^2+y^2)\,dA\right],$$

where $D_1$ denotes the square and $D_2$ the disk.

$$\iint_{D_1} (x^2+y^2)\,dA = \int_{-1}^{1}\int_{-1}^{1}(x^2+y^2)\,dx\,dy = \int_{-1}^{1}\left(\frac{1}{3}x^3+y^2x\right)\Big|_{x=-1}^{1}\,dy$$

$$= \int_{-1}^{1}\left(\frac{2}{3}+2y^2\right)dy = \left(\frac{2}{3}y+\frac{2}{3}y^3\right)\Big|_{-1}^{1} = \frac{4}{3}+\frac{4}{3}=\frac{8}{3}$$

$$\iint_{D_2}(x^2+y^2)\,dA = \int_0^{2\pi}\int_0^1 r^2\cdot r\,dr\,d\theta = \int_0^{2\pi}\frac{1}{4}\,d\theta = \frac{\pi}{2}$$

Therefore the average value is

$$\frac{1}{4-\pi}\left(\frac{8}{3}-\frac{\pi}{2}\right) = \frac{16-3\pi}{24-6\pi} = \frac{3\pi-16}{6\pi-24}\approx 1.2766.$$

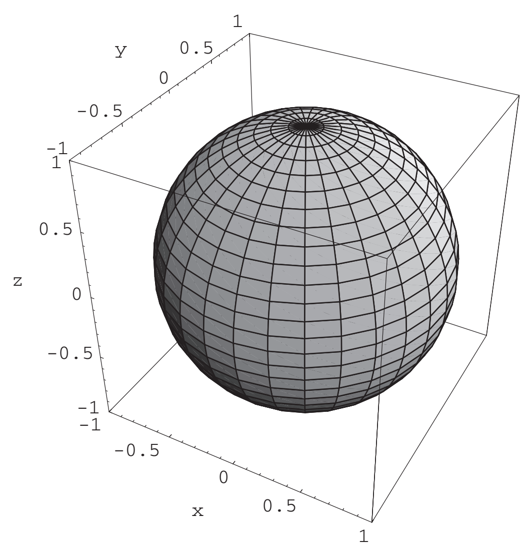
**7.** The volume of $W$ is $8-\dfrac{4}{3}x = \dfrac{24-4\pi}{3}$. Thus the average value is

$$\frac{3}{24-4\pi}\iiint_W (x^2+y^2+z^2)\,dV \quad = \quad \frac{3}{24-4\pi}\left(\iiint_{W_1}(x^2+y^2+z^2)\,dV - \iiint_{W_2}(x^2+y^2+z^2)\,dV\right)$$

where $W_1$ denotes the cube and $W_2$ the ball. Using Cartesian coordinates to integrate over $W_1$ and spherical coordinates to integrate over $W_2$, this may be calculated as

$$\frac{3}{24-4\pi}\left(\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(x^2+y^2+z^2)\,dz\,dy\,dx - \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^4\sin\varphi\,d\rho\,d\varphi\,d\theta\right)$$

$$= \frac{3\pi-30}{5\pi-30}.$$




**8.** We are looking for the average value of the minimum of $x$ and $y$ in the $6\times 6$ box. This is 1/36 times the sum of the average value for $x$ in the region where $x\le y$ and the average value for $y$ in the region where $y\le x$. Because of the symmetry, the average value can be calculated by doubling the result for either region and dividing by 36:

$$[\text{Time}]_{\text{avg}} = \frac{2}{36}\int_0^6\int_0^x y\,dy\,dx = \frac{1}{18}\int_0^6\frac{x^2}{2}\,dx$$

$$= \frac{1}{18}\left(\frac{x^3}{6}\right)\Big|_0^6 = 2.$$

**9.** This is an extension of Exercise 8. The domain is $[0,6]\times[0,6]\times[0,6]$. This time there is six-fold symmetry so we will calculate the average value for $z$ in the region where $z \le y \le x$ and multiply by 6 and then divide by $6^3$ which is the volume of the domain.

$$[\text{Time}]_{\text{avg}} = \frac{6}{216}\int_0^6\int_0^x\int_0^y z\,dz\,dy\,dx = \frac{1}{36}\int_0^6\int_0^x \frac{y^2}{2}\,dy\,dz$$
$$= \frac{1}{36}\int_0^6 \frac{x^3}{6}\,dx = \frac{1}{36}\left(\frac{x^4}{24}\right)\Big|_0^6 = 3/2.$$

So with three train lines the average wait is 90 seconds.

*Exercises 10–24 concern centers of mass. We use the formula:*

$$\textit{Center of mass} = \frac{\displaystyle\int_a^b x\delta(x)\,dx}{\displaystyle\int_a^b \delta(x)\,dx}$$

*and its variants.*

**10. (a)** The curve $y = 8-2x^2$ intersects the $x$-axis at $\pm 2$. So

$$\int_{-2}^2\int_0^{8-2x^2} c\,dy\,dx = c\int_{-2}^2 (8-2x^2)\,dx = c\,(8x-2x^3/3)\Big|_{-2}^2 = 64c/3$$
$$M_y = \int_{-2}^2\int_0^{8-2x^2} cx\,dy\,dx = c\int_{-2}^2 (8x-2x^3)\,dx = c\,(4x^2-x^4/2)\Big|_{-2}^2 = 0 \quad \text{and}$$
$$M_x = \int_{-2}^2\int_0^{8-2x^2} cy\,dy\,dx = (c/2)\int_{-2}^2 (8-2x^2)^2 dx = c\left(\frac{2}{5}x^5 - \frac{16}{3}x^3 + 32x\right)\Big|_{-2}^2 = 1024c/15$$

So $\bar{x} = 0$ and $\bar{y} = \dfrac{1024c/15}{64c/3} = 16/5$.

**(b)** Again, we see the symmetry with respect to $x$ so $\bar{x} = 0$. The following integrals are straightforward so we leave out the details, but

$$\bar{y} = \frac{\displaystyle\int_{-2}^2\int_0^{8-2x^2} 3cy^2\,dy\,dx}{\displaystyle\int_{-2}^2\int_0^{8-2x^2} 3cy\,dy\,dx} = \frac{32768c/35}{1024c/5} = 32/7.$$

**11.** We assume that the plate has uniform density and place it so that the center of the straight border is at the origin and the semicircle is symmetric with respect to the $y$-axis. Once again this means that $\bar{x} = 0$.

$$\bar{y} = \frac{\displaystyle\int_{-a}^a\int_0^{\sqrt{a^2-x^2}} cy\,dy\,dx}{\pi a^2 c/2} = \frac{2a^3c/3}{\pi a^2 c/2} = \frac{4a}{3\pi}.$$

**12.** First calculate

$$M = \int_0^2\int_{x^2}^{2x} (1+x+y)\,dy\,dx = \frac{24}{5}$$
$$M_y = \int_0^2\int_{x^2}^{2x} [x(1+x+y)]\,dy\,dx = \frac{28}{5}$$
$$M_x = \int_0^2\int_{x^2}^{2x} [y(1+x+y)]\,dy\,dx = \frac{328}{35}$$

Thus,

$$\bar{x} = \frac{28/5}{24/5} = 7/6 \quad \text{and} \quad \bar{y} = \frac{328/35}{24/5} = 41/21.$$

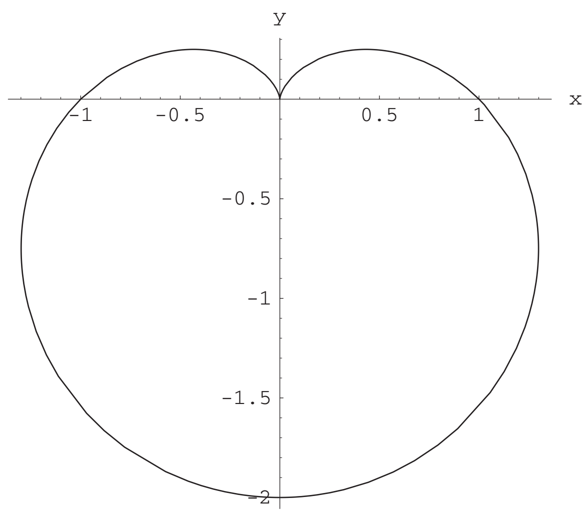
**13.** Again we first calculate

$$M = \int_0^9 \int_0^{\sqrt{x}} (xy)\, dy\, dx = \frac{243}{2}$$
$$M_y = \int_0^9 \int_0^{\sqrt{x}} (x^2 y)\, dy\, dx = \frac{6561}{8}$$
$$M_x = \int_0^9 \int_0^{\sqrt{x}} (xy^2)\, dy\, dx = \frac{1458}{7}$$

so

$$\bar{x} = \frac{6561/8}{243/2} = 27/4 \quad \text{and} \quad \bar{y} = \frac{1458/7}{243/2} = 12/7.$$

**14.** We'll take $\delta$ to be 1. A look at the figure below tells us again that $\bar{x} = 0$. We'll use polar integrals to calculate $\bar{y}$.



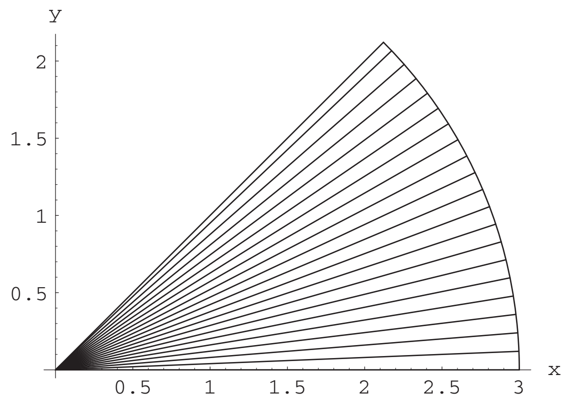

We first calculate $M = \int_0^{2\pi} \int_0^{1-\sin\theta} r\, dr\, d\theta = \frac{3\pi}{2}$ and $M_x = \iint_D y\, dA = \int_0^{2\pi} \int_0^{1-\sin\theta} r^2 \sin\theta\, dr\, d\theta = \frac{-5\pi}{4}$, so $\bar{y} = \frac{-5\pi/4}{3\pi/2} = -5/6$.

**15.** We first calculate

$$M = \int_0^{\pi/3} \int_0^{4\cos\theta} r\, dr\, d\theta = \sqrt{3} + \frac{4\pi}{3}$$
$$M_y = \iint_D x\, dA = \int_0^{\pi/3} \int_0^{4\cos\theta} r^2 \cos\theta\, dr\, d\theta = \frac{7}{\sqrt{3}} + \frac{8\pi}{3} \quad \text{and}$$
$$M_x = \iint_D y\, dA = \int_0^{\pi/3} \int_0^{4\cos\theta} r^2 \sin\theta\, dr\, d\theta = 5,$$

so $\bar{x} = \frac{7\sqrt{3} + 8\pi}{3\sqrt{3} + 4\pi}$ and $\bar{y} = \frac{15}{3\sqrt{3} + 4\pi}$.

**16.** The region is a slice of pie:




$$\text{Total mass } M = \iint_D \delta\, dA = \int_0^{\pi/4} \int_0^3 (4-r) r\, dr\, d\theta = \int_0^{\pi/4} \left(2r^2 - \frac{1}{3}r^3\right)\Big|_{r=0}^{3}$$

$$= \int_0^{\pi/4} (18-9)\, d\theta = \frac{9\pi}{4}$$

$$M_y = \iint_D x\delta\, dA = \int_0^{\pi/4} \int_0^3 (4r^2 - r^3)\cos\theta\, dr\, d\theta = \int_0^{\pi/4} \left(36 - \frac{81}{4}\right)\cos\theta\, d\theta$$

$$= \frac{63}{4\sqrt{2}}$$

$$M_x = \iint_D y\delta\, dA = \int_0^{\pi/4} \int_0^3 (4r^2 - r^3)\sin\theta\, dr d\theta = \int_0^{\pi/4} \left(36 - \frac{81}{4}\right)\sin\theta\, d\theta$$

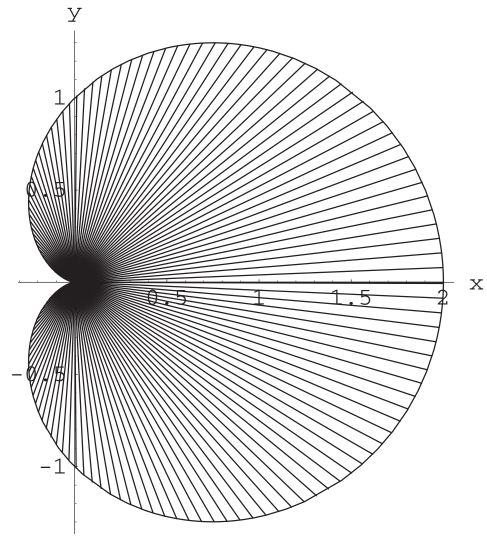
$$= \frac{63}{8}(2-\sqrt{2})$$

Thus

$$\bar{x} = \frac{M_y}{M} = \frac{63}{4\sqrt{2}} \cdot \frac{4}{9\pi} = \frac{7\sqrt{2}}{2\pi}$$

$$\bar{y} = \frac{M_x}{M} = \frac{63(2-\sqrt{2})}{8}\,\frac{4}{9\pi} = \frac{7(2-\sqrt{2})}{2\pi}$$

**17.** The region in question looks as follows:




$$
\begin{aligned}
\text{Total mass } M &= \iint_D \delta\, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^2 dr\, d\theta \\
&= \int_0^{2\pi} \frac{1}{3}(1+\cos\theta)^3 d\theta \\
&= \int_0^{2\pi} \frac{1}{3}(1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta)\, d\theta = \frac{5\pi}{3} \\
M_y &= \iint_D x\delta\, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^3 \cos\theta\, dr\, d\theta = \int_0^{2\pi} \frac{1}{4}(1+\cos\theta)^4 \cos\theta\, d\theta \\
&= \frac{7\pi}{4} \qquad \text{(after some effort!)} \\
M_x &= \iint_D y\delta\, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^3 \sin\theta\, dr\, d\theta \\
&= \int_0^{2\pi} \frac{1}{4}(1+\cos\theta)^4 \sin\theta\, d\theta = -\frac{1}{4}\int_2^2 u^4 du = 0
\end{aligned}
$$

(It's also possible to see this from symmetry.) Thus

$$\bar{x} = \frac{7\pi}{4} \cdot \frac{3}{5\pi} = \frac{21}{20}, \qquad \bar{y} = 0.$$

**18.** Because the volume of the tetrahedron is 1, we can find the centroid by calculating:

$$
\begin{aligned}
\bar{x} &= \int_0^1 \int_0^{2-2x} \int_0^{3-3y/2-3x} x\, dz\, dy\, dx = \frac{1}{4} \\
\bar{y} &= \int_0^1 \int_0^{2-2x} \int_0^{3-3y/2-3x} y\, dz\, dy\, dx = \frac{1}{2} \quad \text{and,} \\
\bar{z} &= \int_0^1 \int_0^{2-2x} \int_0^{3-3y/2-3x} z\, dz\, dy\, dx = \frac{3}{4}.
\end{aligned}
$$

**19. (a)** First calculate:

$$M = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} dz\,dy\,dx = 12$$

$$M_{yz} = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} x\,dz\,dy\,dx = 6$$

$$M_{xz} = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} y\,dz\,dy\,dx = 0$$

$$M_{xy} = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} z\,dz\,dy\,dx = \frac{108}{5}.$$

This means that $(\bar{x}, \bar{y}, \bar{z}) = (1/2, 0, 9/5)$.

**(b)** Next we calculate the center of mass with the given density function.

$$M = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} (z + x^2)\,dz\,dy\,dx = \frac{168}{5}$$

$$M_{yz} = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} x(z + x^2)\,dz\,dy\,dx = \frac{129}{5}$$

$$M_{xz} = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} y(z + x^2)\,dz\,dy\,dx = 0$$

$$M_{xy} = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} z(z + x^2)\,dz\,dy\,dx = \frac{2376}{35}.$$

This means that $(\bar{x}, \bar{y}, \bar{z}) = (43/56, 0, 99/49)$.

*Note that in Exercises 20–22, the symmetry with respect to the z-axis implies that $\bar{x} = 0$ and $\bar{y} = 0$. We only explicitly set up all of the integrals in the solution of Exercise 20.*

**20.** First calculate:

$$M = \int_{0}^{3}\int_{0}^{2\pi}\int_{r^2/3}^{\sqrt{18-r^2}} r\,dz\,d\theta\,dr = 36\sqrt{2}\pi - \frac{63\pi}{2}$$

$$M_{yz} = \int_{0}^{3}\int_{0}^{2\pi}\int_{r^2/3}^{\sqrt{18-r^2}} r\;\cos\theta\,dz\,d\theta\,dr = 0$$

$$M_{xz} = \int_{0}^{3}\int_{0}^{2\pi}\int_{r^2/3}^{\sqrt{18-r^2}} r\;\sin\theta\,dz\,d\theta\,dr = 0$$

$$M_{xy} = \int_{0}^{3}\int_{0}^{2\pi}\int_{r^2/3}^{\sqrt{18-r^2}} rz\,dz\,d\theta\,dr = \frac{189\pi}{4}.$$

This means that $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \dfrac{21}{2(8\sqrt{2} - 7)}\right)$.

**21.** As noted above, $\bar{x} = 0$ and $\bar{y} = 0$. First calculate:

$$M = \int_{0}^{5/2}\int_{0}^{2\pi}\int_{3r^2-16}^{9-r^2} r\,dz\,d\theta\,dr = \frac{625\pi}{8}$$

$$M_{xy} = \int_{0}^{5/2}\int_{0}^{2\pi}\int_{3r^2-16}^{9-r^2} rz\,dz\,d\theta\,dr = -\frac{10625\pi}{96}$$

This means that $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, -\dfrac{17}{12}\right)$.

**22.** Note first that, by symmetry, the centroid must lie along the $z$-axis, so $\bar{x} = \bar{y} = 0$. Since the density is to be assumed constant, we may take it to be equal to 1. Then we have

$$\bar{z} = \frac{\iiint_W z\, dV}{\iiint_W dV}.$$

We use cylindrical coordinates to calculate the integrals. Thus the cone has cylindrical equation $z = 2r$ and the sphere $r^2 + z^2 = 25$. These surfaces intersect where $r^2 + 4r^2 = 25$, or, equivalently, where $r = \sqrt{5}$. Hence

$$\begin{aligned}\iiint_W dV &= \int_0^{2\pi}\int_0^{\sqrt{5}}\int_{2r}^{\sqrt{25-r^2}} r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{5}} \left(\sqrt{25-r^2} - 2r\right) r\, dr\, d\theta \\ &= \int_0^{2\pi} \left(-\frac{1}{2}\cdot\frac{2}{3}(25-r^2)^{3/2} - \frac{2}{3}r^3\right)\Bigg|_{r=0}^{r=\sqrt{5}} d\theta \\ &= \left(-\frac{50\sqrt{5}}{3} + \frac{125}{3}\right)(2\pi) = \frac{(250 - 100\sqrt{5})\pi}{3}.\end{aligned}$$

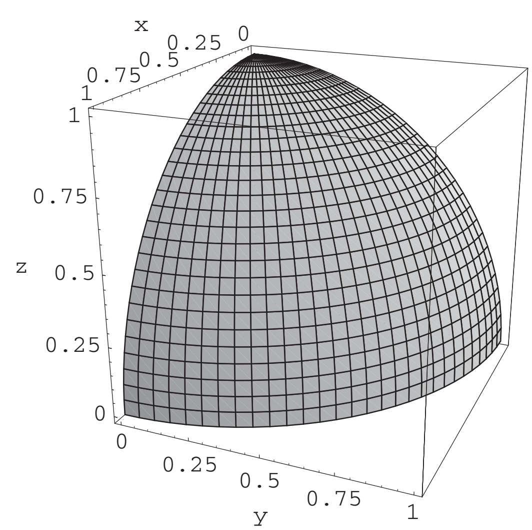
Also,

$$\begin{aligned}\iiint_W z\, dV &= \int_0^{2\pi}\int_0^{\sqrt{5}}\int_{2r}^{\sqrt{25-r^2}} zr\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{5}} \frac{1}{2}\left(25 - r^2 - 4r^2\right) r\, dr\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\int_0^{\sqrt{5}} \left(25r - 5r^3\right) dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\left(\frac{125}{2} - \frac{125}{4}\right) d\theta = \frac{125\pi}{4}.\end{aligned}$$

Therefore,

$$\bar{z} = \frac{125\pi/4}{(250 - 100\sqrt{5})\pi/3} = \frac{15}{8(5 - 2\sqrt{5})} \approx 3.55.$$

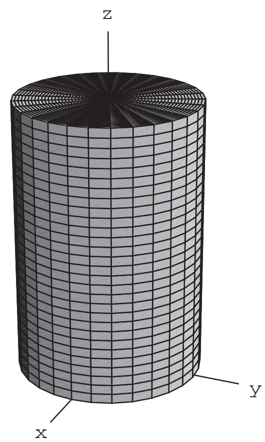
**23.** By symmetry $\bar{x} = \bar{y} = \bar{z}$. $\bar{z}$ is easiest to find. Volume of $W$ is $\frac{1}{8}\left(\frac{4}{3}\pi a^3\right) = \frac{\pi a^3}{6}$.




Thus

$$\begin{aligned}\bar{z} &= \frac{6}{\pi a^3}\iiint_W z\, dV = \frac{6}{\pi a^3}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho\cos\varphi\cdot\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta \\ &= \frac{6}{\pi a^3}\int_0^{\pi/2}\int_0^{\pi/2}\frac{a^4}{4}\cos\varphi\sin\varphi\, d\varphi\, d\theta = \frac{3a}{2\pi}\int_0^{\pi/2}\frac{1}{2}d\theta \\ &= \frac{3a}{8}.\end{aligned}$$

**24.** If we put the bottom of the cylinder in the *xy*-plane, then $\delta(x, y, z) = (h - z)^2$.




Therefore the total mass is

$$M = \iiint_W \delta\, dV = \int_0^{2\pi} \int_0^a \int_0^h (h - z)^2 r\, dz\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^a -\frac{1}{3}(h - z)^3 \Big|_{z=0}^{h} r\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^a \frac{h^3}{3} r\, dr\, d\theta = \frac{\pi h^3 a^2}{3}.$$

$\bar{x} = \bar{y} = 0$ by symmetry, so we compute

$$M_{xy} = \iiint_W z\delta\, dV = \int_0^{2\pi} \int_0^a \int_0^h z(h - z)^2 r\, dz\, dr\, d\theta$$
$$= \int_0^{2\pi} \int_0^a \int_0^h (h^2 z - 2hz^2 + z^3) r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^a r \cdot \frac{1}{12} h^4\, dr\, d\theta = \frac{\pi h^4 a^2}{12}.$$

Thus

$$\bar{z} = \frac{\pi h^4 a^2}{12} \cdot \frac{3}{\pi h^3 a^2} = \frac{h}{4}.$$

**25. (a)** By symmetry we can see that the moment of inertia about each of the coordinate axes is the same.

$$I_x = I_y = I_z = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (y^2 + z^2)\, dz\, dy\, dx = \frac{1}{30}.$$

**(b)** Again, by symmetry we see that the radius of gyration about each of the coordinate axes is the same. We calculate

$$M = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx = \frac{1}{6}.$$

Then $r_x = r_y = r_z = \sqrt{\dfrac{1/30}{1/6}} = 1/\sqrt{5}$.

**26.** By symmetry we can see that the moment of inertia about each of the coordinate axes is the same

$$I_x = I_y = I_z = \int_0^2 \int_0^2 \int_0^2 (y^2 + z^2)(x + y + z + 1)\, dz\, dy\, dx = 96.$$

Again, by symmetry we see that the radius of gyration about each of the coordinate axes is the same. We calculate

$$M = \int_0^2 \int_0^2 \int_0^2 (x + y + z + 1)\, dz\, dy\, dx = 32.$$

Then $r_x = r_y = r_z = \sqrt{\dfrac{96}{32}} = \sqrt{3}$.

**27. (a)** The problem cries out to be solved using cylindrical coordinates. For $I_z$ this means that the $x^2 + y^2$ in the integrand is $r^2$ so

$$\begin{aligned} I_z &= \int_0^3 \int_0^{2\pi} \int_{r^2}^9 2zr^3\,dz\,d\theta\,dr = \frac{6561\pi}{4} \quad \text{and} \\ M &= \int_0^3 \int_0^{2\pi} \int_{r^2}^9 2zr\,dz\,d\theta\,dr = 486\pi, \quad \text{so} \\ r_z &= \frac{3\sqrt{3}}{2\sqrt{2}}. \end{aligned}$$

**(b)** This time

$$\begin{aligned} I_z &= \int_0^3 \int_0^{2\pi} \int_{r^2}^9 r^4\,dz\,d\theta\,dr = \frac{8748\pi}{35} \quad \text{and} \\ M &= \int_0^3 \int_0^{2\pi} \int_{r^2}^9 r^2\,dz\,d\theta\,dr = \frac{324\pi}{5}, \quad \text{so} \\ r_z &= \frac{3\sqrt{3}}{\sqrt{7}}. \end{aligned}$$

**28.** Although it may be tempting to move to spherical coordinates, it is nice to have a $z$-coordinate so we will stay with cylindrical coordinates.

**(a)**

$$\begin{aligned} I_z &= \int_0^a \int_0^{2\pi} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3 c\,dz\,d\theta\,dr = \frac{8\pi c a^5}{15} \quad \text{and} \\ M &= \int_0^a \int_0^{2\pi} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} rc\,dz\,d\theta\,dr = \frac{4\pi c a^3}{3} \quad \text{so} \\ r_z &= a\sqrt{\frac{2}{5}}. \end{aligned}$$

**(b)**

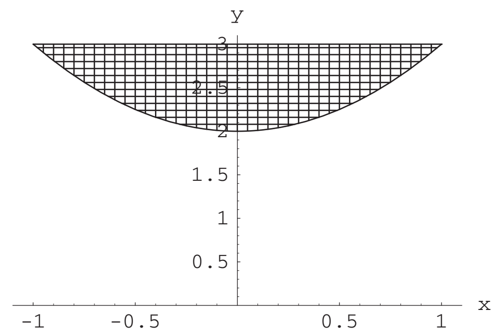
$$\begin{aligned} I_z &= \int_0^a \int_0^{2\pi} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3(r^2+z^2)\,dz\,d\theta\,dr = \frac{8\pi a^7}{21} \quad \text{and} \\ M &= \int_0^a \int_0^{2\pi} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r(r^2+z^2)\,dz\,d\theta\,dr = \frac{4\pi a^5}{5} \quad \text{so} \\ r_z &= a\sqrt{\frac{10}{21}}. \end{aligned}$$

**(c)**

$$\begin{aligned} I_z &= \int_0^a \int_0^{2\pi} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^5\,dz\,d\theta\,dr = \frac{32\pi a^7}{105} \quad \text{and} \\ M &= \int_0^a \int_0^{2\pi} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3\,dz\,d\theta\,dr = \frac{8\pi\delta a^5}{15} \quad \text{so} \\ r_z &= \frac{2a}{\sqrt{7}}. \end{aligned}$$

**29.**

$$\begin{aligned}
I_x &= \iint_D y^2 \delta\, dA = \int_{-1}^{1} \int_{x^2+2}^{3} y^2(x^2+1)\, dy\, dx \\
&= \int_{-1}^{1} \left(9 - \frac{1}{3}(x^2+2)^3\right)(x^2+1)\, dx = \int_{-1}^{1} \frac{1}{3}(19 + 7x^2 - 18x^4 - 7x^6 - x^8)\, dx \\
&= \frac{1}{3}\left(38 + \frac{14}{3} - \frac{36}{5} - 2 - \frac{2}{9}\right) = \frac{1496}{135}
\end{aligned}$$




**30.**

$$\begin{aligned}
I_z &= \iint_{[0,2]\times[0,1]} (x^2+y^2)\delta\, dA = \int_0^2 \int_0^1 (x^2+y^2)(1+y)\, dy\, dx \\
&= \int_0^2 \int_0^1 (x^2 + y^2 + x^2 y + y^3)\, dy\, dx = \int_0^2 \left(x^2 + \frac{1}{3} + \frac{1}{2}x^2 + \frac{1}{4}\right) dx \\
&= \frac{8}{3} + \frac{2}{3} + \frac{4}{3} + \frac{1}{2} = \frac{31}{6}
\end{aligned}$$

**31.** The only adjustment in the formula for $I_x$ is because we are using the square of the distance from the line $y = 3$ and not the formula given in text which squares the distance from the $x$-axis. This is a straightforward application of formula (8).

$$I_{y=3} = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2(3-y)^2))\, dy\, dx = \frac{116\pi}{3}.$$

*What follows is preliminary work for Exercises 32–34. You should probably assign all three together.*

We will be calculating

$$V(0,0,r) = -\iiint_W \frac{Gm\delta(x,y,z)\, dV}{\sqrt{x^2+y^2+(z-r)^2}}.$$

In this special case, $W$ is the shell bound by spheres centered at the origin of radii $a$ and $b$ where $a < b$. The volume of $W$ is therefore $4\pi(b^3 - a^3)/3$. The density is assumed to be constant and so the density is mass divided by volume, so

$$\delta = \frac{M}{[4\pi(b^3-a^3)/3]} = \frac{3M}{4\pi(b^3-a^3)}.$$

So

$$
\begin{aligned}
V(0,0,r) &= -Gm\delta \iiint_W \frac{dV}{\sqrt{x^2+y^2+(z-r)^2}} \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \iiint_W \frac{dV}{\sqrt{x^2+y^2+(z-r)^2}} \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi}\int_a^b\int_0^{\pi} \frac{\rho^2\sin\varphi}{\sqrt{\rho^2+r^2-2r\rho\cos\varphi}}\, d\varphi\, d\rho\, d\theta \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi}\int_a^b \left[\left(\frac{\rho}{r}\right)\sqrt{\rho^2+r^2-2r\rho\cos\varphi}\right]\Bigg|_{\varphi=0}^{\pi} d\rho\, d\theta \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi}\int_a^b \left[\left(\frac{\rho}{r}\right)(|\rho+r|-|\rho-r|)\right] d\rho\, d\theta.
\end{aligned}
$$

Our final note before proceeding to Exercises 32–34 is that

$$
\left(\frac{\rho}{r}\right)(|\rho+r|-|\rho-r|) = \begin{cases} 2\rho & \text{if } \rho \ge r, \text{ and} \\ 2\rho^2/r & \text{if } \rho < r. \end{cases}
$$

**32.** *See preliminary work above*. When $r \ge b$, then in the range $a \le \rho \le b$, we have that $\rho \le r$, so

$$
\begin{aligned}
V(0,0,r) &= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi}\int_a^b \frac{2\rho^2}{r} d\rho\, d\theta \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi} \frac{2\rho^3}{3r}\Bigg|_a^b d\theta \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi} \frac{2(b^3-a^3)}{3r} d\theta \\
&= -\frac{GmM}{2\pi r}\int_0^{2\pi} d\theta = -\frac{GmM}{r}.
\end{aligned}
$$

**33.** *See preliminary work above*. When $r \le a$, then in the range $a \le \rho \le b$, we have that $\rho \ge r$, so

$$
\begin{aligned}
V(0,0,r) &= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi}\int_a^b 2\rho\, d\rho\, d\theta \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi} (b^2-a^2)\, d\theta \\
&= \frac{-3GmM(b^2-a^2)}{2(b^3-a^3)}.
\end{aligned}
$$

What is striking about this result is that $V(0,0,r)$ is independent of $r$. Therefore, since $\mathbf{F} = -\nabla V$, we see that there is no gravitational force.

**34.** **(b)** Students might consider the connection before they explicitly find it in part (a). If $a < r < b$, then we have a combination of the two cases dealt with in Exercises 32 and 33. For $a \le \rho \le r$, we are in a case similar to Exercise 32, and for $r \le \rho \le b$ we are in a case similar to Exercise 33.

**(a)** We must break the integral at $\rho = r$:

$$
\begin{aligned}
V(0,0,r) &= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi}\int_a^r \frac{2\rho^2}{r}\, d\rho\, d\theta - \frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi}\int_r^b 2\rho\, d\rho\, d\theta \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi} \frac{2(r^3-a^3)}{3r}\, d\theta - \frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi} (b^2-r^2)\, d\theta \\
&= -\frac{3GmM}{2(b^3-a^3)}\left(\frac{2(r^3-a^3)}{3r}\right) - \frac{3GmM}{2(b^3-a^3)}(b^2-r^2) \\
&= -\frac{GmM}{2r(b^3-a^3)}(3b^2r - 2a^3 - r^3).
\end{aligned}
$$

## 5.7 Numerical Approximations of Multiple Integrals

**1. (a)** We let $\Delta x = \dfrac{3.1-3}{2} = 0.05$, $\Delta y = \dfrac{2.1-1.5}{3} = 0.2$. Thus, using formula (6) with $f(x,y) = x^2 - 6y^2$, we have

$$\begin{aligned} T_{2,3} = \frac{(0.05)(0.2)}{4} & \left[f(3,1.5) + f(3,2.1) + f(3.1,1.5) + f(3.1,2.1)\right. \\ & + 2\left(f(3,1.7) + f(3,1.9) + f(3.05,1.5) + f(3.05,2.1) + f(3.1,1.7) + f(3.1,1.9)\right) \\ & \left.+ 4\left(f(3.05,1.7) + f(3.05,1.9)\right)\right] \\ = -0.621375 & \end{aligned}$$

**(b)** $\displaystyle\int_3^{3.1}\int_{1.5}^{2.1}(x^2-6y^2)\,dy\,dx = \int_3^{3.1}(x^2y-2y^3)\Big|_{y=1.5}^{y=2.1}dx = \int_3^{3.1}\left[0.6x^2 - 2(2.1^3-1.5^3)\right]\,dx$

$= \left[0.2x^3 - 2(2.1^3-1.5^3)x\right]\Big|_3^{3.1} = 0.2(3.1^3-3^3) - 0.2(2.1^3-1.5^3) = -0.619$

**2. (a)** We let $\Delta x = \dfrac{3.3-3}{2} = 0.15$, $\Delta y = \dfrac{3.3-3}{3} = 0.1$. Thus, using formula (6) with $f(x,y) = xy^2$, we have

$$\begin{aligned} T_{2,3} = \frac{(0.15)(0.1)}{4} & \left[f(3,3) + f(3,3.3) + f(3.3,3) + f(3.3,3.3)\right. \\ & + 2\left(f(3,3.1) + f(3,3.2) + f(3.15,3) + f(3.15,3.3) + f(3.3,3.1) + f(3.3,3.2)\right) \\ & \left.+ 4\left(f(3.15,3.1) + f(3.15,3.2)\right)\right] \\ = 2.81563 & \end{aligned}$$

**(b)** $\displaystyle\int_3^{3.3}\int_3^{3.3}xy^2\,dy\,dx = \int_3^{3.3}\frac{x}{3}y^3\Big|_{y=3}^{y=3.3}dx = \int_3^{3.3}\frac{x}{3}\left(3.3^3-3^3\right)\,dx = 2.979\int_3^{3.3}x\,dx$

$= \left[\dfrac{2.979}{2}x^2\right]\Big|_3^{3.3} = \dfrac{2.979}{2}(3.3^2-3^2) = 2.81516$

**3. (a)** We let $\Delta x = \dfrac{2.2-2}{2} = 0.1$, $\Delta y = \dfrac{1.6-1}{3} = 0.2$. Thus, using formula (6) with $f(x,y) = x/y$, we have

$$\begin{aligned} T_{2,3} = \frac{(0.1)(0.2)}{4} & \left[f(2,1) + f(2,1.6) + f(2.2,1) + f(2.2,1.6)\right. \\ & + 2\left(f(2,1.2) + f(2,1.4) + f(2.1,1) + f(2.1,1.6) + f(2.2,1.2) + f(2.2,1.4)\right) \\ & \left.+ 4\left(f(2.1,1.2) + f(2.1,1.4)\right)\right] \\ = 0.19825 & \end{aligned}$$

**(b)** $\displaystyle\int_2^{2.2}\int_1^{1.6}\frac{x}{y}\,dy\,dx = \int_2^{2.2}x\ln y\Big|_{y=1}^{y=1.6}dx = \int_2^{2.2}x\cdot\ln(1.6)\,dx$

$= \left[\dfrac{\ln(1.6)}{2}x^2\right]\Big|_2^{2.2} = \dfrac{\ln 1.6}{2}(2.2^2-2^2) = 0.42\ln(1.6) = 0.197402$

**4. (a)** We let $\Delta x = \dfrac{1.4-1}{2} = 0.2$, $\Delta y = \dfrac{4.3-4}{3} = 0.1$. Thus, using formula (6) with $f(x,y) = \sqrt{x}+\sqrt{y}$, we have

$$\begin{aligned} T_{2,3} = \frac{(0.2)(0.1)}{4} & \left[f(1,4) + f(1,4.3) + f(1.4,4) + f(1.4,4.3)\right. \\ & + 2\left(f(1,4.1) + f(1,4.2) + f(1.2,4) + f(1.2,4.3) + f(1.4,4.1) + f(1.4,4.2)\right) \\ & \left.+ 4\left(f(1.2,4.1) + f(1.2,4.2)\right)\right] \\ = 0.375666 & \end{aligned}$$

**(b)** $\int_1^{1.4}\int_4^{4.3}\left(\sqrt{x}+\sqrt{y}\right)\,dy\,dx = \int_1^{1.4}\left(\sqrt{x}y+\frac{2}{3}y^{3/2}\right)\Big|_{y=4}^{y=4.3}dx$

$= \int_1^{1.4}(0.3\sqrt{x}+0.61111318)\,dx = \frac{0.6}{3}\left((1.4)^{3/2}-1\right)+(0.61111318)(0.4) = 0.375746$

**5. (a)** We let $\Delta x = \frac{1.1-1}{2} = 0.05$, $\Delta y = \frac{0.6-0}{3} = 0.2$. Thus, using formula (6) with $f(x,y) = e^{x+2y}$, we have

$$\begin{aligned} T_{2,3} = \frac{(0.05)(0.2)}{4} & [f(1,0)+f(1,0.6)+f(1.1,0)+f(1.1,0.6) \\ & + 2\left(f(1,0.2)+f(1,0.4)+f(1.05,0)+f(1.05,0.6)+f(1.1,0.2)+f(1.1,0.4)\right) \\ & + 4\left(f(1.05,0.2)+f(1.05,0.4)\right)] \\ = {} & 0.336123 \end{aligned}$$

**(b)** $\int_1^{1.1}\int_0^{0.6} e^{x+2y}\,dy\,dx = \int_1^{1.1}\int_0^{0.6} e^x e^{2y}\,dy\,dx = \int_1^{1.1} e^x\left(\frac{1}{2}e^{2y}\right)\Big|_{y=0}^{y=0.6}dx$

$= \int_1^{1.1}\left(\frac{e^{1.2}-1}{2}\right)e^x\,dx = \frac{e^{1.2}-1}{2}(e^{1.1}-1) = 0.331642$

**6. (a)** We let $\Delta x = \frac{0.2-1}{2} = 0.1$, $\Delta y = \frac{\pi/3-\pi/6}{3} = \frac{\pi}{18}$. Thus, using formula (6) with $f(x,y) = x\cos y$, we have

$$\begin{aligned} T_{2,3} = \frac{(0.1)(\pi/18)}{4} & [f(0,\pi/6)+f(0,\pi/3)+f(0.2,\pi/6)+f(0.2,\pi/3) \\ & + 2\left(f(0,2\pi/9)+f(0,5\pi/18)+f(0.1,\pi/6)+f(0.1,\pi/3)\right. \\ & \qquad \left.+ f(0.2,2\pi/9)+f(0.2,5\pi/18)\right) \\ & + 4\left(f(0.1,2\pi/9)+f(0.1,5\pi/18)\right)] \\ = {} & 0.00730192 \end{aligned}$$

**(b)** $\int_0^{0.2}\int_{\pi/6}^{\pi/3} x\cos y\,dy\,dx = \int_0^{0.2} x\sin y\Big|_{y=\pi/6}^{y=\pi/3}dx = \int_0^{1/5}\left(\frac{\sqrt{3}-1}{2}\right)x\,dx$

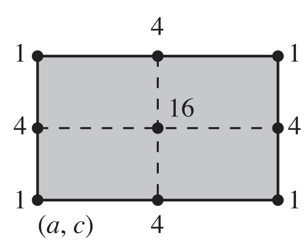
$= \frac{\sqrt{3}-1}{100} = 0.00732051$

*Note that in all of the solutions to Exercises 7–12 below, the rectangle $R = [a,b]\times[c,d]$ is partitioned as in the figure below and the Simpson's rule approximations $S_{2,2}$ may be written as*

$$S_{2,2} = \frac{\Delta x\,\Delta y}{9}\sum_{j=0}^{2}\sum_{i=0}^{2} w_{ij} f(x_i,y_j),$$

*where*

$$w_{ij} = \begin{cases} 1 & \text{if } (x_i,y_j) \text{ is one of the four vertices of } R; \\ 4 & \text{if } (x_i,y_j) \text{ is a point on an edge of } R\text{, but not a vertex;} \\ 16 & \text{if } (x_i,y_j) \text{ is a point in the interior of } R. \end{cases}$$



**7. (a)** We let $\Delta x = \dfrac{0.1-(-0.1)}{2} = 0.1$, $\Delta y = \dfrac{0.3-0}{2} = 0.15$. Hence, with $f(x,y) = y^4 - xy^2$, we have

$$\begin{aligned} S_{2,2} &= \frac{(0.1)(0.15)}{9}\left[f(-0.1,0) + f(-0.1,0.3) + f(0.1,0) + f(0.1,0.3)\right. \\ &\quad \left. + 4\left(f(-0.1,0.15) + f(0,0) + f(0,0.3) + f(0.1,0.15)\right) + 16 f(0,0.15)\right] \\ &= 0.00010125 \end{aligned}$$

**(b)** $\displaystyle\int_{-0.1}^{0.1}\int_0^{0.3}\left(y^4 - xy^2\right)\,dy\,dx = \int_{-0.1}^{0.1}\left(\tfrac{1}{5}y^5 - \tfrac{1}{3}xy^3\right)\Big|_{y=0}^{y=0.3}\,dx$

$\displaystyle = \int_{-0.1}^{0.1}(0.000486 - 0.009x)\,dx = \left(0.000486x - 0.0045x^2\right)\Big|_{-0.1}^{0.1} = 0.0000972$

**8. (a)** We let $\Delta x = \dfrac{0.1-0}{2} = 0.05$, $\Delta y = \dfrac{2-1}{2} = 0.5$. Hence, with $f(x,y) = 1/(1+x^2)$, we have

$$\begin{aligned} S_{2,2} &= \frac{(0.05)(0.5)}{9}\left[f(0,1) + f(0,2) + f(0.1,1) + f(0.1,2)\right. \\ &\quad \left. + 4\left(f(0,1.5) + f(0.05,1) + f(0.05,2) + f(0.1,1.5)\right) + 16 f(0.05,1.5)\right] \\ &= 0.0996687 \end{aligned}$$

**(b)** $\displaystyle\int_0^{0.1}\int_1^2 \frac{1}{1+x^2}\,dy\,dx = \int_0^{0.1}\frac{1}{1+x^2}\,dx = \tan^{-1}x\Big|_0^{0.1} = \tan^{-1}0.1 = 0.0996687$

(Note that this agrees to seven decimal places with our answer in part (a).)

**9. (a)** We let $\Delta x = \dfrac{1.1-1}{2} = 0.05$, $\Delta y = \dfrac{0.6-0}{2} = 0.3$. Hence, with $f(x,y) = e^{x+2y}$, we have

$$\begin{aligned} S_{2,2} &= \frac{(0.05)(0.3)}{9}\left[f(1,0) + f(1,0.6) + f(1.1,0) + f(1.1,0.6)\right. \\ &\quad \left. + 4\left(f(1,0.3) + f(1.05,0) + f(1.05,0.6) + f(1.1,0.3)\right) + 16 f(1.05,0.3)\right] \\ &= 0.331871 \end{aligned}$$

**(b)** In part (b) of Exercise 5 we calculated $\displaystyle\int_1^{1.1}\int_0^{0.6} e^{x+2y}\,dy\,dx$ to be $0.331642$.

**10. (a)** We let $\Delta x = \dfrac{\pi/4-0}{2} = \dfrac{\pi}{8}$, $\Delta y = \dfrac{\pi/2-\pi/4}{2} = \dfrac{\pi}{8}$. Hence, with $f(x,y) = \sin 2x\cos 3y$, we have

$$\begin{aligned} S_{2,2} &= \frac{(\pi/8)(\pi/8)}{9}\left[f(0,\pi/4) + f(0,\pi/2) + f(\pi/4,\pi/4) + f(\pi/4,\pi/2)\right. \\ &\quad \left. + 4\left(f(0,3\pi/8) + f(\pi/8,\pi/4) + f(\pi/8,\pi/2) + f(\pi/4,3\pi/8)\right) + 16 f(\pi/8,3\pi/8)\right] \\ &= -0.288808 \end{aligned}$$

**(b)** $\displaystyle\int_0^{\pi/4}\int_{\pi/4}^{\pi/2}\sin 2x\cos 3y\,dy\,dx = \int_0^{\pi/4}\frac{1}{3}\sin 2x\sin 3y\Big|_{y=\pi/4}^{y=\pi/2}\,dx$

$\displaystyle = \int_0^{\pi/4} -\left(\frac{2+\sqrt{2}}{6}\right)\sin 2x\,dx = \frac{2+\sqrt{2}}{12}\cos 2x\Big|_0^{\pi/4}$

$\displaystyle = \frac{2+\sqrt{2}}{12}(0-1) = -\frac{2+\sqrt{2}}{12} = -0.284518$

**11. (a)** We let $\Delta x = \Delta y = \dfrac{\pi/4 - 0}{2} = \dfrac{\pi}{8}$. Hence, with $f(x, y) = \sin(x + y)$, we have

$$\begin{aligned} S_{2,2} = \frac{(\pi/8)(\pi/8)}{9} & [f(0,0) + f(0,\pi/4) + f(\pi/4, 0) + f(\pi/4, \pi/4) \\ & + 4\left(f(0,\pi/8) + f(\pi/8, 0) + f(\pi/8, \pi/4) + f(\pi/4, \pi/8)\right) + 16 f(\pi/8, \pi/8)] \\ = 0.414325 & \end{aligned}$$

**(b)** $\displaystyle\int_0^{\pi/4}\int_0^{\pi/4} \sin(x+y)\,dy\,dx = \int_0^{\pi/4} -\cos(x+y)\Big|_{y=0}^{y=\pi/4} dx$

$\displaystyle = \int_0^{\pi/4} \left(-\cos\left(x + \frac{\pi}{4}\right) + \cos x\right) dx = \left[\sin x - \sin\left(x + \frac{\pi}{4}\right)\right]\Big|_0^{\pi/4}$

$= \sqrt{2} - 1 = 0.414214$

**12. (a)** We let $\Delta x = \dfrac{1.1 - 1}{2} = 0.05$, $\Delta y = \dfrac{\pi/4 - 0}{2} = \dfrac{\pi}{8}$. Hence, with $f(x, y) = e^x \cos y$, we have

$$\begin{aligned} S_{2,2} = \frac{(0.05)(\pi/8)}{9} & [f(1,0) + f(1,\pi/4) + f(1.1, 0) + f(1.1, \pi/4) \\ & + 4\left(f(1,\pi/8) + f(1.05, 0) + f(1.05, \pi/4) + f(1.1, \pi/8)\right) + 16 f(1.05, \pi/8)] \\ = 0.202178 & \end{aligned}$$

**(b)** $\displaystyle\int_0^{1.1}\int_0^{\pi/4} e^x \cos y\,dy\,dx = \int_0^{1.1} e^x \sin y\Big|_{y=0}^{y=\pi/4} dx = \int_0^{1.1} \frac{\sqrt{2}}{2} e^x\,dx$

$\displaystyle = \frac{\sqrt{2}}{2}\left(e^{1.1} - e\right) = 0.202151$

**13. (a)** The paraboloid is a portion of the graph of $f(x, y) = 4 - x^2 - 3y^2$. We have $\partial f/\partial x = -2x$, $\partial f/\partial y = -6y$ so that the surface area integral we desire is

$$\int_0^1\int_0^1 \sqrt{4x^2 + 36y^2 + 1}\,dy\,dx.$$

**(b)** We let $\Delta x = \Delta y = \dfrac{1 - 0}{4} = 0.25$. With $g(x, y) = \sqrt{4x^2 + 36y^2 + 1}$, we have

$$\begin{aligned} T_{4,4} = \frac{(0.25)(0.25)}{4} & [g(0,0) + g(0,1) + g(1,0) + g(1,1) \\ & + 2\left(g(0,0.25) + g(0,0.5) + g(0,0.75) + g(0.25,0) + g(0.25,1) + g(0.5,0)\right. \\ & \quad \left. + g(0.5,1) + g(0.75,0) + g(0.75,1) + g(1,0.25) + g(1,0.5) + g(1,0.75)\right) \\ & + 4\left(g(0.25,0.25) + g(0.25,0.5) + g(0.25,0.75) + g(0.5,0.25) + g(0.5,0.5)\right. \\ & \quad \left. + g(0.5,0.75) + g(0.75,0.25) + g(0.75,0.5) + g(0.75,0.75)\right)] \\ = 3.52366 & \end{aligned}$$

**14. (a)** We let $\Delta x = \dfrac{1.5 - 1}{2} = 0.25$, $\Delta y = \dfrac{2 - 1.4}{4} = 0.15$. Then

$$\begin{aligned} T_{2,4} = \frac{(0.25)(0.15)}{4} & [\ln(2 + 1.4) + \ln(2 + 2) + \ln(3 + 1.4) + \ln(3 + 2) \\ & + 2\left(\ln(2 + 1.55) + \ln(2 + 1.7) + \ln(2 + 1.85) + \ln(2.5 + 1.4) + \ln(2.5 + 2)\right. \\ & \quad \left. + \ln(3 + 1.55) + \ln(3 + 1.7) + \ln(3 + 1.85)\right) \\ & + 4\left(\ln(2.5 + 1.55) + \ln(2.5 + 1.7) + \ln(2.5 + 1.85)\right)] \\ = 0.429161 & \end{aligned}$$

**(b)** We have

$$\frac{\partial^2}{\partial x^2}\ln(2x+y) = -\frac{4}{(2x+y)^2} \quad \text{and} \quad \frac{\partial^2}{\partial y^2}\ln(2x+y) = -\frac{1}{(2x+y)^2}.$$

The maximum magnitude of both of these expressions on the rectangle $[1, 1.5] \times [1.4, 2]$ occurs at $(1, 1.4)$. Hence, from Theorem 7.3, we have that

$$|E_{2,4}| \le \frac{(0.5)(0.6)}{12}\left((0.25)^2 \cdot \frac{4}{(2+1.4)^2} + (0.15)^2 \cdot \frac{1}{(2+1.4)^2}\right) = 0.00589317.$$

Thus the actual value of the integral lies between 0.428572 and 0.42975.

**(c)** With $\Delta x$ and $\Delta y$ as in part (a), we have

$$\begin{aligned} S_{2,4} = {} & \frac{(0.25)(0.15)}{9}\left[\ln(2+1.4) + \ln(2+2) + \ln(3+1.4) + \ln(3+2)\right. \\ & + 2\left(\ln(2+1.7) + \ln(3+1.7)\right) + 4\left(\ln(2+1.55) + \ln(2+1.85)\right. \\ & \quad \left. + \ln(2.5+1.4) + \ln(2.5+2) + \ln(3+1.55) + \ln(3+1.85)\right) \\ & \left. + 8\ln(2.5+1.7) + 16\left(\ln(2.5+1.55) + \ln(2.5+1.85)\right)\right] \\ = {} & 0.429552 \end{aligned}$$

**(d)** We have

$$\frac{\partial^4}{\partial x^4}\ln(2x+y) = -\frac{96}{(2x+y)^4} \quad \text{and} \quad \frac{\partial^4}{\partial y^4}\ln(2x+y) = -\frac{6}{(2x+y)^4}$$

and, as in part (b), the maximum magnitude of both of these expressions on $[1, 1.5] \times [1.4, 2]$ occurs at $(1, 1.4)$. Hence, from Theorem 7.4, we have that

$$|E_{2,4}| \le \frac{(0.5)(0.6)}{180}\left((0.25)^4 \cdot \frac{96}{(2+1.4)^4} + (0.15)^4 \cdot \frac{6}{(2+1.4)^4}\right) = 6.36068 \times 10^{-6}.$$

Hence the actual value of the integral lies between 0.429546 and 0.429559.

**15.** To answer the question, we compare the errors of the respective methods as given in Theorems 7.3 and 7.4.

First we consider the error $E_{4,4}$ associated with the trapezoidal rule approximation $T_{4,4}$. In this case we have

$$\Delta x = \frac{1.4-1}{4} = 0.1 \qquad \text{and} \qquad \Delta y = \frac{0.7-0.5}{4} = 0.05.$$

In addition,

$$\frac{\partial^2}{\partial x^2}\ln(xy) = -\frac{1}{x^2} \quad \text{and} \quad \frac{\partial^2}{\partial y^2}\ln(xy) = -\frac{1}{y^2}.$$

Theorem 7.3 says that there exist points $(\zeta_1, \eta_1)$ and $(\zeta_2, \eta_2)$ in the rectangle $[1, 1.4] \times [0.5, 0.7]$ such that

$$\begin{aligned} E_{4,4} &= -\frac{(1.4-1)(0.7-0.5)}{12}\left[(0.1)^2\left(-\frac{1}{\zeta_1^2}\right) + (0.05)^2\left(-\frac{1}{\eta_2^2}\right)\right] \\ &= \frac{(0.4)(0.2)}{12}\left[\frac{(0.1)^2}{\zeta_1^2} + \frac{(0.05)^2}{\eta_2^2}\right]. \end{aligned}$$

Now $1 \le \zeta_1 \le 1.4$ and $0.5 \le \eta_2 \le 0.7$ so that, if we choose $\zeta_1 = 1.4$ and $\eta_2 = 0.7$, we can make the value in the brackets as small as possible; hence $E_{4,4} \ge 0.000068027$.

Next we consider the error $E_{2,2}$ associated with the Simpson's rule approximation $S_{2,2}$. Hence we have

$$\Delta x = \frac{1.4-1}{2} = 0.2 \qquad \text{and} \qquad \Delta y = \frac{0.7-0.5}{2} = 0.1;$$

also

$$\frac{\partial^4}{\partial x^4}\ln(xy) = -\frac{6}{x^4} \quad \text{and} \quad \frac{\partial^4}{\partial y^4}\ln(xy) = -\frac{6}{y^4}.$$

Theorem 7.4 says that there exist points $(\zeta_1, \eta_1)$ and $(\zeta_2, \eta_2)$ in $[1, 1.4] \times [0.5, 0.7]$ such that

$$E_{2,2} = -\frac{(1.4-1)(0.7-0.5)}{180}\left[(0.2)^4\left(-\frac{6}{\zeta_1^4}\right) + (0.1)^4\left(-\frac{6}{\eta_2^4}\right)\right]$$
$$= \frac{(0.4)(0.2)}{30}\left[\frac{(0.2)^4}{\zeta_1^4} + \frac{(0.1)^4}{\eta_2^4}\right].$$

By choosing $\zeta_1 = 1$ and $\eta_2 = 0.5$ we make the expression as large as possible; hence $E_{2,2} \leq 8.5\overline{3} \times 10^{-6}$. Since the *maximum* possible error using Simpson's rule is *less* than the *minimum* possible error using the trapezoidal rule, we see that $S_{2,2}$ will be more accurate than $T_{4,4}$.

**16.** We calculate the error $E_{n,n}$ associated with the trapezoidal rule approximation $T_{n,n}$. Note first that

$$\frac{\partial^2}{\partial x^2}\left(e^{x^2+2y}\right) = (4x^2+2)e^{x^2+2y} \quad \text{and} \quad \frac{\partial^2}{\partial y^2}\left(e^{x^2+2y}\right) = 4e^{x^2+2y}.$$

The maximum values of these expressions on the rectangle $[0, 0.2] \times [-0.1, 0.1]$ both occur at the point $(0.2, 0.1)$ and are, respectively, $(2.16)e^{0.24}$ and $4e^{0.24}$. Also note that in calculating $T_{n,n}$, we have $\Delta x = \Delta y = 0.2/n$. Thus, from Theorem 7.3, we have that

$$|E_{n,n}| \leq \frac{(0.2)(0.2)}{12}\left[\left(\frac{0.2}{n}\right)^2(2.16)e^{0.24} + \left(\frac{0.2}{n}\right)^2\left(4e^{0.24}\right)\right] = \frac{(0.2)^4(6.16)e^{0.24}}{12n^2}.$$

For this last expression to be at most $10^{-4}$, we must have

$$\frac{(0.2)^4(6.16)e^{0.24}}{12n^2} \leq 10^{-4} \iff n^2 \geq \frac{10^4(0.2)^4(6.16)e^{0.24}}{12} \iff n > 3.23.$$

Hence, since $n$ must be an integer, we should take $n$ to be at least 4.

**17. (a)** We have

$$\frac{\partial^2}{\partial x^2}\left(e^{x-y}\right) = \frac{\partial^2}{\partial y^2}\left(e^{x-y}\right) = e^{x-y}.$$

The maximum value of $e^{x-y}$ on $[0, 0.3] \times [0, 0.4]$ is $e^{0.3-0} = e^{0.3}$. Furthermore, in computing the approximation $T_{n,n}$ we have $\Delta x = 0.3/n$ and $\Delta y = 0.4/n$. Thus Theorem 7.3 implies that

$$|E_{n,n}| \leq \frac{(0.3)(0.4)}{12}\left[\left(\frac{0.3}{n}\right)^2 e^{0.3} + \left(\frac{0.4}{n}\right)^2 e^{0.3}\right] = \frac{(0.3)(0.4)(0.5)^2 e^{0.3}}{12n^2}.$$

For this expression to be at most $10^{-5}$, we must have

$$\frac{(0.3)(0.4)(0.5)^2e^{0.3}}{12n^2} \leq 10^{-5} \iff n^2 \geq \frac{10^5(0.3)(0.4)(0.5)^2e^{0.3}}{12} \iff n > 18.37.$$

Thus we should take $n$ to be at least 19.

**(b)** In this case, we use Theorem 7.4. First note that we have

$$\frac{\partial^4}{\partial x^4}\left(e^{x-y}\right) = \frac{\partial^4}{\partial y^4}\left(e^{x-y}\right) = e^{x-y},$$

so that, as in part (a), the maximum value of $e^{x-y}$ on $[0, 0.3] \times [0, 0.4]$ is $e^{0.3}$. Moreover, in computing the approximation $S_{2n,2n}$, we have $\Delta x = 0.3/(2n)$ and $\Delta y = 0.4/(2n)$. Therefore, Theorem 7.4 implies that

$$|E_{2n,2n}| \leq \frac{(0.3)(0.4)}{180}\left[\left(\frac{0.3}{2n}\right)^4 e^{0.3} + \left(\frac{0.4}{2n}\right)^4 e^{0.3}\right] = \frac{(0.3)(0.4)((0.3)^4+(0.4)^4)e^{0.3}}{180 \cdot 16n^4}.$$

For this expression to be at most $10^{-5}$, we must have

$$\frac{(0.3)(0.4)((0.3)^4+(0.4)^4)e^{0.3}}{180 \cdot 16n^4} \leq 10^{-5} \iff n^4 \geq \frac{10^5(0.3)(0.4)((0.3)^4+(0.4)^4)e^{0.3}}{180 \cdot 16}$$
$$\iff n > 0.659.$$

Thus, since $n$ must be an integer, we must have $n$ at least 1; that is, $S_{2,2}$ will give an approximation with the desired accuracy.

**18. (a)** Let $\Delta x = \dfrac{2-0}{2} = 1$, $\Delta y = \dfrac{3-0}{2} = 1.5$. With $f(x,y) = 3x + 5y$, we have

$$T_{2,2} = \frac{1(1.5)}{4}\left[f(0,0) + f(0,3) + f(2,0) + f(2,3) \right.$$
$$\left. + 2\left(f(0,1.5) + f(1,0) + f(1,3) + f(2,1.5)\right) + 4f(1,1.5)\right] = 63.$$

**(b)** $\displaystyle\int_0^2 \int_0^3 (3x+5y)\,dy\,dx = \int_0^2 \left(3xy + \tfrac{5}{2}y^2\right)\Big|_{y=0}^{y=3} dx = \int_0^2 \left(9x + \tfrac{45}{2}\right)\,dx$
$= \left(\tfrac{9}{2}x^2 + \tfrac{45}{2}x\right)\Big|_0^2 = 63.$
This is exactly the same result as in part (a).

**(c)** Note that, for all $(x,y)$, we have

$$\frac{\partial^2}{\partial x^2}(3x+5y) = \frac{\partial^2}{\partial y^2}(3x+5y) = 0.$$

Hence Theorem 7.3 shows that the error term $E_{2,2}$ must be zero. Hence it's no surprise that the results in parts (a) and (b) are the same.

**19. (a)** Let $\Delta x = \dfrac{0-(-1)}{2} = 0.5$, $\Delta y = \dfrac{1/2-0}{2} = 0.25$. With $f(x,y) = x^3y^3$, we have

$$S_{2,2} = \frac{(0.5)(0.25)}{9}\left[f(-1,0) + f(-1,\tfrac{1}{2}) + f(0,0) + f(0,\tfrac{1}{2}) \right.$$
$$\left. + 4\left(f(-1,\tfrac{1}{4}) + f(-\tfrac{1}{2},0) + f(-\tfrac{1}{2},\tfrac{1}{2}) + f(0,\tfrac{1}{4})\right) + 16f(-\tfrac{1}{2},\tfrac{1}{4})\right]$$
$$= -0.00390625$$

**(b)** $\displaystyle\int_{-1}^0 \int_0^{1/2} x^3y^3\,dy\,dx = \int_{-1}^0 \frac{x^3}{4}y^4\Big|_{y=0}^{y=1/2} dx = \int_{-1}^0 \frac{x^3}{64}\,dx = \frac{x^4}{256}\Big|_{-1}^0 = -\frac{1}{256}.$

**(c)** The answers in parts (a) and (b) turn out to be the same. Note that, for all $(x,y)$, we have

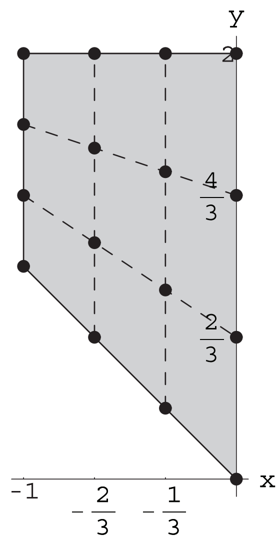
$$\frac{\partial^4}{\partial x^4}\left(x^3y^3\right) = \frac{\partial^4}{\partial y^4}\left(x^3y^3\right) = 0.$$

Hence Theorem 7.4 shows that the error term $E_{2,2}$ must be zero.

**20.** We let $\Delta x = \dfrac{0-(-1)}{3} = \frac{1}{3}$, so that $x_0 = -1$, $x_1 = -\frac{2}{3}$, $x_2 = -\frac{1}{3}$, $x_3 = 0$. Then $\Delta y(x) = \dfrac{2-(-x)}{3} = \dfrac{x+2}{3}$ so that

$$\begin{aligned}
\Delta y(-1) = \tfrac{1}{3} &\implies y_0(x_0) = 1,\ y_1(x_0) = \tfrac{4}{3},\ y_2(x_0) = \tfrac{5}{3},\ y_3(x_0) = 2\\
\Delta y(-\tfrac{2}{3}) = \tfrac{4}{9} &\implies y_0(x_1) = \tfrac{2}{3},\ y_1(x_1) = \tfrac{10}{9},\ y_2(x_1) = \tfrac{14}{9},\ y_3(x_1) = 2\\
\Delta y(-\tfrac{1}{3}) = \tfrac{5}{9} &\implies y_0(x_2) = \tfrac{1}{3},\ y_1(x_2) = \tfrac{8}{9},\ y_2(x_2) = \tfrac{13}{9},\ y_3(x_2) = 2\\
\Delta y(0) = \tfrac{2}{3} &\implies y_0(x_3) = 0\ ,y_1(x_3) = \tfrac{2}{3},\ y_2(x_3) = \tfrac{4}{3},\ y_3(x_3) = 2
\end{aligned}$$

This information is pictured in the figure below.




Therefore, using $f(x, y) = x^3 + 2y^2$, we have

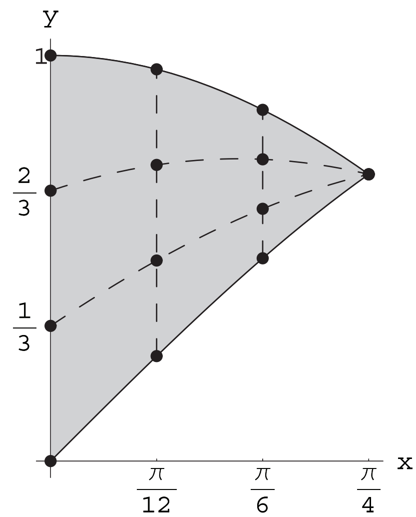
$$\begin{aligned}
T_{3,3} = &\frac{(1/3)(1/3)}{4}\left[f(-1,1) + 2f(-1,\tfrac{4}{3}) + 2f(-1,\tfrac{5}{3}) + f(-1,2)\right]\\
&+ \frac{(1/3)(4/9)}{4}\left[2f(-\tfrac{2}{3},\tfrac{2}{3}) + 4f(-\tfrac{2}{3},\tfrac{10}{9}) + 4f(-\tfrac{2}{3},\tfrac{14}{9}) + 2f(-\tfrac{2}{3},2)\right]\\
&+ \frac{(1/3)(5/9)}{4}\left[2f(-\tfrac{1}{3},\tfrac{1}{3}) + 4f(-\tfrac{1}{3},\tfrac{8}{9}) + 4f(-\tfrac{1}{3},\tfrac{13}{9}) + 2f(-\tfrac{1}{3},2)\right]\\
&+ \frac{(1/3)(2/3)}{4}\left[f(0,0) + 2f(0,\tfrac{2}{3}) + 2f(0,\tfrac{4}{3}) + f(0,2)\right]\\
= &\ 4.97119
\end{aligned}$$

(Note that the exact answer is $24/5 = 4.8$.)

**21.** We let $\Delta x = \dfrac{\pi/4 - 0}{3} = \dfrac{\pi}{12}$, so that $x_0 = 0$, $x_1 = \frac{\pi}{12}$, $x_2 = \frac{\pi}{6}$, $x_3 = \frac{\pi}{4}$. Then $\Delta y(x) = \dfrac{\cos x - \sin x}{3}$, so that

$$\begin{aligned}
\Delta y(0) = \tfrac{1}{3} \quad &\Longrightarrow \quad y_0(x_0) = 0,\ y_1(x_0) = \tfrac{1}{3},\ y_2(x_0) = \tfrac{2}{3},\ y_3(x_0) = 1\\
\Delta y\left(\tfrac{\pi}{12}\right) = \tfrac{1}{3\sqrt{2}} \quad &\text{(by use of the half-angle formula)}\\
&\Longrightarrow \quad y_0(x_1) = \sin\left(\tfrac{\pi}{12}\right),\ y_1(x_1) = \sin\left(\tfrac{\pi}{12}\right) + \tfrac{1}{3\sqrt{2}},\\
&\qquad\quad y_2(x_1) = \sin\left(\tfrac{\pi}{12}\right) + \tfrac{2}{3\sqrt{2}},\ y_3(x_1) = \sin\left(\tfrac{\pi}{12}\right) + \tfrac{1}{\sqrt{2}}\\
\Delta y\left(\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}-1}{6} \quad &\Longrightarrow \quad y_0(x_2) = \tfrac{1}{2},\ y_1(x_2) = \tfrac{\sqrt{3}+2}{6},\ y_2(x_2) = \tfrac{2\sqrt{3}+1}{6},\ y_3(x_2) = \tfrac{\sqrt{3}}{2}\\
\Delta y\left(\tfrac{\pi}{4}\right) = 0 \quad &\Longrightarrow \quad \text{partition points not needed.}
\end{aligned}$$

This information is pictured in the figure below.




Thus, using $f(x, y) = 2x\cos y + \sin^2 x$, we have

$$T_{3,3} = \frac{(\pi/12)(1/3)}{4}\left[f(0,0) + 2f(0,\tfrac{1}{3}) + 2f(0,\tfrac{2}{3}) + f(0,1)\right]$$

$$+ \frac{(\pi/12)(1/(3\sqrt{2}))}{4}\Big[2f(\tfrac{\pi}{12}, \sin\tfrac{\pi}{12}) + 4f(\tfrac{\pi}{12}, \sin\tfrac{\pi}{12} + \tfrac{1}{3\sqrt{2}})$$

$$+ 4f(\tfrac{\pi}{12}, \sin\tfrac{\pi}{12} + \tfrac{2}{3\sqrt{2}}) + 2f(\tfrac{\pi}{12}, \cos\tfrac{\pi}{12})\Big]$$

$$+ \frac{(\pi/12)((\sqrt{3}-1)/6)}{4}\Big[2f(\tfrac{\pi}{6}, \tfrac{1}{2}) + 4f(\tfrac{\pi}{6}, \tfrac{\sqrt{3}+2}{6}) + 4f(\tfrac{\pi}{6}, \tfrac{2\sqrt{3}+1}{6}) + 2f(\tfrac{\pi}{6}, \tfrac{\sqrt{3}}{2})\Big]$$

$$= 0.190978$$

(This approximation turns out to be rather low.)

**22.** Let $\Delta x = \dfrac{0.3-0}{3} = 0.1$, so that $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$. Then $\Delta y(x) = \dfrac{2x - x}{3} = \dfrac{x}{3}$ so that

$$\Delta y(0) = 0 \implies \text{partition points not needed;}$$

$$\Delta y(0.1) = \frac{0.1}{3} \implies y_0(x_1) = 0.1,\ y_1(x_1) = 0.1\overline{3},\ y_2(x_1) = 0.1\overline{6},\ y_3(x_1) = 0.2$$

$$\Delta y(0.2) = \frac{0.2}{3} \implies y_0(x_2) = 0.2,\ y_1(x_2) = 0.2\overline{6},\ y_2(x_2) = 0.\overline{3},\ y_3(x_2) = 0.4$$

$$\Delta y(0.3) = 0.1 \implies y_0(x_3) = 0.3\ , y_1(x_3) = 0.4,\ y_2(x_3) = 0.5,\ y_3(x_1) = 0.6$$

Then, using $f(x, y) = xy - x^2$, we have

$$T_{3,3} = \frac{(0.1)(0.1/3)}{4}\left[2f(0.1, 0.1) + 4f(0.1, 0.1\overline{3}) + 2f(0.1, 0.1\overline{6}) + 2f(0.1, 0.2)\right]$$

$$+ \frac{(0.1)(0.2/3)}{4}\left[2f(0.2, 0.2) + 4f(0.2, 0.2\overline{6}) + 4f(0.2, 0.\overline{3}) + 2f(0.2, 0.4)\right]$$

$$+ \frac{(0.1)(0.1)}{4}\left[f(0.3, 0.3) + 2f(0.3, 0.4) + 2f(0.3, 0.5) + f(0.3, 0.6)\right]$$

$$= 0.001125$$

(Note that the actual value is $0.0010125$.)

**23.** Let $\Delta x = \dfrac{\pi/3 - 0}{3} = \dfrac{\pi}{9}$, so that $x_0 = 0$, $x_1 = \frac{\pi}{9}$, $x_2 = \frac{2\pi}{9}$, $x_3 = \frac{\pi}{3}$. Then $\Delta y(x) = \dfrac{\sin x - 0}{3} = \dfrac{1}{3}\sin x$, so that

$$\begin{aligned}
\Delta y(0) = 0 \quad &\Longrightarrow \quad \text{partition points not needed;}\\
\Delta y\left(\tfrac{\pi}{9}\right) = \tfrac{1}{3}\sin\tfrac{\pi}{9} \quad &\Longrightarrow \quad y_0(x_1) = 0,\ y_1(x_1) = \tfrac{1}{3}\sin\tfrac{\pi}{9},\ y_2(x_1) = \tfrac{2}{3}\sin\tfrac{\pi}{9},\ y_3(x_1) = \sin\tfrac{\pi}{9}\\
\Delta y\left(\tfrac{2\pi}{9}\right) = \tfrac{1}{3}\sin\tfrac{2\pi}{9} \quad &\Longrightarrow \quad y_0(x_2) = 0,\ y_1(x_2) = \tfrac{1}{3}\sin\tfrac{2\pi}{9},\ y_2(x_2) = \tfrac{2}{3}\sin\tfrac{2\pi}{9},\ y_3(x_2) = \sin\tfrac{2\pi}{9}\\
\Delta y\left(\tfrac{\pi}{3}\right) = \tfrac{\sqrt{3}}{6} \quad &\Longrightarrow \quad y_0(x_3) = 0,\ y_1(x_3) = \tfrac{\sqrt{3}}{6},\ y_2(x_3) = \tfrac{\sqrt{3}}{3},\ y_3(x_3) = \tfrac{\sqrt{3}}{2}
\end{aligned}$$

Thus, using $f(x,y) = x/\sqrt{1-y^2}$, we have

$$\begin{aligned}
T_{3,3} &= \frac{(\frac{\pi}{9})(\frac{1}{3}\sin\frac{\pi}{9})}{4}\left[2f(\tfrac{\pi}{9},0) + 4f(\tfrac{\pi}{9},\tfrac{1}{3}\sin\tfrac{\pi}{9}) + 4f(\tfrac{\pi}{9},\tfrac{2}{3}\sin\tfrac{\pi}{9}) + 2f(\tfrac{\pi}{9},\sin\tfrac{\pi}{9})\right]\\
&\quad + \frac{(\frac{\pi}{9})(\frac{1}{3}\sin\frac{2\pi}{9})}{4}\left[2f(\tfrac{\pi}{9},0) + 4f(\tfrac{2\pi}{9},\tfrac{1}{3}\sin\tfrac{2\pi}{9}) + 4f(\tfrac{2\pi}{9},\tfrac{2}{3}\sin\tfrac{2\pi}{9}) + 2f(\tfrac{2\pi}{9},\sin\tfrac{2\pi}{9})\right]\\
&\quad + \frac{(\pi/9)(\sqrt{3}/6)}{4}\left[f(\tfrac{\pi}{3},0) + 2f(\tfrac{\pi}{3},\tfrac{\sqrt{3}}{6}) + 2f(\tfrac{\pi}{3},\tfrac{\sqrt{3}}{3}) + f(\tfrac{\pi}{3},\tfrac{\sqrt{3}}{2})\right]\\
&= 0.412888
\end{aligned}$$

(This actual value is $\pi^3/81 \approx 0.382794$, so our approximation is not especially good here.)

**24.** We must first let $\Delta y = \dfrac{\pi - 1}{3}$, so that $y_0 = 1$, $y_1 = \frac{\pi+2}{3}$, $y_2 = \frac{2\pi+1}{3}$, $y_3 = \pi$. Then $\Delta x(y) = \dfrac{y-0}{3} = \dfrac{y}{3}$, so that

$$\begin{aligned}
\Delta x(1) = \tfrac{1}{3} \quad &\Longrightarrow \quad x_0(y_0) = 0,\ x_1(y_0) = \tfrac{1}{3},\ x_2(y_0) = \tfrac{2}{3},\ x_3(y_0) = 1\\
\Delta x\left(\tfrac{\pi+2}{3}\right) = \tfrac{\pi+2}{9} \quad &\Longrightarrow \quad x_0(y_1) = 0,\ x_1(y_1) = \tfrac{\pi+2}{9},\ x_2(y_1) = \tfrac{2\pi+4}{9},\ x_3(y_1) = \tfrac{\pi+2}{3}\\
\Delta x\left(\tfrac{2\pi+1}{3}\right) = \tfrac{2\pi+1}{9} \quad &\Longrightarrow \quad x_0(y_2) = 0,\ x_1(y_2) = \tfrac{2\pi+1}{9},\ x_2(y_2) = \tfrac{4\pi+2}{9},\ x_3(y_2) = \tfrac{2\pi+1}{3}\\
\Delta x(\pi) = \tfrac{\pi}{3} \quad &\Longrightarrow \quad x_0(y_3) = 0,\ x_1(y_3) = \tfrac{\pi}{3},\ x_2(y_3) = \tfrac{2\pi}{3},\ x_3(y_3) = \pi
\end{aligned}$$

Thus, using $f(x,y) = \sin x$, we have

$$\begin{aligned}
T_{3,3} &= \frac{(\frac{\pi-1}{3})(\frac{1}{3})}{4}\left[f(0,1) + 2f(\tfrac{1}{3},1) + 2f(\tfrac{2}{3},1) + f(1,1)\right]\\
&\quad + \frac{(\frac{\pi-1}{3})(\frac{\pi+2}{9})}{4}\left[2f(0,\tfrac{\pi+2}{3}) + 4f(\tfrac{\pi+2}{9},\tfrac{\pi+2}{3}) + 4f(\tfrac{2\pi+4}{9},\tfrac{\pi+2}{3}) + 2f(\tfrac{\pi+2}{3},\tfrac{\pi+2}{3})\right]\\
&\quad + \frac{(\frac{\pi-1}{3})(\frac{2\pi+1}{9})}{4}\left[2f(0,\tfrac{2\pi+1}{3}) + 4f(\tfrac{2\pi+1}{9},\tfrac{2\pi+1}{3}) + 4f(\tfrac{4\pi+2}{9},\tfrac{2\pi+1}{3}) + 2f(\tfrac{2\pi+1}{3},\tfrac{2\pi+1}{3})\right]\\
&\quad + \frac{(\frac{\pi-1}{3})(\frac{\pi}{3})}{4}\left[f(0,\pi) + 2f(\tfrac{\pi}{3},\pi) + 2f(\tfrac{2\pi}{3},\pi) + f(\pi,\pi)\right]\\
&= 2.78757
\end{aligned}$$

(This actual value is $\sin 1 + \pi - 1 \approx 2.98306$, so this result is quite rough.)

**25.** We let $\Delta y = \dfrac{1.6 - 1}{3} = 0.2$, so that $y_0 = 1$, $y_1 = 1.2$, $y_2 = 1.4$, $y_3 = 1.6$. Then $\Delta x(y) = \dfrac{2y - y}{3} = \dfrac{y}{3}$, so that

$$\begin{aligned}
\Delta x(1) = \tfrac{1}{3} \quad &\Longrightarrow \quad x_0(y_0) = 1,\ x_1(y_0) = \tfrac{4}{3},\ x_2(y_0) = \tfrac{5}{3},\ x_3(y_0) = 2\\
\Delta x(1.2) = 0.4 \quad &\Longrightarrow \quad x_0(y_1) = 1.2,\ x_1(y_1) = 1.6,\ x_2(y_1) = 2,\ x_3(y_1) = 2.4\\
\Delta x(1.4) = 0.4\overline{6} \quad &\Longrightarrow \quad x_0(y_2) = 1.4,\ x_1(y_2) = 1.8\overline{6},\ x_2(y_2) = 2.\overline{3},\ x_3(y_2) = 2.8\\
\Delta x(1.6) = 0.5\overline{3} \quad &\Longrightarrow \quad x_0(y_3) = 1.6,\ x_1(y_3) = 2.1\overline{3},\ x_2(y_3) = 2.\overline{6},\ x_3(y_3) = 3.2
\end{aligned}$$

Thus, using $f(x,y)=\ln(xy)$, we have

$$\begin{aligned}T_{3,3} &= \frac{(0.2)(0.\overline{3})}{4}\left[f(1,1)+2f(\tfrac{4}{3},1)+2f(\tfrac{5}{3},1)+f(2,1)\right]\\ &\quad+\frac{(0.2)(0.4)}{4}\left[2f(1.2,1.2)+4f(1.6,1.2)+4f(2,1.2)+2f(2.4,1.2)\right]\\ &\quad+\frac{(0.2)(0.4\overline{6})}{4}\left[2f(1.4,1.4)+4f(1.8\overline{6},1.4)+4f(2.\overline{3},1.4)+2f(2.8,1.4)\right]\\ &\quad+\frac{(0.2)(0.5\overline{3})}{4}\left[f(1.6,1.6)+2f(2.1\overline{3},1.6)+2f(2.\overline{6},1.6)+f(3.2,1.6)\right]\\ &= 0.724061\end{aligned}$$

(This actual value is closer to $0.724519$.)

**26. (a)** We have

$$\begin{aligned}\iint_R L\,dA &= \int_a^b\int_c^d (Ax+By+C)\,dy\,dx = \int_a^b \left[(Ax+C)y+\tfrac{1}{2}By^2\right]\Big|_{y=c}^{y=d}dx\\ &= \int_a^b \left[(Ax+C)(d-c)+\tfrac{1}{2}B(d^2-c^2)\right]\,dx\\ &= (d-c)\int_a^b \left[Ax+C+\tfrac{1}{2}B(c+d)\right]\,dx\\ &= (d-c)\left[\tfrac{1}{2}A(b^2-a^2)+\left(C+\tfrac{1}{2}B(c+d)\right)(b-a)\right]\\ &= (b-a)(d-c)\left[\tfrac{1}{2}A(a+b)+\tfrac{1}{2}B(c+d)+C\right]\\ &= \frac{(b-a)(d-c)}{4}\left[2A(a+b)+2B(c+d)+4C\right].\end{aligned}$$

(Note that we drew out factors along the way.) The average of the values of $L$ taken at the vertices of $R$ is

$$\begin{aligned}&\frac{1}{4}\left[L(a,c)+L(a,d)+L(b,c)+L(b,d)\right]\\ &\quad=\frac{1}{4}\left[(Aa+Bc+C)+(Aa+Bd+C)+(Ab+Bc+C)+(Ab+Bd+C)\right]\\ &\quad=\frac{1}{4}\left[2A(a+b)+2B(c+d)+4C\right].\end{aligned}$$

If we multiply this expression by $(b-a)(d-c)$, which is the area of $R$, we obtain the expression for $\iint_R L\,dA$ calculated above.

**(b)** To calculate $T_{1,1}$, note that $\Delta x=b-a$, $\Delta y=d-c$, so that $x_0=a$, $x_1=b$, $y_0=c$, $y_1=d$ and formula (6) becomes

$$\begin{aligned}T_{1,1} &= \frac{(b-a)(d-c)}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\\ &= (\text{area of } R)\cdot(\text{average of values of } f \text{ on vertices of } R).\end{aligned}$$

**(c)** By part (b), the approximation $T_{1,1}$ to $\iint_{R_{ij}} f\,dA$ is

$$\frac{\Delta x\,\Delta y}{4}\left[f(x_{i-1},y_{j-1})+f(x_{i-1},y_j)+f(x_i,y_{j-1})+f(x_i,y_j)\right].$$

Thus $\iint_R f\,dA = \sum_{j=1}^{n}\sum_{i=1}^{m}\iint_{R_{ij}} f\,dA$ is approximated by

$$
\begin{aligned}
&\sum_{j=1}^{n}\sum_{i=1}^{m}\frac{\Delta x\,\Delta y}{4}\left[f(x_{i-1},y_{j-1})+f(x_{i-1},y_j)+f(x_i,y_{j-1})+f(x_i,y_j)\right]\\
&=\frac{\Delta x\,\Delta y}{4}\Bigg[\sum_{j=1}^{n}\sum_{i=1}^{m}f(x_{i-1},y_{j-1})+\sum_{j=1}^{n}\sum_{i=1}^{m}f(x_{i-1},y_j)\\
&\qquad+\sum_{j=1}^{n}\sum_{i=1}^{m}f(x_i,y_{j-1})+\sum_{j=1}^{n}\sum_{i=1}^{m}f(x_i,y_j)\Bigg]\\
&=\frac{\Delta x\,\Delta y}{4}\Bigg[f(x_0,y_0)+\sum_{j=1}^{n-1}f(x_0,y_j)+\sum_{i=1}^{m-1}f(x_i,y_0)+\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}f(x_i,y_j)\\
&\qquad+f(x_0,y_n)+\sum_{j=1}^{n-1}f(x_0,y_j)+\sum_{i=1}^{m-1}f(x_i,y_n)+\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}f(x_i,y_j)\\
&\qquad+f(x_m,y_0)+\sum_{j=1}^{n-1}f(x_m,y_j)+\sum_{i=1}^{m-1}f(x_i,y_0)+\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}f(x_i,y_j)\\
&\qquad+f(x_m,y_n)+\sum_{j=1}^{n-1}f(x_m,y_j)+\sum_{i=1}^{m-1}f(x_i,y_n)+\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}f(x_i,y_j)\Bigg]\\
&=\frac{\Delta x\,\Delta y}{4}\Bigg[f(x_0,y_0)+2\sum_{i=1}^{m-1}f(x_i,y_0)+f(x_m,x_0)\\
&\qquad+2\sum_{j=1}^{n-1}f(x_0,y_j)+4\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}f(x_i,y_j)+2\sum_{j=1}^{n-1}f(x_m,y_j)\\
&\qquad+f(x_0,y_n)+2\sum_{i=1}^{m-1}f(x_i,y_n)+f(x_m,y_n)\Bigg]\\
&=T_{m,n}.
\end{aligned}
$$

## True/False Exercises for Chapter 5

**1.** False. (Not all rectangles must have sides parallel to the coordinate axes.)
**2.** True.
**3.** True.
**4.** True.
**5.** False. (Let $f(x,y)=x$, for example.)
**6.** True.
**7.** False. (The integral on the right isn't even a number!)
**8.** True.
**9.** True.
**10.** False. (It's a type 1 region.)
**11.** True.
**12.** True.
**13.** False. (The value of the integral is 3.)
**14.** True. (Use symmetry.)
**15.** True.

**16.** False. (The $y$ part of the integrand gives a nonzero value.)
**17.** True. (The inner integral with respect to $z$ is zero because of symmetry.)
**18.** False. (The triple integral of $y$ is zero because of symmetry, but not the triple integral of $x$.)
**19.** True.
**20.** False. (The area is 30 square units.)
**21.** False. (The integrals are opposites of one another.)
**22.** True.
**23.** False. (A factor of $r$ should appear in the integrand.)
**24.** True.
**25.** False. (A factor of $\rho$ is missing in the integrand.)
**26.** True.
**27.** True.
**28.** False. (The centroid is at $\left(0, 0, \frac{1}{4}h\right)$.)
**29.** True.
**30.** True.

## Miscellaneous Exercises for Chapter 5

**1.** First let's split the integrand:

$$\iiint_B (z^3 + 2)\, dV = \iiint_B z^3 dV + \iiint_B 2\, dV = \iiint_B z^3 dV + 2 \iiint_B dV.$$

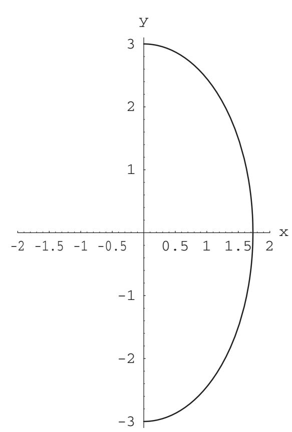
Here $B$ is the ball of radius 3 centered at the origin. The integral of an odd function of $z$ over a region which is symmetric with respect to $z$ is 0. The other integral is twice the volume of a sphere of radius 3 so

$$\iiint_B (z^3 + 2)\, dV = 2\left(\frac{4}{3}\pi 3^3\right) = 72\pi.$$

**2.** As in Exercise 1 we see that our integrand is the sum of odd functions in $x$ and $y$ and a constant which we are integrating over a region which is symmetric with respect to $x$ and $y$. Our answer will be $-3$ times the volume of the hemisphere of radius 2. In symbols,

$$V = \iiint_W (x^3 + y - 3)\, dV = -3 \iiint_W dV = -3\left(\frac{1}{2}\right)\left(\frac{4}{3}\pi 2^3\right) = -16\pi.$$

**3. (a)** We'll use the bounds given for $z$ in both integrals and just reverse the order of integration for $x$ and $y$. We are integrating over the ellipse:



$$\iiint_W 3\,dV = \int_{-3}^{3}\int_0^{\sqrt{3-y^2/3}}\int_{2x^2+y^2}^{9-x^2} 3\,dz\,dx\,dy \quad \text{and}$$
$$= \int_0^{\sqrt{3}}\int_{-\sqrt{9-3x^2}}^{\sqrt{9-3x^2}}\int_{2x^2+y^2}^{9-x^2} 3\,dz\,dy\,dx.$$

**(b)** Using *Mathematica*, the result was $(81\sqrt{3}\pi)/4$ in either order.

**4.** First follow the hint (noting that $x'$ and $y'$ are just dummy variables) and write

$$F(x,y) = \int_a^x g(x',y)\,dx' \quad \text{where} \quad g(x',y) = \int_c^y f(x',y')\,dy'.$$

By the fundamental theorem of calculus,

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\int_a^x g(x',y)\,dx' = g(x,y).$$

Also, again by the fundamental theorem,

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y}[g(x,y)] = \frac{\partial}{\partial y}\int_c^y f(x,y')\,dy' = f(x,y).$$

By Fubini's theorem,

$$\int_a^x\int_b^y f(x',y')\,dy'dx' = \int_c^y\int_a^x f(x',y')\,dx'dy'.$$

As above, write

$$F(x,y) = \int_c^y h(x,y')\,dy' \quad \text{where} \quad h(x,y') = \int_a^x f(x',y')\,dx'.$$

Proceeding as above we see that

$$\frac{\partial F}{\partial y} = h(x,y) \quad \text{and} \quad \frac{\partial^2 F}{\partial x \partial y} = f(x,y).$$

**5.** I think the given form is the easiest to integrate:

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r\sqrt{9-r^2}\,dr\,d\theta$$
$$= \int_0^{2\pi}\left(9 - \frac{16\sqrt{2}}{3}\right)d\theta$$
$$= 2\pi\left(9 - \frac{16\sqrt{2}}{3}\right).$$

**(a)** In Cartesian coordinates, $z$ doesn't really change and for the outer two limits, we are integrating over a unit circle so our answer is

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{9-x^2-y^2}} dz\,dy\,dx.$$

**(b)** The solid is the intersection of the top half of a sphere of radius 3 centered at the origin and a cylinder of radius 1 with axis of symmetry the $z$-axis. In spherical coordinates this means that we have to split the integral into two pieces: one that corresponds to the spherical cap and one that corresponds to the straight sides. The "cone" of intersection is when $\varphi = \sin^{-1} 1/3$. For the integral that corresponds to the "straight sides", $0 \le r \le 1$. In spherical coordinates that is $0 \le \rho\sin\varphi \le 1$ or $0 \le \rho \le \csc\varphi$. The integrals are, therefore,

$$\int_0^{2\pi}\int_0^{\sin^{-1}1/3}\int_0^3 \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta + \int_0^{2\pi}\int_{\sin^{-1}1/3}^{\pi/2}\int_0^{\csc\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

**6. (a)** This solid is similar to that in Exercise 5. It is the intersection of a cylinder over the circle of radius 2 with center $(0,2)$ (i.e., $x^2 = 4y - y^2$) and the plane $x = 0$ with caps on either end that are portions of the sphere of radius 4 centered at the origin ($z = \pm\sqrt{16 - x^2 - y^2}$).

**(b)** In cylindrical coordinates, $-\sqrt{16-r^2} \le z \le \sqrt{16-r^2}$ and we are above the first quadrant so $0 \le \theta \le \pi/2$. Since $x^2+y^2=4y$, $r^2 = 4r\sin\theta$ so in the first quadrant, $r = 4\sin\theta$. The volume is therefore

$$\begin{aligned} V &= \int_0^{\pi/2}\int_0^{4\sin\theta}\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{4\sin\theta} 2r\sqrt{16-r^2}\,dr\,d\theta \\ &= \left(\frac{128}{3}\right)\int_0^{\pi/2}(1-\cos^3\theta)\,d\theta = \frac{64}{9}(3\pi-4). \end{aligned}$$

*Exercises 7 and 8 are a good lesson in the advantage of choosing the right coordinate system in which to work. This simple problem in Cartesian coordinates is a pain using either cylindrical or spherical coordinates.*

**7.** Orient the cube so that a vertex is at the origin and the edges that meet at that vertex lie along the $x$-, $y$- and $z$-axes so that the cube is in the first octant. We'll double the volume of half of the cube. In this case $0 \le z \le a$, $0 \le \theta \le \pi/4$ and the only difficulty is with $r$. The radius varies from 0 to the line $x = a$. In cylindrical coordinates $x = r\cos\theta$ so $r = a\sec\theta$ and our limits for $r$ are $0 \le r \le a\sec\theta$. The volume is

$$V = 2\int_0^a\int_0^{\pi/4}\int_0^{a\sec\theta} r\,dr\,d\theta\,dz = \int_0^a\int_0^{\pi/4} a^2\sec^2\theta\,d\theta\,dz = \int_0^a a^2\,dz = a^3.$$

The above calculation wouldn't change much if you followed the hint in the text and placed the center of the cube at the origin. In this case you would have 1/8 of the figure in the first octant and you would be calculating the volume of a cube with sides $a/2$.

**8.** We again orient the cube so that a vertex is at the origin and the edges that meet at that vertex lie along the $x$-, $y$- and $z$-axes so that the cube is in the first octant. As in Exercise 7 we will double the volume of half of the cube corresponding to $0 \le \theta \le \pi/4$. We will have to split the integral into two pieces: the piece in which $\rho$ is bounded by the top of the cube ($z = a$ or $\rho = a\sec\varphi$) and the piece in which $\rho$ is bounded by the side of the cube ($x = a$ or $\rho = a\ \csc\ \varphi\sec\theta$). The boundary value of $\varphi$ depends on $\theta$. Set $a = \rho\cos\varphi$ equal to $a = \rho\sin\varphi\cos\theta$ and solve to obtain $\varphi = \cot^{-1}\cos\theta$. So the volume is

$$\begin{aligned} V &= 2\int_0^{\pi/4}\int_0^{\cot^{-1}\cos\theta}\int_0^{a\ \sec\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta + 2\int_0^{\pi/4}\int_{\cot^{-1}\cos\theta}^{\pi/2}\int_0^{a\ \csc\ \varphi\sec\theta} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \\ &= 2\left(\frac{a^3}{6}+\frac{a^3}{3}\right) = a^3. \end{aligned}$$

*Exercises 9–17 are examples where a change of variables helps. Exercise 14 depends on Exercise 11 and together they are much less difficult than they may first appear.*

**9.** Here we will let $u = x - 2y$ and $v = x + y$. We calculate

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = 3 \quad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} = 1/3.$$

The three boundary lines $x + y = 1, x = 0$, and $y = 0$ correspond to $v = 1, 2v = -u$, and $u = v$. We have all of the pieces to assemble our integral:

$$\begin{aligned} \iint_D \cos\left(\frac{x-2y}{x+y}\right) dA &= \int_0^1\int_{-2v}^{v} \frac{1}{3}\cos\left(\frac{u}{v}\right) du\,dv = \int_0^1 \frac{v}{3}\sin\left(\frac{u}{v}\right)\Big|_{-2v}^{v} du \\ &= \int_0^1 \frac{v}{3}(\sin 1 - \sin(-2))\,dv = \frac{v^2}{6}(\sin 1 + \sin 2)\Big|_0^1 = \frac{1}{6}(\sin 1 + \sin 2). \end{aligned}$$

**10.** Let $u = y^3$ and $v = x + 2y$. Then $0 \le u \le 216, 0 \le v \le 1$, and

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 0 & 3y^2 \\ 1 & 2 \end{vmatrix} = -3y^2 = -3u^{2/3} \quad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3u^{2/3}}.$$

Then

$$\begin{aligned}\int_0^6 \int_{-2y}^{1-2y} y^3(x+2y)^2 e^{(x+2y)^3}\,dx\,dy &= \int_0^{216}\int_0^1 \left(\frac{u}{3u^{2/3}}v^2 e^{v^3}\right)dv\,du\\ &= \int_0^{216}\int_0^1 \left(\frac{1}{3}u^{1/3}v^2 e^{v^3}\right)dv\,du\\ &= \int_0^{216}\left(\frac{1}{9}u^{1/3}e^{v^3}\right)\Big|_{v=0}^{1}du\\ &= \int_0^{216}\left(\frac{1}{9}u^{1/3}(e-1)\right)du\\ &= \left(\frac{1}{9}\left(\frac{3}{4}u^{4/3}\right)(e-1)\right)\Big|_0^{216}\\ &= 108(e-1).\end{aligned}$$

**11. (a)** As we've seen before, we can write the integral as $\int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} dy\,dx$.

**(b)** When we scale by letting $x = a\bar{x}$ and $y = b\bar{y}$, the ellipse is transformed into the unit circle $E^*$ in the $\bar{x}\bar{y}$-plane. To rewrite the integral we also quickly calculate that $\partial(x,y)/\partial(\bar{x},\bar{y}) = ab$. The transformed integral is $\int_{-1}^{1}\int_{-\sqrt{1-\bar{x}^2}}^{\sqrt{1-\bar{x}^2}} ab\,d\bar{y}\,d\bar{x}$.

**(c)** Because we are integrating over a unit circle, we transform to polar coordinates (o.k., really we do it because the text tells us to):

$$\int_{-1}^{1}\int_{-\sqrt{1-\bar{x}^2}}^{\sqrt{1-\bar{x}^2}} ab\,d\bar{y}\,d\bar{x} = \int_0^{2\pi}\int_0^1 abr\,dr\,d\theta = \int_0^{2\pi}\frac{1}{2}ab\,d\theta = \frac{1}{2}ab(2\pi) = \pi\,ab.$$

**12. (a)** With $u = 2x - y$, $v = x + y$, we see $u + v = 3x$ so $x = \dfrac{u+v}{3}$ which implies $y = \dfrac{2v-u}{3}$. Substituting these expressions into the equation for the ellipse, we obtain

$$13\left(\frac{u+v}{3}\right)^2 + 14\left(\frac{u+v}{3}\right)\left(\frac{2v-u}{3}\right) + 10\left(\frac{2v-u}{3}\right)^2 = 9.$$

Expanding and simplifying, we find

$$\frac{u^2}{9} + v^2 = 1.$$

**(b)** Area $= \iint_E 1\,dA = \iint_E 1\,dx\,dy = \iint_{E^*}\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$ where $E^*$ denotes the corresponding ellipse in the *uv*-plane given above. Now $\dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix}1/3 & 1/3\\ 2/3 & -1/3\end{bmatrix} = -\dfrac{1}{3}$ so

Area $= \iint_{E^*}\dfrac{1}{3}du\,dv = \dfrac{1}{3}(\text{area of } E^*) = \dfrac{1}{3}(\pi\cdot 3\cdot 1) = \pi$ using the result of part (c) of Exercise 11.

**13.** With $u = x - y$, $v = x + y$ we find that $x = \dfrac{u+v}{2}$, $y = \dfrac{v-u}{2}$. Substituting these expressions into the equation for $E$, we find

$$5\left(\frac{u+v}{2}\right)^2 + 6\left(\frac{u+v}{2}\right)\left(\frac{v-u}{2}\right) + 5\left(\frac{v-u}{2}\right)^2 = 4,$$

which simplifies to

$$\frac{u^2}{4} + v^2 = 1.$$

The area of ellipse $E^*$ in the *uv*-plane is $2\pi$. The area of the original ellipse $E$ is

$$\begin{aligned}\iint_E 1\,dA = \iint_E dx\,dy = \iint_{E^*}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv &= \iint_{E^*}\left|\det\begin{bmatrix}1/2 & 1/2\\ -1/2 & 1/2\end{bmatrix}\right| du\,dv\\ = \iint_{E^*}\left|\frac{1}{2}\right| du\,dv = \frac{1}{2}\text{ area of } E^* &= \frac{1}{2}(2\pi) = \pi.\end{aligned}$$

**14.** We follow the steps in Exercise 11, inserting the same letters for ease in locating the corresponding parts.

**(a)** First we write the integral in Cartesian coordinates as

$$\int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}}\int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx.$$

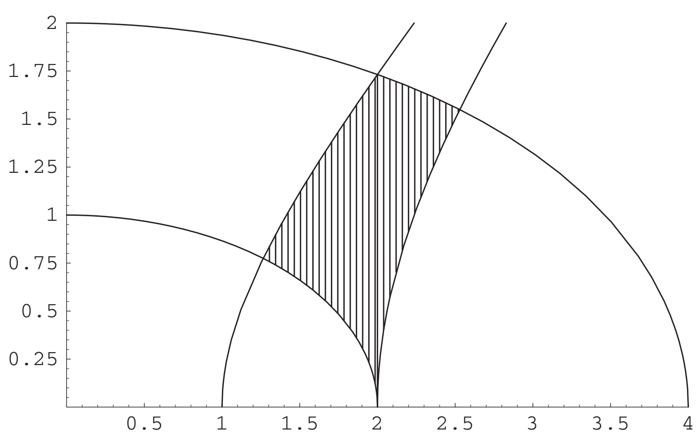
**(b)** We now scale the variables using $x = a\bar{x}, y = b\bar{y}$, and $z = c\bar{z}$. Note that the ellipsoid $E$ is transformed into the unit sphere $E^*$ and that $\partial(x,y,z)/\partial(\bar{x},\bar{y},\bar{z}) = abc$. The transformed integral is:

$$\int_{-1}^{1}\int_{-\sqrt{1-\bar{x}^2}}^{\sqrt{1-\bar{x}^2}}\int_{-\sqrt{1-\bar{x}^2-\bar{y}^2}}^{\sqrt{1-\bar{x}^2-\bar{y}^2}} abc\,d\bar{z}\,d\bar{y}\,d\bar{x}.$$

**(c)** Because we are integrating over a unit sphere, we will transform to spherical coordinates:

$$\begin{aligned}\int_{-1}^{1}\int_{-\sqrt{1-\bar{x}^2}}^{\sqrt{1-\bar{x}^2}}\int_{-\sqrt{1-\bar{x}^2-\bar{y}^2}}^{\sqrt{1-\bar{x}^2-\bar{y}^2}} abc\,d\bar{z}\,d\bar{y}\,d\bar{x} &= \int_0^{2\pi}\int_0^1\int_0^{\pi} abc\,\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta \\ &= \int_0^{2\pi}\int_0^1 \left(-\cos\varphi(abc)\rho^2\right)\Big|_0^{\pi} d\rho\,d\theta = \int_0^{2\pi}\int_0^1 (2abc\,\rho^2)\,d\rho\,d\theta \\ &= \int_0^{2\pi}\frac{2}{3}abc\,d\theta = \frac{4}{3}\pi abc.\end{aligned}$$

**15.** If you didn't first sketch the region you may be tempted to use the numerator and denominator of the integrand as your new variables. The diamond-like shape is bounded on two sides by the hyperbolas $x^2 - y^2 = 1$ and $x^2 - y^2 = 4$ and on the other two sides by the ellipses $x^2/4 + y^2 = 1$ and $x^2/4 + y^2 = 4$.



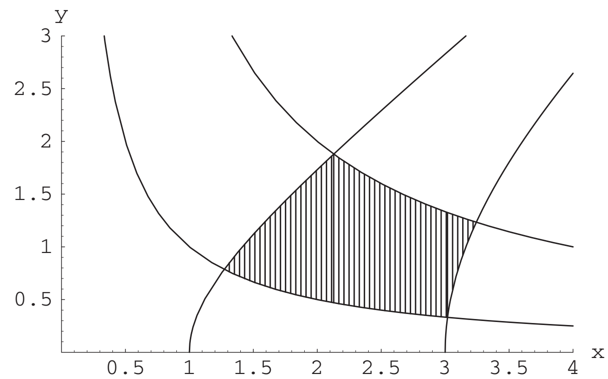

We, therefore, make the change of variables $u = x^2 - y^2$ and $v = x^2/4 + y^2$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ x/2 & 2y \end{vmatrix} = 5xy \quad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} = 1/(5xy).$$

The integral greatly simplifies:

$$\begin{aligned}\iint_D \frac{xy}{y^2-x^2}\,dA &= \int_1^4\int_1^4 \left(\frac{xy}{-u}\frac{1}{5xy}\right) du\,dv = -\frac{1}{5}\int_1^4\int_1^4 \frac{1}{u}\,du\,dv \\ &= -\frac{1}{5}\int_1^4 \ln 4\,dv = -\frac{3}{5}\ln 4.\end{aligned}$$

**16.** The region $D$ is bounded on the left and right by $x^2 - y^2 = 1$ and $x^2 - y^2 = 9$ and on the bottom and top by $xy = 1$ and $xy = 4$. It looks like



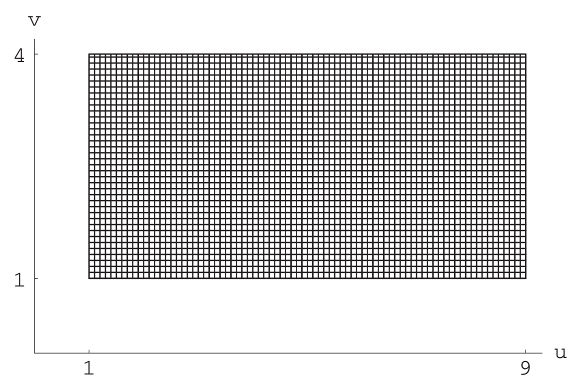

This suggests we try the change of variables $u = x^2 - y^2$, $v = xy$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix} = 2(x^2 + y^2)$$

so that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2(x^2+y^2)}.$$

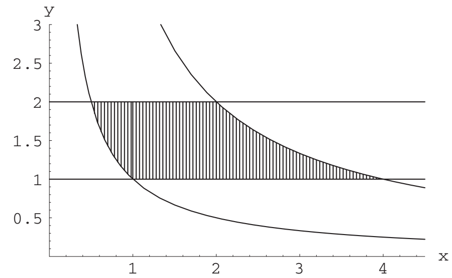
Moreover, the region $D$* in the $uv$-plane corresponding to $D$ is




Thus, using the change of variables theorem, we have

$$\iint_D (x^2+y^2)e^{x^2-y^2}\,dA = \iint_{D^*} \frac{1}{2}e^u\,du\,dv = \int_1^4 \int_1^9 \frac{1}{2}e^u\,du\,dv$$
$$= \int_1^4 \frac{1}{2}(e^9 - e)\,dV = \frac{3}{2}(e^9 - e).$$

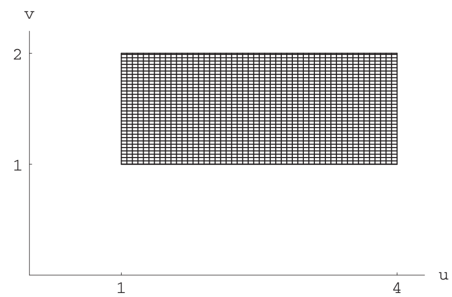
**17.** The region $D$ is bounded on the bottom and top by $y = 1$ and $y = 2$ and on the left and right by $xy = 1$ and $xy = 4$; the region looks like the following figure.




With this in mind, we try the change of variables $u = xy, v = y$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} = y = v \quad \Longrightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{v}.$$

Moreover, the region $D^*$ in the *uv*-plane corresponding to $D$ is the rectangle $[1,4] \times [1,2]$:




The change of variables theorem tells us that

$$\iint_D \frac{1}{x^2y^2+1}\,dA = \iint_{D^*} \frac{1}{u^2+1} \cdot \frac{1}{v}\,du\,dv = \int_1^4 \int_1^2 \frac{1}{u^2+1} \cdot \frac{1}{v}\,dv\,du$$

$$= \int_1^4 \frac{\ln 2}{u^2+1}\,du = \ln 2 \left( \tan^{-1} u \Big|_1^4 \right)$$

$$= \ln 2 \left( \tan^{-1} 4 - \frac{\pi}{4} \right).$$

**18. (a)** Follow the same steps as in defining the double and triple integrals.

- Define a **partition** of $B = [a,b] \times [c,d] \times [p,q] \times [r,s]$ of order $n$ to be four collections of partition points that break up $B$ into a union of $n^4$ subboxes. See Definition 4.1 and add that $r = w_0 < w_1 < \cdots < w_n = s$ and $\Delta w_l = w_l - w_{l-1}$.
- Define a Riemann sum. For a function $f$ defined on $B$, partition $B$ as above and let $\mathbf{c}_{ijkl}$ be any point in the subbox

$$B_{ijkl} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k] \times [w_{l-1}, w_l].$$

- The Riemann sum of $f$ on $B$ corresponding to the partition is

$$S = \sum_{i,j,k,l=1}^{n} f(\mathbf{c}_{ijkl}) \Delta x_i \Delta y_j \Delta z_k \Delta w_l = \sum_{i,j,k,l=1}^{n} f(\mathbf{c}_{ijkl}) \Delta V_{ijkl}.$$

- Define the **quadruple integral** of $f$ on $B$, written

$$\iiiint_B f(x,y,z,w)\,dV = \iiiint_B f(x,y,z,w)\,dx\,dy\,dz\,dw$$

to be

$$\iiiint_B f(x,y,z,w)\,dV = \lim_{\text{all } \Delta x_i, \Delta y_j, \Delta z_k, \Delta w_l \to 0} \sum_{i,j,k,l=1}^{n} f(\mathbf{c}_{ijkl}) \Delta x_i \Delta y_j \Delta z_k \Delta w_l.$$

- We extend the definition to compact non-box regions $W$ by defining the function $f^{ext}$ which is $f$ everywhere in $W$ and is 0 everywhere else. Then if $B$ is a box containing $W$ we can define

$$\iiiint_W f dV = \iiiint_B f^{ext} dV.$$

- As in the cases of the double and triple integrals, Fubini's theorem allows us to evaluate the integral as an iterated integral.

**(b)** We calculate:

$$\begin{aligned}\iiiint_W (x+2y+3z-4w)\,dV &= \int_0^2\int_{-1}^3\int_0^4\int_{-2}^2 (x+2y+3z-4w)\,dw\,dz\,dy\,dx\\ &= 4\int_0^2\int_{-1}^3\int_0^4 (x+2y+3z)\,dz\,dy\,dx\\ &= 4\int_0^2\int_{-1}^3 (4x+8y+24)\,dy\,dx\\ &= 4\int_0^2 (16x+32+96)\,dx = 64\int_0^2 (x+8)\,dx\\ &= 64(2+16) = 1152.\end{aligned}$$

**19. (a)** We are just generalizing what we've done to set up the area of a circle or the volume of a sphere (more recently see Exercises 11 and 14 from this section). Here our integral is:

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2-z^2}}^{\sqrt{a^2-x^2-y^2-z^2}} dw\,dz\,dy\,dx.$$

**(b)** You should get $\pi^2 a^2/4$.

**(c)** For $n = 5$ you should get $8\pi^2 a^5/15$, and for $n = 6$ you should get $\pi^3 a^6/6$. If you include the cases for $n = 2$ and $n = 3$ you may begin to see a pattern for the even exponents. If $n$ is even, the volume of the $n$-sphere of radius $a$ is $\pi^{n/2}a^n/(n/2)!$. Fitting in the odd terms looks really hard and the pattern shouldn't occur to any of your students. In fact, the general formula depends on the Gamma function which is beyond what we would expect the students to know at this point. For kicks, the volume of the $n$-sphere of radius $a$ is

$$\frac{\pi^{n/2}a^n}{\Gamma((n/2)+1)}.$$

Note that the volume of an $n$-sphere of radius $a$ decreases to 0 as $n$ increases.

**20.** Let $x_1 = a\bar{x}_1,\ x_2 = a\bar{x}_2, \ldots, x_n = a\bar{x}_n$. Then, by substitution,

$$\begin{aligned}B &= \{(x_1,\ldots,x_n) \mid x_1^2+\cdots+x_n^2 \le a^2\}\\ &= \{(\bar{x}_1,\ldots,\bar{x}_n) \mid (a\bar{x}_1)^2+\cdots+(a\bar{x}_n)^2 \le a^2\}\\ &= \{(\bar{x}_1,\ldots,\bar{x}_n) \mid \bar{x}_1^2+\cdots+\bar{x}_n^2 \le 1\},\end{aligned}$$

which is the *unit* ball in $(\bar{x}_1,\ldots,\bar{x}_n)$-coordinates. The Jacobian of this change of variables is

$$\frac{\partial(x_1,\ldots,x_n)}{\partial(\bar{x}_1,\ldots,\bar{x}_n)} = \det\begin{bmatrix} a & 0 & \cdots & 0\\ 0 & a & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & a\end{bmatrix} = a^n.$$

Hence

$$V_n(a) = \int\cdots\int_B 1\,dx_1\cdots dx_n = \int\cdots\int_U a^n\,d\bar{x}_1\cdots d\bar{x}_n = C_n a^n.$$

**21. (a)** Since a point $(x_1, \ldots, x_n)$ in $B$ satisfies the inequality $x_1^2+\cdots+x_n^2 \le a^2$, a point in $B$ of the form $(x_1, x_2, 0, \ldots, 0)$ must have $x_1^2 + x_2^2 + 0 \cdots + 0 \le a^2$. Thus $(x_1, x_2)$, considered as a point in $\mathbf{R}^2$, satisfies $x_1^2 + x_2^2 \le a^2$, so that $(x_1, x_2)$ lies in the disk of radius $a$ in $\mathbf{R}^2$.

**(b)** The point $(x_1, x_2, 0, \ldots, 0)$ in $B$ described in part (a) has coordinates that relate to polar coordinates $(r, \theta)$ by $x_1^2 + x_2^2 = r^2 \le a^2$. Hence any point $(r, \theta, x_3, \ldots, x_n)$ in $B$ lying over the specific point $(r, \theta)$ in the disk must satisfy $r^2 + x_3^2 + \cdots + x_n^2 \le a^2 \iff x_3^2 + \cdots + x_n^2 \le a^2 - r^2$. Hence the coordinates $(x_3, \ldots, x_n)$ fill out an $(n-2)$-dimensional ball of radius $\sqrt{a^2 - r^2}$.

**(c)** If $D$ denotes the radius $a$ disk centered at the origin, then, from part (b), we have

$$\begin{aligned} V_n(a) &= \int \cdots \int_B dx_1 \cdots dx_n = \iint_D V_{n-2}\left(\sqrt{a^2 - r^2}\right) dx_1\, dx_2 \\ &= \int_0^{2\pi} \int_0^a V_{n-2}\left(\sqrt{a^2 - r^2}\right) r\, dr\, d\theta. \end{aligned}$$

**22.** By the previous exercise, we have

$$\begin{aligned} V_n(a) &= \int_0^{2\pi} \int_0^a V_{n-2}\left(\sqrt{a^2 - r^2}\right) r\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^a C_{n-2}\left(a^2 - r^2\right)^{(n-2)/2} r\, dr\, d\theta \qquad \text{from Exercise 20,} \\ &= C_{n-2} \int_0^{2\pi} \int_{a^2}^0 u^{(n-2)/2}\left(-\tfrac{1}{2}\, du\right) d\theta \\ &= \frac{C_{n-2}}{2} \int_0^{2\pi} \frac{2}{n} u^{n/2}\Big|_{u=0}^{a^2} d\theta \\ &= \frac{2\pi}{n} C_{n-2} a^n. \end{aligned}$$

Now $V_{n-2}(a) = C_{n-2}a^{n-2}$, so we have

$$V_n(a) = \left(\frac{2\pi}{n} a^2\right)\left(C_{n-2}a^{n-2}\right) = \left(\frac{2\pi}{n} a^2\right) V_{n-2}(a).$$

**23. (a)** The one-dimensional ball of radius $a$ consists of points in $\mathbf{R}$ described as

$$\{x_1 \in \mathbf{R} \mid x_1^2 \le a^2\} = \{x_1 \in \mathbf{R} \mid -a \le x_1 \le a\} = [-a, a].$$

The one-dimensional volume of this "ball" is the length of the interval; thus $V_1(a) = 2a$. The two-dimensional ball of radius $a$ consists of points $(x_1, x_2) \in \mathbf{R}^2$ such that $x_1^2 + x_2^2 \le a^2$. Such points form a disk of radius $a$, so the two-dimensional volume of this disk is its area; hence $V_2(a) = \pi a^2$.

**(b)** By repeatedly using the recursive formula in Exercise 22, we have

$$V_n(a) = \begin{cases} \left(\dfrac{2\pi}{n} a^2\right)\left(\dfrac{2\pi}{n-2} a^2\right) \cdots \left(\dfrac{2\pi}{4} a^2\right) V_2(a) & \text{if } n \text{ is even,} \\ \left(\dfrac{2\pi}{n} a^2\right)\left(\dfrac{2\pi}{n-2} a^2\right) \cdots \left(\dfrac{2\pi}{1} a^2\right) V_1(a) & \text{if } n \text{ is odd.} \end{cases}$$

In the expressions for $V_n(a)$ above, there are $\frac{n}{2} - 1$ factors appearing before $V_2(a)$ when $n$ is even and $\frac{n-1}{2}$ factors appearing before $V_1(a)$ when $n$ is odd. Hence, using the results of part (a), we have

$$V_n(a) = \begin{cases} \dfrac{2^{(n/2)-1}\pi^{(n/2)-1}\left(a^2\right)^{(n/2)-1}\pi a^2}{n(n-2)\cdots 4} & \text{if } n \text{ is even} \\ \dfrac{2^{(n-1)/2}\pi^{(n-1)/2}\left(a^2\right)^{(n-1)/2} 2a}{n!!} & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} \dfrac{2^{n/2}\pi^{n/2}\left(a^2\right)^{n/2}}{n(n-2)\cdots 4\cdot 2} & \text{if } n \text{ is even} \\ \dfrac{2^{(n+1)/2}\pi^{(n-1)/2}\left(a^2\right)^{(n-1)/2} a}{n!!} & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} \dfrac{2^{n/2}\pi^{n/2}a^n}{2(n/2)\cdot 2\left((n/2)-1\right)\cdot 2\left((n/2)-2\right)\cdots(2\cdot 1)} & \text{if } n \text{ is even} \\ \dfrac{2^{(n+1)/2}\pi^{(n-1)/2}a^n}{n!!} & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} \dfrac{\pi^{n/2}a^n}{(n/2)!} & \text{if } n \text{ is even} \\ \dfrac{2^{(n+1)/2}\pi^{(n-1)/2}a^n}{n!!} & \text{if } n \text{ is odd.} \end{cases}$$

**24. (a)** To obtain the mass we compute the following integral (which is straightforward so the details are omitted):

$$M = \int_0^{2\pi}\int_0^{\pi}\int_3^4 ((.12\rho^2)\rho^2\sin\varphi)\,d\rho\,d\varphi\,d\theta = 74.976\pi \approx 235.5440508g.$$

**(b)** Because the shell is sealed, the volume is $V = (4/3)\pi(4^3) = 256\pi/3$ cm$^3$, so the mass of that volume of water is greater, and so the shell would float.

**(c)** If the core of the shell fills with water, then the volume that the shell has to displace is $V = (4/3)\pi(4^3 - 3^3) = (4/3)\pi(37)$. The water for that volume would have mass of about 155 grams so the shell would sink.

**25.** When you average the height of the hemisphere of radius $a$, first you integrate

$$\int_0^{2\pi}\int_0^a zr\,dr\,d\theta = \int_0^{2\pi}\int_0^a r\sqrt{a^2-r^2}\,dr\,d\theta = \frac{2}{3}\pi a^3.$$

For the average height, we divide this by the area of the region over which we are integrating:

$$\frac{(2/3)\pi a^3}{\pi a^2} = \frac{2}{3}a.$$

We now solve to see which values of $r$ correspond to this height: $(2/3)a = \sqrt{a^2-r^2}$ when $(4/9)a^2 = a^2 - r^2$, which is when $r = \sqrt{5}a/3$. Therefore, the pole can be installed at most $\sqrt{5}a/3$ from the center of the floor of the dome.

**26. (a)** By the fundamental theorem of calculus

$$\frac{d}{dy}\int_c^y G(y')\,dy' = G(y) \quad \text{so} \quad \frac{d}{dy}\int_c^y\int_a^b f_y(x,y')\,dx\,dy' = \int_a^b f_y(x,y)\,dx.$$

**(b)** On the other hand, by Fubini's theorem,

$$\begin{aligned}\frac{d}{dy}\int_c^y\int_a^b f_y(x,y')\,dx\,dy' &= \frac{d}{dy}\int_a^b\int_c^y f_y(x,y')\,dy'dx \\ &= \frac{d}{dy}\int_a^b (f(x,y)-f(x,c))\,dx = \frac{d}{dy}\int_a^b f(x,y)\,dx.\end{aligned}$$

Combine parts (a) and (b) to obtain the desired results.

**27. (a)**

$$I(\epsilon,\delta)=\int_\epsilon^{1-\epsilon}\int_\delta^{1-\delta}\frac{1}{\sqrt{xy}}\,dy\,dx=\int_\epsilon^{1-\epsilon}\frac{2}{\sqrt{x}}(\sqrt{1-\delta}-\sqrt{\delta})\,dx$$
$$=4(\sqrt{1-\epsilon}-\sqrt{\epsilon})(\sqrt{1-\delta}-\sqrt{\delta})$$

**(b)** $\lim_{(\epsilon,\delta)\to(0,0)} I(\epsilon,\delta)=4\cdot 1\cdot 1=4$

**28.** For $0<\epsilon<\dfrac{1}{2}, 0<\delta<\dfrac{1}{2}$ we consider $I(\epsilon,\delta)=\iint_{D_{\epsilon,\delta}}\dfrac{1}{x+y}dA$ where $D_{\epsilon,\delta}=[\epsilon,1-\epsilon]\times[\delta,1-\delta]$. Then

$$I(\epsilon,\delta)=\int_\epsilon^{1-\epsilon}\int_\delta^{1-\delta}\frac{1}{x+y}\,dy\,dx=\int_\epsilon^{1-\epsilon}\ln(x+y)\Big|_{y=\delta}^{1-\delta}dx$$
$$=\int_\epsilon^{1-\epsilon}(\ln(x+1-\delta)-\ln(x+\delta))\,dx.$$

Using integration by parts, we find that $\int \ln u\,du=u\ln u-u+C$ so that

$$I(\epsilon,\delta)=[(x+1-\delta)\ln(x+1-\delta)-(x+1-\delta)-(x+\delta)\ln(x+\delta)+(x+\delta)|_{x=\epsilon}^{1-\epsilon}$$
$$=(2-\epsilon-\delta)\ln(2-\epsilon-\delta)-(2-\epsilon-\delta)-(1-\epsilon+\delta)\ln(1-\epsilon+\delta)$$
$$+(1-\epsilon+\delta)-(\epsilon+1-\delta)\ln(\epsilon+1-\delta)+(\epsilon+1-\delta)$$
$$+(\epsilon+\delta)\ln(\epsilon+\delta)-(\epsilon+\delta).$$

To evaluate $\lim_{(\epsilon,\delta)\to(0^+,0^+)} I(\epsilon,\delta)$ we first note that, by l'Hôpital's rule,

$$\lim_{u\to 0^+}u\ln u=\lim_{u\to 0^+}\frac{\ln u}{1/u}=\lim_{u\to 0^+}\frac{1/u}{-1/u^2}=-\lim_{u\to 0^+}u=0.$$

Thus $(\epsilon+\delta)\ln(\epsilon+\delta)\to 0$ as $(\epsilon,\delta)\to(0^+,0^+)$. The other terms in the expression have evident limits so that

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} I(\epsilon,\delta)=2\ln 2-2-\ln 1+1-\ln 1+1+0-0=2\ln 2.$$

**29.** For $0<\epsilon<\frac{1}{2}, 0<\delta<\frac{1}{2}$, let $D_{\epsilon,\delta}=[\epsilon,1-\epsilon]\times[\delta,1-\delta]$ and consider

$$I(\epsilon,\delta)=\iint_{D_{\epsilon,\delta}}\frac{x}{y}\,dA=\int_\delta^{1-\delta}\int_\epsilon^{1-\epsilon}\frac{x}{y}\,dx\,dy$$
$$=\int_\delta^{1-\delta}\frac{\left(\frac{1}{2}-\epsilon\right)}{y}\,dy=\left(\frac{1}{2}-\epsilon\right)(\ln(1-\delta)-\ln\delta).$$

Note that $\lim_{(\epsilon,\delta)\to(0,0)} I(\epsilon,\delta)=-\infty$ since $\frac{1}{2}-\epsilon$ and $\ln(1-\delta)$ remain finite, but $\ln\delta\to-\infty$. Thus the improper integral does not converge.

*In Exercises 30 and 31 the students will need integration by parts and l'Hôpital's rule.*

**30.**

$$\iint_D \ln\sqrt{x^2+y^2}\,dA=\lim_{\epsilon\to 0}\iint_{D_\epsilon}\ln\sqrt{x^2+y^2}\,dA=\lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^1 r\ln r\,dr\,d\theta$$
$$=\lim_{\epsilon\to 0}\int_0^{2\pi}\left[\frac{r^2}{2}\ln r-\frac{r^2}{4}\right]\Big|_\epsilon^1 d\theta=\lim_{\epsilon\to 0}\int_0^{2\pi}\left[-\frac{1}{4}-\frac{\epsilon^2}{2}\ln\epsilon+\frac{\epsilon^2}{4}\right]d\theta$$
$$=\lim_{\epsilon\to 0}2\pi\left[-\frac{1}{4}-\frac{\epsilon^2}{2}\ln\epsilon+\frac{\epsilon^2}{4}\right]=-\pi/2.$$

**31.** Define $B_\epsilon = \{(x, y, z) | \epsilon \le x^2 + y^2 + z^2 \le 1\}$. Then

$$\begin{aligned}\iiint_B \ln\sqrt{x^2+y^2+z^2}\, dV &= \lim_{\epsilon\to 0}\iiint_{B_\epsilon} \ln\sqrt{x^2+y^2+z^2}\, dV\\ &= \lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^1\int_0^\pi ((\ln\rho)\rho^2\sin\varphi)\, d\varphi\, d\rho\, d\theta\\ &= \lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^1 ((2\rho^2\ln\rho))\, d\rho\, d\theta\\ &= \lim_{\epsilon\to 0}\int_0^{2\pi} ((2/3)\rho^3\ln\rho - 2\rho^3/9)\Big|_\epsilon^1 d\theta\\ &= \lim_{\epsilon\to 0} 2\pi\left(-\frac{2}{9} - \frac{2\epsilon^3}{3}\ln\epsilon + \frac{2\epsilon^3}{9}\right) = -4\pi/9.\end{aligned}$$

**32. (a)**

$$\begin{aligned}I(a,b) &= \int_1^a\int_1^b \frac{1}{x^2y^3}\, dy\, dx = \int_1^a -\frac{1}{2y^2x^2}\Big|_{y=1}^b dx\\ &= \int_1^a\left(\frac{1}{2} - \frac{1}{2b^2}\right)\frac{1}{x^2}dx = \left(\frac{1}{2} - \frac{1}{2b^2}\right)\left(1 - \frac{1}{a}\right)\end{aligned}$$

**(b)** As $a, b \to \infty$, $I(a,b) \to \dfrac{1}{2}$.

**33.** Let $D_{a,b} = [1, a] \times [1, b]$ and consider, for $p, q \ne 1$,

$$\begin{aligned}I(a,b) &= \iint_{D_{a,b}} \frac{1}{x^py^q}\, dA = \int_1^b\int_1^a \frac{1}{x^py^q}\, dx\, dy\\ &= \int_1^b \frac{1}{(1-p)y^qx^{p-1}}\Big|_{x=1}^a dy = \frac{1}{1-p}\left(\frac{1}{a^{p-1}} - 1\right)\int_1^b \frac{1}{y^q}\, dy\\ &= \frac{1}{(1-p)(1-q)}\left(\frac{1}{a^{p-1}} - 1\right)\left(\frac{1}{b^{q-1}} - 1\right).\end{aligned}$$

If $p > 1, q > 1$ then as $a, b \to \infty$, $I(a,b) \to \frac{1}{(1-p)(1-q)} = \frac{1}{(p-1)(q-1)}$, so the integral converges in this case. If $p < 1$, then $1/a^{p-1} \to \infty$. Similarly if $q < 1$, $1/b^{p-1} \to \infty$.

If $p = 1, q \ne 1$, then $\int_1^b\int_1^a \frac{1}{xy^q}dx\, dy = \ln a\left(\frac{1}{1-q}\right)\left(\frac{1}{b^{q-1}} - 1\right)$. This becomes infinite as $a, b \to \infty$. Similarly, if $q = 1, p \ne 1$, $I(a,b)$ becomes infinite as $a, b \to \infty$. If $p = q = 1$, then $I(a,b) = \ln a \ln b \to \infty$ as $a, b \to \infty$.

To summarize: the integral converges if and only if $p > 1$ and $q > 1$—in which case the value of the integral is $1/(p-1)(q-1)$.

**34. (a)** We use polar coordinates to make the evaluation. Let

$$\begin{aligned}I(a) &= \iint_{D_a} (1+x^2+y^2)^{-2}\, dA = \int_0^{2\pi}\int_0^a (1+r^2)^{-2} r\, dr\, d\theta\\ &= 2\pi\left(\frac{1}{2}\right)(-(1+r^2)^{-1})\Big|_{r=0}^a = -\pi\left(\frac{1}{1+a^2} - 1\right)\\ &= \pi\left(1 - \frac{1}{1+a^2}\right).\end{aligned}$$

$\lim_{a\to\infty} I(a) = \pi$. Thus the integral converges.

**(b)** Let $I(a) = \displaystyle\iint_{D_a} (1+x^2+y^2)^p dA = \int_0^{2\pi}\int_0^a (1+r^2)^p r\, dr\, d\theta$. If $p \ne -1$, then

$$I(a) = \frac{\pi}{p+1}((1+a^2)^{p+1} - 1).$$

Now $\lim_{a\to\infty}(1+a^2)^{p+1}$ is finite (and equals 0) just in case $p+1<0$ or $p<-1$. In such case, the integral converges and its value is $-\dfrac{\pi}{p+1}$. If $p=-1$, then $I(-1)=\displaystyle\int_0^{2\pi}\int_0^a \frac{r}{1+r^2}dr\,d\theta=\pi\ln(1+a^2)\to\infty$ as $a\to\infty$. So the integral converges if and only if $p<-1$.

**35.** Consider

$$I(a)=\iiint_{B_a}\frac{1}{(1+x^2+y^2+z^2)^{3/2}}\,dV=\int_0^{2\pi}\int_0^{\pi}\int_0^a \frac{\rho^2\sin\varphi}{(1+\rho^2)^{3/2}}\,d\rho\,d\varphi\,d\theta$$
$$=\int_0^a\int_0^{2\pi}\int_0^{\pi}\frac{\rho^2}{(1+\rho^2)^{3/2}}\sin\varphi\,d\varphi\,d\theta\,d\rho=\int_0^a 4\pi\cdot\frac{\rho^2}{(1+\rho^2)^{3/2}}\,d\rho$$

Now let $\rho=\tan u$ so $d\rho=\sec^2 u\,du$. Then

$$I(a)=4\pi\int_0^{\tan^{-1}a}\frac{\tan^2 u\cdot\sec^2 u\,du}{\sec^3 u}=4\pi\int_0^{\tan^{-1}a}\frac{\sin^2 u}{\cos u}du$$
$$=4\pi\int_0^{\tan^{-1}a}\frac{1-\cos^2 u}{\cos u}\,du=4\pi\int_0^{\tan^{-1}a}(\sec u-\cos u)\,du$$
$$=4\pi(\ln|\sec u+\tan u|-\sin u)\Big|_0^{\tan^{-1}a}=4\pi\left(\ln(\sqrt{a^2+1}+a)-\frac{a}{a^2+1}\right).$$

Since $\lim\limits_{a\to\infty} I(a)=\infty$ the integral does not converge.

**36.** Consider

$$I(a)=\iiint_{B_a}e^{-\sqrt{x^2+y^2+z^2}}\,dV=\int_0^{2\pi}\int_0^{\pi}\int_0^a e^{-\rho}\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$
$$=\int_0^a\int_0^{2\pi}\int_0^{\pi}e^{-\rho}\rho^2\sin\varphi\,d\varphi\,d\theta\,d\rho=4\pi\int_0^a\rho^2e^{-\rho}\,d\rho$$

Now use integration by parts twice: First let $u=\rho^2$ and $dv=e^{-\rho}d\rho$. Then

$$I(a)=4\pi\left(-\rho^2e^{-\rho}\Big|_0^a+2\int_0^a\rho e^{-\rho}d\rho\right)=-4\pi a^2e^{-a}+8\pi\int_0^a\rho e^{-\rho}d\rho.$$

Now let $u=\rho$ and $dv=e^{-\rho}d\rho$ so that

$$I(a)=-4\pi a^2e^{-a}+8\pi\left(-\rho e^{-\rho}|_0^a+\int_0^a e^{-\rho}\,d\rho\right)$$
$$=-4\pi a^2e^{-a}-8\pi ae^{-a}-8\pi e^{-a}+8\pi.$$

$\lim\limits_{a\to\infty} I(a)=8\pi$, so the integral converges and has value $8\pi$.

*The importance of Exercise 37 can not be overemphasized. The students have come from a course where they learned one technique of integration after another. They also learned some numerical methods (at least a brief introduction to Riemann sums, the trapezoid rule and Simpson's rule). In a way Exercise 37 is the payoff—it is a chance to mention*:

- *Until now they couldn't calculate $\int_{-\infty}^{\infty}e^{-x^2}dx$. The fact that you need the tools of multivariable calculus (or complex analysis) is pretty cool.*
- *They still can't calculate $\int_a^b e^{-x^2}dx$. There is a need for numerical methods to calculate a function as common as the bell curve (with a constant that stretches in the vertical direction and another constant that stretches in the horizontal direction, this is the normal curve). Many will encounter this function in a course on statistics and use the tables; they should know that this is because we can't find the definite integral over a general finite interval.*
- *The technique is pretty and unexpected and is one of the tricks that they should see some time in their mathematical training. The problem is surprisingly straightforward once someone shows you the trick.*

**37.** **(a)** $\int_{-1}^{1} e^{-x^2}\,dx$ is finite since $e^{-x^2}$ is bounded on $[-1,1]$. Since $0 \le e^{-x^2} \le 1/x^2$ on both $[1,\infty)$ and $(-\infty,-1]$ and the improper integrals $\int_1^\infty (1/x^2)\,dx$ and $\int_{-\infty}^{-1}(1/x^2)\,dx$ converge, we see that $\int_1^\infty e^{-x^2}dx$ and $\int_{-\infty}^{-1} e^{-x^2}dx$ both converge. Hence $\int_{-\infty}^{\infty} e^{-x^2}dx = \int_{-\infty}^{-1} e^{-x^2}dx + \int_{-1}^{1} e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$ converges.

**(b)** We have

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy = \iint_{\mathbf{R}^2} e^{-x^2-y^2}\,dA. \end{aligned}$$

**(c)** We'll use polar coordinates.

$$\begin{aligned} \iint_{D_a} e^{-x^2-y^2}\,dA &= \int_0^{2\pi}\int_0^a e^{-r^2}\cdot r\,dr\,d\theta = -\frac{1}{2}\int_0^{2\pi}(e^{-a^2}-1)\,d\theta \\ &= \pi(1-e^{-a^2}) \end{aligned}$$

**(d)** Note that, as $a \to \infty$, the disk $D_a$ fills out more and more of $\mathbf{R}^2$. Thus $\lim_{a\to\infty}\iint_{D_a} e^{-x^2-y^2}dA = \iint_{\mathbf{R}^2} e^{-x^2-y^2}dA = I^2$.

**(e)** Putting everything together:

$$I^2 = \lim_{a\to\infty}\pi(1-e^{-a^2}) = \pi.$$

Thus $I = \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

**38.** **(a)** First, clearly $f(x) \ge 0$. And second, by symmetry,

$$\int_{-\infty}^{\infty} e^{-2|x|}\,dx = 2\int_0^\infty e^{-2x}\,dx = -e^{-2x}\Big|_0^\infty = 1.$$

**(b)** We will reduce the calculations in Egbert's problem by recentering.

$$\begin{aligned} P(250 \le x \le 350) &= \int_{250}^{350}\frac{1}{2}e^{-|x-300|}\,dx = 2\int_{300}^{350}\frac{1}{2}e^{-(x-300)}\,dx \\ &= \int_0^{50} e^{-x}\,dx = -e^{-x}\Big|_0^{50} = 1-e^{-50}. \end{aligned}$$

**39.** **(a)** Since $\dfrac{2x+y}{140} \ge 0$ on $[0,5]\times[0,4]$, $f(x,y) \ge 0$ for all $(x,y)$. Now

$$\begin{aligned} \iint_{\mathbf{R}^2} f(x,y) &= \int_0^4\int_0^5 \frac{2x+y}{140}\,dx\,dy = \frac{1}{140}\int_0^4(25+5y)\,dy = \frac{1}{140}\left(25y+\frac{5}{2}y^2\right)\Big|_0^4 \\ &= \frac{1}{140}(100+40) = 1. \end{aligned}$$

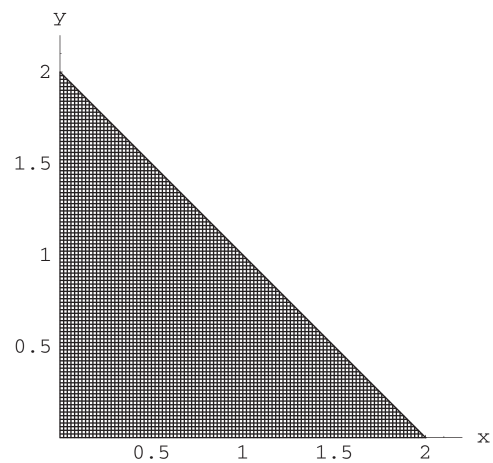
**(b)** Since $f(x,y) = 0$ for $x < 0$ or $y < 0$,

$$\begin{aligned} \text{Prob}(x \le 1, y \le 1) = \text{Prob}((x,y) \in [0,1]\times[0,1]) &= \int_0^1\int_0^1 \frac{2x+y}{140}\,dx\,dy \\ &= \frac{1}{140}\int_0^1(1+y)\,dy = \frac{1}{140}\left(1+\frac{1}{2}\right) = \frac{3}{280} \approx 0.0107. \end{aligned}$$

**40.** **(a)** Since $ye^{-x-y} \geq 0$ for $y \geq 0$ (note the exponential term is strictly positive), we have that $f(x, y) \geq 0$ for all $(x, y)$. Now we check

$$\begin{aligned} \iint_{\mathbf{R}^2} ye^{-x-y}\, dA &= \int_0^\infty \int_0^\infty ye^{-y}e^{-x}\, dx\, dy \\ &= \int_0^\infty ye^{-y}\, dy \\ &= (-ye^{-y} - e^{-y})\Big|_0^\infty = 1. \end{aligned}$$

**(b)** $\text{Prob}(x + y \leq 2) = \text{Prob}((x, y) \in D)$ where $D$ is the triangular region bounded by $x = 0, y = 0$ and $x + y = 2$.




Thus the desired probability is

$$\begin{aligned} &\int_0^2 \int_0^{2-x} ye^{-y}e^{-x}\, dy\, dx = \int_0^2 (-ye^{-y} - e^{-y})|_0^{2-x} e^{-x}\, dx \\ &= \int_0^2 ((x-2)e^{x-2} - e^{x-2} + 1)e^{-x}\, dx = \int_0^2 ((x-2)e^{-2} - e^{-2} + e^{-x})\, dx \\ &= 1 - 5e^{-2}. \end{aligned}$$

**41.** First, we know that $C \geq 0$. Second,

$$\begin{aligned} 1 &= \int_{-\infty}^\infty \int_{-\infty}^\infty Ce^{-a|x|-b|y|}\, dx\, dy = 4\int_0^\infty \int_0^\infty Ce^{-ax-by}\, dx\, dy \\ &= 4C\left[\int_0^\infty e^{-ax}\, dx\right]\left[\int_0^\infty e^{-by}\, dy\right] = 4C\left[-\frac{1}{a}e^{-ax}\Big|_0^\infty\right]\left[-\frac{1}{b}e^{-by}\Big|_0^\infty\right] = \frac{4C}{ab}. \end{aligned}$$

So $C = ab/4$.

**42.** Note that if $C \geq 0$, then $f(x, y) \geq 0$ for all $x$ since $a$ and $b$ are nonnegative. Thus we calculate

$$\begin{aligned} \iint_{\mathbf{R}^2} f(x, y)\, dA &= \int_0^1 \int_0^1 C(ax + by)\, dx\, dy = C\int_0^1 \left(\frac{1}{2}a + by\right)\, dy \\ &= C\left(\frac{1}{2}a + \frac{1}{2}b\right) = C\left(\frac{a+b}{2}\right). \end{aligned}$$

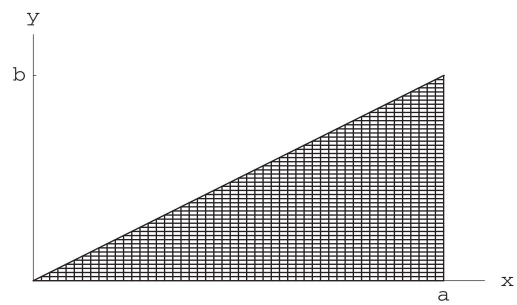
For this to equal 1, we must have $C = \dfrac{2}{a+b}$.

**43. (a)** If $C \geq 0$, then $f(x, y) \geq 0$ for all $(x, y)$. Thus we calculate

$$\iint_{\mathbf{R}^2} f(x, y)\, dA = \int_0^b \int_0^a C\, xy\, dx\, dy = C \int_0^b \frac{a^2}{2} y\, dy$$
$$= C\frac{a^2b^2}{4}.$$

For this to equal 1 we must have $C = \dfrac{4}{a^2b^2}$.

**(b)** Prob$(bx - ay \geq 0)$ is the probability that a point $(x, y)$ falls *below* the line $y = \dfrac{b}{a}x$. Since $f$ is zero outside the rectangle $[0, a] \times [0, b]$, we see that the desired probability is the *same* as the probability Prob$((x, y) \in D)$ where $D$ is the triangular region shown below.




This last probability is calculated as

$$\iint_D f dA = \int_0^a \int_0^{(b/a)x} \frac{4}{a^2b^2} xy\, dy\, dx$$
$$= \int_0^a \frac{4}{a^2b^2} x \cdot \frac{1}{2} \left(\frac{b}{a}x\right)^2 dx = \int_0^a \frac{2}{a^4} x^3\, dx$$
$$= \frac{1}{2a^4} x^4 \Big|_0^a = \frac{1}{2}.$$

**44.** We are integrating over the triangle where $0 \leq x + y \leq 60$. The integral is fairly straightforward so the details are omitted:

$$\frac{1}{250} \int_0^{60} \int_0^{60-x} e^{-x/50} e^{-y/5}\, dy\, dx = -\frac{1}{50} \int_0^{60} (e^{-x/50}[e^{-12+x/5} - 1])\, dx$$
$$= 1 - \frac{10}{9} e^{-6/5} + \frac{1}{9} e^{-12} \approx .665340.$$

**45.** We use polar coordinates:

$$\int_{-1/2}^{1/2} \int_{-\sqrt{(1/4)-x^2}}^{\sqrt{(1/4)-x^2}} \frac{1}{\pi} e^{-x^2-y^2}\, dy\, dx = \int_0^{2\pi} \int_0^{1/2} \left(\frac{1}{\pi} r e^{-r^2}\right) dr\, d\theta$$
$$= \int_0^{2\pi} \left(\frac{1}{\pi}\left(\frac{1}{2}\right)(1 - e^{-1/4})\right) d\theta$$
$$= 1 - e^{-1/4} \approx .22199.$$

**46.** The joint density function of the components is

$$F(x, y) = f(x) \cdot f(y) = \begin{cases} \dfrac{1}{(2000)^2} e^{-(x+y)/2000} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

So we want $\text{Prob}(x \le 2000, y \le 2000) = \int_0^{2000}\int_0^{2000} \frac{1}{2000^2} e^{-x/2000} e^{-y/2000}\,dx\,dy$

$$= \frac{1}{(2000)^2}\int_0^{2000} -2000e^{-x/2000}\Big|_0^{2000} e^{-y/2000}\,dy$$
$$= \frac{1}{2000}\int_0^{2000}\left(1-\frac{1}{e}\right)e^{-y/2000}\,dy = \left(1-\frac{1}{e}\right)^2.$$

**47.** Formula (8) in Section 5.6 is $I = \iiint_W d^2\delta(x,y,z)\,dV$. If we choose the coordinates so that the center of mass is at the origin, then $\iiint_W x\delta(x,y,z)\,dV = 0$ and $\iiint_W y\delta(x,y,z)\,dV = 0$. We can also choose the coordinates so that $\bar{L}$ is the $z$-axis. $L$ is a line parallel to the $z$-axis distance $h$ away, so $L$ is the line corresponding to $x = a$ and $y = b$ where $a^2 + b^2 = h^2$. Then

$$I_{\bar{L}} = \iiint_W (x^2+y^2)\delta(x,y,z)\,dV$$

and

$$I_L = \iiint_W (x^2+y^2+h^2-2ax-2by)\delta(x,y,z)\,dV = \iiint_W (x^2+y^2+h^2)\delta(x,y,z)\,dV$$

so

$$I_L - I_{\bar{L}} = \iiint_W h^2\delta(x,y,z)\,dV = h^2\iiint_W \delta(x,y,z)\,dV = h^2M.$$

**48. (a)** With $\Delta x = (b-a)/m$ and $\Delta y = (d-c)/n$, we have

$$\begin{aligned} T_{m,n} = \frac{\Delta x\Delta y}{4}\Bigg[ & f(a,c) + 2\sum_{i=1}^{m-1} f(x_i,c) + f(b,c) \\ & + 2\sum_{j=1}^{n-1} f(a,y_j) + 4\sum_{j=1}^{n-1}\sum_{i=1}^{m-1} f(x_i,y_j) + 2\sum_{j=1}^{n-1} f(b,y_j) \\ & + f(a,d) + 2\sum_{i=1}^{m-1} f(x_i,d) + f(b,d)\Bigg]. \end{aligned}$$

Now $f(x,y) = F(x)$, so the formula above becomes

$$\begin{aligned} T_{m,n} = \frac{\Delta x\Delta y}{4}\Bigg[ & F(a) + 2\sum_{i=1}^{m-1} F(x_i) + F(b) \\ & + 2\sum_{j=1}^{n-1} F(a) + 4\sum_{j=1}^{n-1}\sum_{i=1}^{m-1} F(x_i) + 2\sum_{j=1}^{n-1} F(b) \\ & + F(a) + 2\sum_{i=1}^{m-1} F(x_i) + F(b)\Bigg]. \end{aligned}$$

Note that the terms in $\sum_{j=1}^{n-1} F(a)$ do not depend on $j$; hence $\sum_{j=1}^{n-1} F(a) = (n-1)F(a)$. Similarly, we have

$\sum_{j=1}^{n-1} \sum_{i=1}^{m-1} F(x_i) = (n-1) \sum_{i=1}^{m-1} F(x_i)$. Therefore,

$$\begin{aligned}
T_{m,n} = \frac{\Delta x \Delta y}{4} \Bigg[ & F(a) + 2\sum_{i=1}^{m-1} F(x_i) + F(b) \\
& + 2(n-1)F(a) + 4(n-1)\sum_{i=1}^{m-1} F(x_i) + 2(n-1)F(b) \\
& + F(a) + 2\sum_{i=1}^{m-1} F(x_i) + F(b) \Bigg] \\
= \frac{\Delta x \Delta y}{4} & \left[ 2nF(a) + 4n\sum_{i=1}^{m-1} F(x_i) + 2nF(b) \right] \\
= \frac{\Delta x \Delta y}{4} & (2n) \left[ F(a) + 2\sum_{i=1}^{m-1} F(x_i) + F(b) \right].
\end{aligned}$$

Since $\Delta y = (d-c)/n$, we thus have

$$\begin{aligned}
T_{m,n} &= \frac{\Delta x}{2} \left( \frac{d-c}{2n} \right) (2n) \left[ F(a) + 2\sum_{i=1}^{m-1} F(x_i) + F(b) \right] \\
&= (d-c)\frac{\Delta x}{2} \left[ F(a) + 2\sum_{i=1}^{m-1} F(x_i) + F(b) \right] \\
&= (d-c)T_m,
\end{aligned}$$

using the formula for the trapezoidal rule approximation $T_m$ of the definite integral $\int_a^b F(x)\,dx$.

**(b)** We proceed in a similar manner. With $\Delta x = (b-a)/(2m)$ and $\Delta y = (d-c)/(2n)$, we have

$$\begin{aligned}
& S_{2m,2n} = \\
& \frac{\Delta x \Delta y}{9} \Bigg[ F(a) + 2\sum_{i=1}^{m-1} F(x_{2i}) + 4\sum_{i=1}^{m} F(x_{2i-1}) + F(b) \\
& + 2\sum_{j=1}^{n-1} F(a) + 4\sum_{j=1}^{n-1}\sum_{i=1}^{m-1} F(x_{2i}) + 8\sum_{j=1}^{n-1}\sum_{i=1}^{m} F(x_{2i-1}) + 2\sum_{j=1}^{n-1} F(b) \\
& + 4\sum_{j=1}^{n} F(a) + 8\sum_{j=1}^{n}\sum_{i=1}^{m-1} F(x_{2i}) + 16\sum_{j=1}^{n}\sum_{i=1}^{m} F(x_{2i-1}) + 4\sum_{j=1}^{n} F(b) \\
& + F(a) + 2\sum_{i=1}^{m-1} F(x_{2i}) + 4\sum_{i=1}^{m} F(x_{2i-1}) + F(b) \Bigg].
\end{aligned}$$

Again, we note that the terms in $\sum_{j=1}^{n-1} F(a)$ do not depend on $j$ so that $\sum_{j=1}^{n-1} F(a) = (n-1)F(a)$. In a similar manner, we have $\sum_{j=1}^{n-1}\sum_{i=1}^{m-1} F(x_{2i}) = (n-1)\sum_{i=1}^{m-1} F(x_{2i})$, etc. Thus,

$$\begin{aligned}
&S_{2m,2n} = \\
&\frac{\Delta x \Delta y}{9}\Bigg[F(a) + 2\sum_{i=1}^{m-1} F(x_{2i}) + F(b) \\
&\quad + 2(n-1)F(a) + 4(n-1)\sum_{i=1}^{m-1} F(x_{2i}) \\
&\quad + 8(n-1)\sum_{i=1}^{m} F(x_{2i-1}) + 2(n-1)F(b) \\
&\quad + 4nF(a) + 8n\sum_{i=1}^{m-1} F(x_{2i}) + 16n\sum_{i=1}^{m} F(x_{2i-1}) + 4nF(b) \\
&\quad + F(a) + 2\sum_{i=1}^{m-1} F(x_{2i}) + 4\sum_{i=1}^{m} F(x_{2i-1}) + F(b)\Bigg] \\
&= \frac{\Delta x \Delta y}{9}\left[6nF(a) + 12n\sum_{i=1}^{m-1} F(x_{2i}) + 24n\sum_{i=1}^{m} F(x_{2i-1}) + 6nF(b)\right] \\
&= \frac{\Delta x \Delta y}{9}(6n)\left[F(a) + 2\sum_{i=1}^{m-1} F(x_{2i}) + 4\sum_{i=1}^{m} F(x_{2i-1}) + F(b)\right].
\end{aligned}$$

Since $\Delta y = (d-c)/(2n)$, we have that

$$\begin{aligned}
S_{2m,2n} &= \frac{\Delta x}{3}\left(\frac{d-c}{6n}\right)(6n)\left[F(a) + 2\sum_{i=1}^{m-1} F(x_{2i}) + 4\sum_{i=1}^{m} F(x_{2i-1}) + F(b)\right] \\
&= (d-c)\frac{\Delta x}{3}\left[F(a) + 2\sum_{i=1}^{m-1} F(x_{2i}) + 4\sum_{i=1}^{m} F(x_{2i-1}) + F(b)\right] \\
&= (d-c)S_{2m},
\end{aligned}$$

using the formula for the Simpson's rule approximation $S_{2m}$ of the definite integral $\int_a^b F(x)\,dx$.

**49.** With $\Delta x = (b-a)/m$ and $\Delta y = (d-c)/n$, the trapezoidal rule approximation to $\iint_{[a,b]\times[c,d]} f(x)g(y)\,dA =$

$\int_a^b \int_c^d f(x)g(y)\,dy\,dx$ is

$$\begin{aligned}
T_{m,n} &= \frac{\Delta x \Delta y}{4}\left[f(a)g(c) + 2\sum_{i=1}^{m-1} f(x_i)g(c) + f(b)g(c)\right. \\
&\quad + 2\sum_{j=1}^{n-1} f(a)g(y_j) + 4\sum_{j=1}^{n-1}\sum_{i=1}^{m-1} f(x_i)g(y_j) + 2\sum_{j=1}^{n-1} f(b)g(y_j) \\
&\quad \left. + f(a)g(d) + 2\sum_{i=1}^{m-1} f(x_i)g(d) + f(b)g(d)\right] \\
&= \frac{\Delta x \Delta y}{4}\left[g(c)\left(f(a) + 2\sum_{i=1}^{m-1} f(x_i) + f(b)\right)\right. \\
&\quad + 2f(a)\sum_{j=1}^{n-1} g(y_j) + 4\left(\sum_{j=1}^{n-1} g(y_j)\right)\left(\sum_{i=1}^{m-1} f(x_i)\right) + 2f(b)\sum_{j=1}^{n-1} g(y_j) \\
&\quad \left. + g(d)\left(f(a) + 2\sum_{i=1}^{m-1} f(x_i) + f(b)\right)\right] \\
&= \frac{\Delta x \Delta y}{4}\left[\left(f(a) + 2\sum_{i=1}^{m-1} f(x_i) + f(b)\right)\left(g(c) + 2\sum_{j=1}^{n-1} g(y_j) + g(d)\right)\right] \\
&= T_m(f)T_n(g),
\end{aligned}$$

where $T_m(f)$ denotes the trapezoidal rule approximation to $\int_a^b f(x)\,dx$ and $T_n(g)$ the trapezoidal rule approximation to $\int_c^d f(y)\,dy$.

## Chapter 6

# Line Integrals

### 6.1 Scalar and Vector Line Integrals

**1. (a)** $\mathbf{x}'(t) = (-3, 4)$ and $\|\mathbf{x}'(t)\| = 5$ so by Definition 1.1,

$$\int_{\mathbf{x}} f\, ds = \int_0^2 (x+2y)(5)\, dt = 5\int_0^2 [(2-3t)+(8t-2)]\, dt = 5\int_0^2 5t\, dt = \frac{25}{2}t^2\Big|_0^2 = 50.$$

**(b)** $\mathbf{x}'(t) = (-\sin t, \cos t)$ and $\|\mathbf{x}'(t)\| = 1$ so by Definition 1.1,

$$\int_{\mathbf{x}} f\, ds = \int_0^{\pi} (x+2y)(1)\, dt = \int_0^{\pi} [\cos t + 2\sin t]\, dt = [\sin t - 2\cos t]|_0^{\pi} = 4.$$

*For Exercises 2–7 we will use Definition 1.1. For each calculate* $\mathbf{x}'$, $\|\mathbf{x}'\|$, *and* $f(\mathbf{x})$.

**2.**

$$\int_{\mathbf{x}} f\, ds = \int_0^2 [(t)(2t)(3t)\sqrt{1^2+2^2+3^2}]\, dt = 6\sqrt{14}\int_0^2 t^3 dt = \frac{6\sqrt{14}}{4}t^4\Big|_0^2 = 24\sqrt{14}.$$

**3.**

$$\int_{\mathbf{x}} f\, ds = \int_1^3 \left[\frac{t+t^{3/2}}{t+t^{3/2}}\sqrt{1+1+\frac{9}{4}t}\right] dt = \int_1^3 \sqrt{2+\frac{9}{4}t}\, dt = \frac{8}{27}\left(2+\frac{9}{4}t\right)^{3/2}\Bigg|_1^3$$
$$= (35\sqrt{35} - 17\sqrt{17})/27.$$

**4.**

$$\int_{\mathbf{x}} f\, ds = \sqrt{16+9}\int_0^{2\pi} (3\cos 4t + \cos 4t \sin 4t + 27t^3)\, dt = 5\int_0^{2\pi}\left(3\cos 4t + \frac{1}{2}\sin 8t + 27t^3\right) dt$$
$$= 5\left(\frac{3}{4}\sin 4t - \frac{1}{16}\cos 8t + \frac{27}{4}t^4\right)\Bigg|_0^{2\pi} = 540\pi^4.$$

**5.**

$$\int_{\mathbf{x}} f\, ds = \int_0^5 \frac{e^{2t}}{e^{4t}}\sqrt{17}e^{2t}\, dt = \int_0^5 \sqrt{17}\, dt = 5\sqrt{17}.$$

**6.**

$$\int_{\mathbf{x}} f\, ds = \int_0^1 2t\cdot 2\, dt + \int_1^2 (3t-1)\cdot 3\, dt + \int_2^3 (2t+1)\cdot 2\, dt$$
$$= 2t^2\Big|_0^1 + \left(\frac{9}{2}t^2 - 3t\right)\Bigg|_1^2 + \left(2t^2+2t\right)\Big|_2^3 = 2 + \frac{21}{2} + 12 = \frac{49}{2}.$$

**7.**

$$\int_{\mathbf{x}} f\, ds = \int_0^1 [(2t-t)\sqrt{1+4t^2}]\, dt + \int_1^3 (2-1+2t^2-4t+2)\, dt$$
$$= (5^{3/2}-1)/12 + 22/3 = (5^{3/2}+87)/12.$$

*For Exercises 8–16 we will use Definition 1.2.*

**8.**

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (2t+1, t, 3t-1) \cdot (2,1,3)\, dt = \int_0^1 (14t-1)\, dt = (7t^2 - t)\Big|_0^1 = 6.$$

**9.**

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{\pi/2} (2-\cos t, \sin t) \cdot (\cos t, \sin t)\, dt = \int_0^{\pi/2} (2\cos t - \cos^2 t + \sin^2 t)\, dt$$

$$= (2\sin t - (\sin 2t)/2)\Big|_0^{\pi/2} = 2.$$

**10.**

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (2t+1, t+2) \cdot (2,1)\, dt = \int_0^1 (5t+4)\, dt = \left(\frac{5}{2}t^2 + 4t\right)\Big|_0^1 = \frac{13}{2}.$$

**11.**

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{-1}^1 (t^3 - t^2, t^{11}) \cdot (2t, 3t^2)\, dt$$

$$= \int_{-1}^1 (2t^4 - 2t^3 + 3t^{13})\, dt = \left(\frac{2}{5}t^5 - \frac{1}{2}t^4 + \frac{3}{14}t^{14}\right)\Big|_{-1}^1 = \frac{4}{5}.$$

**12.**

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} (3\cos t, 6\cos t \sin t, 30t\cos t \sin t) \cdot (-3\sin t, 2\cos t, 5)\, dt$$

$$= \int_0^{2\pi} (-9\cos t \sin t + 12\cos^2 t \sin t + 150t\cos t \sin t)\, dt$$

$$= \int_0^{2\pi} -9\cos t \sin t\, dt + \int_0^{2\pi} 12\cos^2 t \sin t\, dt + \int_0^{2\pi} 75t\sin 2t\, dt.$$

In the first two integrals, let $w = \cos t$; in the last integrate by parts. Thus

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \left(\frac{9}{2}\cos^2 t - 4\cos^3 t - \frac{75}{2}t\cos 2t + \frac{75}{4}\sin 2t\right)\Big|_0^{2\pi}$$

$$= 0 - 0 - 75\pi + 0 = -75\pi.$$

**13.**

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (-3(t^2+t), 2t+1, 3e^{2t}) \cdot (2, 2t+1, e^t)\, dt$$

$$= \int_0^1 (-2t^2 - 2t + 1 + 3e^{3t})\, dt = \left(\tfrac{2}{3}t^3 - t^2 + t + e^{3t}\right)\Big|_0^1$$

$$= \frac{2}{3} - 1 + 1 + e^3 - 1 = \frac{3e^3 - 5}{3}.$$

**14.**

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{-1}^1 (t, 3t^2, -2t^3) \cdot (1, 6t, 6t^2)\, dt = \int_{-1}^1 (t + 18t^3 - 12t^5)\, dt = 0.$$

**15.**

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{4\pi} (t, \sin^2 t, 2t) \cdot (-\sin t, \cos t, 1/3)\, dt = \int_0^{4\pi} (-t\sin t + \sin^2 t\cos t + 2t/3)\, dt$$

$$= \left(t\cos t - \sin t + \frac{\sin^3 t}{3} + \frac{t^2}{3}\right)\Big|_0^{4\pi} = \frac{12\pi + 16\pi^2}{3}.$$

**16.**

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_0^1 (t^2\cos t^3, t\sin t^3, t^3\sin t^6)\cdot(1, 2t, 3t^2)\,dt = \int_0^1 (t^2\cos t^3 + 2t^2\sin t^3 + 3t^5\sin t^6)\,dt$$
$$= \left(\frac{\sin t^3}{3} - \frac{2\cos t^3}{3} - \frac{\cos t^6}{2}\right)\Big|_0^1 = \frac{7 - 7\cos 1 + 2\sin 1}{6}.$$

*Assign at least one of Exercises 17 and 19 so that the students are exposed to the notation before they encounter Green's theorem in the next section.*

**17.** $\int_{\mathbf{x}} x\,dy - y\,dx = \int_0^{\pi} [3(\cos 3t)^2 + 3(\sin 3t)^2]\,dt = 3\int_0^{\pi} dt = 3\pi.$

**18.** $\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_0^2 (2t, 1, 0)\cdot(1, 6t, 0)\,dt = \int_0^2 8t\,dt = 16.$

**19.** The good news is: there is a ton of cancellation.

$$\int_{\mathbf{x}} \frac{x\,dx + y\,dx}{(x^2+y^2)^{3/2}} = \int_0^{2\pi} \frac{e^{2t}\cos 3t(2e^{2t}\cos 3t - 3e^{2t}\sin 3t) + e^{2t}\sin 3t(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)}{(e^{4t}\cos^2 3t + e^{4t}\sin^2 3t)^{3/2}}\,dt$$
$$= \int_0^{2\pi} 2e^{-2t}\,dt = -e^{-2t}\Big|_0^{2\pi} = 1 - e^{-4\pi}.$$

**20.** Note that $\mathbf{x} = (t, 2\sqrt{t})$ and $\mathbf{x}' = (1, t^{-1/2})$, so

$$\int_C 3y\,ds = \int_1^9 6t^{1/2}\sqrt{1 + \frac{1}{t}}\,dt = 40\sqrt{10} - 8\sqrt{2}.$$

**21. (a)**

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_0^1 (2t^2, t^2 - t)\cdot(1, 2t)\,dt = \int_0^1 2t^3\,dt = \frac{1}{2}.$$
$$\int_{\mathbf{y}} \mathbf{F}\cdot d\mathbf{s} = \int_0^{1/2} (2 - 8t + 8t^2, 4t^2 - 2t)\cdot(-2, 8t - 4)\,dt$$
$$= \int_0^{1/2} (32t^3 - 48t^2 + 24t - 4)\,dt = -\frac{1}{2}.$$

**(b)** The path $\mathbf{y}$ is an orientation-reversing reparametrization of $\mathbf{x}$.

**22.** We write the path as $\mathbf{x}(t) = (t+1, -4t+1, 2t+1)$, $0 \le t \le 1$. This means that $\mathbf{x}'(t) = (1, -4, 2)$, therefore

$$\text{Work} = \int_C \mathbf{F}\cdot d\mathbf{s} = \int_0^1 ((1+t)^2(1-4t) - 4(1+2t) + 2(2(1+t) - (1-4t)))\,dt$$
$$= \int_0^1 (-4t^3 - 7t^2 + 2t - 1)\Big|_0^1 = -\frac{10}{3}.$$

**23.** First we organize the information we need for each of the four paths (each is for $0 \le t \le 1$).

| $i$ | $\mathbf{x}_i$ | $\mathbf{x}_i'(t)$ | $\mathbf{F}(\mathbf{x}_i(t))\cdot\mathbf{x}_i'(t)$ |
|---|---|---|---|
| 1 | $(1-2t, 1, 3)$ | $(-2, 0, 0)$ | $-2(486 - 3(1-2t))$ |
| 2 | $(-1, 1-2t, 3)$ | $(0, -2, 0)$ | $-2(-1)$ |
| 3 | $(-1+2t, -1, 3)$ | $(2, 0, 0)$ | $2(486 - 3(1-2t))$ |
| 4 | $(1, -1+2t, 3)$ | $(0, 2, 0)$ | $2(-1)$ |

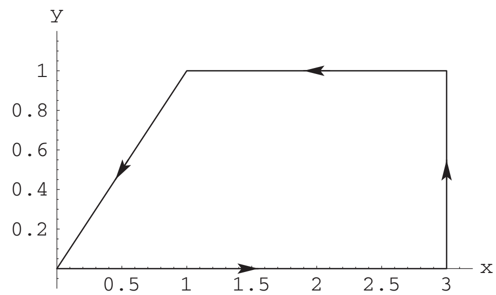
So

$$\text{Work} = \int_C \mathbf{F}\cdot d\mathbf{s} = \sum_{i=1}^{4}\int_{\mathbf{x}_i} \mathbf{F}(\mathbf{x}_i(t))\cdot\mathbf{x}_i'(t)\,dt$$
$$= \int_0^1 (-2(486 - 3(1-2t)))\,dt + \int_0^1 (2)\,dt + \int_0^1 (2(486 - 3(1-2t)))\,dt + \int_0^1 (-2)\,dt$$
$$= 0.$$

**24.** The path is $\mathbf{x}(t) = (2t+1, 4t+1)$, $0 \le t \le 1$. The integral is

$$\int_C (x^2 - y)\,dx + (x - y^2)\,dy = \int_0^1 [4t^2(2) + (-16t^2 - 6t)(4)]\,dt$$
$$= \int_0^1 (-56t^2 - 24t)\,dt = -\frac{92}{3}.$$

**25.** The curve $C$ looks like




Then $\displaystyle\int_C x^2y\,dx - (x+y)\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$

- $C_1$ is the segment from (0, 0) to (3, 0), given as $\mathbf{x}(t) = (t, 0)$, $0 \le t \le 3 \Rightarrow \mathbf{x}'(t) = (1, 0)$.
  Then $\displaystyle\int_{C_1} = \int_0^3 0\,dt - t \cdot 0 = 0.$
- $C_2$ is the segment from (3, 0) to (3, 1), given by $\mathbf{x}(t) = (3, t)$, $0 \le t \le 1 \Rightarrow \mathbf{x}'(t) = (0, 1)$.
  Then $\displaystyle\int_{C_2} = \int_0^1 0 - (3+t)\,dt = \left(-3t - \frac{1}{2}t^2\right)\Big|_0^1 = -7/2.$
- $C_3$ is the segment from (3, 1) to (1, 1), given by $\mathbf{x}(t) = (3-t, 1)$, $0 \le t \le 2 \Rightarrow \mathbf{x}'(t) = (-1, 0)$.
  Then $\displaystyle\int_{C_3} = \int_0^2 (3-t)^2(-1)\,dt - (4-t)\cdot 0 = \frac{1}{3}(3-t)^3\Big|_0^2 = \frac{1}{3}(1-27) = -\frac{26}{3}.$
- $C_4$ is the segment from (1, 1) to (0, 0), given by $\mathbf{x}(t) = (1-t, 1-t)$, $0 \le t \le 1 \Rightarrow \mathbf{x}'(t) = (-1, -1)$.

$$\int_{C_4} = \int_0^1 [(1-t)^3(-1) + (2-2t)]\,dt = \left(\frac{1}{4}(1-t)^4 + 2t - t^2\right)\Big|_0^1$$
$$= 2 - 1 - \frac{1}{4} = \frac{3}{4}.$$

So $\displaystyle\int_C = -\frac{7}{2} - \frac{26}{3} + \frac{3}{4} = -\frac{137}{12}.$

**26.** Parametrize $C$ as $\mathbf{x}(t) = (t^2, t^3)$, $-1 \le t \le 1$, so that $\mathbf{x}'(t) = (2t, 3t^2)$. Then

$$\int_C x^2y\,dx - xy\,dy = \int_{-1}^1 (t^7(2t) - t^5(3t^2))\,dt = \int_{-1}^1 (2t^8 - 3t^7)\,dt = \left(\frac{2}{9}t^9 - \frac{3}{8}t^8\right)\Big|_{-1}^1 = \frac{4}{9}.$$

**27.** Parametrize $C$ as $\mathbf{x}(t) = (3-t, (3-t)^2)$, $0 \le t \le 3$, so that the parabola is oriented correctly. Then

$$\int_C y\,dx - x\,dy = \int_0^3 \left[(3-t)^2(-1) - (3-t)(-2(3-t))\right]\,dt$$
$$= \int_0^3 (3-t)^2\,dt = -\tfrac{1}{3}(3-t)^3\Big|_0^3 = 9.$$

**28.** We parametrize $C$ in two parts: $\mathbf{x}(t) = (t, -t)$ for $-2 \le t \le 0$ and $\mathbf{x}(t) = (t, t)$ for $0 \le t \le 1$. Therefore,

$$\int_C (x-y)^2\,dx + (x+y)^2\,dy = \int_{-2}^{0} ((2t)^2 + 0)\,dt + \int_0^1 (0 + (2t)^2)\,dt$$
$$= \int_{-2}^{1} 4t^2\,dt = \frac{4}{3}t^3\Big|_{-2}^{1} = 12.$$

**29.** In order to obtain the correct direction, we parametrize $C$ as $\mathbf{x}(t) = (2\sin t, 2\cos t)$, $0 \le t \le \pi$. Then

$$\int_C xy^2\,dx - xy\,dy = \int_0^{\pi} \left[(8\sin t\cos^2 t)(2\cos t) - (4\sin t\cos t)(-2\sin t)\right]\,dt$$
$$= \int_0^{\pi} \left[16\cos^3 t\sin t + 8\sin^2 t\cos t\right]\,dt$$
$$= -4\cos^4 t + \frac{8}{3}\sin^3 t\Big|_0^{\pi} = -4 - (-4) = 0.$$

**30.** We parametrize the circle as $\mathbf{x}(t) = (4\cos t, 4\sin t)$, $0 \le t \le 2\pi$. Then the circulation is given by

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} (16\cos^2 t - 4\sin t, 16\cos t\sin t + 4\cos t)\cdot(-4\sin t, 4\cos t)\,dt$$
$$= \int_0^{2\pi} (-64\cos^2 t\sin t + 16\sin^2 t + 64\cos^2 t\sin t + 16\cos^2 t)\,dt$$
$$= \int_0^{2\pi} 16\,dt = 32\pi.$$

**31.** The path is $\mathbf{x}(t) = (4t+1, 2t+1, -t+2)$, $0 \le t \le 1$. The integral is

$$\int_C yz\,dx - xz\,dy + xy\,dz = \int_0^1 [4(-2t^2 + 3t + 2) - 2(-4t^2 + 7t + 2) - (8t^2 + 6t + 1)]\,dt$$
$$= \int_0^1 [-8t^2 - 8t + 3]\,dt = -\frac{11}{3}.$$

**32.** We must parametrize $C$. Along the cylinder we may take $x = 2\cos t$, $y = 2\sin t$. Then $z = x^2$ so we have $z = 4\cos^2 t$. The curve is traced once as $t$ varies from 0 to $2\pi$, so we have

$$\int_C z\,dx + x\,dy + y\,dz = \int_0^{2\pi} [(4\cos^2 t)(-2\sin t) + (2\cos t)(2\cos t) + (2\sin t)8\cos t(-\sin t)]\,dt$$
$$= \int_0^{2\pi} (8\cos^2 t(-\sin t) + 2(1 + \cos 2t) - 16\sin^2 t\cos t)\,dt$$
$$= \left(\frac{8}{3}\cos^3 t + 2t + \sin 2t - \frac{16}{3}\sin^3 t\right)\Big|_0^{2\pi} = 4\pi.$$

**33.** Using formula (3) in §6.1, we have

$$\int_{\mathbf{x}} \mathbf{T}\cdot d\mathbf{s} = \int_{\mathbf{x}} (\mathbf{T}\cdot\mathbf{T})\,ds = \int_{\mathbf{x}} 1\,ds = \text{length of } \mathbf{x}.$$

**34.** Of course it's left to Becky Thatcher to figure out that the path is $\mathbf{x}(t) = (5\cos t, 5\sin t)$, $0 \le t \le \pi/2$, so the area of one side of the fence is

$$\int_C (10 - x - y)\,ds = \int_0^{\pi/2} 5(10 - 5\cos t - 5\sin t)\,dt = 25[\pi - 2] \approx 28.54 \text{ ft}^2.$$

**35.** **(a)** The force that Sisyphus is applying is $50\mathbf{x}'(t)/\|\mathbf{x}'(t)\|$. The path is given as $\mathbf{x}(t) = (5\cos 3t, 5\sin 3t, 10t)$ and so $\mathbf{x}'(t) = (-15\sin 3t, 15\cos 3t, 10)$ and $\|\mathbf{x}'(t)\| = \sqrt{325}$. The total work done is

$$\int_0^{10} \frac{50\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}\cdot\mathbf{x}'(t)\,dt = \int_0^{10} 50\|\mathbf{x}'(t)\|\,dt = \int_0^{10} 50\sqrt{325}\,dt = 2500\sqrt{13} \text{ ft-lb}.$$

**(b)** This time the 75 pounds is applied straight down. The total work done is

$$\int_0^{10} (0,0,75)\cdot(-15\sin 3t, 15\cos 3t, 10)\,dt = \int_0^{10} 750\,dt = 7500 \text{ ft-lb}.$$

**36.** The force is applied in the direction of $(24, 32-14t)$. Force is applied in the opposite direction to the tension. The total work done is

$$\int_0^1 25\frac{(24,32-14t)}{\sqrt{24^2+(32-14t)^2}}\cdot(0,-14)\,dt = -(7)(25)\int_0^1 \frac{32-14t}{\sqrt{400-224t+49t^2}}\,dt = -250 \text{ ft-lb}.$$

**37.** The path is $\mathbf{x}(t) = (t, f(t))$, $a \le t \le b$, and $\mathbf{x}'(t) = (1, f'(t))$. Since $\mathbf{F} = y\mathbf{i}$,

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_a^b (f(t),0)\cdot(1,f'(t))\,dt = \int_a^b f(t)\,dt.$$

**38.** We take the sphere to be of radius $c$, so that $x^2+y^2+z^2=c^2$. Begin with the hint and take the derivative with respect to $t$ of $[x(t)]^2+[y(t)]^2+[z(t)]^2=c^2$. Divide the result by 2 to obtain: $x(t)x'(t)+y(t)y'(t)+z(t)z'(t)=0$. Now we are ready to calculate the integral.

$$\begin{aligned}\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} &= \int_a^b (x(t),y(t),z(t))\cdot(x'(t),y'(t),z'(t))\,dt\\ &= \int_a^b (x(t)x'(t)+y(t)y'(t)+z(t)z'(t))\,dt\\ &= \int_a^b 0\,dt = 0.\end{aligned}$$

**39.** If $\mathbf{x}$ is a parametrization of $C$, then formula (3) of §6.1 gives $\int_C \nabla f\cdot d\mathbf{s} = \int_{\mathbf{x}} (\nabla f\cdot\mathbf{T})\,ds$, where $\mathbf{T} = \mathbf{x}'(t)/\|\mathbf{x}'(t)\|$. But $\mathbf{T}$ is tangent to $C$ and $\nabla f$ is perpendicular to level sets of $f$ (including $C$), so $\nabla f\cdot\mathbf{T}=0$, and thus the integral must be zero.

**40. (a)** We have $\frac{ds}{dt} = v(s)$, so $dt = \frac{ds}{v(s)}$. Hence the total time for the trip is $\int dt = \int \frac{ds}{v(s)} = 2\int_0^{20}\frac{ds}{2s+20}$ (where I've used symmetry) $= \ln(2s+20)\big|_0^{20} = \ln 60 - \ln 20 = \ln 3 \approx 1.0986$ hours or 65.92 min.

**(b)** On a semicircular path you can travel at a maximum constant speed of 60 mph. You must do so for $20\pi$ miles, so the trip will take $\frac{20\pi}{60} = \pi/3 \approx 1.047$ hrs or 62.83 min.

**(c)** Traveling through the center of Cleveland (as in part (a)) will take

$$\begin{aligned}2\int_0^{20}\frac{ds}{s^2/16+25} &= 2\int_0^{20}\frac{16\,ds}{s^2+20^2} = 32\int_0^{\pi/4}\frac{20\sec^2\theta\,d\theta}{20^2\sec^2\theta}\\ &= \frac{32}{20}\int_0^{\pi/4} d\theta = \frac{8}{5}\cdot\frac{\pi}{4} = \frac{2\pi}{5} \approx 1.2566 \text{ hrs} \quad\text{or}\quad 75.40 \text{ min}.\end{aligned}$$

Going around Cleveland will take $\frac{20\pi}{50} = \frac{2\pi}{5}$—same time!

**41. (a)** Newton's second law gives $m\mathbf{a} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$. Take the dot product with $\mathbf{v}$: $m\mathbf{a}\cdot\mathbf{v} = q(\mathbf{E}\cdot\mathbf{v} + (\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}) = q\mathbf{E}\cdot\mathbf{v}$ since $\mathbf{v}\times\mathbf{B}\perp\mathbf{v}$.

**(b)**

$$\begin{aligned}\text{Work} &= \int_{\mathbf{x}} \mathbf{E}\cdot d\mathbf{s} = \int_a^b \mathbf{E}(\mathbf{x}(t))\cdot\mathbf{x}'(t)\,dt = \int_a^b \mathbf{E}(\mathbf{x}(t))\cdot\mathbf{v}(t)\,dt\\ &= \int_a^b m\mathbf{a}(t)\cdot\mathbf{v}(t)\,dt \text{ by part (a).}\end{aligned}$$

If the path has constant speed, then $\|\mathbf{v}(t)\|$ is constant. Hence $\mathbf{v}\cdot\mathbf{v}$ is constant so that $\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = 0 \Leftrightarrow 2\mathbf{a}\cdot\mathbf{v} = 0 \Leftrightarrow \mathbf{a}\cdot\mathbf{v} = 0$. Therefore the integrand of the work integral is zero.

**42. (a)** In this case all $\Delta x_k = \Delta x = \dfrac{1}{4}$, while $\Delta y_1 = \dfrac{1}{16}$, $\Delta y_2 = \dfrac{3}{16}$, $\Delta y_3 = \dfrac{5}{16}$, $\Delta y_4 = \dfrac{7}{16}$. Then

$$T_4 = \left[0^3 + 2\left(\frac{1}{16}\right)^3 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{9}{16}\right)^3 + 1^3\right]\frac{1/4}{2}$$
$$+\left(-0^2 - \left(\frac{1}{4}\right)^2\right)\frac{1/16}{2} + \left(-\left(\frac{1}{4}\right)^2 - \left(\frac{1}{2}\right)^2\right)\frac{3/16}{2} + \left(-\left(\frac{1}{2}\right)^2 - \left(\frac{3}{4}\right)^2\right)\frac{5/16}{2}$$
$$+\left(-\left(\frac{3}{4}\right)^2 - 1^2\right)\frac{7/16}{2} = -\frac{2675}{8192} \approx -0.326538.$$

**(b)** With $y = x^2$ we have $dy = 2x\,dx$ so that

$$\int_C y^3\,dx - x^2\,dy = \int_0^1 (x^2)^3\,dx - x^2(2x\,dx) = \int_0^1 (x^6 - 2x^3)\,dx$$
$$= \left(\frac{1}{7}x^7 - \frac{1}{2}x^4\right)\Big|_0^1 = -5/14 = -0.357143.$$

**43. (a)** We have $\mathbf{x}_0 = (0,0,0)$, $\mathbf{x}_1 = \left(\dfrac{1}{4}, \dfrac{1}{2}, \dfrac{3}{4}\right)$, $\mathbf{x}_2 = \left(\dfrac{1}{2}, 1, \dfrac{3}{2}\right)$, $\mathbf{x}_3 = \left(\dfrac{3}{4}, \dfrac{3}{2}, \dfrac{9}{4}\right)$, $\mathbf{x}_4 = (1,2,3)$. Then all $\Delta x_k = \dfrac{1}{4}$, $\Delta y_k = \dfrac{1}{2}$, $\Delta z_k = \dfrac{3}{4}$. Then

$$T_4 = \left(0 + 2\cdot\frac{3}{8} + 2\cdot\frac{3}{2} + 2\cdot\frac{27}{8} + 6\right)\frac{1/4}{2} + (0 + 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + 4)\frac{1/2}{2}$$
$$+\left(0 + 2\cdot\frac{1}{32} + 2\cdot\frac{1}{4} + 2\cdot\frac{27}{32} + 2\right)\frac{3/4}{2} = \frac{245}{32} = 7.65625.$$

**(b)** Parametrize $C$ as $\begin{cases} x = t \\ y = 2t, \quad 0 \le t \le 1. \\ z = 3t \end{cases}$ Then

$$\int_C yz\,dx + (x+z)\,dy + x^2y\,dz = \int_0^1 (6t^2 + 4t\cdot 2 + 2t^3\cdot 3)\,dt$$
$$= \left(2t^3 + 4t^2 + \frac{3}{2}t^4\right)\Big|_0^1 = \frac{15}{2} = 7.5.$$

**44. (a)** We have: $\Delta x_1 = 1$, $\Delta x_2 = 0$, $\Delta x_3 = 1$, $\Delta x_4 = \Delta x_5 = \Delta x_6 = \Delta x_7 = 0$, $\Delta x_8 = -1$;
$\Delta y_1 = 3$, $\Delta y_2 = 1$, $\Delta y_3 = \Delta y_4 = 0$, $\Delta y_5 = -1$, $\Delta y_6 = 0$, $\Delta y_7 = -1$, $\Delta y_8 = 0$;
$\Delta z_1 = 0$, $\Delta z_2 = 1$, $\Delta z_3 = \Delta z_4 = 1$, $\Delta z_5 = 0$, $\Delta z_6 = \Delta z_7 = -1$, $\Delta z_8 = 0$. Then

$$T_8 = (0+0)\frac{\Delta x_1}{2} + (0+1)\frac{\Delta x_2}{2} + (1+2)\frac{\Delta x_3}{2} + (2+2)\frac{\Delta x_4}{2} + (2+2)\frac{\Delta x_5}{2}$$
$$+(2+3)\frac{\Delta x_6}{2} + (3+4)\frac{\Delta x_7}{2} + (4+4)\frac{\Delta x_8}{2} + (0+1)\frac{\Delta y_1}{2}$$
$$+(1+1)\frac{\Delta y_2}{2} + (1+1)\frac{\Delta y_3}{2} + (1+2)\frac{\Delta y_4}{2} + (2+3)\frac{\Delta y_5}{2}$$
$$+(3+3)\frac{\Delta y_6}{2} + (3+3)\frac{\Delta y_7}{2} + (3+3)\frac{\Delta y_8}{2}$$
$$+(1+2)\frac{\Delta z_1}{2} + (2+2)\frac{\Delta z_2}{2} + (2+2)\frac{\Delta z_3}{2} + (2+2)\frac{\Delta z_4}{2} + (2+3)\frac{\Delta z_5}{2}$$

$$+ (3+3)\frac{\Delta z_6}{2} + (3+3)\frac{\Delta z_7}{2} + (3+4)\frac{\Delta z_8}{2}$$

$$= \frac{3}{2} + (-4) + \frac{3}{2} + 1 + \left(-\frac{5}{2}\right) + (-3) + 2 + 2 + 2 + (-3) + (-3) = -\frac{11}{2}.$$

**(b)** Now

$$\begin{array}{lll} \Delta x_1 = x_2 - x_0 = 1 & \Delta y_1 = 4 & \Delta z_1 = 1 \\ \Delta x_2 = x_4 - x_2 = 1 & \Delta y_2 = 0 & \Delta z_2 = 2 \\ \Delta x_3 = x_6 - x_4 = 0 & \Delta y_3 = -1 & \Delta z_3 = -1 \\ \Delta x_4 = x_8 - x_6 = -1 & \Delta y_4 = -1 & \Delta z_4 = -1 \end{array}$$

Then

$$\begin{aligned} T_4 &= (0+1)\frac{\Delta x_1}{2} + (1+2)\frac{\Delta x_2}{2} + (2+3)\frac{\Delta x_3}{2} + (3+4)\frac{\Delta x_4}{2} \\ &\quad + (0+1)\frac{\Delta y_1}{2} + (1+2)\frac{\Delta y_2}{2} + (2+3)\frac{\Delta y_3}{2} + (3+3)\frac{\Delta y_4}{2} \\ &\quad + (1+2)\frac{\Delta z_1}{2} + (2+2)\frac{\Delta z_2}{2} + (2+3)\frac{\Delta z_3}{2} + (3+4)\frac{\Delta z_4}{2} \\ &= \frac{1}{2} + \frac{3}{2} + \left(-\frac{7}{2}\right) + 2 + \left(-\frac{5}{2}\right) + (-3) + \frac{3}{2} + 4 + \left(-\frac{5}{2}\right) + \left(-\frac{7}{2}\right) \\ &= -\frac{11}{2}. \end{aligned}$$

## 6.2 Green's Theorem

**1.** $M(x,y) = -x^2y$ and $N(x,y) = xy^2$.

- For the line integral the path is $\mathbf{x}(t) = (2\cos t, 2\sin t)$, $0 \le t \le 2\pi$.

$$\begin{aligned} \oint_{\partial D} M\,dx + N\,dy &= \int_0^{2\pi} (-8\cos^2 t \sin t, 8\cos t \sin^2 t) \cdot (-2\sin t, 2\cos t)\,dt \\ &= 32\int_0^{2\pi} \sin^2 t \cos^2 t\,dt \\ &= (4t - \sin 4t)\Big|_0^{2\pi} = 8\pi. \end{aligned}$$

- For the area calculation, we use polar coordinates:

$$\begin{aligned} \iint_D (N_x - M_y)\,dA = \iint_D (y^2 + x^2)\,dA &= \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta \\ &= \int_0^{2\pi} 4\,d\theta = 8\pi. \end{aligned}$$

**2.** $M(x,y) = x^2 - y$ and $N(x,y) = x + y^2$.

- For the line integral, the path is split into four pieces, in each case $0 \le t \le 1$: $\mathbf{x}_1(t) = (2t, 0)$, $\mathbf{x}_2(t) = (2,t)$, $\mathbf{x}_3(t) = (2-2t, 1)$, and $\mathbf{x}_4(t) = (0, 1-t)$.

$$\begin{aligned} \oint_{\partial D} M\,dx + N\,dy &= \int_0^1 [2(4t^2) + (2+t^2) - 2(4t^2 - 8t + 3) - (t^2 - 2t + 1)]\,dt \\ &= \int_0^1 [18t - 5]\,dt = 4. \end{aligned}$$

- The area calculation is straightforward:

$$\iint_D (N_x - M_y)\,dA = \iint_D 2\,dA = \int_0^1\int_0^2 2\,dx\,dy = 4.$$

**3.** $M(x, y) = y$ and $N(x, y) = x^2$.

- For the line integral, the path is again split into four pieces, in each case $0 \le t \le 1$: $\mathbf{x}_1(t) = (1 - 2t, 1)$, $\mathbf{x}_2(t) = (-1, 1 - 2t)$, $\mathbf{x}_3(t) = (-1 + 2t, -1)$, and $\mathbf{x}_4(t) = (1, -1 + 2t)$.

$$\begin{aligned}\oint_{\partial D} M\,dx + N\,dy &= \int_0^1 [-2(1) + -2(1) + 2(-1) + 2(1)]\,dt \\ &= \int_0^1 -4\,dt = -4.\end{aligned}$$

- The area calculation is again straightforward:

$$\iint_D (N_x - M_y)\,dA = \iint_D (2x - 1)\,dA = \int_{-1}^1 \int_{-1}^1 (2x - 1)\,dx\,dy = \int_{-1}^1 -2\,dy = -4.$$

**4.** $M(x, y) = 2y$ and $N(x, y) = x$.

- For the line integral, the path is split into two pieces: $\mathbf{x}_1(t) = (a\cos t, a\sin t)$, $0 \le t \le \pi$, and $\mathbf{x}_2(t) = (-a + 2at, 0)$, $0 \le t \le 1$.

$$\begin{aligned}\oint_{\partial D} M\,dx + N\,dy &= \int_0^\pi (2a\sin t, a\cos t)\cdot(-a\sin t, a\cos t)\,dt + \int_0^1 a(0)\,dt \\ &= a^2\int_0^\pi (-2\sin^2 t + \cos^2 t)\,dt = a^2\int_0^\pi (-2 + 3\cos^2 t)\,dt = -\frac{\pi a^2}{2}.\end{aligned}$$

- We'll use polar coordinates for the area calculation:

$$\iint_D (N_x - M_y)\,dA = \iint_D (1 - 2)\,dA = \int_0^\pi \int_0^a -r\,dr\,d\theta = \int_0^\pi -\frac{a^2}{2}\,d\theta = -\frac{\pi a^2}{2}.$$

**5.** $M(x, y) = 3y$ and $N(x, y) = -4x$.

- For the line integral, the path is $\mathbf{x}(t) = (2\cos t, \sqrt{2}\sin t)$, $0 \le t \le 2\pi$.

$$\begin{aligned}\oint_{\partial D} M\,dx + N\,dy &= \int_0^{2\pi} \left[(3\sqrt{2}\sin t)(-2\sin t) - (8\cos t)(\sqrt{2}\cos t)\right]\,dt \\ &= \int_0^{2\pi} \left(-6\sqrt{2}\sin^2 t - 8\sqrt{2}\cos^2 t\right)\,dt \\ &= -2\sqrt{2}\int_0^{2\pi} \left(3\sin^2 t + 4\cos^2 t\right)\,dt = -2\sqrt{2}\int_0^{2\pi} \left(3 + \cos^2 t\right)\,dt \\ &= -2\sqrt{2}\int_0^{2\pi} \left(3 + \tfrac{1}{2}(1 + \cos 2t)\right)\,dt \\ &= -2\sqrt{2}\left(\frac{7}{2}t + \frac{1}{4}\sin 2t\right)\Big|_0^{2\pi} = -14\sqrt{2}\pi.\end{aligned}$$

- For the double integral calculation, we have:

$$\begin{aligned}\iint_D (N_x - M_y)\,dA &= \iint_D (-4 - 3)\,dA = -7\iint_D dA \\ &= -7\cdot(\text{area of } D) = -7\cdot 2\cdot\sqrt{2} = -14\sqrt{2}\pi.\end{aligned}$$

(See Example 3 in §6.2.) Alternatively, we can let $x = 2u$, $y = \sqrt{2}v$ so that the ellipse $x^2 + 2y^2 = 4$ transforms to $u^2 + v^2 = 1$. The Jacobian of this transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2}\end{bmatrix} = 2\sqrt{2}.$$

Then, using the change of variables theorem from §5.5,

$$-7\iint_D dA = -7\iint_{D^*} 2\sqrt{2}\,du\,dv = -14\sqrt{2}\cdot(\text{area of } D^*) = -14\sqrt{2}\pi,$$

since $D^*$ is just the unit disk.

**6.** $M(x,y) = x^2y + x$ and $N(x,y) = y^3 - xy^2$.

- In order to make the line integral calculation along the boundary of $D$, we need *two* parametrized paths:

$$\mathbf{x}_1(t) = (3\cos t, 3\sin t),\ 0 \le t \le 2\pi \qquad \text{and} \qquad \mathbf{x}_2(t) = (2\cos t, 2\sin t),\ 0 \le t \le 2\pi.$$

Note, however, that the path $\mathbf{x}_2$ goes counterclockwise, which is the wrong orientation for Green's theorem. We must take this into account and compute

$$\oint_{\partial D} (x^2y + x)\,dx + (y^3 - xy^2)\,dy$$
$$= \int_{\mathbf{x}_1} (x^2y + x)\,dx + (y^3 - xy^2)\,dy - \int_{\mathbf{x}_2} (x^2y + x)\,dx + (y^3 - xy^2)\,dy.$$

Thus we calculate

$$\int_{\mathbf{x}_1} (x^2y + x)\,dx + (y^3 - xy^2)\,dy$$
$$= \int_0^{2\pi} \left[(27\cos^2 t\sin t + 3\cos t)(-3\sin t) + (27\sin^3 t - 27\cos t\sin^2 t)(3\cos t)\right]\,dt$$
$$= \int_0^{2\pi} \left(-162\cos^2 t\sin^2 t - 9\cos t\sin t + 81\sin^3 t\cos t\right)\,dt$$
$$= \int_0^{2\pi} \left(-\frac{81}{2}(1+\cos 2t)(1-\cos 2t) - 9\cos t\sin t + 81\sin^3 t\cos t\right)\,dt$$
$$= \int_0^{2\pi} \left(-\frac{81}{2}(1-\cos^2 2t) - 9\cos t\sin t + 81\sin^3 t\cos t\right)\,dt$$
$$= \int_0^{2\pi} \left(-\frac{81}{2} + \frac{81}{4}(1+\cos 4t) - 9\cos t\sin t + 81\sin^3 t\cos t\right)\,dt$$
$$= \left(-\frac{81}{4}t + \frac{81}{16}\sin 4t - \frac{9}{2}\sin^2 t + \frac{81}{4}\sin^4 t\right)\Big|_0^{2\pi} = -\frac{81\pi}{2}.$$

Similarly, we have

$$\int_{\mathbf{x}_2} (x^2y + x)\,dx + (y^3 - xy^2)\,dy$$
$$= \int_0^{2\pi} \left[(8\cos^2 t\sin t + 2\cos t)(-2\sin t) + (8\sin^3 t - 8\cos t\sin^2 t)(2\cos t)\right]\,dt$$
$$= \int_0^{2\pi} \left(-32\cos^2 t\sin^2 t - 4\cos t\sin t + 16\sin^3 t\cos t\right)\,dt$$
$$= \int_0^{2\pi} \left(-8(1+\cos 2t)(1-\cos 2t) - 4\cos t\sin t + 16\sin^3 t\cos t\right)\,dt$$
$$= \int_0^{2\pi} \left(-8(1-\cos^2 2t) - 4\cos t\sin t + 16\sin^3 t\cos t\right)\,dt$$
$$= \int_0^{2\pi} \left(-8 + 4(1+\cos 4t) - 4\cos t\sin t + 16\sin^3 t\cos t\right)\,dt$$
$$= \left(-4t + \sin 4t - 2\sin^2 t + 4\sin^4 t\right)\Big|_0^{2\pi} = -8\pi.$$

Therefore,

$$\oint_{\partial D} (x^2y + x)\,dx + (y^3 - xy^2)\,dy = -\frac{81\pi}{2} + 8\pi = -\frac{65\pi}{2}.$$

- For the double integral calculation, making use of polar coordinates, we have:

$$\iint_D (N_x - M_y)\,dA = \iint_D (-y^2 - x^2)\,dA = \int_0^{2\pi}\int_2^3 -r^2 \cdot r\,dr\,d\theta$$
$$= \int_0^{2\pi} -\tfrac{1}{4}(3^4 - 2^4)\,d\theta = -\frac{\pi}{2}(81 - 16) = -\frac{65\pi}{2}.$$

**7. (a)** By Green's theorem, we have

$$\oint_C y^2\,dx + x^2\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\right] dA = \int_0^1\int_0^1 (2x - 2y)\,dx\,dy$$
$$= \int_0^1 (x^2 - 2xy)\Big|_{x=0}^1 dy = \int_0^1 (1 - 2y)\,dy = (y - y^2)\Big|_0^1 = 0.$$

**(b)** Our path is made up of four straight-line pieces with $0 \le t \le 1$ on each: $\mathbf{x}_1(t) = (t, 0)$ (so $dx = dt$, $dy = 0$), $\mathbf{x}_2(t) = (1, t)$ (so $dx = 0$, $dy = dt$), $\mathbf{x}_3(t) = (1 - t, 1)$ (so $dx = -dt$, $dy = 0$), and $\mathbf{x}_4(t) = (0, 1 - t)$ (so $dx = 0$, $dy = -dt$). Therefore,

$$\oint_C y^2\,dx + x^2\,dy = \int_{\mathbf{x}_1} + \int_{\mathbf{x}_2} + \int_{\mathbf{x}_3} + \int_{\mathbf{x}_4}$$
$$= \int_0^1 (0 + t^2(0))\,dt + \int_0^1 (t^2(0) + 1)\,dt$$
$$+ \int_0^1 (1(-1) + (1 - t)^2(0))\,dt + \int_0^1 ((1 - t)^2(0) + 0)\,dt$$
$$= 0 + 1 - 1 + 0 = 0.$$

**8.** $M(x, y) = 3xy$ and $N(x, y) + 2x^2$.

- For the line integral, the path is split into four pieces: $\mathbf{x}_1(t) = (0, -2t)$, $\mathbf{x}_2(t) = (2t, -2)$, and $\mathbf{x}_3(t) = (2, -2 + 2t)$, with $0 \le t \le 1$, and $\mathbf{x}_4(t) = (\cos t + 1, \sin t)$ with $0 \le t \le \pi$. So

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 [-2(0) + 2(-12t) + 2(8)]\,dt + \int_0^\pi [-3\sin^2 t(\cos t + 1) + 2\cos t(\cos t + 1)^2]\,dt$$
$$= \int_0^1 [-24t + 16]\,dt + \int_0^\pi [2\cos t + 4\cos^2 t + 2\cos^3 t - 3\sin^2 t - 3\cos t\sin^2 t]\,dt$$
$$= 4 + \pi/2.$$

- If $D$ is the region bounded by $C$, then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D (4x - 3x)\,dA = \iint_D x\,dA$$
$$= \int_{-2}^0\int_0^2 x\,dx\,dy + \int_0^\pi\int_0^1 r(r\cos\theta + 1)\,dr\,d\theta$$
$$= \int_{-2}^0 2\,dy + \int_0^\pi \left[\frac{1}{3}\cos\theta + \frac{1}{2}\right] d\theta = 4 + \pi/2.$$

**9.** Note that the curve is oriented clockwise so the square lies on the right side of the curve.

$$\oint_C (x^2 - y^2)\,dx + (x^2 + y^2)\,dy = -\int_0^1\int_0^1 (2x + 2y)\,dy\,dx = -\int_0^1 (2x + 1)\,dx = -2.$$

**10.** As we saw in Section 6.1, Work $= \oint_C \mathbf{F} \cdot d\mathbf{s}$. If $D$ is the ellipse $x^2 + 4y^2 = 4$ and its boundary is $C$, then by Green's theorem

$$\oint_C (4y - 3x, x - 4y) \cdot d\mathbf{s} = \iint_D (1 - 4)\, dA = \int_{-2}^{2} \int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} -3\, dy\, dx$$
$$= \int_{-2}^{2} \left[-6\sqrt{1 - x^2/4}\right] dx = -6\pi.$$

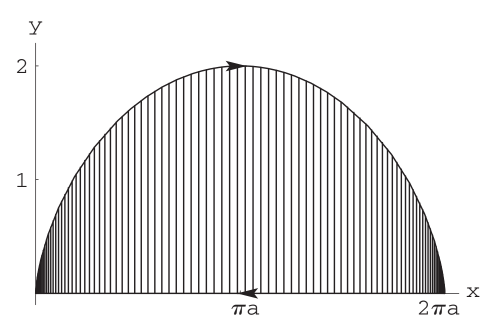
*For Exercises 11, 12, 14, 15, 16 we will calculate the area of a region using formula (1):*

$$\textit{area of } D = \frac{1}{2} \oint_{\partial D} -y\, dx + x\, dy.$$

**11.** By formula (1), the area is $\frac{1}{2} \oint_{\partial R} -y\, dx + x\, dy$. We can work this out, as in the case of Exercises 2 and 3, by enumerating the paths along the four sides and calculating the integral. We can, however, eliminate a lot of work by first noting that $dy = 0$ along both horizontal parts of the path and that $x = 0$ along the left vertical portion of the path. Also, $dx = 0$ along both vertical parts of the path and $y = 0$ along the bottom portion. So

$$\frac{1}{2} \oint_{\partial R} -y\, dx + x\, dy = \frac{1}{2} \left[\int_0^a b\, dx + \int_0^b a\, dy\right] = ab.$$

**12.** One arch of the cycloid is produced from $t = 0$ to $t = 2\pi$.



Because of the orientation shown,

$$\text{Area} = \frac{1}{2}\oint_C y\,dx - x\,dy = \frac{1}{2}\int_{C_1} + \frac{1}{2}\int_{C_2}.$$

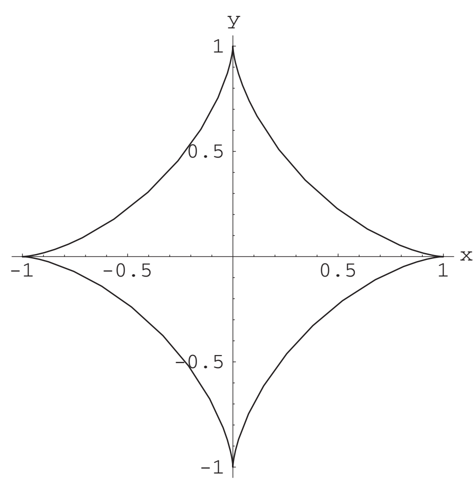
$$\begin{aligned}
\frac{1}{2}\int_{C_1} &= \frac{1}{2}\int_0^{2\pi} (a(1-\cos t)\cdot a(1-\cos t) - a(t-\sin t)\cdot a\sin t)\,dt \\
&= \frac{a^2}{2}\int_0^{2\pi} ((1-\cos t)^2 - t\sin t + \sin^2 t)\,dt = \frac{a^2}{2}\int_0^{2\pi} (1 - 2\cos t + \cos^2 t - t\sin t + \sin^2 t)\,dt \\
&= \frac{a^2}{2}\int_0^{2\pi} (2 - 2\cos t - t\sin t)\,dt = \frac{a^2}{2}(2t - 2\sin t + t\cos t - \sin t)\Big|_0^{2\pi} = \frac{a^2}{2}(4\pi + 2\pi) = 3\pi a^2.
\end{aligned}$$

**13.** By Green's theorem:

$$\begin{aligned}
\oint_C (x^4y^5 - 2y)\,dx + (3x + x^5y^4)\,dy &= -\iint_D ((3 + 5x^4y^4) - (5x^4y^4 - 2))\,dA \\
&= -\iint_D 5\,dA = -5\cdot \text{area of } D = -5(2+3+4) = -45.
\end{aligned}$$

(Note the minus sign because of the orientation of the curve.)

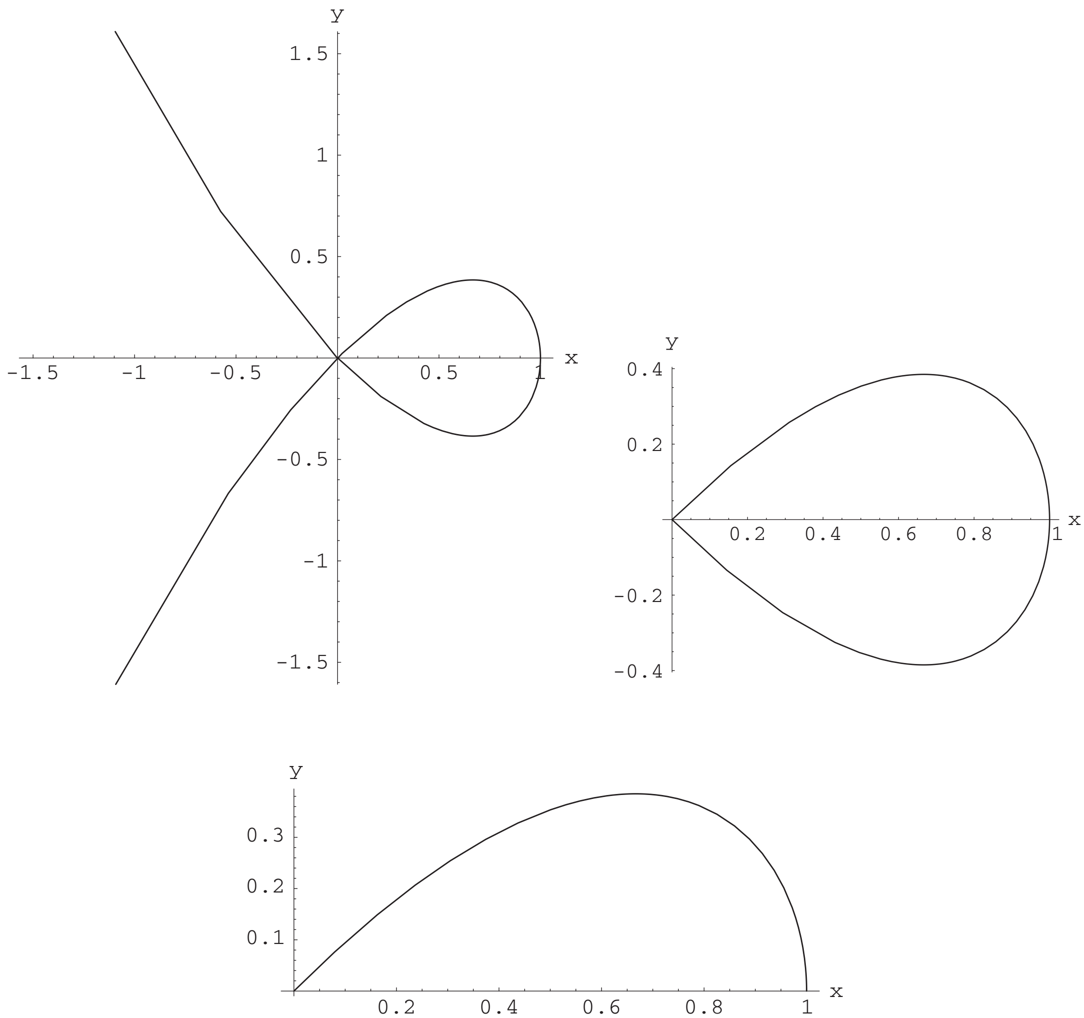
**14.** A sketch of a hypocycloid with $a = 1$ is:




Let $D$ be the interior of the hypocycloid and let $C$ be the bounding curve traced by the path $\mathbf{x}(t) = (a\cos^3 t, a\sin^3 t)$. Then by Green's theorem,

$$\begin{aligned}
\iint_D dy\,dx = \frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy &= \int_0^{2\pi} [a\sin^3 t(3a\cos^3 t\sin t) + a\cos^3 t(3a\sin^2 t\cos t)]\,dt \\
&= \frac{3a^2}{2}\int_0^{2\pi} (\cos^2 t\sin^4 t + \cos^4 t\sin^2 t)\,dt = \frac{3\pi a^2}{8}.
\end{aligned}$$

**15. (a)** Shown below are three views of the curve $\mathbf{x}(t) = (1 - t^2, t^3 - t)$.



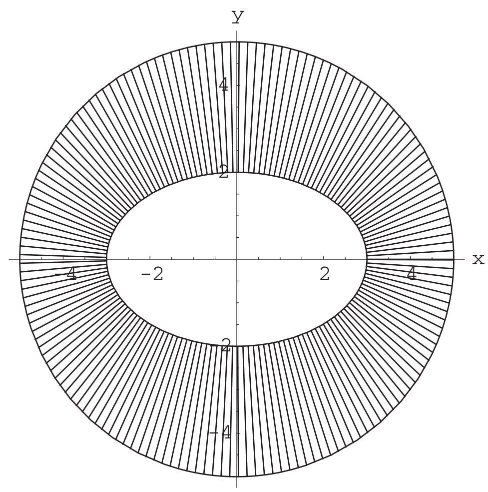

The figure on the top left is for $-3 \le t \le 3$, the top right figure is for $-1 \le t \le 1$, and the figure on the bottom is for $-1 \le t \le 0$. The first gives a feel for the curve, the second isolates the closed portion of the curve and the third gives us the orientation: that as $t$ increases from $-1$ to 1 the path moves clockwise.

**(b)** We again must make an adjustment because the path moves clockwise. The area is

$$\frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy = -\int_{-1}^{1} [(t^3 - t)(2t) + (1 - t^2)(3t^2 - 1)]\,dt$$

$$= \int_{-1}^{1} (t^4 - 2t^2 + 1)\,dt = \frac{8}{15}.$$

**16.** In this exercise, we are finding the area of the region $D$ that is outside the ellipse and inside the circle.




We need to orient the boundary curve $C$ so that the path travels counterclockwise around the circle and clockwise around the ellipse. We split this path into two pieces, each with $0 \le t \le 2\pi$, $\mathbf{x}_1(t) = (5\cos t, 5\sin t)$ and $\mathbf{x}_2(t) = (3\cos t, -2\sin t)$. By Green's theorem,

$$\begin{aligned}\iint_D dA &= \frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy \\ &= \frac{1}{2}\int_0^{2\pi} [(-5\sin t)(-5\sin t) + (5\cos t)(5\cos t)]\,dt \\ &\quad + \frac{1}{2}\int_0^{2\pi} [(2\sin t)(-3\sin t) + (3\cos t)(-2\cos t)]\,dt \\ &= \frac{1}{2}\int_0^{2\pi} 25\,dt + \frac{1}{2}\int_0^{2\pi}(-6)\,dt = 19\pi.\end{aligned}$$

**17.** It is easier if we work from the line integral to the double integral. By Green's theorem,

$$\oint_{\partial D} x\,dy = \iint_D \left(\frac{\partial x}{\partial x}\right) dA = \iint_D dA = \text{Area of } D.$$

Similarly, also by Green's theorem,

$$-\oint_{\partial D} y\,dx = -\iint_D \left(-\frac{\partial y}{\partial y}\right) dA = \iint_D dA = \text{Area of } D.$$

*Note: Assign Exercises 17 and 18 together to point out that the students cannot mix the two line integrals given in Exercise 17. The quadrilateral in Exercise 18 has one vertical side and one horizontal side so there is a temptation to use the integral with a dx in it along the vertical side and the integral with a dy in it along the horizontal side so that they disappear. You must choose one or the other for the entire problem.*

**18.** We'll use the results of Exercise 17. If we use the formula $\oint_{\partial D} x\,dy$, then for the side connecting $(1, 1)$ to $(-1, 1)$, since there is no change in $y$, this integral is 0. Therefore,

$$\text{Area of } D = \oint_C x\,dy = \int_0^1 ((2-t)(2) + 1(-1) + 0 + (-1+3t)(-1))\,dt = \int_0^1 (4-5t)\,dt = \frac{3}{2}.$$

**19.** The area inside the polygon may be computed from

$$\frac{1}{2}\oint_C -y\,dx + x\,dy.$$

The key is to parametrize the boundary $C$ of the polygon. This may be done in $n$ line segment pieces. For $k = 1, \ldots, n-1$, the line segment from $(a_k, b_k)$ to $(a_{k+1}, b_{k+1})$ is

$$\mathbf{x}_k(t) = ((a_{k+1} - a_k)t + a_k, (b_{k+1} - b_k)t + b_k), \quad 0 \le t \le 1,$$

while the last segment from $(a_n, b_n)$ to $(a_1, b_1)$ is

$$\mathbf{x}_n(t) = ((a_1 - a_n)t + a_n, (b_1 - b_n)t + b_n), \quad 0 \le t \le 1.$$

Thus, for $k = 1, \ldots, n-1$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{x}_k} - y\,dx + x\,dy \\
&\quad = \frac{1}{2}\int_0^1 [((b_k - b_{k+1})t - b_k)(a_{k+1} - a_k) + ((a_{k+1} - a_k)t + a_k)(b_{k+1} - b_k)]\,dt \\
&\quad = \frac{1}{2}\int_0^1 [(b_k - b_{k+1})(a_{k+1} - a_k)t - b_k(a_{k+1} - a_k) \\
&\qquad\qquad + (a_{k+1} - a_k)(b_{k+1} - b_k)t + a_k(b_{k+1} - b_k)]\,dt \\
&\quad = \frac{1}{2}\int_0^1 (-a_{k+1}b_k + a_k b_k + a_k b_{k+1} - a_k b_k)\,dt \\
&\quad = \frac{1}{2}\int_0^1 (-a_{k+1}b_k + a_k b_{k+1})\,dt = \frac{1}{2}(-a_{k+1}b_k + a_k b_{k+1}) = \frac{1}{2}\begin{vmatrix} a_k & b_k \\ a_{k+1} & b_{k+1} \end{vmatrix}.
\end{aligned}$$

For the last segment, the calculation is very similar, so we abbreviate the steps:

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{x}_k} - y\,dx + x\,dy \\
&\quad = \frac{1}{2}\int_0^1 [((b_n - b_1)t - b_n)(a_1 - a_n) + ((a_1 - a_n)t + a_n)(b_1 - b_n)]\,dt \\
&\quad = \frac{1}{2}\int_0^1 (-a_1 b_n + a_n b_1)\,dt = \frac{1}{2}(-a_1 b_n + a_n b_1) = \frac{1}{2}\begin{vmatrix} a_n & b_n \\ a_1 & b_1 \end{vmatrix}.
\end{aligned}$$

Adding the results of these calculations, we obtain

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\left(\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + \cdots + \begin{vmatrix} a_{n-1} & b_{n-1} \\ a_n & b_n \end{vmatrix} + \begin{vmatrix} a_n & b_n \\ a_1 & b_1 \end{vmatrix}\right),$$

as desired.

**20. (a)** Using the hint, we see that $\|\mathbf{x}(t)\|^2 = a^2$ when

$$((a+1)\cos t - \cos(a+1)t)^2 + ((a+1)\sin t - \sin(a+1)t)^2 = a^2.$$

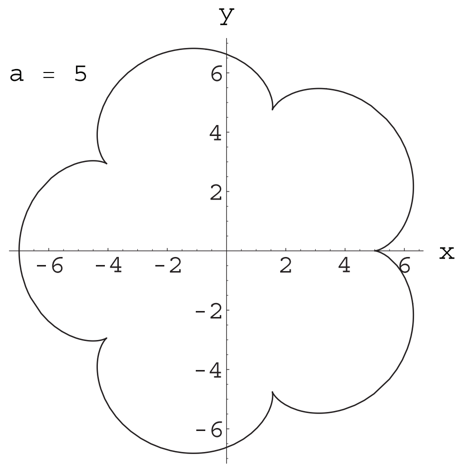
Expanding and simplifying the left side gives

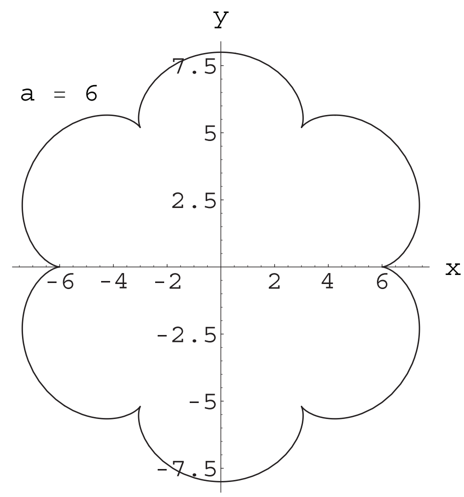
$$\begin{aligned}
(a+1)^2 + 1 - 2(a+1)\cos t\cos(a+1)t - 2(a+1)\sin t\sin(a+1)t &= a^2 \\
\iff 2a + 2 - 2(a+1)(\cos t\cos(a+1)t + \sin t\sin(a+1)t) &= 0 \\
\iff \cos t\cos(a+1)t + \sin t\sin(a+1)t &= 1.
\end{aligned}$$

Using the subtraction formula for cosine, this last equation becomes

$$\cos(a+1)t - t = 1 \iff \cos at = 1 \iff t = \frac{2\pi n}{a}.$$

The graphs for of the epicycloids for $a = 5$ and $a = 6$ are shown.




**(b)** When $a$ is an integer larger than 1, the epicycloid traces its complete image once for $t$ in $[0, 2\pi)$. To compute the enclosed area, we use the line integral $\frac{1}{2}\oint_C -y\,dx + x\,dy$. Thus the area is

$$\begin{aligned}
&\frac{1}{2}\int_0^{2\pi} [((a+1)\sin t - \sin(a+1)t)\,(-(a+1)\sin t + (a+1)\sin(a+1)t) \\
&\qquad + ((a+1)\cos t - \cos(a+1)t)\,((a+1)\cos t - (a+1)\cos(a+1)t)]\ dt \\
&= \frac{1}{2}\int_0^{2\pi} \left[(a+1)^2 - \left((a+1) + (a+1)^2\right)(\cos(a+1)t\cos t + \sin(a+1)t\sin t) \right. \\
&\qquad \left. + (a+1)\right]\ dt \\
&= \frac{(a+1)}{2}\int_0^{2\pi} [(a+1) - (a+2)(\cos(a+1)t\cos t + \sin(a+1)t\sin t) + 1]\ dt \\
&= \frac{(a+1)(a+2)}{2}\int_0^{2\pi} [1 - (\cos(a+1)t\cos t + \sin(a+1)t\sin t)]\ dt
\end{aligned}$$

after expansion and some simplification. Next we use the subtraction formula for cosine:

$$\begin{aligned}
\text{Area} &= \frac{(a+1)(a+2)}{2}\int_0^{2\pi} (1 - \cos at)\ dt \\
&= \frac{(a+1)(a+2)}{2}\left(t - \frac{1}{a}\sin at\right)\Big|_0^{2\pi} = \pi(a+1)(a+2).
\end{aligned}$$

(Note that we used the fact that $a$ is an integer when evaluating the final integral.)

**(c)** The area of the fixed circle is $\pi a^2$. Thus

$$\lim_{a\to\infty} \frac{\pi(a+1)(a+2)}{\pi a^2} = \lim_{a\to\infty}\left(1 + \frac{1}{a}\right)\left(1 + \frac{2}{a}\right) = 1.$$

Hence, in the limit, the epicycloid's area approaches that of the fixed circle.

**21.** By Green's theorem,

$$\oint_C 5y\,dx - 3x\,dy = \iint_D (-3-5)\,dA = \iint_D -8\,dA = -8(\text{area of } D),$$

where $D$ is the region in the plane enclosed by the cardioid. We may evaluate the double integral using polar coordinates.

$$\begin{aligned}\iint_D -8\,dA &= -8\int_0^{2\pi}\int_0^{1-\sin\theta} r\,dr\,d\theta \\ &= -8\int_0^{2\pi} \tfrac{1}{2}r^2\Big|_{r=0}^{r=1-\sin\theta} d\theta = -4\int_0^{2\pi}(1-\sin\theta)^2\,d\theta \\ &= -4\int_0^{2\pi}(1-2\sin\theta+\sin^2\theta)\,d\theta = -4\int_0^{2\pi}\left(1-2\sin\theta+\tfrac{1}{2}(1-\cos 2\theta)\right)d\theta \\ &= -4\left(\tfrac{1}{2}\theta+2\cos\theta-\tfrac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} = -12\pi.\end{aligned}$$

**22. (a)** Note that we have

$$\frac{\partial}{\partial x}\frac{y}{x^2+y^2} = -\frac{2xy}{x^2+y^2} = \frac{\partial}{\partial y}\frac{x}{x^2+y^2}.$$

Therefore, Green's theorem implies that, for the region $D$ enclosed by $C$

$$\oint_C \frac{x\,dx+y\,dy}{x^2+y^2} = \pm\iint_D\left(-\frac{2xy}{x^2+y^2}-\left(-\frac{2xy}{x^2+y^2}\right)\right)dA = \pm\iint_D 0\,dA = 0.$$

(The $\pm$ sign is due to the fact that we do not know the orientation of $C$, not that it ultimately matters.)

**(b)** Green's theorem does *not* apply since $M$ and $N$ are not defined at the origin, which is in the region $D$ enclosed by $C$.

**(b)** If $C_1$ and $C_2$ both enclose the origin and don't cross or touch, then one of the curves must lie entirely inside the other. Let's assume that $C_2$ lies inside $C_1$. Together, $C_1$ and $C_2$ make up the boundary of a region $D$ that does *not* contain the origin. Thus we may apply Green's theorem to $D$ and its boundary:

$$\begin{aligned}0 &= \iint_D\left(\frac{\partial}{\partial x}\frac{y}{x^2+y^2}-\frac{\partial}{\partial y}\frac{x}{x^2+y^2}\right)dA = \oint_{\partial D}\frac{x\,dx+y\,dy}{x^2+y^2} \\ &= \oint_{C_1}\frac{x\,dx+y\,dy}{x^2+y^2} - \oint_{C_2}\frac{x\,dx+y\,dy}{x^2+y^2}.\end{aligned}$$

Hence the desired result follows. Note that the orientation of $\partial D$ requires a counterclockwise orientation on the outer curve, but a *clockwise* orientation on the inner curve.

**(c)** Find a circle $C'$ of some small radius $a$ so that $C'$ lies entirely inside $C$ and is oriented the same way that $C$ is. Then, from part (c), we know that

$$\oint_C\frac{x\,dx+y\,dy}{x^2+y^2} = \oint_{C'}\frac{x\,dx+y\,dy}{x^2+y^2}.$$

We may evaluate this last integral using the parametrization $\mathbf{x}(t) = (a\cos t, a\sin t)$, $0\le t\le 2\pi$. Thus

$$\oint_{C'}\frac{x\,dx+y\,dy}{x^2+y^2} = \pm\int_0^{2\pi}\frac{(a\cos t)(-a\sin t)+(a\sin t)(a\cos t)}{a^2} = \pm\int_0^{2\pi}0\,dt = 0.$$

(Once again the $\pm$ sign is due to the fact that we do not know the actual orientation of $C'$.)

**23. (a)** By the divergence theorem:

$$\oint_C(2y\mathbf{i}-3x\mathbf{j})\cdot\mathbf{n}\,ds = \iint_D[(2y)_x+(-3x)_y]\,dA = \iint_D 0\,dA = 0.$$

**(b)** For direct computation, $\mathbf{n} = (\cos\theta, \sin\theta)$ and $x = \cos\theta$ and $y = \sin\theta$. Therefore,

$$\mathbf{F}\cdot\mathbf{n} = (2y, -3x)\cdot(\cos\theta,\sin\theta) = 2\cos\theta\sin\theta - 3\cos\theta\sin\theta = -\cos\theta\sin\theta = -\frac{1}{2}\sin 2\theta.$$

Then

$$\oint_C\mathbf{F}\cdot\mathbf{n}\,ds = -\frac{1}{2}\int_0^{2\pi}\sin 2\theta\,d\theta = 0.$$

**24.** Similar to what was done in the proof of the divergence theorem, we will calculate the line integral $\oint_{\partial D} \mathbf{F} \cdot \mathbf{T}\, ds$ along a $C^1$ segment of $\partial D$. Recall that $\mathbf{T}(t) = \mathbf{x}'(t)/\|\mathbf{x}'(t)\|$.

$$\int_a^b (\mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{T}(t))\, \|\mathbf{x}'(t)\|\, dt = \int_a^b (\mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t))\, dt$$
$$= \int_a^b ((M(x(t), y(t))\, x'(t) + N(x(t), y(t))\, y'(t))\, dt = \int_{\mathbf{x}} M\, dx + N\, dy.$$

We extend this result to the entire curve and apply Green's theorem.

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{T}\, ds = \oint_{\partial D} M\, dx + N\, dy = \iint_D (N_x - M_y)\, dA.$$

**25.** By Green's Theorem, if $D$ is the region bounded by $C$,

$$\oint_C 3x^2 y\, dx + x^3\, dy = \iint_D (3x^2 - 3x^2)\, dA = 0.$$

(Note that in this case the orientation of $C$ is not important as $-(3x^2 - 3x^2) = 3x^2 - 3x^2$.)

**26.** If $C$ is oriented as required and $D$ is the region bounded by $C$, then by Green's Theorem,

$$\oint_C -y^3\, dx + (x^3 + 2x + y)\, dy = \iint_D (3x^2 + 2 + 3y^2)\, dA > 0.$$

**27.** Let $\delta = 1$ if $C$ is oriented counterclockwise and $\delta = -1$ if $C$ is oriented clockwise. Let $D$ be the region bounded by $C$. Then by Green's Theorem,

$$\oint_C (x^2y^3 - 3y)\, dx + x^3y^2\, dy = \delta \iint_D (3x^2y^2 - 3x^2y^2 + 3)\, dA = 3\delta \text{ (the area of the rectangle).}$$

**28.**

$$\text{Flux} = \oint_C (\mathbf{r} \cdot \mathbf{n})\, ds = \int_a^b (x\mathbf{i} + y\mathbf{j}) \cdot \frac{y'\mathbf{i} - x'\mathbf{j}}{\sqrt{x'^2 + y'^2}} \sqrt{x'^2 + y'^2}\, dt$$
$$= \int_a^b \left( x\frac{dy}{dt} - y\frac{dx}{dt} \right) dt = \int_C -y\, dx + x\, dy$$
$$= \iint_D (1 - (-1))\, dA \quad \text{by Green's theorem}$$
$$= \iint_D 2\, dA = 2 \cdot \text{(area inside } C).$$

**29.** We have $u\, \nabla v = \left( u\frac{\partial v}{\partial x}, u\frac{\partial v}{\partial y} \right)$ so that

$$\oint_C (u\, \nabla v) \cdot d\mathbf{s} = \oint_C u\frac{\partial v}{\partial x}\, dx + u\frac{\partial v}{\partial y}\, dy$$
$$= \iint_D \left( \frac{\partial}{\partial x}\left( u\frac{\partial v}{\partial y} \right) - \frac{\partial}{\partial y}\left( u\frac{\partial v}{\partial x} \right) \right) dA \quad \text{by Green's theorem}$$
$$= \iint_D \left( \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial x\, \partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - u\frac{\partial^2 v}{\partial y\, \partial x} \right) dA$$
$$= \iint_D \left( \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \right) dA \quad \text{since } v \text{ is of class } C^2$$
$$= \iint_D \frac{\partial(u, v)}{\partial(x, y)}\, dA.$$

**30.** Let $D$ be the region bounded by $C$. By Green's theorem,

$$\oint_C \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dx = -\iint_D \left(\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial x^2}\right) dA = -\iint_D 0\,dA = 0.$$

**31.** First, $\oint_{\partial D} \frac{\partial f}{\partial n}\,ds = \oint_{\partial D} \nabla f \cdot \mathbf{n}\,ds = \oint_{\partial D} \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \cdot \mathbf{n}\,ds$. You can continue the calculation or note that this is the same computation done in the proof of the divergence theorem with $\mathbf{F} = M\mathbf{i} + N\mathbf{j} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$. Therefore, applying Green's theorem,

$$\oint_{\partial D} \frac{\partial f}{\partial n}\,ds = \oint_{\partial D} \frac{\partial f}{\partial y}\,dx + \frac{\partial f}{\partial x}\,dy = \iint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dA = \iint_D \nabla^2 f\,dA.$$

## 6.3 Conservative Vector Fields

**1. (a)** Let $C$ be the path parametrized by $\mathbf{x}(t) = (t, t, t)$ with $0 \le t \le 1$. Then

$$\int_C z^2\,dx + 2y\,dy + xz\,dz = \int_0^1 (t^2 + 2t + t^2)\,dt = 2\int_0^1 (t^2 + t)\,dt = \frac{5}{3}.$$

**(b)** Let $\mathbf{x}(t) = (t, t^2, t^3)$ with $0 \le t \le 1$. Then

$$\int_C z^2\,dx + 2y\,dy + xz\,dz = \int_0^1 (t^6 + 2t^2(2t) + t^4(3t^2))\,dt = 4\int_0^1 (t^6 + t^3)\,dt = \frac{11}{7}.$$

**(c)** Parts (a) and (b) show that line integrals are not path-independent. By Theorem 3.3, therefore, **F** is not conservative.

**2. (a)** Let $C$ be the path parameterized by $\mathbf{x}(t) = (t^2, t^3, t^5)$ with $0 \le t \le 1$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (2t^5, t^4 + t^{10}, 2t^8) \cdot (2t, 3t^2, 5t^4)\,dt = \int_0^1 (7t^6 + 13t^{12})\,dt = 2.$$

**(b)** Let $C$ be comprised of the two paths: $\mathbf{x}_1(t) = (t, 0, 0)$ and $\mathbf{x}_2(t) = (1, t, t)$ each with $0 \le t \le 1$. The integral along $\mathbf{x}_1$ is easily seen to be zero ($y$, $z$, and $dx$ are all identically zero along $\mathbf{x}_1$). We have that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (2t, 1 + t^2, 2t) \cdot (0, 1, 1)\,dt = 2.$$

**(c)** Obviously the fact that our answers to parts (a) and (b) are the same is not enough to convince us that **F** is conservative. We can, however, easily see that $\mathbf{F} = \nabla\,(x^2y + yz^2)$ so **F** is conservative.

*In Exercises 3–9, we will check to see whether $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is conservative by checking to see whether $\partial N/\partial x = \partial M/\partial y$ (formula (1)).*

**3.** $\frac{\partial N}{\partial x} = ye^{xy} \neq e^{x+y} = \frac{\partial M}{\partial y}$, so **F** is not conservative.

**4.** $\frac{\partial N}{\partial x} = 2x\cos y = \frac{\partial M}{\partial y}$, so **F** is conservative. We want to find $f$ where $\mathbf{F} = \nabla f(x, y)$. We find that the indefinite integral of $2x\sin y$ with respect to $x$ is $x^2\sin y$. To see whether any adjustments need to be made, we check to make certain that $\frac{\partial}{\partial y}(x^2\sin y) = x^2\cos y$. It does, so we conclude that $f(x, y) = \nabla(x^2\sin y)$.

**5.** $\frac{\partial N}{\partial x} = 3x^2\sin y + \frac{1 - x^2y^2}{(1 + x^2y^2)^2} \neq -3x^2\sin y + \frac{1 - x^2y^2}{(1 + x^2y^2)^2} = \frac{\partial M}{\partial y}$, so **F** is not conservative.

**6.** $\frac{\partial N}{\partial x} = \frac{2xy}{(1 + x^2)^2} = \frac{\partial M}{\partial y}$, so **F** is conservative. $\mathbf{F} = \nabla\left(\frac{x^2y^2}{2(1 + x^2)}\right)$.

**7.** Note that $\frac{\partial}{\partial y}(e^{-y} - y\sin(xy)) = -e^{-y} - \sin xy - xy\cos xy = \frac{\partial}{\partial x}(-xe^{-y} - x\sin xy)$. Since the domain of **F** is all of $\mathbf{R}^2$, the vector field is conservative. Thus $\mathbf{F} = \nabla f$, so $\frac{\partial f}{\partial x} = e^{-y} - y\sin xy \Rightarrow f(x, y) = xe^{-y} + \cos xy + g(y)$ for some $g$. Hence $\frac{\partial f}{\partial y} = -xe^{-y} - x\sin xy + g'(y) = -xe^{-y} - x\sin xy$ so $g'(y) = 0$. Thus $f(x, y) = xe^{-y} + \cos xy + C$ is a potential for any $C$.

**8.** $\dfrac{\partial N}{\partial x} = 12xy - y \neq 12xy + 6y = \dfrac{\partial M}{\partial y}$, so **F** is *not* conservative.

**9.** $\dfrac{\partial N}{\partial x} = 12xy = \dfrac{\partial M}{\partial y}$, so **F** is conservative. $\mathbf{F} = \nabla\left(3x^2y^2 - x^3 + \frac{1}{3}y^3\right)$.

*In Exercises 10–18, we will check to see whether $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative by checking whether $\nabla \times \mathbf{F} = \mathbf{0}$. This amounts to checking whether $\partial N/\partial x = \partial M/\partial y, \partial P/\partial x = \partial M/\partial z$, and $\partial P/\partial y = \partial N/\partial z$. We also need to check that the domain of **F** is simply-connected. This last condition is only an issue in Exercise 16.*

**10.** $\nabla \times \mathbf{F} = (6x^2 - 2yz - 6x^2z^2 - 2y)\,\mathbf{i} + (3xyz^2 - 2z - 12xy)\,\mathbf{j} + (3xz^3 - x)\,\mathbf{i} \neq \mathbf{0}$. Hence **F** is *not* conservative.

**11.** We see that $\partial N/\partial x = 4xz^3 - 2x = \partial M/\partial y$, $\partial P/\partial x = 12xyz^2 = \partial M/\partial z$, and $\partial P/\partial y = 6x^2z^2 + 2y = \partial N/\partial z$. Thus **F** is conservative. $\mathbf{F} = \nabla\left(2x^2yz^3 - x^2y + y^2z\right)$.

**12.** $\nabla \times \mathbf{F} = (2xe^{xyz} + 2x^2yze^{xyz})\mathbf{i} - (2ze^{xyz} + 2xyz^2e^{xyz})\mathbf{k} \neq \mathbf{0}$. Hence **F** is *not* conservative.

**13.** We see that $\partial N/\partial x = 1 = \partial M/\partial y$, $\partial P/\partial x = 0 = \partial M/\partial z$, and $\partial P/\partial y = \cos yz - yz\sin yz = \partial N/\partial z$. So **F** is conservative. $\mathbf{F} = \nabla(x^2 + xy + \sin yz)$.

**14.** Here, $\partial N/\partial x = 0 \neq 1 = \partial M/\partial y$, so **F** is not conservative.

**15.** We see that $\partial N/\partial x = e^x\cos y = \partial M/\partial y, \partial P/\partial x = 0 = \partial M/\partial z$, and $\partial P/\partial y = 0 = \partial N/\partial z$. So **F** is conservative. $\mathbf{F} = \nabla(e^x\sin y + z^3 + 2z)$.

**16.** We see that $\partial N/\partial x = 0 = \partial M/\partial y, \partial P/\partial x = 0 = \partial M/\partial z$, and $\partial P/\partial y = 2z/y = \partial N/\partial z$. So **F** is conservative in each of the simply-connected regions on which it is defined: $\{(x,y,z)|y>0\}$ and $\{(x,y,z)|y<0\}$. On each, $\mathbf{F} = \nabla(x^3 + z^2\ln|y|)$.

**17.** We see that $\partial N/\partial x = ze^{-yz} + e^{xyz}(xyz^2 + z) = -\partial M/\partial y$, so **F** is not conservative.

**18.** We see that, for **G**, $\partial N/\partial x = 2x = \partial M/\partial y, \partial P/\partial x = 0 = \partial M/\partial z$, and $\partial P/\partial y = 2y = \partial N/\partial z$. So $\mathbf{G} = (2xy, x^2 + 2yz, y^2)$ is conservative and $\mathbf{G} = \nabla(x^2y + y^2z)$. We know, therefore, that **F** is not conservative because the wording of the problem assured us that exactly one of **F** and **G** was conservative. It may be more satisfying to verify that for **F**, $\partial M/\partial y = 2xyz^3$ while $\partial N/\partial x = 4xy$. These are different so **F** is not conservative.

**19.** **(a)** We have, for $i = 1, \ldots, n$, that $f_{x_i}(\mathbf{x}) = 0$ for all $\mathbf{x}$ in the domain of $f$. Taking these results one at a time, we have

$$f_{x_1}(\mathbf{x}) = 0 \Longrightarrow f \text{ is a function of } x_2, \ldots, x_n \text{ only.}$$
$$f_{x_2}(\mathbf{x}) = 0 \Longrightarrow \text{in addition } f \text{ is a function of } x_3, \ldots, x_n \text{ only.}$$
$$\vdots$$

Continuing in this way, we see that $f$ must be independent of all variables, and so must be a constant function.

**(b)** We have $\nabla g = \nabla h = \mathbf{F}$. Consider $f = g - h$. Then

$$\nabla f = \nabla g - \nabla h = \mathbf{F} - \mathbf{F} = \mathbf{0}.$$

Therefore, by part (a), $f = g - h$ is constant.

**20.** For **F** to be conservative, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x\sin y - y\cos x) = \sin y + y\sin x.$$

Thus

$$M(x,y) = -\cos y + \tfrac{1}{2}y^2\sin x + u(x),$$

where $u$ is any $C^1$ function of $x$.

**21.** For **F** to be conservative, we must have $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial y}(ye^{2x} + 3x^2e^y) = e^{2x} + 3x^2e^y$. Thus $N(x,y) = \dfrac{1}{2}e^{2x} + x^3e^y + u(y)$ where $u$ is any $C^1$ function of $y$.

**22.** Note that the constant function $g(x) = 0$ is a trivial solution. Otherwise, we must have

$$\frac{\partial}{\partial y}\left[(xe^x + y^2)g(x)\right] = \frac{\partial}{\partial x}\left[xyg(x)\right].$$

Thus means that

$$2yg(x) = yg(x) + xyg'(x) \iff \frac{g'(x)}{g(x)} = \frac{1}{x}.$$

Integrating this last equation, we have

$$\ln|g(x)| = \ln|x| + C \qquad \text{or} \qquad \ln\left|\frac{g(x)}{x}\right| = C.$$

Exponentiating, we have

$$\left|\frac{g(x)}{x}\right| = k,$$

where $k = e^C$. Thus $g(x) = \pm kx$. If we allow $k$ to be completely arbitrary (i.e., positive, negative, or zero), then we may simply say $g(x) = kx$ for *any* constant $k$ gives a solution.

**23.** For $\mathbf{F}$ to be conservative, we must have $\nabla \times \mathbf{F} = \mathbf{0}$. Thus we demand

$$\mathbf{0} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3y - 3x^2z & N(x,y,z) & 2yz - x^3 \end{vmatrix}$$

$$= (2z - N_z)\,\mathbf{i} + (-3x^2 + 3x^2)\,\mathbf{j} + (N_x - x^3)\,\mathbf{k}.$$

From this, we see that $N$ must satisfy $\partial N/\partial x = x^3$ and $\partial N/\partial z = 2z$. The first equation implies that $N(x,y,z) = \frac{1}{4}x^4 + g(y,z)$, and so $2z = \partial N/\partial z = \partial g/\partial z$, which in turn implies that $g(y,z) = z^2 + h(y)$. From here it is easy to check that the curl condition above is satisfied when $N(x,y,z) = \frac{1}{4}x^4 + z^2 + h(y)$, where $h$ is any function of class $C^1$ defined on a simply-connected domain.

**24.** For $\mathbf{F}$ to be conservative, we must have $\nabla \times \mathbf{F} = \mathbf{0}$. Thus we impose

$$\mathbf{0} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2 + 3y^2z\sin xz & ay\cos xz + bz & 3xy^2\sin xz + 5y \end{vmatrix}$$

$$= (6xy\sin xz + 5 + axy\sin xz - b)\mathbf{i}$$
$$+ (3y^2\sin xz + 3xy^2z\cos xz - 3y^2\sin xz - 3xy^2z\cos xz)\mathbf{j}$$
$$+ (-ayz\sin xz - 6yz\sin xz)\mathbf{k}.$$

From this, it is easy to see that only the choices $a = -6$, $b = 5$ will work. Moreover, the resulting vector field is clearly defined on all of $\mathbf{R}^3$ (a simply-connected region), so the vanishing of the curl is enough to guarantee that $\mathbf{F}$ is conservative.

**25.** **(a)** As above we check that $\partial N/\partial x = 0 = \partial M/\partial y$, $\partial P/\partial x = 0 = \partial M/\partial z$, and $\partial P/\partial y = \cos y\cos z = \partial N/\partial z$. So $\mathbf{F}$ is conservative. $\mathbf{F} = \nabla(x^3/3 + \sin y\sin z)$.

**(b)** By Theorem 3.3,

$$\int_x \mathbf{F}\cdot d\mathbf{s} = f(\mathbf{x}(1)) - f(\mathbf{x}(0)) = f(2, e, e^2) - f(1,1,1) = 7/3 + \sin e\sin(e^2) - \sin^2 1.$$

**26.** Since $(7y - 5x)_x = -5 = (3x - 5y)_y$, $\mathbf{F} = (3x - 5y)\mathbf{i} + (7y - 5x)\mathbf{j}$ is conservative and the integral is path independent. We'll integrate along the path $\mathbf{x}(t) = (4t + 1, -t + 3)$ for $0 \le t \le 1$.

$$\int_C (3x - 5y)\,dx + (7y - 5x)\,dy = \int_0^1 [4(3(4t+1) - 5(-t+3)) - ((7(-t+3) - 5(4t+1))]\,dt$$
$$= \int_0^1 (95t - 64)\,dt = -\frac{33}{2}.$$

Using Theorem 3.3,

$$\int_C (3x - 5y)\,dx + (7y - 5x)\,dy = f(5,2) - f(1,3), \quad \text{where } \mathbf{F} = \nabla f.$$

In this case, $f(x,y) = 3x^2/2 - 5xy + 7y^2/2$. Therefore,

$$\int_C (3x - 5y)\,dx + (7y - 5x)\,dy = \left(\frac{3}{2}(25) - 5(5)(2) + \frac{7}{2}(4)\right) - \left(\frac{3}{2}(1) - 5(1)(3) + \frac{7}{2}(9)\right) = -\frac{33}{2}.$$

**27.** Here

$$\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = -\frac{xy}{(x^2+y^2)^{3/2}} = \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right).$$

So **F** is conservative so long as we restrict the domain. Our domain must be simply-connected and must contain the upper half of the circle of radius 2 centered at the origin. Our domain also must not contain the origin as **F** is not defined at the origin. We can choose, for example, the upper half disk of radius 3 centered at the origin minus the upper half disk of radius one centered at the origin. This "semi-annular" region meets all of our conditions. Therefore, the given integral is path independent. We'll integrate along the path $\mathbf{x}(t) = (2\cos t, 2\sin t)$, $0 \le t \le \pi$. The integral

$$\int_C \frac{x\,dy + y\,dx}{\sqrt{x^2+y^2}} = \int_0^\pi \frac{4\cos^2 t - 4\sin^2 t}{4\cos^2 t + 4\sin^2 t}\,dt = \int_0^\pi \cos 2t\,dt = \left.\frac{\sin 2t}{2}\right|_0^\pi = 0.$$

Using Theorem 3.3, and the fact that $\mathbf{F} = \nabla f$ where $f(x,y) = \sqrt{x^2+y^2}$,

$$\int_C \frac{x\,dy + y\,dx}{\sqrt{x^2+y^2}} = f(-2,0) - f(2,0) = \sqrt{(-2)^2+0} - \sqrt{2^2+0} = 0.$$

**28.** This time we check that three pairs of partial derivatives are equal:

$$\frac{\partial}{\partial x}(2x+z) = 2 = \frac{\partial}{\partial y}(2y-3z)$$
$$\frac{\partial}{\partial x}(y-3x) = -3 = \frac{\partial}{\partial z}(2y-3z)$$
$$\frac{\partial}{\partial y}(y-3z) = 1 = \frac{\partial}{\partial z}(2x+z).$$

We conclude that **F** is conservative, because the domain of **F** is all of $\mathbf{R}^3$. The given integral, therefore, is path independent. We'll integrate along the paths $\mathbf{x}_1(t) = (0,t,t)$, $0 \le t \le 1$, and $\mathbf{x}_2(t) = (t, t+1, 2t+1)$, $0 \le t \le 1$. The integral

$$\int_C (2y-3z)\,dx + (2x+z)\,dy + (y-3x)\,dz$$
$$= \int_0^1 (0(-t) + 1(t) + 1(t))\,dt + \int_0^1 ((-4t-1) + (4t+1) + 2(-2t+1))\,dt$$
$$= \int_0^1 2t\,dt + \int_0^1 (-4t+2)\,dt = 1 + 0 = 0.$$

Using Theorem 3.3, and the fact that $\mathbf{F} = \nabla f$ where $f(x,y,z) = 2xy - 3xz + yz$, we obtain

$$\int_C (2y-3z)\,dx + (2x+z)\,dy + (y-3x)\,dz = f(1,2,3) - f(0,0,0) = 1.$$

*In Exercises 29–32, to determine the work, we need to calculate line integrals of the form $\int_C \mathbf{F}\cdot d\mathbf{s}$, where $C$ is an appropriate curve from $A$ to $B$. To do this, we use the result of Theorem 3.3, since all of the vector fields in these exercises are conservative.*

**29.** A potential function for **F** is easily calculated to be $f(x,y) = x^3y - xy^2$. Thus, for any curve $C$ from $(0,0)$ to $(2,1)$ the work is

$$\int_C \mathbf{F}\cdot d\mathbf{s} = f(2,1) - f(0,0) = 6 - 0 = 6.$$

**30.** A potential function for **F** is $f(x,y) = 2x^{3/2}y$. Thus the work is

$$f(9,1) - f(1,2) = 54 - 4 = 50.$$

**31.** A potential function for **F** is $f(x,y,z) = x^2yz - xy^2z^3$. Therefore, the work is

$$f(6,4,2) - f(1,1,1) = -480 - 0 = -480.$$

**32.** A potential function for **F** is $f(x,y,z) = x^2y\cos z$. Hence the work is

$$f(2,3,0) - f(1,1,\pi/2) = 12 - 0 = 12.$$

**33. (a)** We'll check to see where $N_x = M_y$.

$$\frac{\partial}{\partial x}\left(\frac{x^2+1}{y^3}\right) = \frac{2x}{y^3} = \frac{\partial}{\partial y}\left(\frac{x+xy^2}{y^2}\right),$$

therefore, **F** is conservative on each of the two simply-connected sets on which it is defined. More precisely, **F** is conservative on $\{(x,y)|y>0\}$ and on $\{(x,y)|y<0\}$.

**(b)** The scalar potential is $f(x,y) = \dfrac{x^2+x^2y^2+1}{2y^2}$.

**(c)** As the particle moves from (0, 1) to (1, 1) along the parabola $y = 1 + x - x^2$ we note that $y > 0$ and so the path lies entirely in one of the simply-connected regions. We can, therefore, apply Theorem 3.3 and calculate the work done as $f(1,1) - f(0,1) = 3/2 - 1/2 = 1$.

**34. (a)** We need to check that three pairs of partial derivatives are equal:

$$\begin{aligned}\frac{\partial}{\partial x}(x+g(y)+z) &= 1 = \frac{\partial}{\partial y}(f(x)+y+z)\\ \frac{\partial}{\partial x}(x+y+h(z)) &= 1 = \frac{\partial}{\partial z}(f(x)+y+z)\\ \frac{\partial}{\partial y}(x+y+h(z)) &= 1 = \frac{\partial}{\partial z}(x+g(y)+z).\end{aligned}$$

**(b)** $\mathbf{F} = \nabla\phi(x,y,z)$ where, for constants $a$, $b$, and $c$,

$$\phi(x,y,z) = xy + xz + yz + \int_a^x f(t)\,dt + \int_b^y g(t)\,dt + \int_c^z h(t)\,dt.$$

**(c)** Using Theorem 3.3,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{s} &= \phi(x_1,y_1,z_1) - \phi(x_0,y_0,z_0)\\ &= x_1y_1 - x_0y_0 + x_1z_1 - x_0z_0 + y_1z_1 - y_0z_0 + \int_{x_0}^{x_1} f(t)\,dt + \int_{y_0}^{y_1} g(t)\,dt + \int_{z_0}^{z_1} h(t)\,dt.\end{aligned}$$

**35. (a)** **F** is conservative since $\mathbf{F} = \nabla f$ where $f(x,y,z) = \sin(x^2+xz) + \cos(y+yz)$.

**(b)** Since we have a potential function,

$$\begin{aligned}\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{x}} \nabla f\cdot d\mathbf{s} &= f(\mathbf{x}(1)) - f(\mathbf{x}(0))\\ &= f(1,1,\pi-1) - f(0,0,0) = -2.\end{aligned}$$

**36. (a)** $\mathbf{G} = \mathbf{F} + x\mathbf{j}$, where **F** is given in Exercise 35. Now $\nabla\times\mathbf{G} = \nabla\times\mathbf{F} + \nabla\times(x\mathbf{j}) = \mathbf{k} \neq \mathbf{0}$, so **G** is not conservative.

**(b)** Here we have that

$$\int_{\mathbf{x}} \mathbf{G}\cdot d\mathbf{s} = \int_{\mathbf{x}} (\mathbf{F}+x\mathbf{j})\cdot d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathbf{x}} x\mathbf{j}\cdot d\mathbf{s}.$$

From Exercise 35, we have that $\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = -2$, so

$$\begin{aligned}\int_{\mathbf{x}} \mathbf{G}\cdot d\mathbf{s} &= -2 + \int_0^1 (0,t^3,0)\cdot\left(3t^2, 2t, \pi - \frac{\pi}{2}\cos\frac{\pi t}{2}\right)dt\\ &= -2 + \int_0^1 2t^4\,dt = -2 + 2/5 = -\frac{8}{5}.\end{aligned}$$

**37.** You could check that **F** is conservative by confirming that $\nabla\times\mathbf{F} = \mathbf{0}$ on any simply-connected region that misses the origin. It is, however, easy enough to find the scalar potential for **F** is $f(x,y,z) = GMm(x^2+y^2+z^2)^{-1/2}$. So the work done by **F** as a particle of mass $m$ moves from $\mathbf{x}_0$ to $\mathbf{x}_1$ is

$$f(x_1,y_1,z_1) - f(x_0,y_0,z_0) = \frac{GMm}{\sqrt{x_1^2+y_1^2+z_1^2}} - \frac{GMm}{\sqrt{x_0^2+y_0^2+z_0^2}} = GMm\left(\frac{1}{\|\mathbf{x}_1\|} - \frac{1}{\|\mathbf{x}_0\|}\right).$$

### True/False Exercises for Chapter 6

1. True.
2. False. (The value is 2.)
3. False. (The integral is negative.)
4. True.
5. False. (The integral is 0.)
6. True.
7. False. (There is equality only up to sign.)
8. False.
9. True.
10. True.
11. True.
12. False. ($\nabla f$ is everywhere normal to $C$.)
13. True.
14. False. (The work is at most 3 times the length of $C$.)
15. False. (The line integral could be $\pm \int_C \|\mathbf{F}\|\, ds$, depending on whether **F** points in the same or the opposite direction as $C$.)
16. True. (Just use Green's theorem.)
17. False. (Let $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ and consider Green's theorem.)
18. True. (Use the divergence theorem in the plane.)
19. False. (Under appropriate conditions, the integral is $f(B) - f(A)$.)
20. True.
21. True.
22. False. (There's a negative sign missing.)
23. False. (For the vector field to be conservative, the line integral must be zero for *all* closed curves, not just a particular one.)
24. True.
25. False. (The vector field $(e^x \cos y \sin z, e^x \sin y \sin z, e^x \cos y \cos z)$ is not conservative.)
26. False. (**F** must be of class $C^1$ on a simply-connected region.)
27. False. (The domain is not simply-connected.)
28. True.
29. False. ($f$ is only defined up to a constant.)
30. True.

### Miscellaneous Exercises for Chapter 6

1. Partition the curve into $n$ pieces each of length $\Delta s_k = (\text{length of } C)/n$. The right side of the given formula is just our calculation of arclength for a rectifiable curve:

$$\frac{\int_C f\, ds}{\text{length of } C} = \frac{\int_C f\, ds}{\int_C ds}.$$

Now, if on the $k$th sub-interval we choose any $\mathbf{c}_k$, then on the interval $f(\mathbf{x}) \approx f(\mathbf{c}_k)$. Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} f(\mathbf{c}_k)\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{f(\mathbf{c}_k)}{n} = \frac{\lim_{n\to\infty}\sum_{k=1}^{n} f(\mathbf{c}_k)\Delta s_k}{\text{length of } C} = \frac{\int_C f\, ds}{\text{length of } C}.$$

For each value of $n$ we are calculating an average of $n$ values of $f$ at points on the curve. As $n$ grows large, if this limit exists, it is reasonable to define it as the **average value** of $f$ along $C$.

2. Here $f(\mathbf{x}(t)) = 2 + 2t^2$, and $\|\mathbf{x}'(t)\| = \sqrt{2}$. Therefore,

$$[f]_{\text{avg}} = \frac{\int_C f\, ds}{\int_C ds} = \frac{\int_0^{3\pi}((2t^2+2)\sqrt{2})\, dt}{\int_0^{3\pi}\sqrt{2}\, dt} = \frac{2}{3\pi}\int_0^{3\pi}(t^2+1)\, dt = \frac{2}{3\pi}(9\pi^3 + 3\pi) = 6\pi^2 + 2.$$

**3.** We may parametrize the semicircle as $\mathbf{x}(t) = (a\cos t, a\sin t)$, where $0 \le t \le \pi$. Therefore, $\|\mathbf{x}'(t)\| = a$. The length of the semicircle is $\pi a$ and so the average $y$-coordinate may be found by calculating

$$\frac{1}{\pi a}\int_0^\pi a\sin t \cdot a\,dt = \frac{a}{\pi}\int_0^\pi \sin t\,dt = \frac{2a}{\pi}.$$

**4.** Calculate $[z]_{\text{avg}}$ as $\dfrac{\int_C z\,ds}{\text{length of } C}$. The total length of $C$ is just the sum of the lengths of four straight segments:

$$2 + 1 + \sqrt{4+0+1} + \sqrt{1+1+1} = 3 + \sqrt{5} + \sqrt{3}.$$

Now $\int_C z\,ds = \int_{C_1} z\,ds + \cdots + \int_{C_4} z\,ds$, but $z$ is clearly zero on two of the four segments.

The segment $C_3$ joining (2, 1, 0) and (0, 1, 1) may be parametrized as

$$\begin{aligned}\mathbf{x}(t) &= (1-t)(2,1,0) + t(0,1,1), \quad 0 \le t \le 1\\ &= (2-2t, 1, t).\end{aligned}$$

Thus $\mathbf{x}'(t) = (-2, 0, 1)$ and $\|\mathbf{x}'(t)\| = \sqrt{5}$. Therefore,

$$\int_{C_3} z\,ds = \int_0^1 t\cdot\sqrt{5}\,dt = \frac{\sqrt{5}}{2}.$$

The segment $C_4$ joining (0, 1, 1) and (1, 0, 2) may be parametrized as

$$\begin{aligned}\mathbf{x}(t) &= (1-t)(0,1,1) + t(1,0,2), \quad 0 \le t \le 1\\ &= (t, 1-t, t+1).\end{aligned}$$

Thus $\mathbf{x}'(t) = (1, -1, 1)$ and $\|\mathbf{x}'(t)\| = \sqrt{3}$. Hence

$$\int_{C_4} z\,ds = \int_0^1 (t+1)\sqrt{3}\,dt = \sqrt{3}\left(\frac{t^2}{2} + t\right)\Big|_0^1 = \frac{3\sqrt{3}}{2}.$$

Putting all this together, we find

$$[z]_{\text{avg}} = \frac{\sqrt{5}/2 + 3\sqrt{3}/2}{3+\sqrt{5}+\sqrt{3}} = \frac{\sqrt{5}+3\sqrt{3}}{2(3+\sqrt{5}+\sqrt{3})} \approx 0.5333.$$

**5.** The curve may be parametrized as $\mathbf{x}(t) = (\sqrt{5}\cos t, \sin t, 2\sin t), 0 \le t \le 2\pi$. Then $\|\mathbf{x}'(t)\| = \sqrt{(-\sqrt{5}\sin t)^2 + (\cos t)^2 + (2\cos t)^2} = \sqrt{5}$ so the length of $C$ is $\int_0^{2\pi}\sqrt{5}\,dt = 2\pi\sqrt{5}$. Now

$$\begin{aligned}\int_C f\,ds &= \int_0^{2\pi}(4\sin^2 t + \sqrt{5}\cos t\cdot e^{\sin t})\sqrt{5}\,dt\\ &= \sqrt{5}\int_0^{2\pi}(2(1-\cos 2t) + \sqrt{5}\,e^{\sin t}\cdot\cos t)\,dt\\ &= \sqrt{5}(2t - \sin 2t + \sqrt{5}\,e^{\sin t})\Big|_0^{2\pi} = \sqrt{5}(4\pi + \sqrt{5} - 0 + 0 - \sqrt{5})\\ &= 4\pi\sqrt{5}.\end{aligned}$$

Hence $[f]_{\text{avg}} = \dfrac{4\sqrt{5}\pi}{2\sqrt{5}\pi} = 2.$

**6. (a)** For the total mass we integrate the density along the curve:

$$\int_C (3-y)\,ds = \int_0^\pi 2(3 - 2\sin t)\,dt = 6\pi - 8.$$

**(b)** The density depends only on $y$ and the wire is symmetric with respect to $x$ so $\bar{x} = 0$ (if you write out the formula you'll see that the numerator is an integral of an odd function of $x$ over a curve that is symmetric with respect to $x$). Also, since $z \equiv 0$, we quickly conclude that $\bar{z} = 0$. What remains is to calculate

$$\bar{y} = \frac{\int_C y\delta(x,y,z)\,ds}{\int_C \delta(x,y,z)\,ds} = \frac{\int_0^\pi (2\sin t(3-2\sin t)2)\,dt}{6\pi-8} = \frac{24-4\pi}{6\pi-8} = \frac{12-2\pi}{3\pi-4}.$$

**7.** Locate the wire in the first quadrant of the $xy$-plane. Then the center is at $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$ and $\delta(x,y,z) = \left(x-\frac{a}{\sqrt{2}}\right)^2 + \left(y-\frac{a}{\sqrt{2}}\right)^2$. From symmetry considerations, we must have $\overline{x} = \overline{y}$. Now parametrize the quarter circle as

$$\begin{cases} x = a\cos t \\ y = a\sin t, \quad 0 \le t \le \pi/2. \end{cases}$$

Then $\|\mathbf{x}'(t)\| = a$. We have

$$\begin{aligned} M &= \int_C \delta\,ds = \int_0^{\pi/2} \left(\left(a\cos t - \frac{a}{\sqrt{2}}\right)^2 + \left(a\sin t - \frac{a}{\sqrt{2}}\right)^2\right) a\,dt \\ &= a^3 \int_0^{\pi/2} \left(\cos^2 t - \sqrt{2}\cos t + \frac{1}{2} + \sin^2 t - \sqrt{2}\sin t + \frac{1}{2}\right) dt \\ &= a^3 \int_0^{\pi/2} (2 - \sqrt{2}\cos t - \sqrt{2}\sin t)\,dt = (\pi - 2\sqrt{2})a^3 \\ M_{yz} &= \int_C x\delta\,ds = \int_0^{\pi/2} a\cos t \left(\left(a\cos t - \frac{a}{\sqrt{2}}\right)^2 + \left(a\sin t - \frac{a}{\sqrt{2}}\right)^2\right) \cdot a\,dt \\ &= a^4 \int_0^{\pi/2} \cos t(2 - \sqrt{2}\cos t - \sqrt{2}\sin t)\,dt = a^4 \int_0^{\pi/2} \left(2\cos t - \frac{\sqrt{2}}{2}(1+\cos 2t) - \sqrt{2}\sin t\cos t\right) dt \\ &= a^4 \left(2\sin t - \frac{\sqrt{2}}{2}t - \frac{\sqrt{2}}{4}\sin 2t - \frac{\sqrt{2}}{2}\sin^2 t\right)\Big|_0^{\pi/2} \\ &= a^4 \left(2 - \frac{\sqrt{2}\pi}{4} - 0 - \frac{\sqrt{2}}{2} - 0\right) \\ &= \left(\frac{8 - \sqrt{2}\pi - 2\sqrt{2}}{4}\right) a^4. \end{aligned}$$

Hence

$$\bar{x} = \bar{y} = \frac{(8 - \sqrt{2}\pi - 2\sqrt{2})a^4}{4} \Big/ (\pi - 2\sqrt{2})a^3 = \left(\frac{8 - \sqrt{2}\pi - 2\sqrt{2}}{4(\pi - 2\sqrt{2})}\right) a.$$

**8. (a)** By symmetry $\bar{x} = \bar{y} = 0$ and $\bar{z} = 8\pi$.

**(b)** $\|\mathbf{x}'\| = \sqrt{9+16} = 5$; $\delta(x,y,z) = x^2 + y^2 + z^2$. Hence the mass of the wire is

$$\begin{aligned} M &= \int_{\mathbf{x}} \delta\,ds = \int_0^{4\pi} (9 + 16t^2) \cdot 5\,dt = \frac{20\pi}{3}(27 + 256\pi^2) \\ \bar{x} &= \frac{1}{M}\int_{\mathbf{x}} x\delta\,ds = \frac{1}{M}\int_0^{4\pi} 3\cos t(9+16t^2) \cdot 5\,dt = \frac{1}{M} \cdot 1920\pi \\ \bar{y} &= \frac{1}{M}\int_{\mathbf{x}} y\delta\,ds = \frac{1}{M}\int_0^{4\pi} 3\sin t(9+16t^2) \cdot 5\,dt = \frac{1}{M}(-3840\pi^2) \\ \bar{z} &= \frac{1}{M}\int_{\mathbf{x}} z\delta\,ds = \frac{1}{M}\int_0^{4\pi} 4t(9+16t^2) \cdot 5\,dt = \frac{1}{M}(160\pi^2)(9 + 128\pi^2). \end{aligned}$$

Thus

$$\bar{x} = \frac{3 \cdot 1920\pi}{20\pi(27 + 256\pi^2)} = \frac{288}{27 + 256\pi^2} \approx 0.112781$$

$$\bar{y} = \frac{3(-3840\pi^2)}{20\pi(27 + 256\pi^2)} = -\frac{576\pi}{27 + 256\pi^2} \approx -0.708625$$

$$\bar{z} = \frac{3 \cdot 160\pi^2(9 + 128\pi^2)}{20\pi(27 + 256\pi^2)} = \frac{24\pi(9 + 128\pi^2)}{27 + 256\pi^2} \approx 37.5662.$$

**9. (a)** Parametrize the wire as $\mathbf{x}(t) = (2\cos t, 2\sin t), 0 \le t \le \pi$. Then $\|\mathbf{x}'\| = 2$ and

$$\begin{aligned} I_y &= \int_C x^2 \delta\, ds = \int_0^\pi 4\cos^2 t(3 - 2\sin t) \cdot 2\, dt \\ &= \int_0^\pi (24\cos^2 t - 16\cos^2 t \sin t)\, dt = \int_0^\pi (12(1 + \cos 2t) + 16\cos^2 t(-\sin t))\, dt \\ &= 12\pi + \frac{16}{3}(-1) - \frac{16}{3}(1) = 12\pi - \frac{32}{3} = \frac{36\pi - 32}{3}. \end{aligned}$$

**(b)** The square of the distance between a point on the wire and the $z$-axis is $x^2 + y^2$. Thus $I_z = \int_C (x^2 + y^2)\delta(x, y, z)\, ds$. Using the given information,

$$I_z = \int_0^\pi 4 \cdot (3 - 2\sin t)2\, dt = 8(3\pi - 4) = 24\pi - 32.$$

The total mass was found in Exercise 6 to be $6\pi - 8$. Hence the radius of gyration is

$$r_z = \sqrt{\frac{24\pi - 32}{6\pi - 8}} = 2.$$

**10.** Parametrize the curve as $\mathbf{x}(t) = \left(t, \frac{t}{2} + 2\right), -2 \le t \le 2$. Then $\|\mathbf{x}'\| = \frac{\sqrt{5}}{2}$ and

$$\begin{aligned} I_x &= \int_C y^2 \delta\, ds = \int_{-2}^2 \left(\frac{t}{2} + 2\right)^2 \left(\frac{t}{2} + 2\right) \cdot \frac{\sqrt{5}}{2}\, dt = \sqrt{5} \int_{-2}^2 \left(\frac{t}{2} + 2\right)^3 \cdot \frac{1}{2}\, dt \\ &= \sqrt{5} \int_1^3 u^3\, du = \frac{\sqrt{5}}{4} u^4 \Big|_1^3 = 20\sqrt{5} \\ M &= \int_C \delta\, ds = \int_{-2}^2 \left(\frac{t}{2} + 2\right) \frac{\sqrt{5}}{2}\, dt = \frac{\sqrt{5}}{2}\left(\frac{1}{4}t^2 + 2t\right)\Big|_{-2}^2 = 4\sqrt{5} \end{aligned}$$

Hence

$$r_x = \sqrt{I_x/M} = \sqrt{\frac{20\sqrt{5}}{4\sqrt{5}}} = \sqrt{5}.$$

**11.** We use $x$ as parameter so $\mathbf{x}(t) = (t, t^2),\ 0 \le t \le 2$, and $\|\mathbf{x}'\| = \sqrt{1 + 4t^2}$. Then

$$I_x = \int_C y^2 \delta\, ds = \int_0^2 t^4 \cdot t\sqrt{1 + 4t^2}\, dt = \int_0^2 t^5 \sqrt{1 + 4t^2}\, dt.$$

Now let $2t = \tan\theta$ so $dt = \dfrac{1}{2}\sec^2\theta\, d\theta$. Then

$$\begin{aligned}
I_x &= \int_0^{\tan^{-1}4} \frac{1}{32}\tan^5\theta\ \sec\theta\left(\frac{1}{2}\sec^2\theta\, d\theta\right)\\
&= \frac{1}{64}\int_0^{\tan^{-1}4} \tan^4\theta\ \sec^2\theta(\sec\theta\ \tan\theta\, d\theta)\\
&= \frac{1}{64}\int_0^{\tan^{-1}4} (\sec^2\theta - 1)^2\sec^2\theta(\sec\theta\ \tan\theta\, d\theta)\\
&= \frac{1}{64}\int_0^{\tan^{-1}4} (\sec^6\theta - 2\sec^4\theta + \sec^2\theta)\, d(\sec\theta)\\
&= \frac{1}{64}\left(\frac{1}{7}\sec^7\theta - \frac{2}{5}\sec^5\theta + \frac{1}{3}\sec^3\theta\right)\Big|_0^{\tan^{-1}4}\\
&= \frac{1}{64}\left(\frac{1}{7}\,17^{7/2} - \frac{2}{5}\,17^{5/2} + \frac{1}{3}\,17^{3/2} - \frac{1}{7} + \frac{2}{5} - \frac{1}{3}\right)\\
&= \frac{7769\sqrt{17}-1}{840} \approx 38.1326.
\end{aligned}$$

We also have

$$\begin{aligned}
M = \int_C \delta\, ds = \int_0^2 t\sqrt{1+4t^2}\, dt &= \frac{1}{8}\cdot\frac{2}{3}(1+4t^2)^{3/2}\Big|_0^2\\
&= \frac{1}{12}(17^{3/2}-1) \approx 5.75773.
\end{aligned}$$

Hence

$$\begin{aligned}
r_x = \sqrt{\frac{I_x}{M}} &= \sqrt{\frac{7769\sqrt{17}-1}{840}\cdot\frac{12}{17\sqrt{17}-1}} = \sqrt{\frac{7769\sqrt{17}-1}{1190\sqrt{17}-70}}\\
&\approx 2.57349.
\end{aligned}$$

**12.** **(a)** $I_x = \int_C (y^2+z^2)\delta(x,y,z)\, ds$, $I_y = \int_C (x^2+z^2)\delta(x,y,z)\, ds$, $I_z = \int_C (x^2+y^2)\delta(x,y,z)\, ds$
**(b)** For the given parametrization, $\|\mathbf{x}'\| = \sqrt{9+16} = 5$.

$$\begin{aligned}
I_x &= 5\delta\int_0^{4\pi}(9\sin^2 t + 16t^2)\, dt = 5\delta\int_0^{4\pi}\left(\frac{9}{2}(1-\cos 2t) + 16t^2\right)\, dt\\
&= 5\delta\left(18\pi + \frac{1024\pi^3}{3}\right) = \frac{10\pi(27+512\pi^2)\delta}{3}\\
I_y &= 5\delta\int_0^{4\pi}(9\cos^2 t + 16t^2)\, dt = 5\delta\int_0^{4\pi}\left(\frac{9}{2}(1+\cos 2t) + 16t^2\right)\, dt\\
&= \frac{10\pi(27+512\pi^2)\delta}{3}\\
I_z &= 5\delta\int_0^{4\pi} 9\, dt = 180\pi\delta
\end{aligned}$$

Now $M = \displaystyle\int_C \delta\, ds = \int_0^{4\pi} 5\delta\, dt = 20\delta$. Thus

$$r_x = r_y = \sqrt{\frac{\pi(27+512\pi^2)}{6}},\quad r_z = \sqrt{\frac{180\pi\delta}{20\delta}} = 3\sqrt{\pi}.$$

**13.** We may parametrize the segment as $\mathbf{x}(t) = (1-t)(-1,1,2) + t(2,2,3)$, $0 \le t \le 1$, or $\mathbf{x}(t) = (3t-1, t+1, t+2)$. Then $\|\mathbf{x}'\| = \sqrt{9+1+1} = \sqrt{11}$.

$$\begin{aligned} I_z &= \int_C (x^2+y^2)\delta\,ds = \int_0^1 [(3t-1)^2+(t+1)^2][1+(t+2)^2]\cdot\sqrt{11}\,dt \\ &= \sqrt{11}\int_0^1 (10t^4+36t^3+36t^2-12t+10)\,dt = \sqrt{11}(2t^5+9t^4+12t^3-6t^2+10t)\Big|_0^1 \\ &= \sqrt{11}(2+9+12-6+10) = 27\sqrt{11} \\ M &= \int_C \delta\,ds = \int_0^1 (1+(t+2)^2)\sqrt{11}\,dt = \sqrt{11}\left(t+\frac{1}{3}(t+2)^3\right)\Big|_0^1 \\ &= \sqrt{11}\left(1+9-\frac{8}{3}\right) = \frac{22\sqrt{11}}{3}. \end{aligned}$$

Hence $r_z = \sqrt{I_z/M} = \sqrt{\frac{81}{22}} = \frac{9\sqrt{22}}{22}$.

*Exercises 14, 15, and 23 explore polar versions of results we've seen in this Chapter.*

**14. (a)** The path is

$$\mathbf{x}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta).$$

Using the product rule, we find that

$$\mathbf{x}'(\theta) = (f'(\theta)\cos\theta - f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta).$$

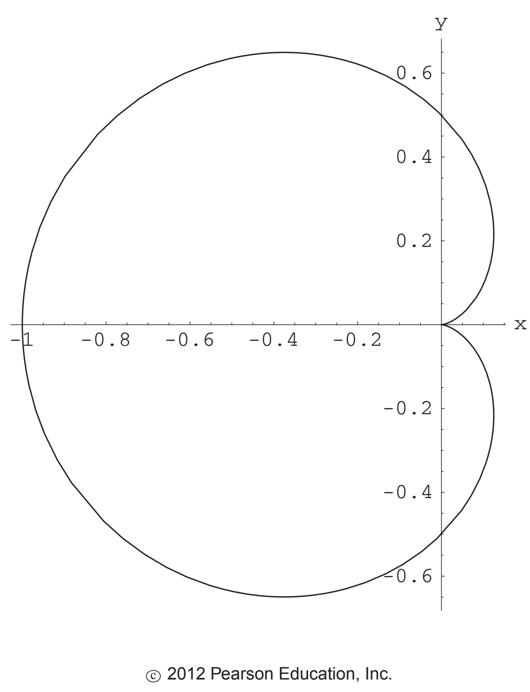
The length of $\mathbf{x}'(\theta)$ is a straightforward calculation:

$$\|\mathbf{x}'(t)\| = \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2} = \sqrt{(f'(\theta))^2 + (f(\theta))^2}.$$

We conclude that the arclength of the curve between $(f(a), a)$ and $(f(b), b)$ is

$$\int_C ds = \int_{\mathbf{x}(\theta)} \|\mathbf{x}'(\theta)\|\,d\theta = \int_a^b \sqrt{(f(\theta))^2 + (f'(\theta))^2}\,d\theta.$$

**(b)** The sketch of $r = \sin^2(\theta/2)$ is



The length is

$$\int_0^{2\pi} \sqrt{\sin^4(\theta/2) + \sin^2(\theta/2)\cos^2(\theta/2)}\, d\theta = \int_0^{2\pi} \sin(\theta/2)\, d\theta = -2(-1-1) = 4.$$

**15. (a)** $\oint_C g(x, y)\, ds = \int_{\mathbf{x}(\theta)} g(\mathbf{x}(\theta))\|\mathbf{x}'(\theta)\|\, d\theta = \int_a^b g(f(\theta)\cos\theta, f(\theta)\sin\theta)\sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta.$

**(b)** We'll use the formula from part (a).

$$\begin{aligned}\int_C g\, ds &= \int_0^{2\pi} ([(e^{3\theta}\cos\theta)^2 + (e^{3\theta}\sin\theta)^2 - 2(e^{3\theta}\cos\theta)]\sqrt{e^{6\theta} + 9e^{6\theta}})\, d\theta \\ &= \int_0^{2\pi} ([e^{6\theta} - 2e^{3\theta}\cos\theta]\sqrt{10}e^{3\theta})\, d\theta \\ &= \frac{\sqrt{10}}{333}(37e^{18\pi} - 108e^{12\pi} + 71).\end{aligned}$$

*In this text $\kappa$ is always non-negative. In cases where the curvature is signed, differential geometers are often interested in the* **total squared curvature**: $\int_C \kappa^2\, ds$.

**16.** In Section 3.2 it was shown that

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}.$$

Here $\mathbf{v} = \mathbf{x}'$ and $\mathbf{a} = \mathbf{x}''$. So

$$K = \int_C \kappa\, ds = \int_a^b \left(\frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}\|\mathbf{x}'\|\right) dt = \int_a^b \left(\frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^2}\right) dt.$$

**17.** We use the results of Exercise 16:

$$\begin{aligned}K &= \int_0^{10\pi} \frac{\|(-3\sin t, 3\cos t, 4) \times (-3\cos t, -3\sin t, 0)\|}{\|(-3\sin t, 3\cos t, 4)\|^2}\, dt \\ &= \int_0^{10\pi} \frac{\|(-12\sin t, -12\cos t, 9)\|}{25}\, dt = \int_0^{10\pi} \frac{3}{5}\, dt = 6\pi.\end{aligned}$$

**18.** We use the results of Exercise 16 with $\mathbf{x}(t) = (t, At^2, 0)$:

$$\begin{aligned}K &= \int_a^b \frac{\|(1, 2At, 0) \times (0, 2A, 0)\|}{\|(1, 2At, 0)\|^2}\, dt = \int_a^b \frac{\|(0, 0, 2A)\|}{1 + 4A^2t^2}\, dt \\ &= \int_a^b \frac{2A}{1 + 4A^2t^2}\, dt = \tan^{-1}(2At)\Big|_a^b = \tan^{-1}(2Ab) - \tan^{-1}(2Aa).\end{aligned}$$

**19.** We parameterize the ellipse by the path $\mathbf{x}(t) = (a\cos t, b\sin t, 0)$ for $0 \le t \le 2\pi$. Then, using Exercise 16,

$$\begin{aligned}K &= \int_0^{2\pi} \frac{\|(-a\sin t, b\cos t, 0) \times (-a\cos t, -b\sin t, 0)\|}{\|(-a\sin t, b\cos t, 0)\|^2}\, dt = \int_0^{2\pi} \frac{\|(0, 0, ab)\|}{a^2\sin^2 t + b^2\cos^2 t}\, dt \\ &= \int_0^{2\pi} \frac{ab}{a^2\sin^2 t + b^2\cos^2 t}\, dt = 2\pi.\end{aligned}$$

This verifies Fenchel's Theorem for the given ellipse. This final integral was calculated using *Mathematica*. With work it can also be done by hand.

**20. (a)** By Fenchel's theorem (see Exercise 19), we know that for $C$ (a simple, closed $C^1$ curve in $\mathbf{R}^3$), $K \ge 2\pi$, so

$$K = \int_C \kappa\, ds \ge 2\pi.$$

But $0 \le \kappa \le 1/a$, therefore

$$K = \int_C \kappa\, ds \le \int_C \frac{1}{a}\, ds = \frac{1}{a}\int_C ds = \frac{L}{a}.$$

Putting these two inequalities together we see that

$$\frac{L}{a} \ge K \ge 2\pi \quad \text{so} \quad L \ge 2\pi a.$$

**(b)** To conclude that $L = 2\pi a$ we would need both of the preliminary inequalities to be equalities. As we saw in Exercise 19, we have $K = 2\pi$ when $C$ is also a plane convex curve. Also, as we saw above, $K = L/a$ when $\kappa = 1/a$. Together, these two conditions imply that $C$ is a circle of radius $a$.

**21.** The work done is

$$\int_{\mathbf{x}} \mathbf{F}\cdot ds = \int_0^1 (\sin(t^3), \cos(-t^2), t^4)\cdot(3t^2, -2t, 1)\, dt = \int_0^1 (3t^2\sin(t^3) - 2t\cos(-t^2) + t^4)\, dt$$

$$= (-\cos(t^3) + \sin(-t^2) + t^5/5)\Big|_0^1 = 6/5 - \cos 1 - \sin 1.$$

**22.** The first thing to note is that we are traversing the path in the wrong direction to apply Green's theorem. If $C_1$ is the triangular path described in the problem from the origin, to (0, 1), to (1, 0), back to the origin, then let $C_2$ be the path traversed in the opposite direction and let $D$ be the region bounded by $C_1$ and $C_2$. Then

$$\oint_{C_1} x^2y\, dx + (x+y)y\, dy = -\oint_{C_2} x^2y\, dx + (x+y)y\, dy = -\iint_D (y - x^2)\, dA$$

$$= -\int_0^1\int_0^{1-x} (y - x^2)\, dy\, dx = -\int_0^1 (y^2/2 - x^2y)\Big|_0^{1-x}$$

$$= -\int_0^1 \left(\frac{1}{2} - x - \frac{x^2}{2} + x^3\right) dx = -\frac{1}{2}\left(x - x^2 - \frac{x^3}{3} + \frac{x^4}{2}\right)\Big|_0^1 = -\frac{1}{12}.$$

**23.** In Section 6.2 we saw that Green's theorem implied the formula

$$\text{Area} = \frac{1}{2}\oint_{\partial D} -y\, dx + x\, dy.$$

In general the boundary $\partial D$ of the region $D$ consists of the curve $r = f(\theta)$, which may be parametrized by $\mathbf{x}(\theta) = (x(\theta), y(\theta)) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$, and possibly straight line segments along $\theta = a$ and $\theta = b$. The line $\theta = a$ may be parametrized by $\mathbf{y}(r) = (x(r), y(r)) = (r\cos a, r\sin a)$ and the line $\theta = b$ may be parametrized similarly. Note that, along the straight segment $C_1$ given by $\theta = a$, we have

$$\frac{1}{2}\int_{C_1} -y\, dx + x\, dy = \frac{1}{2}\int_0^{f(a)} (-r\sin a\cos a + r\cos a\sin a)\, dr = 0.$$

An identical result holds for the straight segment $C_2$ given by $\theta = b$. Therefore, the area of $D$ may be evaluated by computing the line integral over the path $\mathbf{x}$ described above:

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_{\mathbf{x}} -y\, dx + x\, dy\\ &= \frac{1}{2}\int_a^b ((-f(\theta)\sin\theta)(f'(\theta)\cos\theta - f(\theta)\sin\theta)\\ &\qquad + (f(\theta)\cos\theta)(f'(\theta)\sin\theta + f(\theta)\cos\theta))\, d\theta\\ &= \frac{1}{2}\int_a^b (f(\theta))^2\, d\theta.\end{aligned}$$

**24.** By Green's theorem, if $D$ is the region with $C = \partial D$,

$$\oint_C f(x)\, dx + g(y)\, dy = \iint_D \left(\frac{\partial}{\partial x}(g(y)) - \frac{\partial}{\partial y}(f(x))\right) dx\, dy = \iint_D 0\, dx\, dy = 0.$$

**25.** We begin by applying Green's theorem (here $D$ has constant density $\delta$):

$$\frac{1}{2 \cdot \text{area of } D} \oint_{\partial D} x^2\, dy = \frac{1}{2 \cdot \text{area of } D} \iint_D 2x\, dx\, dy = \frac{\iint_D x\, dx\, dy}{\iint_D dx\, dy}$$
$$= \frac{\iint_D x\, \delta\, dx\, dy}{\iint_D \delta\, dx\, dy} = \bar{x}.$$

Similarly,

$$\frac{1}{\text{area of } D} \oint_{\partial D} xy\, dy = \frac{1}{\text{area of } D} \iint_D y\, dx\, dy = \frac{\iint_D y\, dx\, dy}{\iint_D dx\, dy}$$
$$= \frac{\iint_D y\, \delta\, dx\, dy}{\iint_D \delta\, dx\, dy} = \bar{y}.$$

For the second pair of formulas, we proceed in an entirely similar manner with Green's theorem.

$$-\frac{1}{\text{area of } D} \oint_{\partial D} xy\, dx = -\frac{1}{\text{area of } D} \iint_D -x\, dx\, dy$$
$$= \frac{1}{\text{area of } D} \iint_D x\, dx\, dy = \frac{\iint_D x\, dx\, dy}{\iint_D dx\, dy}$$
$$= \frac{\iint_D x\, \delta\, dx\, dy}{\iint_D \delta\, dx\, dy} = \bar{x}.$$

Also,

$$-\frac{1}{2 \cdot \text{area of } D} \oint_{\partial D} y^2\, dx = -\frac{1}{2 \cdot \text{area of } D} \iint_D -2y\, dx\, dy$$
$$= \frac{1}{\text{area of } D} \iint_D y\, dx\, dy = \frac{\iint_D y\, \delta\, dx\, dy}{\iint_D \delta\, dx\, dy} = \bar{y}.$$

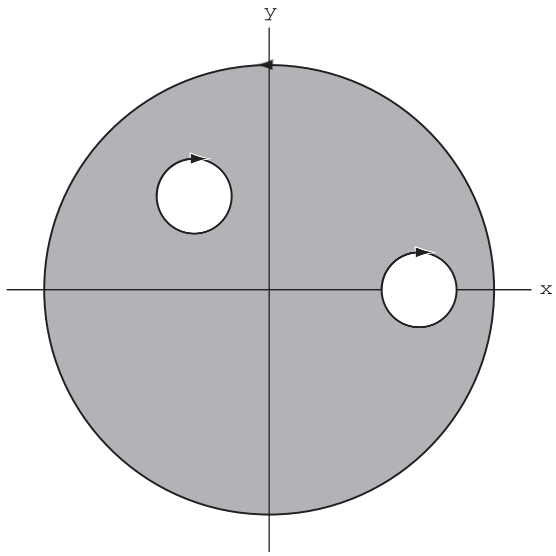
**26.** Along the bottom of the triangle, $dy$ is zero and along the left side $x$ is zero, so the first pair of line integrals in Exercise 25 must be zero except along the side connecting (1, 0) to (0, 2). Parametrize this side by $\mathbf{x}(t) = (1 - t, 2t)$ with $0 \leq t \leq 1$. Note also that the area of the triangle is 1. Using the results of Exercise 25, we have

$$\bar{x} = \frac{1}{2} \oint_{\partial D} x^2\, dy = \frac{1}{2} \int_0^1 (1 - t)^2 2\, dt = \int_0^1 (t^2 - 2t + 1)\, dt$$
$$= \frac{1}{3} - 1 + 1 = \frac{1}{3}$$

and

$$\bar{y} = \oint_{\partial D} xy\, dy = \int_0^1 (1 - t)(2t) 2\, dt = 2 \int_0^1 (-2t^2 + 2t)\, dt$$
$$= 2\left(-\frac{2}{3} + 1\right) = \frac{2}{3}.$$

**27.** The region in question looks like:




The area of this region is

$$36\pi - \pi - \pi = 34\pi.$$

Using the result of Exercise 25, we calculate

$$\bar{x} = \frac{1}{2 \cdot \text{area of } D} \oint_{\partial D} x^2\, dy \quad \text{and} \quad \bar{y} = -\frac{1}{2 \cdot \text{area of } D} \oint_{\partial D} y^2\, dx$$

(other computations are possible).

We may parametrize the outer boundary of the region by

$$\mathbf{x}(t) = (6\cos t, 6\sin t), \quad 0 \le t \le 2\pi$$

and the inner two circles by

$$\mathbf{y}(t) = (4 + \sin t, \cos t), 0 \le t \le 2\pi \text{ and}$$
$$\mathbf{z}(t) = (\sin t - 2, \cos t + 2), 0 \le t \le 2\pi.$$

Hence

$$\begin{aligned}
\bar{x} &= \frac{1}{68\pi} \oint_{\partial D} x^2\, dy \\
&= \frac{1}{68\pi}\left[\int_0^{2\pi} 36\cos^2 t \cdot 6\cos t\, dt + \int_0^{2\pi} (\sin t + 4)^2(-\sin t)\, dt + \int_0^{2\pi} (\sin t - 2)^2(-\sin t)\, dt\right] \\
&= \frac{1}{68\pi}\left[\int_0^{2\pi} 216(1 - \sin^2 t)\cos t\, dt - \int_0^{2\pi} (\sin^3 t + 8\sin^2 t + 16\sin t)\, dt \right. \\
&\qquad \left. - \int_0^{2\pi} (\sin^3 t - 4\sin^2 t + 4\sin t)\, dt\right] \\
&= \frac{1}{68\pi}\left[\int_0^{2\pi} 216(1 - \sin^2 t)\cos t\, dt - \int_0^{2\pi} (2\sin^3 t + 4\sin^2 t + 20\sin t)\, dt\right] \\
&= \frac{1}{68\pi}\left[\left(216\sin t - 72\sin^3 t\right)\Big|_0^{2\pi} - \int_0^{2\pi} (2(1 - \cos^2 t)\sin t + 2(1 - \cos 2t) + 20\sin t)\, dt\right]
\end{aligned}$$

$$= \frac{1}{68\pi}\left[0 + \left(2\cos t - \frac{2}{3}\cos^3 t - 2t + \sin 2t + 20\cos t\right)\bigg|_0^{2\pi}\right]$$

$$= \frac{1}{68\pi}(-4\pi) = -\frac{1}{34}$$

and

$$\bar{y} = -\frac{1}{68\pi}\oint_{\partial D} y^2\,dx = -\frac{1}{68\pi}\left[\int_0^{2\pi} 36\sin^2 t\cdot(-6\sin t)\,dt + \int_0^{2\pi}\cos^2 t\cdot\cos t\,dt + \int_0^{2\pi}(\cos t + 2)^2\cos t\,dt\right]$$

$$= -\frac{1}{68\pi}\left[\int_0^{2\pi} 216(1-\cos^2 t)(-\sin t)\,dt + \int_0^{2\pi}\cos^3 t\,dt + \int_0^{2\pi}(\cos^3 t + 4\cos^2 t + 4\cos t)\,dt\right]$$

$$= -\frac{1}{68\pi}\left[(216\cos t - 72\cos^3 t)\bigg|_0^{2\pi} + \int_0^{2\pi}(2(1-\sin^2 t)\cos t + 2(1+\cos 2t) + 4\cos t)\,dt\right]$$

$$= -\frac{1}{68\pi}\left[0 + \left(2\sin t - \frac{2}{3}\sin^3 t + 2t + \sin 2t + 4\sin t\right)\bigg|_0^{2\pi}\right]$$

$$= -\frac{1}{68\pi}[4\pi] = -\frac{1}{34}.$$

**28.** We can write $f\nabla g$ as $(f\partial g/\partial x, f\partial g/\partial y)$. Now apply the divergence theorem and collect the appropriate terms.

$$\oint_C f\nabla g\cdot\mathbf{n}\,ds = \iint_D \left(\frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right)\right)dA$$

$$= \iint_D \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial y^2}\right)dA$$

$$= \iint_D \left(\left[\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right] + \left[f\frac{\partial^2 g}{\partial x^2} + f\frac{\partial^2 g}{\partial y^2}\right]\right)dA$$

$$= \iint_D (\nabla f\cdot\nabla g + f\nabla^2 g)\,dA.$$

**29.** Apply the results of Exercise 28 to both parts of the line integral.

$$\oint_C (f\nabla g - g\nabla f)\cdot\mathbf{n}\,ds = \oint_C f\nabla g\cdot\mathbf{n}\,ds - \oint_C g\nabla f\cdot\mathbf{n}\,ds$$

$$= \iint_D (f\nabla^2 g + \nabla f\cdot\nabla g)\,dA - \iint_D (g\nabla^2 f + \nabla g\cdot\nabla f)\,dA$$

$$= \iint_D (f\nabla^2 g - g\nabla^2 f)\,dA$$

**30.** With $f(x,y)\equiv 1$ in Green's first identity, we have $\nabla f\equiv\mathbf{0}$, so

$$\iint_D (f\nabla^2 g + \nabla f\cdot\nabla g)\,dA = \iint_D \nabla^2 g\,dA = \oint_{\partial D}\nabla g\cdot\mathbf{n}\,ds = \oint_{\partial D}\frac{\partial g}{\partial n}\,ds.$$

But if $g$ is harmonic, $\nabla^2 g = 0$, so $\oint_{\partial D}\frac{\partial g}{\partial n}\,ds = 0$.

**31.** Now use Green's first identity with $f = g$ and $f$ harmonic to obtain $\iint_D(\nabla f\cdot\nabla f)\,dA = \oint_C(f\nabla f\cdot\mathbf{n})\,ds$. Since $C = \partial D$ and $\nabla f\cdot\mathbf{n} = \frac{\partial f}{\partial n}$, the desired result follows.

**32.** If $f$ is zero on the boundary of $D$, then Exercise 31 implies that $0 = \oint_{\partial D} f\frac{\partial f}{\partial n}\,ds = \iint_D \nabla f\cdot\nabla f\,dA$. But $\nabla f\cdot\nabla f = \|\nabla f\|^2 \geq 0$. Thus the right integral is of a nonnegative, continuous integrand. For it to be zero, the integrand must be identically zero. That is, $\nabla f\cdot\nabla f$ vanishes an $D$. We conclude that $\nabla f$ is zero on $D$ and so $f$ must be constant. Since $f(x,y) = 0$ on $\partial D$ and $f$ is constant on $D$, we must have $f\equiv 0$ on $D$.

**33.** Let $f = f_1 - f_2$. Then since $f_1 = f_2$ on $\partial D$, $f = 0$ on $\partial D$. Also $f$ is harmonic if $f_1$ and $f_2$ are. Hence, by Exercise 32, $f \equiv 0$ on $D$ so $f_1 = f_2$ on $D$.

**34. (a)** Exercise 37 from Section 6.3 is a particularly nice example of a nontrivial radially symmetric vector field because there is a compelling physical reason for the field **F** to be radially symmetric. There **F** is the gravitational force field of a mass $M$ on a particle of mass $m$.

$$\mathbf{F} = -\frac{GMm}{(x^2+y^2+z^2)^{3/2}}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) = -\frac{GMm}{(x^2+y^2+z^2)}\frac{(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{\|(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\|} = -\frac{GMm}{\rho^2}\mathbf{e}_\rho.$$

**(b)** Apply the formula for the curl in spherical coordinates found in Theorem 4.6 in Chapter 3.

$$\nabla \times \mathbf{F} = \frac{1}{\rho^2\sin\varphi}\begin{vmatrix} \mathbf{e}_\rho & \rho\,\mathbf{e}_\varphi & \rho\sin\varphi\,\mathbf{e}_\theta \\ \partial/\partial\rho & \partial/\partial\varphi & \partial/\partial\theta \\ f(\rho) & 0 & 0 \end{vmatrix} = \left(0, 0, \frac{1}{\rho}\mathbf{e}_\theta \begin{vmatrix} \partial/\partial\rho & \partial/\partial\varphi \\ f(\rho) & 0 \end{vmatrix}\right) = (0,0,0).$$

*When students get to complex analysis and learn to integrate around poles, texts often refer to their experience with Green's theorem in multivariable calculus. At least one of Exercises 35 and 36 should be assigned so that this reference might ring a bell.*

**35. (a)** The boundary is in two pieces which we separately parametrize as $\mathbf{x}_1(\theta) = (\cos\theta, \sin\theta)$ and $\mathbf{x}_2(\theta) = (a\cos\theta, -a\sin\theta)$, each for $0 \le \theta \le 2\pi$. The line integral is then

$$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \int_0^{2\pi}(\sin^2\theta + \cos^2\theta)\,d\theta + \int_0^{2\pi}\left(-\frac{a^2\sin^2\theta}{a^2} - \frac{a^2\cos^2\theta}{a^2}\right)d\theta$$
$$= \int_0^{2\pi}(1-1)\,d\theta = 0.$$

The double integral is

$$\iint_D \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right]dx\,dy = \iint_D \left[\frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{-y^2+x^2}{(x^2+y^2)^2}\right)dx\,dy$$
$$= \iint_D 0\,dx\,dy = 0.$$

Thus the conclusion of Green's theorem holds for **F** in the given annular region.

**(b)** This time the line integral is only taken over the outer boundary and so

$$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 2\pi.$$

The same cancellation takes place in the double integral as in part (a), so

$$\iint_D \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right]dx\,dy = 0.$$

The problem is that **F** is not defined at the origin.

**(c)** Let $D$ be the region so that $\partial D$ consists of the given curve $C$ oriented counterclockwise and also the curve $C_a$, the circle of radius $a$ centered at the origin oriented clockwise. Then **F** is defined everywhere in the region $D$. Green's theorem holds so

$$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy + \oint_{C_a} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \oint_{C\cup C_a} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$
$$= \iint_D \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right)dx\,dy = 0, \quad \text{but}$$

$$\oint_{C_a} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = -2\pi. \quad \text{Therefore,} \quad \oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 2\pi.$$

**36. (a)**

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix} = (0,0,0).$$

**(b)** Here the path is $\mathbf{x}(\theta) = (\cos\theta, \sin\theta)$ for $0 \le \theta \le 2\pi$. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} ((-\sin\theta)(-\sin\theta) + (\cos\theta)(\cos\theta))\, d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

**(c)** We saw in part (b) that the line integral around a closed path is not zero, so **F** cannot be conservative on its domain.

**(d)** The conditions are not met for the theorem as the domain of **F** is not a simply-connected region.

**37. (a)** By the divergence theorem, the flux

$$\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_0^1 \int_0^5 \left( \frac{\partial}{\partial x}[e^y] + \frac{\partial}{\partial y}[x^4] \right) dy\, dx = 0.$$

**(b)** Again, by the divergence theorem, the flux

$$\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \iint_D \left( \frac{\partial}{\partial x}[f(y)] + \frac{\partial}{\partial y}[f(x)] \right) dy\, dx = 0.$$

**38.** Over the path $\mathbf{x}(t)$,

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{x}} m\mathbf{a}\cdot d\mathbf{s} = \int_a^b m\mathbf{x}''(t)\cdot\mathbf{x}'(t)\, dt = m\int_a^b \frac{1}{2}\frac{d}{dt}[\mathbf{x}'(t)\cdot\mathbf{x}'(t)]\, dt$$

$$= \frac{1}{2}m\|\mathbf{x}'(t)\|^2 \Big|_a^b = \frac{1}{2}m[v(b)]^2 - \frac{1}{2}m[v(a)]^2.$$

**39.** We'll first replace **F** with $-\nabla V$ and apply Theorem 3.3.

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{x}} -\nabla V\cdot d\mathbf{s} = -V(B) + V(A),$$

where $A = \mathbf{x}(a)$ and $B = \mathbf{x}(b)$. However, in Exercise 38 we showed that

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \frac{1}{2}m[v(b)]^2 - \frac{1}{2}m[v(a)]^2.$$

Therefore,

$$\begin{aligned} -V(B) + V(A) &= \frac{1}{2}m[v(b)]^2 - \frac{1}{2}m[v(a)]^2, \text{ or} \\ V(A) + \frac{1}{2}m[v(a)]^2 &= V(B) + \frac{1}{2}m[v(b)]^2. \end{aligned}$$

We see, therefore, that the sum of the potential and kinetic energies of the particle remains constant.

## Chapter 7

# Surface Integrals and Vector Analysis

### 7.1 Parametrized Surfaces

**1.** **(a)** To find a normal vector we calculate

$$\mathbf{T}_s(s,t) = (2s, 1, 2s) \quad \text{so} \quad \mathbf{T}_s(2,-1) = (4,1,4)$$
$$\mathbf{T}_t(s,t) = (-2t, 1, 3) \quad \text{so} \quad \mathbf{T}_t(2,-1) = (2,1,3).$$

Then a normal vector is

$$\mathbf{N}(2,-1) = \mathbf{T}_s(2,-1) \times \mathbf{T}_t(2,-1) = (-1,-4,2).$$

**(b)** We find an equation for the tangent plane using

$$0 = \mathbf{N}(2,-1) \cdot (\mathbf{x} - (3,1,1)) = (-1,-4,2) \cdot (\mathbf{x} - (3,1,1)) = -x + 3 - 4y + 4 + 2z - 2.$$

This is equivalent to $x + 4y - 2z = 5$.

**2.** First we figure that since $2\sin t = 1$, either $t = \pi/6$ or $5\pi/6$. Since $2\cos t < 0$ we know that $t = 5\pi/6$. Then we can see that $\sin s = \sqrt{2}/2$ so $s = \pi/4$. Next, find a normal vector to the surface at the given point by calculating

$$\mathbf{T}_s(s,t) = (-(5+2\cos t)\sin s, (5+2\cos t)\cos s, 0) \quad \text{and}$$
$$\mathbf{T}_t(s,t) = (-2\sin t\cos s, -2\sin t\sin s, 2\cos t) \quad \text{so}$$
$$\mathbf{N}(s,t) = \mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t) = 2(5+2\cos t)(\cos s\cos t, \sin s\cos t, \sin t). \quad \text{Therefore,}$$
$$\mathbf{N}(\pi/4, 5\pi/6) = \frac{\sqrt{3}-5}{\sqrt{2}}(\sqrt{3}, \sqrt{3}, -\sqrt{2}).$$

We calculate an equation for the tangent plane by writing $\mathbf{N} \cdot (\mathbf{x} - (x_0, y_0, z_0)) = 0$ or, equivalently in this case,

$$0 = (\sqrt{3}, \sqrt{3}, -\sqrt{2}) \cdot \left(\mathbf{x} - \left(\frac{5-\sqrt{3}}{\sqrt{2}}, \frac{5-\sqrt{3}}{\sqrt{2}}, 1\right)\right) \quad \text{or} \quad \sqrt{3}x + \sqrt{3}y - \sqrt{2}z = 5\sqrt{6} - 4\sqrt{2}.$$

**3.** Since $x = e^s$ at $x = 1$, we know that $s = 0$. Also since $z = 2e^{-s} + t$, when $z = 0$ and $s = 0$, we have $t = -2$. As above we calculate,

$$\mathbf{T}_s(s,t) = (e^s, 2t^2e^{2s}, -2e^{-s}) \quad \text{and} \quad \mathbf{T}_t(s,t) = (0, 2te^{2s}, 1).$$

Thus, $\mathbf{N}(0,-2) = \mathbf{T}_s(0,-2) \times \mathbf{T}_t(0,-2) = (1,8,-2) \times (0,-4,1) = (0,-1,-4)$. Then an equation of the tangent plane is $0 = \mathbf{N}(0,-2) \cdot (\mathbf{x} - (1,4,0)) = (0,-1,-4) \cdot (\mathbf{x} - (1,4,0))$. We can simplify this to $y + 4z = 4$.

**4.** **(a)** $\mathbf{T}_s(s,t) = (2s\cos t, 2s\sin t, 1)$ so $\mathbf{T}_s(-1,0) = (-2,0,1)$. Also, $\mathbf{T}_t(s,t) = (-s^2\sin t, s^2\cos t, 0)$ so $\mathbf{T}_t(-1,0) = (0,4,0)$. Therefore, $\mathbf{N}(-1,0) = (-2,0,1) \times (0,4,0) = (-4,0,-8)$.

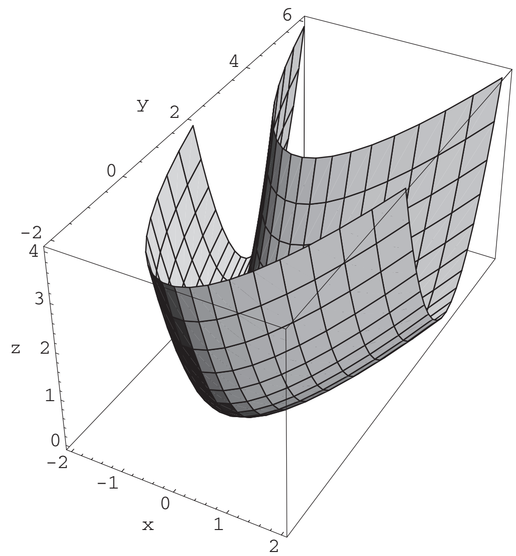
**(b)** An equation of the tangent plane is $(-4,0,-8) \cdot (\mathbf{x} - (1,0,-1)) = 0$. This simplifies to $x + 2z = -1$.

**(c)** Note that the $x$-component of $\mathbf{X}$ is $s^2\cos t$ and the $y$-component is $s^2\sin t$ and the $z$-component is a function of $s$. We can eliminate the $t$ by looking at $x^2 + y^2$. So without much work we have found that an equation for the image of $\mathbf{X}$ is $x^2 + y^2 - z^4 = 0$.

**5.** **(a)** Using *Mathematica* and the command:

ParametricPlot3D[{s, s^2 + t, t^2}, {s, −2, 2}, {t, −2, 2}, AxesLabel → {x,y,z}],

we obtain the image



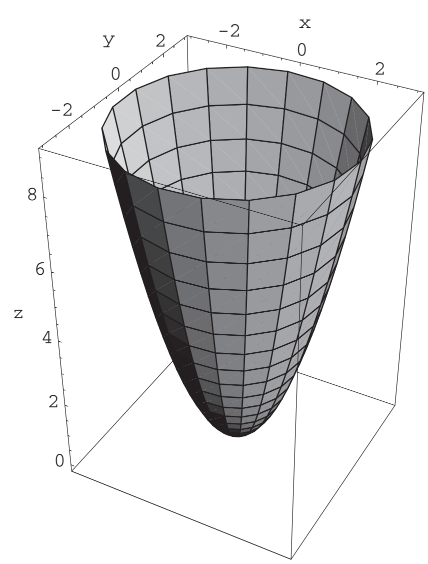

**(b)** To determine whether the surface is smooth we need to calculate $\mathbf{N}$. First, $\mathbf{T}_s(s,t) = (1, 2s, 0)$, and $\mathbf{T}_t(s,t) = (0, 1, 2t)$ so $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = (4st, -2t, 1)$. We conclude that $\mathbf{N} \neq \mathbf{0}$ for any $(s,t)$ so $\mathbf{N}$ is smooth.

**(c)** If $(s, s^2 + t, t^2) = (1, 0, 1)$, then $s = 1$ and $t = -1$. So $\mathbf{N}(1, -1) = (-4, 2, 1)$ and an equation of the tangent plane at this point is $(-4, 2, 1) \cdot (\mathbf{x} - (1, 0, 1)) = 0$ or more simply, $4x - 2y - z = 3$.

**6.** In Exercise 1, $x = s^2 - t^2, y = s + t$, and so if we note that $x = (s+t)(s-t) = y(s-t)$, then $x/y = s - t$. This allows us to solve for $s$ and $t$ separately: $2s = y + x/y$ and $2t = y - x/y$. This means that $z = s^2 + 3t$ can be written as $z = (y + x/y)^2/4 + 3(y - x/y)$.

**7. (a)** For the surface, we have $\mathbf{X}(s,t) = (s\cos t, s\sin t, s^2)$, where $s \geq 0$ and $0 \leq t \leq 2\pi$. This means that $\mathbf{T}_s(s,t) = (\cos t, \sin t, 2s)$ and $\mathbf{T}_t(s,t) = (-s\sin t, s\cos t, 0)$. Then a normal vector is given by $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = (-2s^2\cos t, -2s^2\sin t, s)$. This means that the surface is smooth except when $s = 0$. In other words, $S$ is smooth (as a parametrized surface) except at the origin. Note that the point $(1, \sqrt{3}, 4) = \mathbf{X}(2, \frac{\pi}{3})$. Thus $\mathbf{N}(2, \frac{\pi}{3}) = (-8\cos\frac{\pi}{3}, -8\sin\frac{\pi}{3}, 2) = (-4, -4\sqrt{3}, 2)$ and thus an equation of the tangent plane is given by $(-4, -4\sqrt{3}, 2)\cdot(x - 1, y - \sqrt{3}, z - 4) = 0$ or, equivalently, by $2x + 2\sqrt{3}y - z = 4$.

**(b)** See the figure below and note that $z = x^2 + y^2$ so we see that $S$ is a paraboloid.




**(c)** Again, $z = x^2 + y^2$.

**(d)** Part (a) above takes care of every point except the origin. At the origin $\mathbf{N} = \mathbf{0}$, but we easily see that the tangent plane there is the horizontal plane $z = 0$. Thus smoothness in the sense defined in Section 7.1 depends on the parametrization as well as the geometry of the underlying surface.

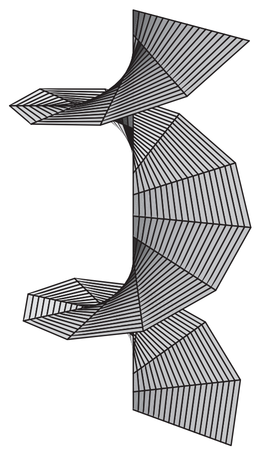
**8.** Really there's not much to show. You know that if the image of the parametrized surface is to be an ellipsoid, you need $a(2\sin s\cos t)^2 + b(3\sin s\sin t)^2 + c(\cos s)^2 = 1$. So $a = 1/4$, $b = 1/9$, and $c = 1$. Therefore the image satisfies $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$.

**9.** For $t = t_0, \mathbf{X}(s, t_0) = ((a + b\cos t_0)\cos s, (a + b\cos t_0)\sin s, b\sin t_0)$. So $z$ is constant and $x^2 + y^2 = (a + b\cos t_0)^2$. This is a circle of radius $a + b\cos t_0$ centered at $(0, 0, b\sin t_0)$.

**10. (a)** When $\theta = \pi/3$, the $r$-coordinate curve is given by $(r/2, r\sqrt{3}/2, \pi/3)$ where $r \geq 0$. This is the ray $y = \sqrt{3}x$ where $x \geq 0$ and $z = \pi/3$. In general, the $r$-coordinate curve when $\theta = \theta_0$ is a ray in the $z = \theta_0$ plane. The solution is simpler than the following four cases make it seem. If $\cos\theta_0 \neq 0$ then $y = (\tan\theta_0)x$ where $x \geq 0$ if $\cos\theta_0 > 0$ and $x \leq 0$ if $\cos\theta_0 < 0$. If $\cos\theta_0 = 0$, then the ray is $x = 0$ with $y \geq 0$ if $\sin\theta_0 > 0$ and $y \leq 0$ if $\sin\theta_0 < 0$.

**(b)** When $r = 1$ the $\theta$-coordinate curve is the helix $(\cos\theta, \sin\theta, \theta)$. In general, when $r = r_0$ the $\theta$-coordinate curve is the helix $(r_0\cos\theta, r_0\sin\theta, \theta)$.

**(c)** You can see that the helicoids are made up of the helices that are the $\theta$-coordinate curves.



**11. (a)** First we consider the sphere as the graph of the function $f(x, y) = \sqrt{4 - (x-2)^2 - (y+1)^2}$. The partial derivatives are

$$f_x = \frac{-(x-2)}{\sqrt{4 - (x-2)^2 - (y+1)^2}} \quad f_y = \frac{-(y+1)}{\sqrt{4 - (x-2)^2 - (y+1)^2}}.$$

So $f_x(1, 0, \sqrt{2}) = 1/\sqrt{2}$ and $f_y(1, 0, \sqrt{2}) = -1/\sqrt{2}$. By Theorem 3.3 of Chapter 2, $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$. In this case, this is $z = \sqrt{2} + (1/\sqrt{2})(x - 1) - (1/\sqrt{2})y$, or equivalently, $-x + y + \sqrt{2}z = 1$.

**(b)** Now we look at the sphere as a level surface of $F(x, y, z) = (x-2)^2 + (y+1)^2 + z^2$. The gradient $\nabla F(x, y, z) = 2(x - 2, y + 1, z)$ and, therefore, $\nabla F(1, 0, \sqrt{2}) = (-2, 2, 2\sqrt{2})$. By formula (5) of Section 2.6, the tangent plane is given by

$$0 = \nabla F(1, 0, \sqrt{2}) \cdot (\mathbf{x} - (1, 0, \sqrt{2})) = (-2, 2, 2\sqrt{2}) \cdot (\mathbf{x} - (1, 0, \sqrt{2})).$$

This too is equivalent to $-x + y + \sqrt{2}z = 1$.

**(c)** Now we'll use the results of this section. Considering the $z$-component, we see $2\cos s = \sqrt{2}$ so $\cos s = \sqrt{2}/2$. Considering the $y$- and $x$-components, $2\sin s\sin t = 1$ and $\sin s\cos t = -1$. Thus we have that $s = \pi/4$ and $t = 3\pi/4$. Also $\mathbf{T}_s(s, t) = (2\cos s\cos t, 2\cos s\sin t, -2\sin s)$ and $\mathbf{T}_t(s, t) = (-2\sin s\sin t, 2\sin s\cos t, 0)$. A normal vector to the sphere at the specified point is

$$\mathbf{N}(\pi/4, 3\pi/4) = \mathbf{T}_s(\pi/4, 3\pi/4) \times \mathbf{T}_t(\pi/4, 3\pi4) = (-1, 1, -\sqrt{2}) \times (-1, -1, 0) = (-\sqrt{2}, \sqrt{2}, 2).$$

The tangent plane is given by $(-\sqrt{2}, \sqrt{2}, 2) \cdot (\mathbf{x} - (1, 0, \sqrt{2})) = 0$ which is also equivalent to $-x + y + \sqrt{2}z = 1$.

**12.** The sphere of radius 3 is parametrized as $\mathbf{X}(s, t) = (3\cos s\sin t, 3\sin s\sin t, 3\sin t)$, where $0 \leq s < 2\pi$ and $0 \leq t \leq \pi$. To obtain the lower hemisphere, we need the $z$-coordinate to be nonpositive. Thus we may use the same expression for $\mathbf{X}(s, t)$, only with $0 \leq s < 2\pi$ and $\pi/2 \leq t \leq \pi$.

**13.** We may let $x = 2\cos s$, $z = 2\sin s$, and $y = t$, where $0 \leq s < 2\pi$, to parametrize the entire, infinitely long cylinder. To obtain the desired finite cylinder, we just let $D = \{(s, t) \mid 0 \leq s \leq 2\pi,\ -1 \leq t \leq 3\}$ and define $\mathbf{X}\colon D \to \mathbf{R}^3$, $\mathbf{X}(s, t) = (2\cos s, t, 2\sin s)$.

**14.** Note that the region we are describing is the part of the plane having equation $5x+10y+2z=10$, or $z=5-\frac{5}{2}x-5y$ lying in the first octant. The projection of the triangle in the $xy$-plane is the triangular region

$$\{(x,y)\mid x\geq 0,\ y\geq 0,\ 5x+10y\leq 10\}=\left\{(x,y)\mid 0\leq y\leq 1-\frac{x}{2},\ 0\leq x\leq 2\right\}.$$

(This was found by setting $z=0$ in the equation for the plane.) Hence the desired surface may be parametrized as $\mathbf{X}\colon D\to\mathbf{R}^3$, $\mathbf{X}(s,t)=\left(s,t,5-\frac{5}{2}s-5t\right)$, where $D=\left\{(s,t)\mid 0\leq t\leq 1-\frac{s}{2},\ 0\leq s\leq 2\right\}$.

**15.** If we rewrite the equation for the hyperboloid as $z^2=x^2+y^2+1$, then we see that we must have $z=\pm\sqrt{x^2+y^2+1}$. Therefore, the hyperboloid may be parametrized with two maps as $\mathbf{X}_1\colon\mathbf{R}^2\to\mathbf{R}^3$, $\mathbf{X}_1(s,t)=\left(s,t,\sqrt{s^2+t^2+1}\right)$ and $\mathbf{X}_2\colon\mathbf{R}^2\to\mathbf{R}^3$, $\mathbf{X}_2(s,t)=\left(s,t,-\sqrt{s^2+t^2+1}\right)$.

**16.** **(a)** $\mathbf{X}(1,-1)=(1,-1,-1)$ and we have $\mathbf{T}_s=(3s^2,0,t),\mathbf{T}_t=(0,3t^2,s)$. Hence the normal at (1, −1, −1), which is when $s=1,t=-1$, is $\mathbf{N}(1,-1)=\mathbf{T}_s(1,-1)\times\mathbf{T}_t(1,-1)=(3,0,-1)\times(0,3,1)=(3,-3,9)$. So an equation for the tangent plane is

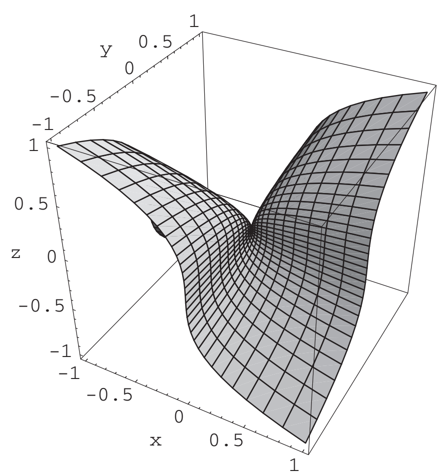
$$3(x-1)-3(y+1)+9(z+1)=0\quad\text{or}\quad x-y+3z=-1.$$

**(b)** In general we have that the standard normal is given by

$$\mathbf{N}(s,t)=\mathbf{T}_s\times\mathbf{T}_t=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3s^2&0&t\\0&3t^2&s\end{vmatrix}=(-3t^3,-3s^3,9s^2t^2).$$

Note that $\mathbf{N}=\mathbf{0}$ when $s=t=0$, i.e., at (0, 0, 0). So the surface fails to be smooth there.

**(c)** A computer graph is shown.




**(d)** With $x=s^3$, $y=t^3$, $z=st$, $\sqrt[3]{xy}=\sqrt[3]{s^3t^3}=st=z$. Sometimes a computer will graph $z=\sqrt[3]{xy}$ for points only where $x$ and $y$ are nonnegative (or sometimes where $xy\geq 0$).

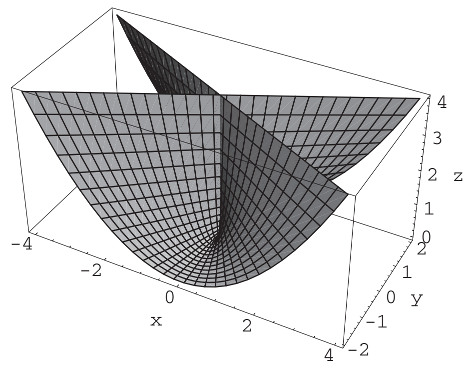
**17.** **(a)** $y^2z=t^2\cdot s^2=(st)^2=x^2$

**(b)** The standard normal is

$$\mathbf{N}(s,t)=\mathbf{T}_s\times\mathbf{T}_t=(t,0,2s)\times(s,1,0)$$
$$=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\t&0&2s\\s&1&0\end{vmatrix}=(-2s,2s^2,t).$$

So $\mathbf{N}=\mathbf{0}$ when $s=t=0$, i.e., at (0, 0, 0). At this point $\mathbf{X}$ fails to be smooth.

**(c)** A computer graph is shown.




**(d)** $\mathbf{X}(s_1, t_1) = \mathbf{X}(s_2, t_2)$ when $s_1t_1 = s_2t_2$, $t_1 = t_2$, $s_1^2 = s_2^2$. Thus if $t_1 = t_2 = 0$ and $s_1 = \pm s_2$ we get the same image—i.e., $\mathbf{X}(s, 0) = \mathbf{X}(-s, 0) = (0, 0, s^2)$. Thus the positive $z$-axis (which lies on the image of **X**) is *not* uniquely determined.

**(e)** Note that $(2, 1, 4) = \mathbf{X}(2, 1)$. From work in part (b), $\mathbf{N}(2, 1) = (-4, 8, 1)$ so an equation for the tangent plane is $-4(x-2) + 8(y-1) + 1(z-4) = 0$ or $-4x + 8y + z = 4$.

**(f)** $(0, 0, 1) = \mathbf{X}(-1, 0) = \mathbf{X}(1, 0)$.

$$\mathbf{N}(-1, 0) = (2, 2, 0) \quad \mathbf{N}(1, 0) = (-2, 2, 0)$$

So the corresponding tangent planes have equations $x + y = 0$ and $x - y = 0$ respectively.
(If you look at the graph in part (c), you can see two parts of the surface intersecting, so this makes sense.)

**18.** Here we generalize the results of parts (a) and (c) of Exercise 11. If we view $S$ as the graph of a function $f(x, y)$ then we can apply formula (4) of Section 2.3:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We can rewrite this equation as $0 = (f_x(a, b), f_y(a, b), -1) \cdot (\mathbf{x} - (a, b, f(a, b)))$, where $a = x(s_0, t_0), b = y(s_0, t_0)$, and $f(a, b) = z(s_0, t_0)$. In other words, we are also considering $S$ to be a surface that is parametrized by $\mathbf{X}(s, t) = (x(s, t), y(s, t), z(s, t))$ and so, using the Chain Rule,

$$\mathbf{T}_s(s, t) = (x_s(s, t), y_s(s, t), f_x(x, y)x_s(s, t) + f_y(x, y)y_s(s, t)) \quad \text{and}$$
$$\mathbf{T}_t(s, t) = (x_t(s, t), y_t(s, t), f_x(x, y)x_t(s, t) + f_y(x, y)y_t(s, t)).$$

We calculate the normal vector **N** by taking the cross product $\mathbf{T}_s \times \mathbf{T}_t$ and simplifying to obtain

$$\mathbf{N}(s, t) = [x_t(s, t)y_s(s, t) - x_s(s, t)y_t(s, t)](f_x(x, y), f_y(x, y), -1).$$

So an equation of the tangent plane at $(s_0, t_0)$ is $\mathbf{N} \cdot (\mathbf{x} - \mathbf{X}(s_0, t_0)) = 0$ which in this case is

$$[x_t(s_0, t_0)y_s(s_0, t_0) - x_s(s_0, t_0)y_t(s_0, t_0)](f_x(a, b), f_y(a, b), -1) \cdot (\mathbf{x} - (a, b, f(a, b))) = 0 \quad \text{or}$$
$$(f_x(a, b), f_y(a, b), -1) \cdot (\mathbf{x} - (a, b, f(a, b))) = 0.$$

So we see that in this case the results of the two methods agree.

**19. (a)** To find an equation for the tangent plane to a surface described by the equation $y = g(x, z)$ at the point $(a, g(a, c), c)$ we basically permute the case detailed in the text and in Exercise 18 to obtain either

$$(g_x(a, c), -1, g_z(a, c)) \cdot (\mathbf{x} - (a, g(a, c), c)) = 0 \quad \text{or}$$
$$g_x(a, c)(x - a) - (y - g(a, c)) + g_z(a, c)(z - c) = 0.$$

**(b)** Similarly, an equation for the tangent plane to a surface described by the equation $x = h(y, z)$ at the point $(h(b, c), b, c)$ is either

$$(-1, h_y(b, c), h_z(b, c)) \cdot (\mathbf{x} - (h(b, c), b, c)) = 0 \quad \text{or}$$
$$-(x - h(b, c)) + h_y(b, c)(y - b) + h_z(b, c)(z - c) = 0.$$

**20.** We have $\mathbf{X}\colon D \to \mathbf{R}^3$ and by Definition 3.8 of Chapter 2, the linear approximation is given by

$$\mathbf{x} = \mathbf{X}(s_0, t_0) + D\mathbf{X}(s_0, t_0) \begin{bmatrix} s - s_0 \\ t - t_0 \end{bmatrix}.$$

Here $D\mathbf{X}(s_0, t_0)$ is the matrix

$$D\mathbf{X}(s_0, t_0) = \begin{bmatrix} x_s(s_0, t_0) & x_t(s_0, t_0) \\ y_s(s_0, t_0) & y_t(s_0, t_0) \\ z_s(s_0, t_0) & z_t(s_0, t_0) \end{bmatrix} = [(\mathbf{T}_s(s_0, t_0))^T \quad (\mathbf{T}_t(s_0, t_0))^T].$$

Thus the tangent plane to the surface is given by

$$\begin{aligned} (x, y, z) &= \mathbf{X}(s_0, t_0) + [(\mathbf{T}_s(s_0, t_0))^T \quad (\mathbf{T}_t(s_0, t_0))^T] \begin{bmatrix} s - s_0 \\ t - t_0 \end{bmatrix} \\ &= \mathbf{X}(s_0, t_0) + \mathbf{T}_s(s_0, t_0)(s - s_0) + \mathbf{T}_t(s_0, t_0)(t - t_0). \end{aligned}$$

**21.** By Exercise 20,

$$\begin{aligned} (x, y, z) &= (1, 0, 1) + \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} s - 1 \\ t + 1 \end{bmatrix} \\ (x, y, z) &= (s, 2s + t - 1, -2t - 1). \end{aligned}$$

We check this against our result for Exercise 5(c):

$$4x - 2y - z = 4s - 2(2s + t - 1) - (-2t - 1) = 3.$$

**22.** In Exercise 3 we parametrized a cylinder of radius $a$ and height $h$ by $\mathbf{X}(s, t) = (a\cos s, a\sin s, t)$ for $0 \le t \le h$ and $0 \le s < 2\pi$. Then $\mathbf{T}_s(s, t) = (-a\sin s, a\cos s, 0)$, $\mathbf{T}_t(s, t) = (0, 0, 1)$, and $\mathbf{T}_s(s, t) \times \mathbf{T}_t(s, t) = (a\cos s, a\sin s, 0)$. Then, by formula (6), the surface area of $S$ is

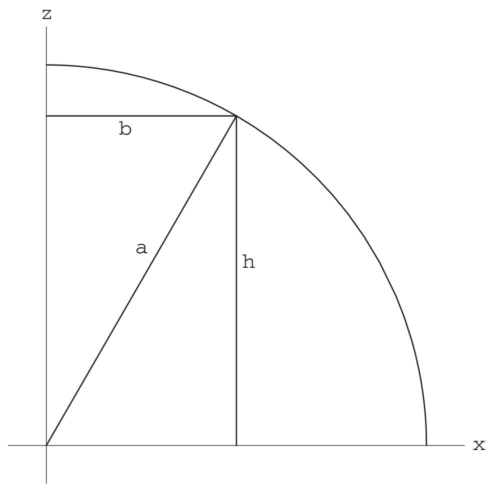
$$\int_0^{2\pi} \int_0^h \|\mathbf{T}_s(s, t) \times \mathbf{T}_t(s, t)\|\, dt\, ds = \int_0^{2\pi} \int_0^h a\, dt\, ds = 2\pi a h.$$

**23.** As in Exercise 22 we need to calculate $\|\mathbf{T}_s(s, t) \times \mathbf{T}_t(s, t)\|$. We have that $\mathbf{X}(s, t) = (s + t, s - t, s)$ for $-1 \le s \le 1$ and $-\sqrt{1 - s^2} \le t \le \sqrt{1 - s^2}$. Therefore, $\mathbf{T}_s(s, t) = (1, 1, 1), \mathbf{T}_t(s, t) = (1, -1, 0), \mathbf{T}_s(s, t) \times \mathbf{T}_t(s, t) = (1, 1, -2)$ and $\|\mathbf{T}_s(s, t) \times \mathbf{T}_t(s, t)\| = \sqrt{6}$. So we are integrating $\sqrt{6}$ over the unit disk in the *st*-plane. Therefore, the surface area of $\mathbf{X}(D) = \iint_D \sqrt{6}\, dt\, ds = \sqrt{6}\pi$.

**24.** For the parametrization of the helicoid, $\mathbf{X}(r, \theta) = (r\cos\theta, r\sin\theta, \theta)$ so $\mathbf{T}_r(r, \theta) = (\cos\theta, \sin\theta, 0)$, $\mathbf{T}_\theta(r, \theta) = (-r\sin\theta, r\cos\theta, 1)$, $\mathbf{T}_r(r, \theta) \times \mathbf{T}_\theta(r, \theta) = (\sin\theta, -\cos\theta, r)$ and $\|\mathbf{T}_r(r, \theta) \times \mathbf{T}_\theta(r, \theta)\| = \sqrt{1 + r^2}$. Then the surface area of $n$ "turns" of the helicoid is

$$\begin{aligned} \int_0^{2\pi n} \int_0^1 \sqrt{1 + r^2}\, dr\, d\theta &= \int_0^{2\pi n} \frac{1}{2}[\sqrt{2} + \sinh^{-1}(1)]\, d\theta = \int_0^{2\pi n} \frac{1}{2}[\sqrt{2} + \ln(\sqrt{2} + 1)]\, d\theta \\ &= [\sqrt{2} + \ln(\sqrt{2} + 1)]\pi n. \end{aligned}$$

**25.** A quick look at the figure below shows a cutaway of a quarter of the *xz*-plane intersection of the cylindrical hole of radius $b$ bored in a sphere of radius $a$. The height of the hole is $2\sqrt{a^2 - b^2}$. The top half of the ring is the region swept out by the portion of the diagram containing the letter 'h'.




If $\mathbf{X}(s,t) = (a\sin s\cos t, a\sin s\sin t, a\cos s)$, then $\mathbf{T}_s(s,t) = (a\cos s\cos t, a\cos s\sin t, -a\sin s)$, $\mathbf{T}_t(s,t) = (-a\sin s\sin t, a\sin s\cos t, 0)$, $\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = a^2\sin s(\sin s\cos t, \sin s\sin t, \cos s)$ and $\|\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t)\| = a^2\sin s$. Notice that the angle $s$ made with the $z$-axis has lower limit $\cos^{-1}(h/a) = \cos^{-1}(\sqrt{a^2-b^2}/a)$ and upper limit $\pi/2$. So the surface area is

$$2\int_0^{2\pi}\int_{\cos^{-1}(\sqrt{a^2-b^2}/a)}^{\pi/2} a^2\sin s\,ds\,dt = 2\int_0^{2\pi} a^2\left(\frac{\sqrt{a^2-b^2}}{a}\right)dt = 4\pi a\sqrt{a^2-b^2}.$$

**26.** The parametrization of the paraboloid is $\mathbf{X}(s,t) = (s\cos t, s\sin t, 9-s^2)$ where $0\le t\le 2\pi$ and $0\le s\le 3$. So $\mathbf{T}_s(s,t) = (\cos t, \sin t, -2s)$, $\mathbf{T}_t(s,t) = (-s\sin t, s\cos t, 0)$, $\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = (2s^2\cos t, 2s^2\sin t, s)$, and $\|\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t)\| = s\sqrt{4s^2+1}$. The surface area is then

$$\int_0^{2\pi}\int_0^3 s\sqrt{4s^2+1}\,ds\,dt = \frac{1}{12}\int_0^{2\pi}\left[(1+4s^2)^{3/2}\right]\Big|_0^3 dt = \frac{\pi}{6}(37^{3/2}-1).$$

**27.** We'll parametrize the surface by $\mathbf{X}(s,t) = (s\cos t, s\sin t, 2s^2)$ for $0\le t\le 2\pi$ and $1\le s\le 2$. So $\mathbf{T}_s(s,t) = (\cos t, \sin t, 4s)$, $\mathbf{T}_t(s,t) = (-s\sin t, s\cos t, 0)$, $\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = (4s^2\cos t, 4s^2\sin t, s)$, and $\|\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t)\| = s\sqrt{16s^2+1}$. So the surface area is

$$\int_0^{2\pi}\int_1^2 s\sqrt{16s^2+1}\,ds\,dt = \frac{1}{48}\int_0^{2\pi}(65^{3/2}-17^{3/2})\,dt = \frac{\pi}{24}(65^{3/2}-17^{3/2}).$$

**28.** **(a)** First we use the parametrization $\mathbf{X}(s,t) = (s, t, a-s-t)$ and calculate $\mathbf{T}_s(s,t) = (1,0,-1)$, $\mathbf{T}_t(s,t) = (0,1,-1)$, $\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = (1,1,1)$ and $\|\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t)\| = \sqrt{3}$. The surface area is then the integral of $\sqrt{3}$ over the disk of radius $a$, which is $\iint_D \sqrt{3}\,ds\,dt = \sqrt{3}\pi a^2$.

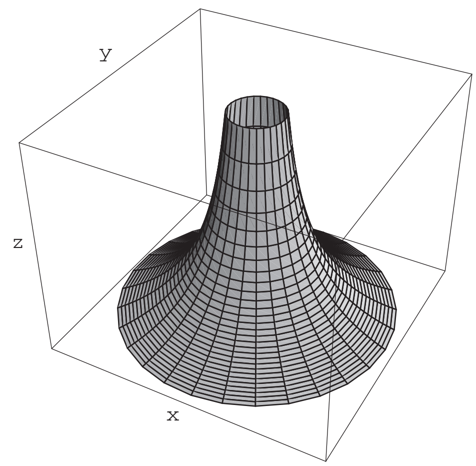
**(b)** To use formula (9), we view the surface as $z = f(x,y) = a-x-y$, so $f_x(x,y) = -1$ and $f_y(x,y) = -1$. Therefore, formula (9) gives the surface area as

$$\iint_D \sqrt{(-1)^2+(-1)^2+1}\,dx\,dy = \iint_D \sqrt{3}\,dx\,dy = \sqrt{3}\pi a^2.$$

**29.** We have $z = f(x,y)$ and $f_x^2+f_y^2 = a$ so, by formula (9), the surface area is

$$\iint_D \sqrt{f_x^2+f_y^2+1}\,dx\,dy = \iint_D \sqrt{a+1}\,dx\,dy = \sqrt{a+1}(\text{area of } D).$$

**30.** **(a)** Here is a sketch of the surface for $z \geq 1$.




**(b)** We can calculate the volume under the infinite funnel by disks:

$$\int_1^\infty \frac{\pi}{z^2}\,dz = \lim_{b\to\infty} -\frac{\pi}{z}\Big|_1^b = \pi.$$

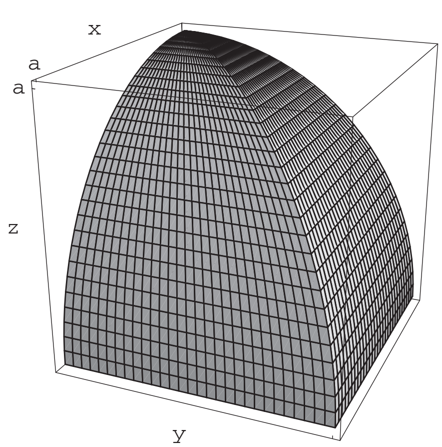
**(c)** To calculate the surface area, we'll parametrize the funnel as $\mathbf{X}(s,t) = \left(s\cos t, s\sin t, \frac{1}{s}\right)$, where $0 < s \leq 1$ and $0 \leq t < 2\pi$. Then $\mathbf{T}_s(s,t) = \left(\cos t, \sin t, -\frac{1}{s^2}\right)$, $\mathbf{T}_t(s,t) = (-s\sin t, s\cos t, 0)$, $\mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t) = \left(\frac{1}{s}\cos t, \frac{1}{s}\sin t, s\right)$ and $\|\mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t)\| = \sqrt{\frac{1}{s^2} + s^2}$. Therefore, using tables or a computer algebra system, we see that the surface area is given by

$$\int_0^{2\pi}\int_0^1 \sqrt{\frac{1}{s^2}+s^2}\,ds\,dt = \lim_{a\to 0+}\int_0^{2\pi}\int_0^1 \sqrt{\frac{1}{s^2}+s^2}\,ds\,dt$$

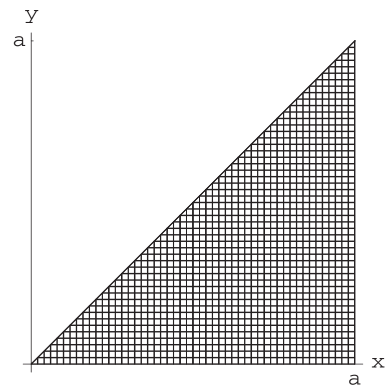
$$\pi \lim_{a\to 0+}[\sqrt{2} - \ln(\sqrt{2}+1) - (\sqrt{a^4+1} - \ln(1+\sqrt{1+a^4}) + \ln(a^2))].$$

Each term in this last expression possesses a finite limit except $\ln(a^2)$. Since $\lim_{a\to 0+}\ln(a^2) = -\infty$, we see that the surface area is infinite.

**31.** The first octant portion of the intersection is shown below.



Note that 1/16 of the total surface area is that of the graph of $z = \sqrt{a^2 - x^2}$ lying over the triangular region bounded by $y = x, x = a$, and $y = 0$.




For $z = \sqrt{a^2 - x^2}$, $\dfrac{\partial z}{\partial x} = -\dfrac{x}{\sqrt{a^2 - x^2}}$ and $\dfrac{\partial z}{\partial y} = 0$. Hence

$$\begin{aligned}\text{Surface area} &= 16 \int_0^a \int_0^x \sqrt{\frac{x^2}{a^2 - x^2} + 0 + 1}\, dy\, dx = 16 \int_0^a x \sqrt{\frac{x^2 + a^2 - x^2}{a^2 - x^2}}\, dx \\ &= 16 \int_0^a \frac{ax}{\sqrt{a^2 - x^2}}\, dx.\end{aligned}$$

Let $u = a^2 - x^2$ so $du = -2x\, dx$. Then

$$\begin{aligned}\text{Surface area} &= -8a \int_{a^2}^0 \frac{du}{\sqrt{u}} = 8a \int_0^{a^2} u^{-1/2}\, du = 8a \cdot 2u^{1/2} \Big|_0^{a^2} \\ &= 16a^2.\end{aligned}$$

**32.** We have $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = f(r,\theta) \end{cases}$ $(r,\theta) \in D$. Therefore,

$$\mathbf{T}_r = \left(\cos\theta, \sin\theta, \frac{\partial f}{\partial r}\right), \quad \mathbf{T}_\theta = \left(-r\sin\theta, r\cos\theta, \frac{\partial f}{\partial \theta}\right).$$

So

$$\mathbf{N}(r,\theta) = \mathbf{T}_r \times \mathbf{T}_\theta = \left(\sin\theta \frac{\partial f}{\partial \theta} - r\cos\theta \frac{\partial f}{\partial r}, -r\sin\theta \frac{\partial f}{\partial r} - \cos\theta \frac{\partial f}{\partial \theta}, r\cos^2\theta + r\sin^2\theta\right).$$

Hence

$$\begin{aligned}\|\mathbf{N}\|^2 &= \sin^2\theta \left(\frac{\partial f}{\partial \theta}\right)^2 - 2r\sin\theta\cos\theta \frac{\partial f}{\partial r}\frac{\partial f}{\partial \theta} + r^2\cos^2\theta \left(\frac{\partial f}{\partial r}\right)^2 \\ &\quad + \cos^2\theta \left(\frac{\partial f}{\partial \theta}\right)^2 + 2r\sin\theta\cos\theta \frac{\partial f}{\partial r}\frac{\partial f}{\partial \theta} + r^2\sin^2\theta \left(\frac{\partial f}{\partial r}\right)^2 + r^2 \\ &= \left(\frac{\partial f}{\partial \theta}\right)^2 + r^2\left(\left(\frac{\partial f}{\partial r}\right)^2 + 1\right).\end{aligned}$$

Thus we have

$$\|\mathbf{N}\| = r\sqrt{\frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 + \left(\frac{\partial f}{\partial r}\right)^2 + 1} \quad \text{and}$$

Surface area $= \iint_D r\sqrt{\frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2 + \left(\frac{\partial f}{\partial r}\right)^2 + 1}\, dr\, d\theta$, using formula (6) in §7.1.

**33.** We have $\begin{cases} x = f(\varphi,\theta)\sin\varphi\cos\theta \\ y = f(\varphi,\theta)\sin\varphi\sin\theta \\ z = f(\varphi,\theta)\cos\varphi \end{cases}$ from spherical/Cartesian conversions. From this,

$$\mathbf{T}_\varphi = (f_\varphi \sin\varphi\cos\theta + f\cos\varphi\cos\theta, f_\varphi\sin\varphi\sin\theta + f\cos\varphi\sin\theta, f_\varphi\cos\varphi - f\sin\varphi)$$
$$\mathbf{T}_\theta = (f_\theta\sin\varphi\cos\theta - f\sin\varphi\sin\theta, f_\theta\sin\varphi\sin\theta + f\sin\varphi\cos\theta, f_\theta\cos\varphi).$$

After some careful computation and using $\cos^2\alpha + \sin^2\alpha = 1$, we find $\mathbf{N}(\varphi,\theta) = \mathbf{T}_\varphi \times \mathbf{T}_\theta = (ff_\theta\sin\theta - ff_\varphi\sin\varphi\cos\varphi\cos\theta + f^2\sin^2\varphi\cos\theta, f^2\sin^2\varphi\sin\theta - ff_\varphi\sin\varphi\cos\varphi\sin\theta - ff_\theta\cos\theta, f^2\sin\varphi\cos\varphi + ff_\varphi\sin^2\varphi)$. After still more computation, one finds $\|\mathbf{N}\|^2 = (ff_\theta)^2 + (ff_\varphi)^2\sin^2\varphi + f^4\sin^2\varphi$, so that using formula (6) in §7.1,

$$\begin{aligned}\text{Surface area} &= \iint_D \sqrt{(ff_\theta)^2 + (ff_\varphi)^2\sin^2\varphi + f^4\sin^2\varphi}\, d\varphi\, d\theta \\ &= \iint_D f(\varphi,\theta)\sqrt{f_\theta^2 + \sin^2\varphi(f_\varphi + f^2)}\, d\varphi\, d\theta.\end{aligned}$$

## 7.2 Surface Integrals

*Many of your students will apply the formulas and techniques introduced in this section by first finding a parametrization for the given surface. In many cases, as was shown in the text, if they examine the geometry of the surface, an easier solution might present itself. In several of the solutions below, each approach is outlined.*

**1.** We will use Definition 2.1 to calculate the integral: $\iint_{\mathbf{X}} f\, dS = \iint_D f(\mathbf{X}(s,t))\|\mathbf{T}_s \times \mathbf{T}_t\|\, ds\, dt$. Here $\mathbf{X}(s,t) = (s, s+t, t)$, $\mathbf{T}_s(s,t) = (1,1,0)$, $\mathbf{T}_t(s,t) = (0,1,1)$, $\mathbf{N}(s,t) = \mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = (1,-1,1)$, and $\|\mathbf{N}(s,t)\| = \sqrt{3}$. Also, $f(\mathbf{X}(s,t)) = s^2 + (s+t)^2 + t^2 = 2(s^2 + st + t^2)$. So

$$\begin{aligned}\iint_{\mathbf{X}} (x^2+y^2+z^2)\, dS &= 2\sqrt{3}\int_0^2\int_0^1 (s^2+st+t^2)\, ds\, dt = 2\sqrt{3}\int_0^2\left(\frac{1}{3} + \frac{t}{2} + t^2\right) dt \\ &= 2\sqrt{3}\left(\frac{2}{3} + 1 + \frac{8}{3}\right) = \frac{26}{\sqrt{3}}.\end{aligned}$$

**2. (a)** Since $\mathbf{X}(s,t) = (s+t, s-t, st)$, we can calculate $\mathbf{T}_s(s,t) = (1,1,t)$, $\mathbf{T}_t(s,t) = (1,-1,s)$, $\mathbf{N}(s,t) = \mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t) = (s+t, t-s, -2)$, and $\|\mathbf{N}(s,t)\| = \sqrt{2s^2+2t^2+4}$. Using polar coordinates in the double integral, we obtain

$$\begin{aligned}\iint_{\mathbf{X}} 4\, dS &= \iint_D 4\sqrt{2s^2+2t^2+4}\, ds\, dt = \int_0^{\pi/2}\int_0^1 4r\sqrt{2r^2+4}\, dr\, d\theta \\ &= \frac{2}{3}\int_0^{\pi/2}[6\sqrt{6}-8]\, d\theta = \frac{\pi}{3}[6\sqrt{6}-8].\end{aligned}$$

**(b)** By Definition 2.2, $\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{X}(s,t))\cdot\mathbf{N}(s,t)\, ds\, dt$. Here $\mathbf{F}(\mathbf{X}(s,t)) = (s+t, s-t, st)$ and so, from part (a), we know that $\mathbf{N}(s,t) = (s+t, t-s, -2)$. This means that $\mathbf{F}\cdot\mathbf{N} = (s+t)^2 - (s-t)^2 - 2st = 2st$. Therefore,

$$\begin{aligned}\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S} &= \iint_D 2st\, ds\, dt = \int_0^1\int_0^{\sqrt{1-t^2}} 2st\, ds\, dt \\ &= \int_0^1 (s^2 t)\Big|_0^{\sqrt{1-t^2}} dt = \int_0^1 (t - t^3)\, dt = \left(\frac{t^2}{2} - \frac{t^4}{4}\right)\Big|_0^1 = \frac{1}{4}.\end{aligned}$$

**3.** We need to calculate $\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S}$. The surface is given by a level set of $f(x,y,z) = 2x - 2y + z$. Since $\nabla f = (2,-2,1)$, the upward-pointing unit normal is $\frac{1}{3}(2,-2,1)$. So, since $2x-2y+z=2$,

$$\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S} = \frac{1}{3}\iint_S (x,y,z)\cdot(2,-2,1)\,dS = \frac{1}{3}\iint_S (2x-2y+z)\,dS$$
$$= \frac{1}{3}\iint_S (2)dS = \frac{2}{3}\iint_S dS = \frac{2}{3}\|\mathbf{N}\|(\text{area of } D) = 2(\text{area of } D).$$

Here $D$ is the "shadow" of $S$ in the *xy*-plane. $D$ is a right triangle in the *xy*-plane with legs each of length 1. Hence

$$\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S} = 2(\text{area of } D) = (2)((1/2)(1)(1)) = 1.$$

**4.** **(a)** You can easily verify that both **X** and **Y** parametrize the surface $z=3x^2+3y^2$ for $0\le x^2+y^2\le 4$. The major difference is that **X** covers the surface once while **Y** covers the surface twice.

**(b)** For **X**, the standard normal **N** is

$$(\cos t,\sin t,6s)\times(-s\sin t, s\cos t, 0) = (-6s^2\cos t, -6s^2\sin t, s)$$

so

$$\iint_{\mathbf{X}} (y\mathbf{i}-x\mathbf{j}+z^2\mathbf{k})\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^2 (s\sin t, -s\cos t, 9s^4)\cdot(-6s^2\cos t, -6s^2\sin t, s)\,ds\,dt$$
$$= \int_0^{2\pi}\int_0^2 9s^5\,ds\,dt = \int_0^{2\pi} \left.\frac{9s^6}{6}\right|_0^2 dt = \int_0^{2\pi} 96\,dt = 192\pi.$$

For **Y**, the standard normal **N** is

$$(2\cos t, 2\sin t, 24s)\times(-2s\sin t, 2s\cos t, 0) = (-48s^2\cos t, -48s^2\sin t, 4s)$$

so

$$\iint_{\mathbf{Y}} (y\mathbf{i}-x\mathbf{j}+z^2\mathbf{k})\cdot d\mathbf{S} = \int_0^{4\pi}\int_0^1 (2s\sin t, -2s\cos t, 144s^4)\cdot(-48s^2\cos t, -48s^2\sin t, 4s)\,ds\,dt$$
$$= \int_0^{4\pi}\int_0^1 576s^5\,ds\,dt = \int_0^{4\pi} \left.\frac{576s^6}{6}\right|_0^1 dt = \int_0^{4\pi} 96\,dt = 384\pi.$$

As noted in part (a), the integral over **Y** should be twice the integral over **X** since they both parametrize the same space but **Y** covers the space twice.

**5.** We will parametrize the six faces of the cube as follows (in each case $-2\le s,t\le 2$):

| $i$ | $\mathbf{X}(s,t)$ for $S_i$ | face |
|---|---|---|
| 1 | $(s,t,2)$ | top |
| 2 | $(s,t,-2)$ | bottom |
| 3 | $(s,2,t)$ | right |
| 4 | $(s,-2,t)$ | left |
| 5 | $(2,s,t)$ | front |
| 6 | $(-2,s,t)$ | back |

Note that in each case $\|\mathbf{N}(s,t)\| = 1$, so $\iint_{S_i} [x(s,t)]^2\|\mathbf{N}(s,t)\|\,ds\,dt = \iint_{S_i} [x(s,t)]^2 ds\,dt$ for $1\le i\le 6$. Also,

$\iint_{S_i} [x(s,t)]^2\, ds\, dt = \int_{-2}^{2}\int_{-2}^{2} s^2\, ds\, dt$ for $i = 1, 2, 3, 4$ and $\iint_{S_i} [x(s,t)]^2\, ds\, dt = \int_{-2}^{2}\int_{-2}^{2} 4\, ds\, dt$ for $i = 5, 6$. Then

$$\begin{aligned}\iint_S x^2\, dS &= \sum_{i=1}^{6} \iint_{S_i} [x(s,t)]^2 \|\mathbf{N}(s,t)\|\, ds\, dt \\ &= 4\int_{-2}^{2}\int_{-2}^{2} s^2\, ds\, dt + 2\int_{-2}^{2}\int_{-2}^{2} 4\, ds\, dt \\ &= 4\int_{-2}^{2} \frac{s^3}{3}\bigg|_{-2}^{2} dt + 8\int_{-2}^{2} s\big|_{-2}^{2}\, dt \\ &= 4\int_{-2}^{2} \frac{16}{3}\, dt + 8\int_{-2}^{2} 4\, dt = \frac{256}{3} + 128 = \frac{640}{3}.\end{aligned}$$

**6.** We parametrize the lateral surface of the cylinder by $\mathbf{X}(s,t) = (a\cos s, a\sin s, t)$ where $0 \le s \le 2\pi$ and $0 \le t \le h$. So we have $\mathbf{T}_s(s,t) = (-a\sin s, a\cos s, 0)$, $\mathbf{T}_t(s,t) = (0,0,1)$, $\mathbf{N}(s,t) = \mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t) = (a\cos s, a\sin s, 0)$, and $\|\mathbf{N}(s,t)\| = a$. So

$$\iint_S (x^2 + y^2)\, dS = \int_0^{2\pi}\int_0^h (a^2\cos^2 s + a^2\sin^2 s) a\, dt\, ds = \int_0^{2\pi}\int_0^h a^3\, dt\, ds = 2\pi h a^3.$$

A quicker approach is to note that on the cylinder $x^2 + y^2 = a^2$, so

$$\iint_S (x^2 + y^2)\, dS = \iint_S a^2\, dS = a^2 \cdot \text{area of } S = a^2(2\pi a h) = 2\pi h a^3.$$

**7. (a)** Because $x^2 + y^2 + z^2 = a^2$ on the surface,

$$\iint_S (x^2 + y^2 + z^2)\, dS = a^2(\text{surface area of } S) = 4\pi a^4.$$

**(b)** Here we note that by part (a)

$$\iint_S x^2\, dS + \iint_S y^2\, dS + \iint_S z^2\, dS = 4\pi a^4$$

and by the symmetries of the sphere

$$\iint_S x^2\, dS = \iint_S y^2\, dS = \iint_S z^2\, dS. \quad \text{So} \quad \iint_S y^2\, dS = 4\pi a^4/3.$$

**8. (a)** The sphere is symmetric about the plane $x = 0$. Hence $\iint_S x\, dS = 0$ as for each small piece of the sphere with coordinate $x > 0$ (and $x \le a$), there is a corresponding piece with coordinate $x < 0$. Hence contributions in an appropriate Riemann sum will cancel.

**(b)** For $x^2 + y^2 + z^2 = a^2$ the outward unit normal is given by $\mathbf{n} = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$. Thus

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \frac{1}{a}(x + y + z)\, dS \\ &= \frac{1}{a}\left(\iint_S x\, dS + \iint_S y\, dS + \iint_S z\, dS\right) = 0\end{aligned}$$

since each surface integral is zero via reasoning as in part (a).

**9. (a)** We parametrize the cylinder as $\begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = s \end{cases} \quad 0 \le t < 2\pi,\ -2 \le s \le 2.$

Then

$$\begin{aligned}\|\mathbf{T}_s \times \mathbf{T}_t\| &= \|(0,0,1) \times (-2\sin t, 2\cos t, 0)\| = \|(-2\cos t, -2\sin t, 0)\| \\ &= 2.\end{aligned}$$

Hence

$$\iint_S (z - x^2 - y^2)\,dS = \int_0^{2\pi}\int_{-2}^{2}(s-4)\cdot 2\,ds\,dt = \int_0^{2\pi}(s^2-8s)\Big|_{s=-2}^{2}\,dt$$
$$= \int_0^{2\pi} -32\,dt = -64\pi.$$

**(b)** $\iint_S (z - x^2 - y^2)\,dS = \iint_S z\,dS - \iint_S (x^2+y^2)\,dS$. $S$ is symmetric about the $z=0$ plane and $x^2+y^2=4$ on $S$. Hence $\iint_S z\,dS = 0$ and $-\iint_S (x^2+y^2)\,dS = -\iint_S 4\,dS = -4\cdot(\text{surface area of } S) = -4(4\pi\cdot 4) = -64\pi$.

*The following calculations are useful for Exercises 10–18. Let's parametrize the surface of the cylinder in three pieces:*

- $S_1$ = the lateral surface, $\mathbf{X}(s,t) = (3\cos s, 3\sin s, t)$ for $0\le s\le 2\pi$ and $0\le t\le 4$.
- $S_2$ = the bottom surface, $\mathbf{X}(s,t) = (t\cos s, t\sin s, 0)$ for $0\le s\le 2\pi$ and $0\le t\le 3$.
- $S_3$ = the top surface, $\mathbf{X}(s,t) = (t\cos s, t\sin s, 4)$ for $0\le s\le 2\pi$ and $0\le t\le 3$.

For $S_1$, $\mathbf{T}_s(s,t) = (-3\sin s, 3\cos s, 0)$, $\mathbf{T}_t(s,t) = (0,0,1)$, $\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = (3\cos s, 3\sin s, 0)$, and $\|\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t)\| = 3$. For both $S_2$ and $S_3$, $\mathbf{T}_s(s,t) = (-t\sin s, t\cos s, 0)$, $\mathbf{T}_t(s,t) = (\cos s, \sin s, 0)$, $\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = (0,0,-t)$, and $\|\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t)\| = t$. Because we are orienting with outward normals, $\mathbf{N}(s,t) = (0,0,-t)$ on $S_2$ and $\mathbf{N}(s,t) = (0,0,t)$ on $S_3$.

*In Exercises 10–13 we use Definition 2.1:* $\iint_{\mathbf{X}} f\,dS = \iint_D f(\mathbf{X}(s,t))\|\mathbf{N}(s,t)\|\,ds\,dt$. And we'll break down the integral as $\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$.

**10.** $\iint_S z\,dS = \int_0^{2\pi}\int_0^4 3t\,dt\,ds + \int_0^{2\pi}\int_0^3 0\,dt\,ds + \int_0^{2\pi}\int_0^3 4t\,dt\,ds = 48\pi + 36\pi = 84\pi$.

**11.** $\iint_S y\,dS = \int_0^4\int_0^{2\pi} 9\cos s\,ds\,dt + 2\int_0^3\int_0^{2\pi} t^2\cos s\,ds\,dt = 0 + 0 = 0$. Alternatively, you could notice that we are integrating an odd function of $y$ over a region that is symmetric with respect to $y$.

**12.** $\iint_S xyz\,dS = \int_0^4\int_0^{2\pi} 27t\cos s\sin s\,ds\,dt + \int_0^3\int_0^{2\pi} 0\,ds\,dt + \int_0^3\int_0^{2\pi} 4t^3\cos s\sin s\,ds\,dt = 0$. Use the substitution $u = \sin s$. Again, alternatively, you could use a symmetry argument. We are again integrating an odd function of $y$ over a region that is symmetric with respect to $y$.

**13.**

$$\iint_S x^2\,dS = \int_0^4\int_0^{2\pi} 27\cos^2 s\,ds\,dt + 2\int_0^3\int_0^{2\pi} t^3\cos^2 s\,ds\,dt$$
$$= 27\int_0^4 \left[\frac{s}{2} + \frac{1}{9}\sin 2s\right]\Big|_0^{2\pi} dt + 2\int_0^3 t^3\left[\frac{s}{2} + \frac{1}{4}\sin 2s\right]\Big|_0^{2\pi} dt = 27\int_0^4 \pi\,dt + 2\int_0^3 \pi t^3\,dt$$
$$= 108\pi + \frac{81\pi}{2} = \frac{297\pi}{2}.$$

*For Exercises 14–18, we use Definition 2.2:* $\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathbf{X}} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D \mathbf{F}(\mathbf{X}(s,t))\cdot\mathbf{N}(s,t)\,ds\,dt$. *For another way of solving these exercises, recall from Section 2.6, that if $S$ is a surface in $\mathbf{R}^3$ defined by an equation of the form $f(x,y,z) = c$, then if $\mathbf{x}_0 \in X$, the gradient vector $\nabla f(\mathbf{x}_0)$ is a vector normal to the plane tangent to $S$ at $\mathbf{x}_0$. Therefore the unit normal to $S_1$ (a surface given by $x^2+y^2=9$) is $\mathbf{n} = (x\mathbf{i} + y\mathbf{j})/3$, while the unit normal to $S_2$ is $-\mathbf{k}$ and the unit normal to $S_3$ is $\mathbf{k}$.*

**14.**

$$\iint_S (x\mathbf{i} + y\mathbf{j})\cdot d\mathbf{S} = \int_0^4\int_0^{2\pi} (3\cos s, 3\sin s, 0)\cdot(3\cos s, 3\sin s, 0)\,ds\,dt$$
$$+ \int_0^3\int_0^{2\pi} (t\cos s, t\sin s, 0)\cdot(0,0,-t)\,ds\,dt + \int_0^3\int_0^{2\pi} (t\cos s, t\sin s, 0)\cdot(0,0,t)\,ds\,dt$$
$$= \int_0^4\int_0^{2\pi} 9\,ds\,dt = 72\pi.$$

A different approach would be to observe that as the unit normals for $S_2$ and $S_3$ are $\pm\mathbf{k}$ then $\mathbf{F}\cdot\mathbf{n} = 0$ on $S_2$ and $S_3$. On $S_1$ the unit normal is $(x\mathbf{i}+y\mathbf{j})/3$ So $\mathbf{F}\cdot\mathbf{n} = (x^2+y^2)/3 = 9/3 = 3$. Therefore we obtain, $\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS = 3(\text{area of } S_1) = 3(2\pi(3)(4)) = 72\pi$.

**15.**

$$\iint_S (z\mathbf{k})\cdot d\mathbf{S} = \int_0^4\int_0^{2\pi}(0,0,t)\cdot(3\cos s, 3\sin s, 0)\,ds\,dt + \int_0^3\int_0^{2\pi}(0,0,0)\cdot(0,0,-t)\,ds\,dt$$
$$+\int_0^3\int_0^{2\pi}(0,0,4)\cdot(0,0,t)\,ds\,dt = \int_0^3\int_0^{2\pi}4t\,ds\,dt = \int_0^3 8\pi t\,dt = 36\pi.$$

A different approach would have been to notice that, since the unit normal vector to the lateral surface $S_1$ has no $\mathbf{k}$ component, $\iint_{S_1} z\mathbf{k}\cdot d\mathbf{S} = 0$. Also, $z = 0$ on $S_2$ so $\iint_{S_2} z\mathbf{k}\cdot d\mathbf{S} = 0$. Finally, $z = 4$ on $S_3$ and therefore

$$\iint_S z\mathbf{k}\cdot d\mathbf{S} = \iint_{S_3} z\mathbf{k}\cdot d\mathbf{S} = \iint_{S_3} 4\mathbf{k}\cdot\mathbf{k}\,dS = \iint_{S_3} 4\,dS = 4\cdot(\text{area of } S_3) = 4(\pi 3^2) = 36\pi.$$

**16.**

$$\iint_S (y^3\mathbf{i})\cdot d\mathbf{S} = \int_0^4\int_0^{2\pi}(27\sin^3 s, 0, 0)\cdot(3\cos s, 3\sin s, 0)\,ds\,dt$$
$$+\int_0^3\int_0^{2\pi}(t^3\sin^3 s, 0, 0)\cdot(0,0,-t)\,ds\,dt + \int_0^3\int_0^{2\pi}(t^3\sin^3 s, 0, 0)\cdot(0,0,t)\,ds\,dt$$
$$= 81\int_0^4\int_0^{2\pi}\sin^3 s\cos s\,ds\,dt = \frac{81}{4}\int_0^4 \sin^4 s\Big|_0^{2\pi} dt = 0.$$

Again, a careful student should have noticed that there is no $\mathbf{k}$ component and so the integrals over $S_2$ and $S_3$ are each 0.

**17.**

$$\iint_S (-y\mathbf{i}+x\mathbf{j})\cdot d\mathbf{S} = \int_0^4\int_0^{2\pi}(27\sin^3 s, 0, 0)\cdot(3\cos s, 3\sin s, 0)\,ds\,dt$$
$$+\int_0^3\int_0^{2\pi}(t^3\sin^3 s, 0, 0)\cdot(0,0,-t)\,ds\,dt + \int_0^3\int_0^{2\pi}(t^3\sin^3 s, 0, 0)\cdot(0,0,t)\,ds\,dt$$
$$= 81\int_0^4\int_0^{2\pi}\sin^3 s\cos s\,ds\,dt = \frac{81}{4}\int_0^4 \sin^4 s\Big|_0^{2\pi} dt = 0.$$

Again, a careful student should have noticed that there is no $\mathbf{k}$ component and so the integrals over $S_2$ and $S_3$ are each 0. Therefore, a different approach would be to calculate

$$\iint_S (-y\mathbf{i}+x\mathbf{j})\cdot d\mathbf{S} = \iint_{S_1}(-y\mathbf{i}+x\mathbf{j})\cdot(x\mathbf{i}+y\mathbf{j})/3\,dS = \iint_{S_1} 0\,dS = 0.$$

**18.**

$$\iint_S (x^2\mathbf{i})\cdot d\mathbf{S} = \int_0^4\int_0^{2\pi}(9\cos^2 s, 0, 0)\cdot(3\cos s, 3\sin s, 0)\,ds\,dt$$
$$+\int_0^3\int_0^{2\pi}(t^2\cos^2 s, 0, 0)\cdot(0,0,-t)\,ds\,dt + \int_0^3\int_0^{2\pi}(t^2\cos^2 s, 0, 0)\cdot(0,0,t)\,ds\,dt$$
$$= 27\int_0^4\int_0^{2\pi}\cos^3 s\,ds\,dt = 27\int_0^4\int_0^{2\pi}(1-\sin^2 s)\cos s\,ds\,dt = 27\int_0^4 [\sin s - (\sin^3 s)/3]\Big|_0^{2\pi} dt = 0.$$

Again, a careful student should have noticed that there is no $\mathbf{k}$ component and so the integrals over $S_2$ and $S_3$ are each 0.

*We calculate the flux from* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S \mathbf{F}(\mathbf{X}(s,t))\cdot\mathbf{N}(s,t)\,ds\,dt$. *For Exercises 19–22 we have that*

$$\begin{aligned}
\mathbf{X}(\varphi,\theta) &= (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi) \quad \text{for } 0\le\theta<2\pi \text{ and } 0\le\varphi\le\pi,\\
\mathbf{T}_\varphi(\varphi,\theta) &= (a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, -a\sin\varphi),\\
\mathbf{T}_\theta(\varphi,\theta) &= (-a\sin\varphi\sin\theta, a\sin\varphi\cos\theta, 0), \quad \text{and}\\
\mathbf{N}(\varphi,\theta) &= \mathbf{T}_\varphi(\varphi,\theta)\times\mathbf{T}_\theta(\varphi,\theta) = (a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\cos\varphi\sin\varphi)\\
&= a^2\sin\varphi(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).
\end{aligned}$$

**19.**

$$\begin{aligned}
\iint_S (y\mathbf{j})\cdot\mathbf{n}\,dS &= \int_0^{2\pi}\int_0^{\pi/2}(0, a\sin\varphi\sin\theta, 0)\cdot(a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\cos\varphi\sin\varphi)\,d\varphi\,d\theta\\
&= \int_0^{2\pi}\int_0^{\pi/2}(a^3\sin^3\varphi\sin^2\theta)\,d\varphi\,d\theta = a^3\int_0^{2\pi}\int_0^{\pi/2}(1-\cos^2\varphi)\sin\varphi\sin^2\theta\,d\varphi\,d\theta\\
&= a^3\int_0^{2\pi}\left[-\cos\varphi+\frac{\cos^3\varphi}{3}\right]\Bigg|_0^{\pi/2}\sin^2\theta\,d\varphi\,d\theta = \frac{2a^3}{3}\int_0^{2\pi}\sin^2\theta\,d\theta\\
&= \frac{2a^3}{3}\int_0^{2\pi}\frac{1-\cos 2\theta}{2}\,d\theta = \frac{2a^3}{3}\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]\Bigg|_0^{2\pi} = \frac{2\pi a^3}{3}.
\end{aligned}$$

**20.**

$$\begin{aligned}
&\iint_S (y\mathbf{i}-x\mathbf{j})\cdot\mathbf{n}\,dS\\
&= \int_0^{2\pi}\int_0^{\pi/2}(a\sin\varphi\sin\theta, -a\sin\varphi\cos\theta, 0)\cdot(a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\cos\varphi\sin\varphi)\,d\varphi\,d\theta\\
&= a^3\int_0^{2\pi}\int_0^{\pi/2}(\sin^3\varphi\sin\theta\cos\theta - \sin^3\varphi\cos\theta\sin\theta)\,d\varphi\,d\theta = 0.
\end{aligned}$$

Actually it is simpler not to resort to the parametrization. Since $\mathbf{n} = (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/a$ for the sphere we see that $(y\mathbf{i}-x\mathbf{j})\cdot\mathbf{n} = 0$ and so $\iint_S (y\mathbf{i}-x\mathbf{j})\cdot\mathbf{n}\,dS = 0$.

**21.**

$$\begin{aligned}
\iint_S(-y\mathbf{i}+x\mathbf{j}-\mathbf{k})\cdot\mathbf{n}\,dS &= -\iint_S \mathbf{k}\cdot\mathbf{n}\,dS - \iint_S(y\mathbf{i}-x\mathbf{j})\cdot\mathbf{n}\,dS\\
&= -\iint_S \mathbf{k}\cdot\mathbf{n}\,dS \quad \left(\text{since, by Exercise 20, } \iint_S (y\mathbf{i}-x\mathbf{j})\cdot\mathbf{n}\,dS = 0\right)\\
&= -\int_0^{2\pi}\int_0^{\pi/2}(0,0,1)\cdot(a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\cos\varphi\sin\varphi)\,d\varphi\,d\theta\\
&= -a^2\int_0^{2\pi}\int_0^{\pi/2}(\cos\varphi\sin\varphi)\,d\varphi\,d\theta = -a^2\int_0^{2\pi}\frac{\sin^2\varphi}{2}\Bigg|_0^{\pi/2}d\theta\\
&= -\frac{a^2}{2}\int_0^{2\pi}d\theta = -\pi a^2.
\end{aligned}$$

**22.**

$$\begin{aligned}
&\iint_S (x^2\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}) \cdot \mathbf{n}\, dS \\
&= \int_0^{\pi/2}\int_0^{2\pi} [(a^2\sin^2\varphi\cos^2\theta, a^2\sin^2\varphi\cos\theta\sin\theta, a^2\cos\varphi\sin\varphi\cos\theta) \\
&\qquad\qquad \cdot (a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\cos\varphi\sin\varphi)]\, d\theta\, d\varphi \\
&= a^4\int_0^{\pi/2}\int_0^{2\pi} (\sin^4\varphi\cos^3\theta + \sin^4\varphi\cos\theta\sin^2\theta + \cos^2\varphi\sin^2\varphi\cos\theta)\, d\theta\, d\varphi \\
&= a^4\int_0^{\pi/2}\int_0^{2\pi} (\sin^4\varphi\cos\theta + \sin^2\varphi\cos^2\varphi\cos\theta)\, d\theta\, d\varphi \\
&= a^4\int_0^{\pi/2}\int_0^{2\pi} (\sin^2\varphi\cos\theta)\, d\theta\, d\varphi = a^4\int_0^{\pi/2} [\sin^2\varphi\sin\theta]\Big|_0^{2\pi} d\theta = 0.
\end{aligned}$$

A different approach would be to see that

$$\begin{aligned}
\iint_S (x^2\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}) \cdot \mathbf{n}\, dS &= \iint_S x(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}\, dS \\
&= \iint_S x\frac{a^2}{a}\, dS = a\iint_S x\, dS.
\end{aligned}$$

The integrand is an odd function of $x$ which is being integrated over a region which is symmetric with respect to $x$; therefore $\iint_S (x^2\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}) \cdot \mathbf{n}\, dS = 0$.

**23.** We have $\mathbf{T}_s = (\cos t, \sin t, 0)$ and $\mathbf{T}_t = (-s\sin t, s\cos t, 1)$, so that the standard normal is

$$\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 1 \end{vmatrix} = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + s\,\mathbf{k}.$$

Therefore, the flux of $\mathbf{F}$ is given by

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^2 \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}(s,t)\, ds\, dt \\
&= \int_0^{2\pi}\int_0^2 \left(s\sin t, s\cos t, t^3\right) \cdot (\sin t, -\cos t, s)\; ds\, dt \\
&= \int_0^{2\pi}\int_0^2 \left(s(\sin^2 t - \cos^2 t) + st^3\right)\; ds\, dt \\
&= \int_0^{2\pi}\int_0^2 \left(st^3 - s\cos 2t\right)\; ds\, dt = \int_0^{2\pi} \left(\tfrac{1}{2}s^2t^3 - \tfrac{1}{2}s^2\cos 2t\right)\Big|_{s=0}^2\; dt \\
&= \int_0^{2\pi} \left(2t^3 - 2\cos 2t\right)\; dt = \left(\tfrac{1}{2}t^4 - \sin 2t\right)\Big|_0^{2\pi} = 8\pi^4.
\end{aligned}$$

**24.** We may parametrize the cone by $\mathbf{X}(s,t) = (s\cos t, s\sin t, s)$, where $-2 \le s \le 1$, $0 \le t \le 2\pi$. Then the standard normal

$$\mathbf{T}_s \times \mathbf{T}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 1 \\ -s\sin t & s\cos t & 0 \end{vmatrix} = -s\cos t\,\mathbf{i} - s\sin t\,\mathbf{j} + s\,\mathbf{k}$$

points the wrong way. (It points upward when $z = s > 0$ and downward when $z = s < 0$.) Thus we take $\mathbf{N}$ to be

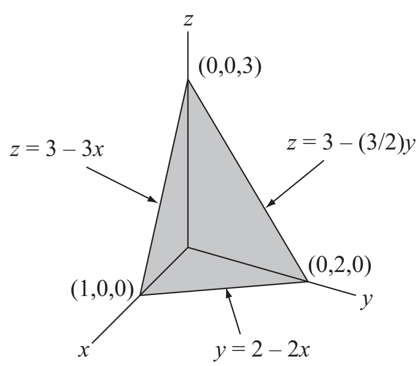
$s\cos t\,\mathbf{i} + s\sin t\,\mathbf{j} - s\,\mathbf{k}$. Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_{-2}^{1} \left(2s\cos t, 2s\sin t, s^2\right)\cdot(s\cos t, s\sin t, -s)\;ds\,dt$$
$$= \int_0^{2\pi}\int_{-2}^{1} \left(2s^2\sin^2 t + 2s^2\cos^2 t - s^3\right)\;ds\,dt$$
$$= \int_0^{2\pi}\int_{-2}^{1} \left(2s^2 - s^3\right)\;ds\,dt = \frac{39\pi}{2}.$$

**25.** The surface $z = g(x, y) = ye^x$ has upward normal $\mathbf{N} = -g_x(x, y)\,\mathbf{i} - g_y(x, y)\,\mathbf{j} + \mathbf{k} = -ye^x\,\mathbf{i} - e^x\,\mathbf{j} + \mathbf{k}$. Therefore, the flux of $\mathbf{F} = y^3 z\,\mathbf{i} - xy\,\mathbf{j} + (x + y + z)\,\mathbf{k}$ is given by

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \int_0^1\int_0^1 \left(y^4e^x, -xy, x + y + ye^x\right)\cdot(-ye^x, -e^x, 1)\;dy\,dx$$
$$= \int_0^1\int_0^1 \left(-y^5e^{2x} + yxe^x + x + y + ye^x\right)\;dy\,dx$$
$$= \int_0^1\int_0^1 \left(-\tfrac{1}{6}e^{2x} + \tfrac{1}{2}xe^x + x + \tfrac{1}{2} + \tfrac{1}{2}e^x\right)\;dx$$
$$= \left(-\tfrac{1}{12}e^{2x} + \tfrac{1}{2}(xe^x - e^x) + \tfrac{1}{2}x^2 + \tfrac{1}{2}x + \tfrac{1}{2}e^x\right)\Big|_0^1$$
$$= \tfrac{13}{12} - \tfrac{1}{12}e^2 + \tfrac{1}{2}e.$$

**26.** The tetrahedron has four triangular faces; we must consider surface integrals over each of them and then add the results.




The top slanted face is the first octant part of the plane through the points $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$. This plane has equation $6x + 3y + 2z = 6$, or $z = 3 - 3x - \frac{3}{2}y$ and upward normal $\mathbf{N} = (3, 3/2, 1)$. The "shadow" of this region in the $xy$-plane is the triangular region $\{(x, y, 0) \mid 0 \le y \le 2 - 2x,\ 0 \le x \le 1\}$; the shadow in the $yz$-plane is $\{(0, y, z) \mid 0 \le z \le 3 - \frac{3}{2}y,\ 0 \le y \le 2\}$; the shadow in the $xz$-plane is $\{(x, 0, z) \mid 0 \le z \le 3 - 3x,\ 0 \le x \le 1\}$. These three shadow regions determine the other three faces of the tetrahedron.

Now we calculate. For the top face $S_1$, we have $z = 3 - 3x - \frac{3}{2}y$, so that

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \int_0^1\int_0^{2-2x} \left(x^2, 12 - 12x - 6y, y - x\right)\cdot\left(3, \tfrac{3}{2}, 1\right)\;dy\,dx$$
$$= \int_0^1\int_0^{2-2x} \left(3x^2 + 18 - 19x - 8y\right)\;dy\,dx$$
$$= \int_0^1 \left((3x^2 - 19x + 18)(2 - 2x) - 4(2 - 2x)^2\right)\;dx$$
$$= 2\int_0^1 \left(-3x^3 + 22x^2 - 37x + 18 - 8(1 - x)^2\right)\;dx = \frac{41}{6}.$$

The bottom face $S_2$ is the portion of the plane $z = 0$ over the triangular region $\{(x, y, 0) \mid 0 \le y \le 2-2x,\ 0 \le x \le 1\}$. To have an overall outward normal, we must take the normal here to be $\mathbf{N} = -\mathbf{k}$. Therefore, with $z = 0$, we have

$$\begin{aligned}\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} &= \int_0^1 \int_0^{2-2x} \left(x^2, 0, y-x\right)\cdot(0,0,-1)\ dy\,dx \\ &= \int_0^1 \int_0^{2-2x} (x-y)\,dy\,dx \\ &= \int_0^1 \left(x(2-2x) - \tfrac{1}{2}(2-2x)^2\right)\ dx = \int_0^1 \left(2x - 2x^2 - 2(1-x)^2\right)\ dx \\ &= \left(x^2 - \tfrac{2}{3}x^3 + \tfrac{2}{3}(1-x)^3\right)\Big|_0^1 = -\frac{1}{3}.\end{aligned}$$

The left face $S_3$ is the portion of the plane $y = 0$ over the triangular region $\{(x, 0, z) \mid 0 \le z \le 3-3x,\ 0 \le x \le 1\}$. To have an overall outward normal, we must here take the normal to be $\mathbf{N} = -\mathbf{j}$. Therefore, with $y = 0$, we have

$$\begin{aligned}\iint_{S_3} \mathbf{F}\cdot d\mathbf{S} &= \int_0^1 \int_0^{3-3x} \left(x^2, 4z, -x\right)\cdot(0,-1,0)\ dz\,dx \\ &= \int_0^1 \int_0^{3-3x} -4z\,dz\,dx \\ &= \int_0^1 -2(3-3x)^2\,dx = -6.\end{aligned}$$

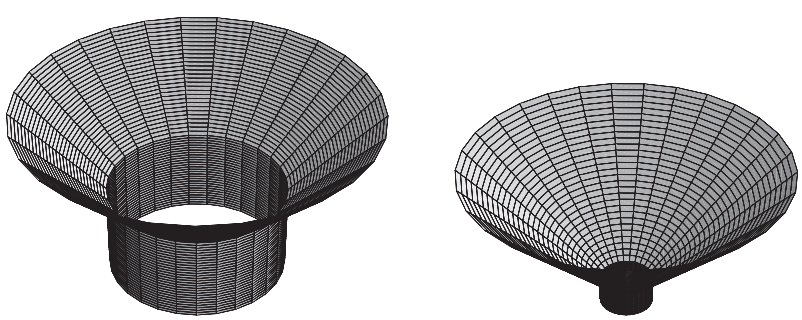
Finally, the right face $S_4$ is the portion of the plane $x = 0$ over the triangular region $\{(0, y, z) \mid 0 \le z \le 3 - \frac{3}{2}y,\ 0 \le y \le 2\}$. For an overall outward normal, we must take the normal to be $\mathbf{N} = -\mathbf{i}$. Therefore, with $x = 0$, we have

$$\begin{aligned}\iint_{S_4} \mathbf{F}\cdot d\mathbf{S} &= \int_0^2 \int_0^{3-(3/2)y} (0, 4z, -x)\cdot(-1,0,0)\ dz\,dy \\ &= \int_0^2 \int_0^{3-(3/2)y} 0\,dz\,dy = 0.\end{aligned}$$

Thus our final result is

$$\begin{aligned}\iint_{S} \mathbf{F}\cdot d\mathbf{S} &= \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_3} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_4} \mathbf{F}\cdot d\mathbf{S} \\ &= \frac{41}{6} - \frac{1}{3} - 6 + 0 = \frac{1}{2}.\end{aligned}$$

**27. (a)** Below left is just the portion of $S$ for $0 \le z \le 2$ so that you can more clearly see the funnel shape. Below right is a sketch of $S$.



**(b)** For the cylindrical portion of $S$, $\mathbf{X}(s,t) = (\cos s, \sin s, t)$ for $0 \le s < 2\pi$ and $0 \le t \le 1$. In that case $\mathbf{T}_s(s,t) = (-\sin s, \cos s, 0)$, $\mathbf{T}_t(s,t) = (0,0,1)$, $\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = (\cos s, \sin s, 0)$ and so the outward pointing unit normal for this portion is $\mathbf{n} = (\cos s, \sin s, 0) = x\mathbf{i} + y\mathbf{j}$.

For the conical portion of $S$, $\mathbf{X}(s,t) = (t\cos s, t\sin s, t)$ for $0 \le s < 2\pi$ and $1 \le t \le 9$. In that case $\mathbf{T}_s(s,t) = (-t\sin s, t\cos s, 0)$, $\mathbf{T}_t(s,t) = (\cos s, \sin s, 1)$, $\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = (t\cos s, t\sin s, -t)$ and so the outward pointing unit normal for this portion is $\mathbf{n} = (1/\sqrt{2})(\cos s, \sin s, -1) = (1/\sqrt{2})((x/z)\mathbf{i} + (y/z)\mathbf{j} - \mathbf{k})$.

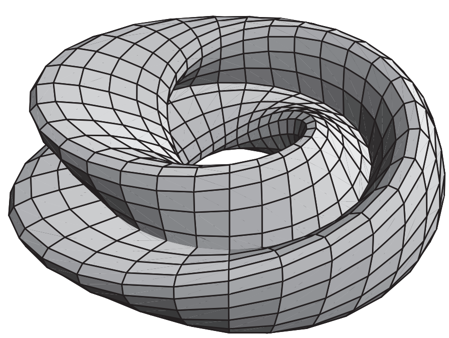
**(c)**

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S (-y\mathbf{i}+x\mathbf{j}+z\mathbf{k})\cdot d\mathbf{S} = \int_0^{2\pi}\int_1^9 (-t\sin s, t\cos s, t)\cdot(t\cos s, t\sin s, -t)\,dt\,ds$$
$$+\int_0^{2\pi}\int_0^1 (-\sin s, \cos s, t)\cdot(\cos s, \sin s, 0)\,dt\,ds = \int_0^{2\pi}\int_1^9 -t^2\,dt\,ds + \int_0^{2\pi}\int_0^1 0\,dt\,ds$$
$$= \int_0^{2\pi} -\frac{t^3}{3}\bigg|_1^9 ds = \int_0^{2\pi}\left[-\frac{729}{3}+\frac{1}{3}\right]ds = -\frac{1456\pi}{3}.$$

**28.** We know that the heat flux density $\mathbf{H} = -k\nabla T = -k(2x, 2y, 6z-12)$. On the ground $k = 3$ and $\mathbf{X}(s,t) = (t\cos s, t\sin s, 0)$ for $0 \le t \le 2$ and $0 \le s \le 2\pi$. Also, $\mathbf{T}_s(s,t) = (-t\sin s, t\cos s, 0), \mathbf{T}_t(s,t) = (\cos s, \sin s, 0)$ and so $\mathbf{N}(s,t) = (0,0,-t)$. Along the glass we have $k = 1$ and $\mathbf{X}(s,t) = (t\cos s, t\sin s, 8-2t^2)$ for $0 \le t \le 2$ and $0 \le s \le 2\pi$. Also, $\mathbf{T}_s(s,t) = (-t\sin s, t\cos s, 0), \mathbf{T}_t(s,t) = (\cos s, \sin s, -4t)$ and therefore $\mathbf{N}(s,t) = (-4t^2\cos s, -4t^2\sin s, -t)$. The outward normal must be $-\mathbf{N}(s,t) = (4t^2\cos s, 4t^2\sin s, t)$.

$$\begin{aligned}\iint_S \mathbf{H}\cdot d\mathbf{S} &= \iint_{S_1}\mathbf{H}\cdot d\mathbf{S} + \iint_{S_2}\mathbf{H}\cdot d\mathbf{S}\\
&= \int_0^{2\pi}\int_0^2 -3(2t\cos s, 2t\sin s, -12)\cdot(0,0,-t)\,dt\,ds\\
&\quad -\int_0^{2\pi}\int_0^2 (2t\cos s, 2t\sin s, 36-12t^2)\cdot(4t^2\cos s, 4t^2\sin s, t)\,dt\,ds\\
&= \int_0^{2\pi}\int_0^2 -36t\,dt\,ds + \int_0^{2\pi}\int_0^2 (-8t^3-36t+12t^3)\,dt\,ds\\
&= \int_0^{2\pi}\int_0^2 (4t^3-72t)\,dt\,ds = \int_0^{2\pi}[t^4-36t^2]\bigg|_0^2 ds\\
&= \int_0^{2\pi}[16-144]\,ds = -256\pi.\end{aligned}$$

**29. (a)** A sketch of the surface for $a = 2$ using *Mathematica* is:



**(b)** At $t = 0$ we have that $\sin t = \sin 2t = 0$ and so the $s$-coordinate curve is given by $(x,y,z) = (a\cos s, a\sin s, 0)$. This is a circle of radius $a$ in the *xy*-plane.

**(c)** A computer algebra system would help the following calculation. It is not difficult; it is just very easy to drop a term here or there.

$$\mathbf{T}_s(s,t) = \left(\cos s\left[-\frac{1}{2}\sin\frac{s}{2}\sin t - \frac{1}{2}\cos\frac{s}{2}\sin 2t\right] - \sin s\left[a + \cos\frac{s}{2}\sin t - \sin\frac{s}{2}\sin 2t\right],\right.$$
$$\sin s\left[-\frac{1}{2}\sin\frac{s}{2}\sin t - \frac{1}{2}\cos\frac{s}{2}\sin 2t\right] + \cos s\left[a + \cos\frac{s}{2}\sin t - \sin\frac{s}{2}\sin 2t\right],$$
$$\left.\frac{1}{2}\cos\frac{s}{2}\sin t - \frac{1}{2}\sin\frac{s}{2}\sin 2t\right), \quad \text{so}$$
$$\mathbf{T}_s(s,0) = (-a\sin s, a\cos s, 0).$$
$$\mathbf{T}_t(s,t) = \left(\cos s\left[\cos\frac{s}{2}\cos t - 2\cos 2t\sin\frac{s}{2}\right], \sin s\left[\cos\frac{s}{2}\cos t - 2\cos 2t\sin\frac{s}{2}\right],\right.$$
$$\left.2\cos\frac{s}{2}\cos 2t + \cos t\sin\frac{s}{2}\right) \quad \text{so}$$
$$\mathbf{T}_t(s,0) = \left(\cos s\left[\cos\frac{s}{2} - 2\sin\frac{s}{2}\right], \sin s\left[\cos\frac{s}{2} - 2\sin\frac{s}{2}\right], 2\cos\frac{s}{2} + \sin\frac{s}{2}\right).$$

Calculate the cross product $\mathbf{T}_s(s,0) \times \mathbf{T}_t(s,0)$ to obtain

$$\mathbf{N}(s,0) = \left(a\cos s\left[2\cos\frac{s}{2} + \sin\frac{s}{2}\right], a\sin s\left[2\cos\frac{s}{2} + \sin\frac{s}{2}\right], 2a\sin\frac{s}{2} - a\cos\frac{s}{2}\right).$$

We note that

$$\mathbf{X}(0,0) = (a,0,0) = \mathbf{X}(2\pi,0)$$

but

$$\mathbf{N}(0,0) = (2a,0,-a) \quad \text{while} \quad \mathbf{N}(2\pi,0) = (-2a,0,a).$$

When you travel around the $s$-coordinate curve at $t = 0$ once, you find that the normal vector is now pointing in the opposite direction. The conclusion is that the Klein bottle cannot be orientable.

## 7.3 Stokes's and Gauss's Theorems

*Exercises 1–4 are similar to Example 1 from the text. Recall from Section 2.6 that if $S$ is a surface in $\mathbf{R}^3$ defined by an equation of the form $f(x,y,z) = c$, then if $\mathbf{x}_0 \in X$, the gradient vector $\nabla f(\mathbf{x}_0)$ is a vector normal to the plane tangent to $S$ at $\mathbf{x}_0$.*

**1.** Calculate

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & yz & x^2+y^2 \end{vmatrix} = (2y-y)\mathbf{i} + (-2x+x)\mathbf{j} = y\mathbf{i} - x\mathbf{j}.$$

By symmetry we can see that the integral will be zero; however, let's follow the instructions. View the surface as a level set at height 1 of $f(x,y,z) = x^2 + y^2 + 5z$ Then $\mathbf{N} = \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 5\mathbf{k}$. So,

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D (y\mathbf{i} - x\mathbf{j}) \cdot (2x\mathbf{i} + 2y\mathbf{j} + 5\mathbf{k})\,dx\,dy$$
$$= \iint_D (2xy - 2xy)\,dx\,dy = 0.$$

On the other hand, $\partial S$ consists of $C = \{(x,y,z) | x^2+y^2 = 1 \text{ and } z = 0\}$ which we parametrize by $\mathbf{x}(t) = (\cos t, \sin t, 0)$. Then,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\,dt = \int_0^{2\pi} (0,0,1) \cdot (-\sin t, \cos t, 0)\,dt = 0.$$

These two answers agree.

**2.** $S$ is a helicoid. We begin by calculating

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

We calculated a normal vector in Exercise 24 of Section 7.1: $\mathbf{N} = (\sin t, -\cos t, s)$. So,

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_D (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\sin t\mathbf{i} - \cos t\mathbf{j} + s\mathbf{k})\, dt\, ds \\ &= \int_0^1 \int_0^{\pi/2} (\sin t - \cos t + s)\, dt\, ds = \int_0^1 \frac{\pi}{2} s\, ds \\ &= \frac{\pi}{4} s^2 \Big|_0^1 = \frac{\pi}{4}.\end{aligned}$$

On the other hand, $\partial S$ consists of four pieces which we parametrize by $\mathbf{x}_1(s) = (s, 0, 0)$ for $0 \le s \le 1$, $\mathbf{x}_2(t) = (\cos t, \sin t, t)$ for $0 \le t \le \pi/2$, $\mathbf{x}_3(s) = (0, 1-s, \pi/2)$ for $0 \le s \le 1$, and $\mathbf{x}_4(t) = (0, 0, \pi/2 - t)$ for $0 \le t \le \pi/2$. Then,

$$\begin{aligned}\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 (0, s, 0) \cdot (1, 0, 0)\, ds + \int_0^{\pi/2} (t, \cos t, \sin t) \cdot (-\sin t, \cos t, 1)\, dt \\ &\quad + \int_0^1 (\pi/2, 0, 1-s) \cdot (0, -1, 0)\, ds + \int_0^{\pi/2} (\pi/2 - t, 0, 0) \cdot (0, 0, -1)\, dt \\ &= \int_0^1 0\, ds + \int_0^{\pi/2} (-t \sin t + \cos^2 t + \sin t)\, dt + \int_0^1 0\, ds + \int_0^{\pi/2} 0\, dt \\ &= \frac{\pi}{4}.\end{aligned}$$

These two answers agree.

**3.** We see that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \mathbf{0} \quad \text{so} \quad \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0.$$

On the other hand, $\partial S$ consists of $C = \{(x, y, z) | y^2 + z^2 = 16 \text{ and } x = 0\}$ which we parametrize by $\mathbf{x}(t) = (0, 4\cos t, 4\sin t)$ for $0 \le t \le 2\pi$. Then,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt = \int_0^{2\pi} (0, 4\cos t, 4\sin t) \cdot (0, -4\sin t, 4\cos t)\, dt = 0.$$

These two answers agree.

**4.** For $S$,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y - z & x + y^2 - z & 4y - 3x \end{vmatrix} = (4 - (-1))\mathbf{i} + (3 - 1)\mathbf{j} + (1 - 2)\mathbf{k} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

If we parametrize $S$ by $\mathbf{X}(s, t) = (2\cos s \sin t, 2\sin s \sin t, 2\cos t)$, a downward normal vector is given by $\mathbf{N} = (4\cos s \sin^2 t, 4\sin s \sin^2 t, 4\sin t \cos t)$. So,

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_D (5\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (4\cos s \sin^2 t\mathbf{i} + 4\sin s \sin^2 t\mathbf{j} + 4\sin t \cos t\mathbf{k})\, ds\, dt \\ &= \int_{\pi/2}^{\pi} \int_0^{2\pi} (20\cos s \sin^2 t\mathbf{i} + 8\sin s \sin^2 t\mathbf{j} - 4\sin t \cos t\mathbf{k})\, ds\, dt \\ &= \int_{\pi/2}^{\pi} (4\pi \sin(2t))\, dt = 4\pi.\end{aligned}$$

On the other hand, $\partial S$ consists of $C = \{(x, y, z) | y^2 + z^2 = 4 \text{ and } z = 0\}$ which we parametrize by $\mathbf{x}(t) = (2\cos t, -2\sin t, 0)$.

Then,

$$\begin{aligned}\oint_{\partial S} \mathbf{F}\cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t))\cdot \mathbf{x}'(t)\,dt\\ &= \int_0^{2\pi} (-4\sin t, 2\cos t + 4\sin^2 t, -8\sin t - 6\cos t)\cdot(-2\sin t, -2\cos t, 0)\,dt\\ &= \int_0^{2\pi} (8\sin^2 t - 4\cos^2 t - 8\sin^2 t\cos t)\,dt\\ &= \int_0^{2\pi} (8 - 6(1+\cos 2t) - 8\sin^2 t\cos t)\,dt\\ &= \int_0^{2\pi} (2 - 6\cos 2t - 8\sin^2 t\cos t)\,dt = 4\pi.\end{aligned}$$

These two answers agree.

**5.** Stokes's Theorem implies that we don't need to be concerned that $S$ is defined as the union of $S_1$ and $S_2$ if we choose the calculation along the boundary. Then $\partial S$ is parametrized by $\mathbf{x}(t) = (3\cos t, 3\sin t, 0)$ where $0 \le t \le 2\pi$, and so

$$\begin{aligned}\oint_{\partial S} \mathbf{F}\cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t))\cdot \mathbf{x}'(t)\,dt\\ &= \int_0^{2\pi} (27\cos^3 t\mathbf{i} + 3^7\sin^7 t\mathbf{j})\cdot(-3\sin t, 3\cos t, 0)\,dt\\ &= -3^4\int_0^{2\pi}\cos^3 t\sin t\,dt + 3^8\int_0^{2\pi}\sin^7 t\cos t\,dt = 0.\end{aligned}$$

**6.** Note that $\nabla\cdot\mathbf{F} = 3$ so

$$\begin{aligned}\iiint_D \nabla\cdot\mathbf{F}\,dV = 3\iiint_D dV &= 3\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (9 - x^2 - y^2)\,dy\,dx\\ &= 3\int_{-3}^{3} \frac{4}{3}(9-x^2)^{3/2}\,dx = \frac{243\pi}{2}.\end{aligned}$$

On the other hand, the boundary of $D$ is in two pieces: $S_1$ = the disk at height $z = 0$ and $S_2$ = the portion of the paraboloid about the *xy*-plane. Parametrize $S_1$ by $\mathbf{X}_1(s,t) = (t\cos s, t\sin s, 0)$ for $0 \le s \le 2\pi$ and $0 \le t \le 3$. Then $\mathbf{N}_1(s,t) = (0,0,-t)$. Also parametrize $S_2$ by $\mathbf{X}_2(s,t) = (t\cos s, t\sin s, 9-t^2)$. Then $\mathbf{N}_2(s,t) = (2t^2\cos s, 2t^2\sin s, t)$. So

$$\begin{aligned}\oiint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_2}\mathbf{F}\cdot d\mathbf{S}\\ &= \int_0^{2\pi}\int_0^3 (t\cos s, t\sin s, 0)\cdot(0,0,-t)\,dt\,ds\\ &\quad + \int_0^{2\pi}\int_0^3 (t\cos s, t\sin s, 9-t^2)\cdot(2t^2\cos s, 2t^2\sin s, t)\,dt\,ds\\ &= \int_0^{2\pi}\int_0^3 (9t + t^3)\,dt\,ds = \int_0^{2\pi}\left[\frac{9t^2}{2} + \frac{t^4}{4}\right]\Big|_0^3 ds\\ &= \int_0^{2\pi}\frac{243}{4}\,ds = \frac{243\pi}{2}.\end{aligned}$$

These two answers agree.

**7.** Here $\nabla\cdot\mathbf{F} = 0$ so $\iiint_D \nabla\cdot\mathbf{F}\,dV = 0$. As for the integral over the surface, because the normal vectors of each of the three opposite pairs of sides are equal and opposite, everything will cancel. So

$$\begin{aligned}
&\oiint_S \mathbf{F}\cdot d\mathbf{S} \\
&= \iint_{\text{top}} (y-x, y-1, x-y)\cdot(0,0,1)\,dS + \iint_{\text{bottom}} (y-x, y-(-1), x-y)\cdot(0,0,-1)\,dS \\
&\quad + \iint_{\text{front}} (y-1, y-z, 1-y)\cdot(1,0,0)\,dS + \iint_{\text{back}} (y-(-1), y-z, -1-y)\cdot(-1,0,0)\,dS \\
&\quad + \iint_{\text{right}} (1-x, 1-z, x-1)\cdot(0,1,0)\,dS + \iint_{\text{left}} (-1-x, -1-z, x-(-1))\cdot(0,-1,0)\,dS \\
&= \iint_{\text{top}} (x-y)\,dS + \iint_{\text{bottom}} (y-x)\,dS + \iint_{\text{front}} (-1)\,dS + \iint_{\text{back}} (-1)\,dS \\
&\quad + \iint_{\text{right}} (1)\,dS + \iint_{\text{left}} (1)\,dS = 0.
\end{aligned}$$

These two answers agree.

**8.** Note that $\nabla\cdot\mathbf{F} = 2x+2$ so

$$\begin{aligned}
\iiint_D \nabla\cdot\mathbf{F}\,dV &= 2\iiint_D (x+1)\,dV \\
&= 2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2+1}^{5} (x+1)\,dz\,dy\,dx \\
&= 2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [(x+1)(4-x^2-y^2)]\,dy\,dx \\
&= \frac{8}{3}\int_{-2}^{2} [(x+1)(4-x^2)^{3/2}]\,dx = \frac{8}{3}(6\pi) = 16\pi.
\end{aligned}$$

On the other hand, the boundary of $D$ can be split into two pieces: the flat top piece $S_1$ and the surface of the paraboloid $S_2$. A parametrization of $S_1$ is $\mathbf{X}_1(s,t) = (t\cos s, t\sin s, 5)$ for $0\le s\le 2\pi$ and $0\le t\le 2$. Then a normal vector is $\mathbf{N}_1(s,t) = (0,0,t)$. A parametrization of $S_2$ is $\mathbf{X}_2(s,t) = (t\cos s, t\sin s, t^2+1)$ for $0\le s\le 2\pi$ and $0\le t\le 2$. Then a normal vector is $\mathbf{N}_2(s,t) = (2t^2\cos s, 2t^2\sin s, -t)$. So,

$$\begin{aligned}
\oiint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} \\
&= \int_0^{2\pi}\int_0^2 (t^2\cos^2 s, t\sin s, 5)\cdot(0,0,t)\,dt\,ds \\
&\quad + \int_0^{2\pi}\int_0^2 (t^2\cos^2 s, t\sin s, t^2+1)\cdot(2t^2\cos s, 2t^2\sin s, -t)\,dt\,ds \\
&= \int_0^{2\pi}\int_0^2 (2t^4\cos^3 s + 2t^3\sin^2 s - t^3 + 4t) = \int_0^{2\pi}\left[8 + \frac{64\cos^3 s}{5} - 4\cos 2s\right] ds = 16\pi.
\end{aligned}$$

These two answers agree.

**9.** Since $\dfrac{\partial}{\partial x}\left(\dfrac{x}{\sqrt{x^2+y^2+z^2}}\right) = \dfrac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$ we see that

$$\nabla\cdot\mathbf{F} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{\sqrt{x^2+y^2+z^2}}, \quad \text{and}$$

$$\iiint_D \nabla\cdot\mathbf{F}\,dV = \iiint_D \frac{2}{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{2}{\rho}(\rho^2\sin\varphi)\,d\rho\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_a^b (2\rho\sin\varphi)\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi}[\rho^2\sin\varphi]\Big|_a^b\,d\varphi\,d\theta$$
$$= (b^2-a^2)\int_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi\,d\theta = (b^2-a^2)\int_0^{2\pi}2\,d\theta = 4\pi(b^2-a^2).$$

On the other hand the boundary consists of two pieces: $S_1$ is the sphere of radius $a$ and $S_2$ is the sphere of radius $b$. Parametrize $S_1$ by $\mathbf{X}_1(s,t) = (a\sin s\cos t, a\sin s\sin t, a\cos s)$ for $0 \le s \le \pi$ and $0 \le t \le 2\pi$. Then a normal vector is $\mathbf{N}_1(s,t) = -a^2\sin s(\sin s\cos t, \sin s\sin t, \cos s)$. A similar calculation for $S_2$ yields $\mathbf{N}_2(s,t) = b^2\sin s(\sin s\cos t, \sin s\sin t, \cos s)$. Note that $\mathbf{N}_1$ is oriented pointing inward and $\mathbf{N}_2$ is oriented pointing outward. Then,

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_2}\mathbf{F}\cdot d\mathbf{S}$$
$$= \int_0^{2\pi}\int_0^{\pi}\frac{1}{a}((a\sin s\cos t, a\sin s\sin t, a\cos s)\cdot[-a^2\sin s(\sin s\cos t, \sin s\sin t, \cos s)]\,ds\,dt$$
$$+ \int_0^{2\pi}\int_0^{\pi}\frac{1}{b}(b\sin s\cos t, b\sin s\sin t, b\cos s)\cdot[b^2\sin s(\sin s\cos t, \sin s\sin t, \cos s)]\,ds\,dt$$
$$= \int_0^{2\pi}\int_0^{\pi}[-a^2\sin s]\,ds\,dt + \int_0^{2\pi}\int_0^{\pi}[b^2\sin s]\,ds\,dt$$
$$= \int_0^{2\pi}[-2a^2]\,dt + \int_0^{2\pi}[2b^2]\,dt = 4\pi(b^2-a^2).$$

These two answers agree.

**10.** For Stokes's theorem we assume that $S$ is a bounded, piecewise smooth, oriented surface in $\mathbf{R}^3$. To specialize to Green's theorem we must further assume that $S$ is in the *xy*-plane. In each case we assume that the boundary $C = \partial S$ consists of finitely many simple, closed curves which are oriented so that $S$ is on the left as you traverse $C$. In each case, $\mathbf{F}$ is a vector field of class $C^1$ whose domain includes $S$. In general, this would mean that $\mathbf{F}(x,y,z) = m(x,y,z)\mathbf{i} + n(x,y,z)\mathbf{j} + p(x,y,z)\mathbf{k}$ but because $S$ is planar we assume that $\mathbf{F}$ is independent of $z$ and that its $\mathbf{k}$-component is identically zero. In other words, we take $\mathbf{F}(x,y,z) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$. Then

$$\iint_S\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}.$$

But by Stokes's theorem,

$$\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = \oint_{\partial S}\mathbf{F}\cdot d\mathbf{s}.$$

Then, by the formula for the differential form of the line integral given in Section 6.1,

$$\oint_{\partial S}\mathbf{F}\cdot d\mathbf{s} = \int_C M\,dx + N\,dy.$$

And so we get Green's theorem from Stokes's theorem.

**11.** Begin by calculating

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz+5z & e^x\cos(yz) & x^2y \end{vmatrix} = (x^2 + e^x y\sin(yz))\mathbf{i} + 5\mathbf{j} + (e^x\cos(yz) - 2xz)\mathbf{k}.$$

As in Example 2, we see that this looks difficult, but that Stokes's theorem implies that

$$\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\nabla\times\mathbf{F}\cdot d\mathbf{S}$$

where $S$ and $S_1$ have the same boundary. So let $S_1$ be the disk in the $y = 1$ plane bounded by the circle $x^2 + z^2 = 9$. The rightward pointing unit normal to $S_1$ is (0, 1, 0) and so

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S} \\ &= \iint_{S_1} ((x^2 + e^x y \sin(yz))\mathbf{i} + 5\mathbf{j} + (e^x \cos(yz) - 2xz)\mathbf{k}) \cdot (0, 1, 0)\, dS \\ &= \iint_{S_1} 5\, dS = 5(\text{area of } S_1) = 5(\pi 3^2) = 45\pi.\end{aligned}$$

**12.** The boundary of $S$ is the ellipse $4x^2 + y^2 = 4$ in the $z = 0$ plane. By Stokes's theorem

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

where $S'$ is any piecewise smooth, orientable surface with $\partial S' = \partial S$ (subject to appropriate orientation). One computes that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3 & e^{y^2} & ze^{xy} \end{vmatrix} = xze^{xy}\mathbf{i} - yze^{xy}\mathbf{j}.$$

This has no **k**-component. So let us take for $S'$ the portion of the $z = 0$ plane inside the ellipse. Hence $\mathbf{n} = \mathbf{k}$ so that

$$\begin{aligned}\iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_{S'} (xze^{xy}\mathbf{i} - yze^{xy}\mathbf{j}) \cdot \mathbf{k}\, dS \\ &= \iint_{S'} 0\, dS = 0.\end{aligned}$$

**13. (a)** By the double angle formula we have $z = \sin 2t = 2 \sin t \cos t = 2xy$.

**(b)** $\oint_C (y^3 + \cos x)\, dx + (\sin y + z^2)\, dy + x\, dz = \oint_C \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{F} = (y^3 + \cos x)\mathbf{i} + (\sin y + z^2)\mathbf{j} + x\mathbf{k}$. By Stokes's theorem we may calculate the line integral by evaluating $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ where $S$ is the portion of $z = 2xy$ bounded by $C$. (Note that $S$ lies over the unit disk in the *xy*-plane.) Now $\nabla \times \mathbf{F} = -4xy\mathbf{i} - \mathbf{j} - 3y^2\mathbf{k} = -2z\mathbf{i} - \mathbf{j} - 3y^2\mathbf{k}$ on $S$. Note that the orientation of $C$ is compatible with an upward orientation of $S$. So we may take for normal

$$\mathbf{n} = \frac{-2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}} \qquad \text{(unit normal of } S\text{)}.$$

Hence $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D (8xy^2 + 2x - 3y^2)\, dx\, dy$ ($D$ = unit disk in $xy$-plane).
Now use polar coordinates, so that the integral becomes

$$\begin{aligned}&\int_0^{2\pi} \int_0^1 (8r^3 \sin^2\theta \cos\theta + 2r\cos\theta - 3r^2 \sin^2\theta) r\, dr\, d\theta \\ &= \int_0^{2\pi} \left( \frac{8}{5}\sin^2\theta\cos\theta + \frac{2}{3}\cos\theta - \frac{3}{4}\left(\frac{1}{2}(1 - \cos 2\theta)\right)\right) d\theta \\ &= \left( \frac{8}{15}\sin^3\theta + \frac{2}{3}\sin\theta - \frac{3}{8}\theta + \frac{3}{16}\sin 2\theta \right)\Bigg|_0^{2\pi} = -\frac{3\pi}{4}.\end{aligned}$$

**14.** First note that $\nabla\times\mathbf{F} = \left(xze^x \cos yz, 3x^2yz^2 - (1 + x)e^x \sin yz, 2xy - x^2z^3\right)$. Stokes's theorem implies

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S},$$

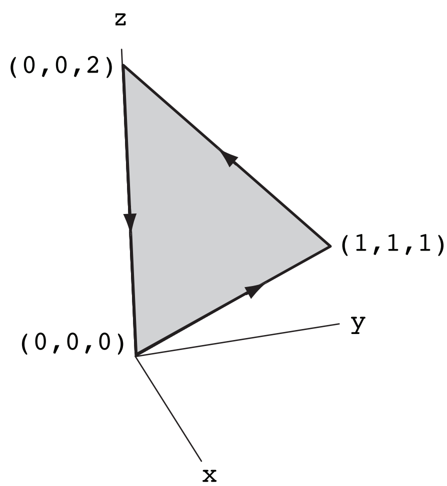
where $S'$ is the top face ($z = a$) of the cube, oriented by downward normal $-\mathbf{k}$. This gives

$$\begin{aligned}\iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \int_{-a}^{a} \int_{-a}^{a} (2xy - a^3x^2)(-1)\, dx\, dy \\ &= \int_{-a}^{a} \left( \frac{a^3}{3}x^3 - yx^2 \right)\Bigg|_{x=-a}^{a} dy = \int_{-a}^{a} \frac{2a^6}{3}\, dy = \frac{4a^7}{3}.\end{aligned}$$

**15.** Note that the path lies in the plane $x = y$. Thus, by Stokes's theorem

$$\text{Work} = \oint_C \mathbf{F}\cdot d\mathbf{s} = \iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S},$$

where $S$ is the triangular part of the plane $x = y$ enclosed by $C$. The configuration looks as follows:




Thus $S$ is given by

$$\begin{cases} x = s \\ y = s \\ z = t \end{cases},$$

where $s \le t \le 2 - s$ and $0 \le s \le 1$. The appropriate normal vector to $S$ is

$$\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j}.$$

Direct calculation reveals that $\nabla\times\mathbf{F} = (xy + x^2z)\mathbf{i} + (xy - 2xyz)\mathbf{j} - (xz + yz)\mathbf{k}$, so that

$$\begin{aligned}\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} &= \int_0^1\int_s^{2-s} (\nabla\times\mathbf{F})(s,s,t)\cdot\mathbf{N}\,dt\,ds \\ &= \int_0^1\int_s^{2-s} (s^2 + s^2t, s^2 - 2s^2t, -2st)\cdot(1,-1,0)\,dt\,ds \\ &= \int_0^1\int_s^{2-s} 3s^2t\,dt\,ds = \int_0^1 \frac{3}{2}s^2t^2\Big|_{t=s}^{2-s} ds \\ &= \int_0^1 \frac{3}{2}s^2\left((2-s)^2 - s^2\right)\,ds = \frac{3}{2}\int_0^1 \left(4s^2 - 4s^3\right)\,ds \\ &= \frac{3}{2}\left(\frac{4}{3} - 1\right) = \frac{1}{2}.\end{aligned}$$

**16.** Let $\mathbf{F} = (3\cos x + z)\mathbf{i} + (5x - e^y)\mathbf{j} - 3y\,\mathbf{k}$. Then, by Stokes's theorem

$$\oint_C (3\cos x + z)\,dx + (5x - e^y)\,dy - 3y\,dz = \oint_C \mathbf{F}\cdot d\mathbf{s} = \iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S},$$

where $S$ is the portion of the plane $2x - 3y + 5z = 17$ enclosed by $C$, oriented consistently with the orientation of $C$. A unit normal to $S$ is given by $\mathbf{n} = (2,-3,5)/\sqrt{38}$ and

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3\cos x + z & 5x - e^y & -3y \end{vmatrix} = (-3, 1, 5).$$

Therefore, we have

$$\begin{aligned}
&\oint_C (3\cos x+z)\,dx + (5x - e^y)\,dy - 3y\,dz \\
&\qquad = \pm \iint_S (-3,1,5)\cdot\frac{(2,-3,5)}{\sqrt{38}}\,dS = \pm\iint_S \frac{-6-3+25}{\sqrt{38}}\,dS \\
&\qquad = \pm\frac{16}{\sqrt{38}}(\text{area of } S) = \pm\frac{16}{\sqrt{38}}(\text{area inside } C),
\end{aligned}$$

where the $\pm$ sign depends on the orientation of $C$.

**17.** The key to this problem is to recall that the volume of a solid region $W$ may be calculated using a surface integral:

$$\text{Volume of } W = \frac{1}{3}\oiint_{\partial W} (x\,\mathbf{i} + y\,\mathbf{j} + z\mathbf{k})\cdot d\mathbf{S}.$$

Now we calculate. The top of the solid is bounded by the paraboloid given by $z = 9 - x^2 - y^2$; if we write $\mathbf{X}_1(x,y) = (x, y, 9 - x^2 - y^2)$, then the standard (upward) normal is given by $\mathbf{N}_1 = (2x, 2y, 1)$. The bottom of the solid is bounded by the paraboloid given by $z = 3x^2 + 3y^2 - 16$; if we write $\mathbf{X}_2(x,y) = (x, y, 3x^2 + 3y^2 - 16)$, then the standard normal is given by $(-6x, -6y, 1)$. However, to put top and bottom surfaces $S_1$ and $S_2$ together to give $\partial W$ a consistent, outward-pointing normal, we need to take $\mathbf{N}_2 = (6x, 6y, -1)$ for the correct orientation. Now the paraboloids intersect when $3x^2 + 3y^2 - 16 = 9 - x^2 - y^2$, or when $x^2 + y^2 = 25/4$; hence we have that $\partial W = S_1 \cup S_2$, where $S_1$ and $S_2$ are the respective portions of the top and bottom paraboloids with $x$- and $y$-coordinates in the disk $D = \{(x,y) \mid x^2 + y^2 \le 25/4\}$. Thus, with $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\mathbf{k}$, we have

$$\oiint_{\partial W} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_2}\mathbf{F}\cdot d\mathbf{S}.$$

For the top boundary, we have

$$\begin{aligned}
\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{X}_1(x,y))\cdot\mathbf{N}_1(x,y)\,dx\,dy \\
&= \iint_D (x, y, 9 - x^2 - y^2)\cdot(2x, 2y, 1)\,dx\,dy \\
&= \iint_D \left(x^2 + y^2 + 9\right)\,dx\,dy.
\end{aligned}$$

This last integral is most easily calculated using polar coordinates. Therefore,

$$\begin{aligned}
\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \iint_D \left(x^2 + y^2 + 9\right)\,dx\,dy \\
&= \int_0^{5/2}\int_0^{2\pi}\left(r^2 + 9\right) r\,d\theta\,dr = 2\pi\left(\frac{1}{4}r^4 + \frac{9}{2}r^2\right)\Bigg|_0^{5/2} = \frac{2425\pi}{32}.
\end{aligned}$$

We make similar calculations for the bottom boundary:

$$\begin{aligned}
\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{X}_2(x,y))\cdot\mathbf{N}_2(x,y)\,dx\,dy \\
&= \iint_D (x, y, 3x^2 + 3y^2 - 16)\cdot(6x, 6y, -1)\,dx\,dy \\
&= \iint_D \left(3x^2 + 3y^2 + 16\right)\,dx\,dy = \int_0^{5/2}\int_0^{2\pi}\left(3r^2 + 16\right) r\,d\theta\,dr \\
&= 2\pi\left(\frac{3}{4}r^4 + 8r^2\right)\Bigg|_0^{5/2} = \frac{5075\pi}{32}.
\end{aligned}$$

Hence

$$\text{Volume of } W = \frac{1}{3}\left(\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S}\right)$$
$$= \frac{1}{3}\left(\frac{2425\pi}{32} + \frac{5075\pi}{32}\right) = \frac{625\pi}{8}.$$

**18.** $S$ is the portion of the "bell" surface for which $z = e^{1-x^2-y^2}$ and $z \geq 1$. Take $S_2$ to be the disk in the plane $z = 1$ bounded by the circle $x^2 + y^2 = 1$. Then $S \bigcup S_2$ is the boundary of a solid $V$. $S$ is oriented with an upward pointing normal and $S_2$ is oriented with a downward pointing normal.

$$\nabla\cdot\mathbf{F} = 0 \quad \text{so} \quad \iiint_V \nabla\cdot\mathbf{F}\, dV = 0.$$

Also,

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} (x, y, 2-2z)\cdot(0,0,-1)\, dS = \iint_{S_2} (2z-2)\, dS.$$

But along $S_2$, $z = 1$, so $\displaystyle\iint_{S_2} (2z-2)\, dS = \iint_{S_2} (2-2)\, dS = 0$. So

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_V \nabla\cdot\mathbf{F}\, dV - \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = 0 - 0 = 0.$$

**19.** Let $\mathbf{X}\colon D \to \mathbf{R}^3$, $\mathbf{X}(u,v) = (x(u,v), y(u,v), z(u,v))$ parametrize $S$ and $(u(t), v(t))$, $a \leq t \leq b$ parametrize $\partial D$ so that $\mathbf{X}(u(t), v(t))$ parametrizes $\partial S$. (Note the assumption that $\partial D$ can be parametrized by a single path—this is not a problem.) Write $\mathbf{F}$ as $M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$. We need to show that

$$(*) \qquad \oint_{\partial S} (M\mathbf{i} + N\mathbf{j} + P\mathbf{k})\cdot d\mathbf{s} = \iint_S \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\cdot d\mathbf{S}.$$

Consider the line integral in $(*)$. We may write it in differential form as $\displaystyle\oint_{\partial S} M\, dx + N\, dy + P dz$. Consider, for the moment, just the piece $\displaystyle\oint_{\partial S} M\, dx$. By the chain rule, $\dfrac{dx}{dt} = \dfrac{\partial x}{\partial u}\dfrac{du}{dt} + \dfrac{\partial x}{\partial v}\dfrac{dv}{dt}$. Hence,

$$\oint_{\partial S} M\, dx = \int_a^b M(\mathbf{X}(u(t), v(t)))\left(\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}\right) dt = \int_{\partial D} M\circ\mathbf{X}\frac{\partial x}{\partial u}\, du + M\circ\mathbf{X}\frac{\partial x}{\partial v}\, dv.$$

The last line integral in just an integral in the *uv*-plane and so we may apply Green's theorem to find

$$\oint_{\partial S} M\, dx = \iint_D \left[\frac{\partial}{\partial u}\left(M\circ\mathbf{X}\frac{\partial x}{\partial v}\right) - \frac{\partial}{\partial v}\left(M\circ\mathbf{X}\frac{\partial x}{\partial u}\right)\right] du\, dv.$$

We need to apply the chain rule again, along with the product rule:

$$\frac{\partial}{\partial u}\left(M\circ\mathbf{X}\frac{\partial x}{\partial v}\right) = \left(\frac{\partial M}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial v} + M\circ\mathbf{X}\frac{\partial^2 x}{\partial u\,\partial v}$$
$$\frac{\partial}{\partial v}\left(M\circ\mathbf{X}\frac{\partial x}{\partial u}\right) = \left(\frac{\partial M}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial u} + M\circ\mathbf{X}\frac{\partial^2 x}{\partial v\,\partial u}.$$

Since the exercise allows us to assume that $\mathbf{X}$ is of class $C^2$, the mixed partials are equal: $\dfrac{\partial^2 x}{\partial u \partial v} = \dfrac{\partial^2 x}{\partial v \partial u}$. Therefore, our double integral becomes, after cancellation,

$$(**) \qquad \iint_D \left[\frac{\partial M}{\partial y}\left(\frac{\partial x}{\partial v}\frac{\partial y}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\right) + \frac{\partial M}{\partial z}\left(\frac{\partial x}{\partial v}\frac{\partial z}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)\right] du\, dv.$$

Now consider the surface integral in $(*)$. Using the parametrization $\mathbf{X}$, and calculating the normal, we have that it is equal to

$$\iint_D (P_y - N_z, M_z - P_x, N_x - M_y)\cdot(y_u z_v - y_v z_u, z_u x_v - z_v x_u, x_u y_v - x_v y_u)\, du\, dv.$$

Next, calculate the dot product and isolate just those terms that contain $M$. Then the piece of the surface integral in $(*)$ that involves just $M$ is

$$\iint_D \left[\frac{\partial M}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) - \frac{\partial M}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\right] du\,dv.$$

This is the same as the double integral in $(**)$.

In an entirely analogous way, we may show that $\oint_{\partial S} N\,dy$ and $\oint_{\partial S} P dz$ are equal to the remaining pieces of the surface integral in $(*)$, completing the proof.

**20.** We will calculate $\iiint_V \nabla \cdot \mathbf{F}\,dV$ for the closed cube and then subtract $\oiint_{S_2} \mathbf{F} \cdot d\mathbf{S}$ where $S_2$ is the bottom. Orient all of the faces of the cube with an outward pointing normal. In particular this means that the normal to $S_2$ is downward pointing.

$$\begin{aligned}\iiint_V \nabla \cdot \mathbf{F}\,dV &= \int_0^1\int_0^1\int_0^1 (2xze^{x^2} + 3 - 7yz^6)\,dx\,dy\,dz \\ &= \int_0^1\int_0^1 [ze^{x^2} + 3x - 7xyz^6]\Big|_{x=0}^{x=1} dy\,dz = \int_0^1\int_0^1 [ez - z + 3 - 7yz^6]\,dy\,dz \\ &= \int_0^1 \left[ezy - zy + 3y - \frac{7}{2}y^2z^6\right]\Big|_{y=0}^{y=1} dz = \int_0^1 \left[ez - z + 3 - \frac{7}{2}z^6\right] dz \\ &= \left[\frac{z^2}{2}e - \frac{z^2}{2} + 3z - \frac{1}{2}z^7\right]\Big|_0^1 = \frac{e}{2} + 2.\end{aligned}$$

Also, $\oiint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_0^1\int_0^1 (0, 3y, 2) \cdot (0, 0, -1)\,dy\,dx = \int_0^1\int_0^1 (-2)\,dy\,dx = -2$. Therefore,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \left(\frac{e}{2} + 2\right) - (-2) = \frac{e}{2} + 4.$$

**21. (a)** If $\mathbf{F} = f\mathbf{a}$, then

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(fa_1) + \frac{\partial}{\partial y}(fa_2) + \frac{\partial}{\partial z}(fa_3) \\ &= a_1\frac{\partial f}{\partial x} + a_2\frac{\partial f}{\partial y} + a_3\frac{\partial f}{\partial z} \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot (a_1, a_2, a_3) = \nabla f \cdot \mathbf{a}.\end{aligned}$$

**(b)** With $\mathbf{F} = f\,\mathbf{i}$, we may apply Gauss's theorem:

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\,dV.$$

The left side is

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \oiint_S (\mathbf{F} \cdot \mathbf{n})\,dS = \oiint_S (f\,\mathbf{i}) \cdot \mathbf{n})\,dS = \oiint_S (fn_1)\,dS.$$

Using part (a) with $\mathbf{a} = \mathbf{i}$, the right side is

$$\iiint_D \nabla \cdot \mathbf{F}\,dV = \iiint_D \nabla f \cdot \mathbf{i}\,dV = \iiint_D \frac{\partial f}{\partial x}\,dV.$$

Similarly, with $\mathbf{a} = \mathbf{j}$, we have

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \oiint_S (f\,\mathbf{j}) \cdot \mathbf{n})\,dS = \oiint_S (fn_2)\,dS$$

and

$$\iiint_D \nabla \cdot \mathbf{F}\,dV = \iiint_D \nabla f \cdot \mathbf{j}\,dV = \iiint_D \frac{\partial f}{\partial y}\,dV.$$

Finally, with $\mathbf{a} = \mathbf{k}$ we obtain

$$\oiint_S (fn_3)\,dS = \iiint_D \frac{\partial f}{\partial z}\,dV.$$

**(c)** Using part (b),

$$\oiint_S f\mathbf{n}\,dS = \left(\oiint_S fn_1\,dS, \oiint_S fn_2\,dS, \oiint_S fn_3\,dS\right)$$
$$= \left(\iiint_D \frac{\partial f}{\partial x}\,dV, \iiint_D \frac{\partial f}{\partial y}\,dV, \iiint_D \frac{\partial f}{\partial z}\,dV\right) = \iiint_D \nabla f\,dV.$$

**22.** Using the previous exercise, we have

$$\mathbf{B} = -\oiint_{\partial D} p\mathbf{n}\,dS$$
$$= -\iiint_D \nabla p\,dV = -\iiint_D \delta g\mathbf{k}\,dV = -\left(\iiint_D 1\,dV\right)(\delta g\mathbf{k})$$
$$= -(\text{volume of } D)(\delta g\mathbf{k}) = -(\text{mass of liquid displaced})(g\mathbf{k})$$
$$= -(\text{weight of liquid displaced})\mathbf{k}.$$

Note that the negative sign is correct—the buoyant force should point *upwards* and it does, since the $z$-axis is oriented down.

**23.** The proof is outlined in the proof of Theorem 3.5 of Chapter 6. One direction has already been proved in Theorem 4.3 of Chapter 3. There it was established that if $\mathbf{F} = \nabla f$, then $\nabla \times \mathbf{F} = \mathbf{0}$. Now suppose that $\nabla \times \mathbf{F} = \mathbf{0}$. We show that then $\oint_C \mathbf{F}\cdot d\mathbf{s} = 0$ where $C$ is any piecewise $C^1$, simple closed curve in $R \subseteq \mathbf{R}^3$. The idea is to "fill in $C$", that is, to find a surface $S \subseteq R$ whose boundary is $C$. Since $R$ is simply-connected, this is possible. If we orient $S$ consistently with $C$, then we may apply Stokes's theorem to conclude

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{0}\cdot d\mathbf{S} = 0.$$

This shows, among other things, that $\mathbf{F}$ has path-independent integrals over curves in $R$. Therefore, by Theorem 3.3 of Chapter 6, $\mathbf{F} = \nabla f$ for some function $f$ on $R$.

**24. (a)** Note that the boundary of $D$ is made up of two components:

- $S_5 =$ the sphere centered at the origin of radius 5 oriented with outward pointing normal and
- $S_7 =$ the sphere centered at the origin of radius 7 oriented with outward pointing normal.

Then by Gauss's theorem

$$\iiint_D \nabla\cdot\mathbf{F}\,dV = \oiint_{S_7} \mathbf{F}\cdot d\mathbf{S} - \oiint_{S_5} \mathbf{F}\cdot d\mathbf{S} = (7a+b) - (5a+b) = 2a.$$

**(b)** By Theorem 4.4 of Section 3.4, $\nabla\cdot(\nabla\times\mathbf{G}) = 0$. So if $D$ is the solid sphere centered at the origin with radius $r$ then, since $\mathbf{F} = \nabla\times\mathbf{G}$,

$$ar + b = \oiint_{S_r} \mathbf{F}\cdot d\mathbf{S} \quad \text{(next apply Gauss's theorem)}$$
$$= \iiint_D \nabla\cdot\mathbf{F}\,dV = \iiint_D \nabla\cdot(\nabla\times\mathbf{G})\,dV = 0.$$

Therefore $ar + b = 0$ for all values of $r$. We conclude that $a = b = 0$.

**25. (a)** If $f(x, y, z) = \ln(x^2 + y^2 + z^2)$ then $\nabla f(x, y, z) = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{x^2 + y^2 + z^2} = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{a^2}$ on $S$. Also, the unit normal to the sphere that points away from the origin is $\mathbf{n}(x, y, z) = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4(x^2 + y^2 + z^2)}} = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2a} =$

$\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}$. So,

$$\begin{aligned}\iint_S \frac{\partial f}{\partial n}\,dS &= \iint_S \nabla f\cdot \mathbf{n}\,dS \\ &= \iint_S \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{x^2+y^2+z^2}\cdot\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}\,dS \\ &= \iint_S \frac{2}{a}\,dS = \frac{2}{a}(\text{surface area of } S) \\ &= \frac{2}{a}\left(\frac{4\pi a^2}{8}\right) = \pi a.\end{aligned}$$

**(b)** First calculate that $\nabla\cdot(\nabla f) = \frac{2}{x^2+y^2+z^2}$. We'll use spherical coordinates to integrate.

$$\begin{aligned}\iiint_D \nabla\cdot(\nabla f)\,dV &= \iiint_D \frac{2}{x^2+y^2+z^2}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \left(\frac{2}{\rho^2}\right)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \\ &= 2\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \sin\varphi\,d\rho\,d\varphi\,d\theta = 2a\int_0^{\pi/2}\int_0^{\pi/2}\sin\varphi\,d\varphi\,d\theta \\ &= 2a\int_0^{\pi/2} d\theta = \pi a.\end{aligned}$$

**(c)** By Gauss's theorem, $\iiint_D \nabla\cdot(\nabla f)\,dV = \oiint_{\partial D}(\nabla f)\cdot d\mathbf{S}$. The boundary of $D$ consists of four pieces: $S$, the surface from part (a); $S_x$, the intersection of $D$ and the plane $x=0$; $S_y$, the intersection of $D$ and the plane $y=0$; and $S_z$, the intersection of $D$ and the plane $z=0$. On $S_x$ we know that $\nabla f(0,y,z) = \frac{2y\mathbf{j}+2z\mathbf{k}}{y^2+z^2}$ and $\mathbf{n}=(-1,0,0)$ so

$$\iint_{S_x}\nabla f\cdot d\mathbf{S} = \iint_{S_x}\nabla f\cdot\mathbf{n}\,dS = \iint_{S_x} 0\,dS = 0.$$

A similar analysis gives us $\iint_{S_y}\nabla f\cdot d\mathbf{S}=0$ and $\iint_{S_z}\nabla f\cdot d\mathbf{S}=0$. Therefore,

$$\iiint_D \nabla\cdot(\nabla f)\,dV = \oiint_{\partial D}(\nabla f)\cdot d\mathbf{S} = \iint_S \nabla f\cdot d\mathbf{S} = \iint_S \frac{\partial f}{\partial n}\,dS.$$

**26.** By Gauss's theorem, $\iiint_D \nabla\cdot(\nabla f)dV = \oiint_{\partial D}(\nabla f)\cdot d\mathbf{S}$. Here the boundary of $D$ consists of finitely many piecewise smooth, closed orientable surfaces $S_i$. By assumption, $\oiint_{S_i}(\nabla f)\cdot d\mathbf{S}=0$ and so $\iiint_D \nabla\cdot(\nabla f)dV = 0$. This is true for any solid $D$, so $\nabla\cdot(\nabla f)=0$. As we saw earlier in the text $\nabla\cdot(\nabla f) = \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$. So $f$ is harmonic.

**27.** We will shrink the region $D$ specified in the problem down to a point $P$. The volume decreases monotonically as we shrink the solid. Let $D_V$ be the shrunken version of $D$ which is the solid of volume $V$ and let $S_V=\partial D_V$ for $0\le V\le$ the volume of $D$. Then, by Gauss's theorem,

$$\oiint_{S_V}\mathbf{F}\cdot d\mathbf{S} = \iiint_{D_V}\nabla\cdot\mathbf{F}\,dV.$$

By the mean value for triple integrals, there exists a $Q_V\in D_V$ so that

$$\iiint_{D_V}\nabla\cdot\mathbf{F}\,dV = \iiint_{D_V}\nabla\cdot\mathbf{F}(Q_V)\,dV = \nabla\cdot\mathbf{F}(Q_V)(\text{volume of } D).$$

So

$$\lim_{V\to 0}\frac{1}{V}\oiint_{S_V}\mathbf{F}\cdot d\mathbf{S} = \lim_{V\to 0}\nabla\cdot\mathbf{F}(Q_V) = \nabla\cdot\mathbf{F}(P) = \text{div}\,\mathbf{F}(P).$$

**28.** The six faces of the cube $S$ are given by planes with equations

$$x = x_0 \pm \frac{a}{2} \text{ (front and back)}, \qquad y = y_0 \pm \frac{a}{2} \text{ (right and left)}, \qquad z = z_0 \pm \frac{a}{2} \text{ (top and bottom)}.$$

The respective outward unit normal vectors to these faces are $\pm\mathbf{i}$, $\pm\mathbf{j}$, $\pm\mathbf{k}$.

From Exercise 27, we have that the divergence of $\mathbf{F}$ at $P$ may be computed as

$$\operatorname{div}\mathbf{F}(P) = \lim_{V\to 0}\frac{1}{V}\oiint_S \mathbf{F}\cdot d\mathbf{S} = \lim_{a\to 0^+}\frac{1}{a^3}\oiint_S \mathbf{F}\cdot d\mathbf{S}.$$

To calculate $\oiint_S \mathbf{F}\cdot d\mathbf{S}$, we add the contributions of the six surface integrals over each of the six square faces. Consider first just the integrals over the faces given by $x = x_0 + \frac{a}{2}$ and $x = x_0 - \frac{a}{2}$. These integrals contribute

$$\begin{aligned}\iint_{\text{front}} \mathbf{F}\cdot d\mathbf{S} + \iint_{\text{back}} \mathbf{F}\cdot d\mathbf{S} &= \iint_{\text{front}} (\mathbf{F}\cdot\mathbf{i})\, dS + \iint_{\text{back}} (\mathbf{F}\cdot(-\mathbf{i}))\, dS \\ &= \iint_{\text{front}} F_1\, dS + \iint_{\text{back}} -F_1\, dS.\end{aligned}$$

The faces are parametrized as

$$\text{front: } \mathbf{X}_1(y,z) = \left(x_0 + \tfrac{a}{2}, y, z\right) \qquad \text{back: } \mathbf{X}_2(y,z) = \left(x_0 - \tfrac{a}{2}, y, z\right),$$

where $(y, z)$ varies over the square $D = \left[y_0 - \frac{a}{2}, y_0 + \frac{a}{2}\right] \times \left[z_0 - \frac{a}{2}, z_0 + \frac{a}{2}\right]$. Hence

$$\begin{aligned}\iint_{\text{front}} F_1\, dS + \iint_{\text{back}} -F_1\, dS &= \iint_{\mathbf{X}_1} F_1\, dS + \iint_{\mathbf{X}_2} -F_1\, dS \\ &= \iint_D F_1\left(x_0 + \tfrac{a}{2}, y, z\right)\, dy\, dz + \iint_D -F_1\left(x_0 - \tfrac{a}{2}, y, z\right)\, dy\, dz \\ &= \iint_D \left[F_1\left(x_0 + \tfrac{a}{2}, y, z\right) - F_1\left(x_0 - \tfrac{a}{2}, y, z\right)\right]\, dy\, dz.\end{aligned}$$

By the mean value theorem for double integrals, there is a point $(y_1, z_1) \in D$ such that

$$\begin{aligned}&\iint_D \left[F_1\left(x_0 + \tfrac{a}{2}, y, z\right) - F_1\left(x_0 - \tfrac{a}{2}, y, z\right)\right]\, dy\, dz \\ &\qquad = \left[F_1\left(x_0 + \tfrac{a}{2}, y_1, z_1\right) - F_1\left(x_0 - \tfrac{a}{2}, y_1, z_1\right)\right] (\text{area of } D) \\ &\qquad = a^2\left[F_1\left(x_0 + \tfrac{a}{2}, y_1, z_1\right) - F_1\left(x_0 - \tfrac{a}{2}, y_1, z_1\right)\right].\end{aligned}$$

In a similar manner, the two surface integrals over the faces given by $y = y_0 + \frac{a}{2}$ and $y = y_0 - \frac{a}{2}$ contribute

$$a^2\left[F_2\left(x_2, y_0 + \tfrac{a}{2}, z_2\right) - F_1\left(x_2, y_0 - \tfrac{a}{2}, z_2\right)\right]$$

to $\oiint_S \mathbf{F}\cdot d\mathbf{S}$, where $(x_2, z_2)$ is a suitable point in the square $\left[x_0 - \frac{a}{2}, x_0 + \frac{a}{2}\right] \times \left[z_0 - \frac{a}{2}, z_0 + \frac{a}{2}\right]$. And, finally, the two surface integrals over the faces given by $z = z_0 + \frac{a}{2}$ and $z = z_0 - \frac{a}{2}$ contribute

$$a^2\left[F_3\left(x_3, y_3, z_0 + \tfrac{a}{2}\right) - F_1\left(x_3, y_3, z_0 - \tfrac{a}{2}\right)\right],$$

where $(x_3, y_3)$ is a suitable point in the square $\left[x_0 - \frac{a}{2}, x_0 + \frac{a}{2}\right] \times \left[y_0 - \frac{a}{2}, y_0 + \frac{a}{2}\right]$.

Putting all of this together, we have

$$\begin{aligned}
\operatorname{div}\mathbf{F}(P) &= \lim_{a\to 0^+}\frac{1}{a^3}\oiint_S \mathbf{F}\cdot d\mathbf{S}\\
&= \lim_{a\to 0^+}\frac{1}{a^3}\left\{a^2\left[F_1\left(x_0+\tfrac{a}{2},y_1,z_1\right)-F_1\left(x_0-\tfrac{a}{2},y_1,z_1\right)\right]\right.\\
&\qquad\qquad + a^2\left[F_2\left(x_2,y_0+\tfrac{a}{2},z_2\right)-F_1\left(x_2,y_0-\tfrac{a}{2},z_2\right)\right]\\
&\qquad\qquad \left.+ a^2\left[F_3\left(x_3,y_3,z_0+\tfrac{a}{2}\right)-F_1\left(x_3,y_3,z_0-\tfrac{a}{2}\right)\right]\right\}\\
&= \lim_{a\to 0^+}\frac{F_1\left(x_0+\frac{a}{2},y_1,z_1\right)-F_1\left(x_0-\frac{a}{2},y_1,z_1\right)}{a}\\
&\quad + \lim_{a\to 0^+}\frac{F_2\left(x_2,y_0+\frac{a}{2},z_2\right)-F_1\left(x_2,y_0-\frac{a}{2},z_2\right)}{a}\\
&\quad + \lim_{a\to 0^+}\frac{F_3\left(x_3,y_3,z_0+\frac{a}{2}\right)-F_1\left(x_3,y_3,z_0-\frac{a}{2}\right)}{a}.
\end{aligned}$$

Note that as $a\to 0^+$, each of the square faces shrinks down to the point $P(x_0,y_0,z_0)$. In particular, we have $(y_1,z_1)\to(y_0,z_0)$, $(x_2,z_2)\to(x_0,z_0)$, and $(x_3,y_3)\to(x_0,y_0)$. Thus, using the remark about partial derivatives, we see that the sum of the limits above is

$$\frac{\partial F_1}{\partial x}(x_0,y_0,z_0)+\frac{\partial F_2}{\partial y}(x_0,y_0,z_0)+\frac{\partial F_3}{\partial z}(x_0,y_0,z_0),$$

as desired.

**29. (a)** $\mathbf{F}=F_r\mathbf{e}_r+F_\theta\mathbf{e}_\theta+F_z\mathbf{e}_z$. The area of the top face is

$$(\Delta\theta/2\pi)[\pi(r+\Delta r/2)^2-\pi(r-\Delta r/2)^2]=(\Delta\theta/2)(2r\Delta r)=r\Delta\theta\Delta r.$$

Therefore,

$$\begin{aligned}
\iint_{\text{top}}\mathbf{F}\cdot d\mathbf{S} &= \iint_{\text{top}}\mathbf{F}\cdot\mathbf{n}\,dS=\iint_{\text{top}}\mathbf{F}\cdot\mathbf{e}_z\,dS=\iint_{\text{top}}F_z\,dS\\
&\approx F_z(r,\theta,z+\Delta z/2)(\text{area of top})=F_z(r,\theta,z+\Delta z/2)r\Delta\theta\Delta r.
\end{aligned}$$

The calculation for the bottom face is similar. The differences are that the normal vector points down and $F_z$ is evaluated at a different point. The result is that

$$\iint_{\text{bottom}}\mathbf{F}\cdot d\mathbf{S}\approx -F_z(r,\theta,z-\Delta z/2)r\Delta\theta\Delta r.$$

The area of the outer face is

$$(\Delta z)(\Delta\theta/2\pi)[2\pi(r+\Delta r/2)]=\Delta\theta\Delta z(r+\Delta r/2).$$

Therefore,

$$\begin{aligned}
\iint_{\text{outer}}\mathbf{F}\cdot d\mathbf{S} &= \iint_{\text{outer}}\mathbf{F}\cdot\mathbf{n}\,dS=\iint_{\text{outer}}\mathbf{F}\cdot\mathbf{e}_r\,dS=\iint_{\text{outer}}F_r\,dS\\
&\approx F_r(r+\Delta r/2,\theta,z)(\text{area of outer})=F_r(r+\Delta r/2,\theta,z)(r+\Delta r/2)\Delta\theta\Delta z.
\end{aligned}$$

The calculation for the inner face is similar. The differences are that the normal vector points inward, $F_r$ is evaluated at a different point, and the area of the face is slightly different. The result is that

$$\iint_{\text{inner}}\mathbf{F}\cdot d\mathbf{S}\approx -F_r(r-\Delta r/2,\theta,z)(r-\Delta r/2)\Delta\theta\Delta z.$$

The area of either the left or right face is just $\Delta r\Delta z$. Therefore, the integral along the left face (looking from the origin out at the solid) is

$$\begin{aligned}
\iint_{\text{left}}\mathbf{F}\cdot d\mathbf{S} &= \iint_{\text{left}}\mathbf{F}\cdot\mathbf{n}\,dS=\iint_{\text{left}}\mathbf{F}\cdot\mathbf{e}_\theta\,dS=\iint_{\text{left}}F_\theta\,dS\\
&\approx F_\theta(r,\theta+\Delta\theta/2,z)(\text{area of left})=F_\theta(r,\theta+\Delta\theta/2,z)\Delta r\Delta z.
\end{aligned}$$

The calculation for the right face is similar. The differences are that the normal vector points the opposite direction and $F_\theta$ is evaluated at a different point. The result is that

$$\iint_{\text{right}} \mathbf{F}\cdot d\mathbf{S} \approx -F_\theta(r,\theta-\Delta\theta/2,z)\Delta r\Delta z.$$

We sum these to obtain

$$\begin{aligned}\oiint_S \mathbf{F}\cdot d\mathbf{S} &\approx F_z(r,\theta,z+\Delta z/2)r\Delta\theta\Delta r - F_z(r,\theta,z-\Delta z/2)r\Delta\theta\Delta r\\ &+ F_r(r+\Delta r/2,\theta,z)(r+\Delta r/2)\Delta\theta\Delta z - F_r(r-\Delta r/2,\theta,z)(r-\Delta r/2)\Delta\theta\Delta z\\ &+ F_\theta(r,\theta+\Delta\theta/2,z)\Delta r\Delta z - F_\theta(r,\theta-\Delta\theta/2,z)\Delta r\Delta z.\end{aligned}$$

**(b)** To calculate the divergence using the results of Exercise 27 we will divide the answer to part (a) by $V \approx r\Delta\theta\Delta r\Delta z$ and take the limit as $V \to 0$. Two notes before the calculation: 1) We can replace $\approx$ with $=$ because in the limit our approximation assumptions are true and 2) in evaluating each of the limits we use the remark given in the text at the end of Exercise 28 (although you may want to break the argument of the middle limit down further to see what is going on).

$$\begin{aligned}\text{div}\,\mathbf{F}(P) &= \lim_{V\to 0}\frac{1}{V}\oiint_S \mathbf{F}\cdot d\mathbf{S}\\ &= \lim_{V\to 0}\left[\frac{F_z(r,\theta,z+\Delta z/2)r\Delta\theta\Delta r - F_z(r,\theta,z-\Delta z/2)r\Delta\theta\Delta r}{r\Delta\theta\Delta r\Delta z}\right]\\ &+ \lim_{V\to 0}\left[\frac{F_r(r+\Delta r/2,\theta,z)(r+\Delta r/2)\Delta\theta\Delta z - F_r(r-\Delta r/2,\theta,z)(r-\Delta r/2)\Delta\theta\Delta z}{r\Delta\theta\Delta r\Delta z}\right]\\ &+ \lim_{V\to 0}\left[\frac{F_\theta(r,\theta+\Delta\theta/2,z)\Delta r\Delta z - F_\theta(r,\theta-\Delta\theta/2,z)\Delta r\Delta z}{r\Delta\theta\Delta r\Delta z}\right]\\ &= \lim_{\Delta z\to 0}\left[\frac{F_z(r,\theta,z+\Delta z/2) - F_z(r,\theta,z-\Delta z/2)}{\Delta z}\right]\\ &+ \lim_{\Delta r\to 0}\left[\frac{F_r(r+\Delta r/2,\theta,z)(r+\Delta r/2) - F_r(r-\Delta r/2,\theta,z)(r-\Delta r/2)}{r\Delta r}\right]\\ &+ \lim_{\Delta\theta\to 0}\left[\frac{F_\theta(r,\theta+\Delta\theta/2,z) - F_\theta(r,\theta-\Delta\theta/2,z)}{r\Delta\theta}\right]\\ &= \left[\frac{\partial F_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta}\right]\Bigg|_P.\end{aligned}$$

**30.** Follow the steps from Exercise 29. This time $\mathbf{F} = F_\rho\mathbf{e}_\rho + F_\theta\mathbf{e}_\theta + F_\varphi\mathbf{e}_\varphi$. Again, for each face, $\iint_S \mathbf{F}\cdot d\mathbf{S}$ is approximately the product of the component of **F** in the normal direction evaluated at the center point of the face and the area of that face. So summing up we have that

$$\begin{aligned}\oiint_S \mathbf{F}\cdot d\mathbf{S} &\approx F_\varphi(\rho,\theta,\varphi+\Delta\varphi/2)\rho\sin(\varphi+\Delta\varphi/2)\Delta\theta\Delta\rho - F_\varphi(\rho,\theta,\varphi-\Delta\varphi/2)\rho\sin(\varphi-\Delta\varphi/2)\Delta\theta\Delta\rho\\ &+ F_\rho(\rho+\Delta\rho/2,\theta,\varphi)(\rho+\Delta\rho/2)^2\sin\varphi\Delta\theta\Delta\varphi - F_\rho(\rho-\Delta\rho/2,\theta,\varphi)(\rho-\Delta\rho/2)^2\sin\varphi\Delta\theta\Delta\varphi\\ &+ F_\theta(\rho,\theta+\Delta\theta/2,\varphi)\rho\Delta\rho\Delta\varphi - F_\theta(\rho,\theta-\Delta\theta/2,\varphi)\rho\Delta\rho\Delta\varphi.\end{aligned}$$

Divide through by $V \approx \rho^2\sin\varphi\Delta\rho\Delta\theta\Delta\varphi$ and simplify to obtain

$$\begin{aligned}\frac{1}{V}\oiint_S \mathbf{F}\cdot d\mathbf{S} &\approx \left[\frac{F_\varphi(\rho,\theta,\varphi+\Delta\varphi/2)\sin(\varphi+\Delta\varphi/2) - F_\varphi(\rho,\theta,\varphi-\Delta\varphi/2)\sin(\varphi-\Delta\varphi/2)}{\rho\sin\varphi\Delta\varphi}\right]\\ &+ \left[\frac{F_\rho(\rho+\Delta\rho/2,\theta,\varphi)(\rho+\Delta\rho/2)^2 - F_\rho(\rho-\Delta\rho/2,\theta,\varphi)(\rho-\Delta\rho/2)^2}{\rho^2\Delta\rho}\right]\\ &+ \left[\frac{F_\theta(\rho,\theta+\Delta\theta/2,\varphi) - F_\theta(\rho,\theta-\Delta\theta/2,\varphi)}{\rho\sin\varphi\Delta\theta}\right].\end{aligned}$$

Take the limit as $V \to 0$ to conclude

$$\begin{aligned}
\text{div}\,\mathbf{F}(P) &= \lim_{V\to 0}\frac{1}{V}\oiint_S \mathbf{F}\cdot d\mathbf{S}\\
&= \lim_{\Delta\varphi\to 0}\left[\frac{F_\varphi(\rho,\theta,\varphi+\Delta\varphi/2)\sin(\varphi+\Delta\varphi/2) - F_\varphi(\rho,\theta,\varphi-\Delta\varphi/2)\sin(\varphi-\Delta\varphi/2)}{\rho\sin\varphi\Delta\varphi}\right]\\
&\quad + \lim_{\Delta\rho\to 0}\left[\frac{F_\rho(\rho+\Delta\rho/2,\theta,\varphi)(\rho+\Delta\rho/2)^2 - F_\rho(\rho-\Delta\rho/2,\theta,\varphi)(\rho-\Delta\rho/2)^2}{\rho^2\Delta\rho}\right]\\
&\quad + \lim_{\Delta\theta\to 0}\left[\frac{F_\theta(\rho,\theta+\Delta\theta/2,\varphi) - F_\theta(\rho,\theta-\Delta\theta/2,\varphi)}{\rho\sin\varphi\Delta\theta}\right]\\
&= \left[\frac{1}{\rho\sin\varphi}\frac{\partial}{\partial\varphi}(\sin\varphi F_\varphi) + \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2F_\rho) + \frac{1}{\rho\sin\varphi}\frac{\partial F_\theta}{\partial\theta}\right]\Bigg|_P.
\end{aligned}$$

**31.** Let $\mathbf{F}$, $P$, $\mathbf{n}$, $S$ and $C$ be as described in the text. As in Exercise 27, we will assume that $C$ shrinks down to the point $P$ so that the area of the surface bounded decreases monotonically. We will then refer to $S_A$ and $C_A$ as the surface and bounding curve that corresponds to area $A$. Then by Stokes's theorem,

$$\oint_{C_A}\mathbf{F}\cdot d\mathbf{s} = \iint_{S_A}\nabla\times\mathbf{F}\cdot d\mathbf{S} = \iint_{S_A}(\nabla\times\mathbf{F}\cdot\mathbf{n})\,dS.$$

By the mean value theorem for surface integrals, there is some point $Q_A \in S_A$ such that

$$\iint_{S_A}(\nabla\times\mathbf{F}\cdot\mathbf{n})\,dS = (\nabla\times\mathbf{F}(Q_A)\cdot\mathbf{n})(\text{area of } S_A) = (\nabla\times\mathbf{F}(Q_A)\cdot\mathbf{n})\,A.$$

Therefore,

$$\begin{aligned}
\lim_{A\to 0}\frac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s} &= \lim_{A\to 0}\frac{1}{A}[(\nabla\times\mathbf{F}(Q_A)\cdot\mathbf{n})A] = \lim_{A\to 0}(\nabla\times\mathbf{F}(Q_A)\cdot\mathbf{n})\\
&= \mathbf{n}\cdot(\nabla\times\mathbf{F}(P)) = \mathbf{n}\cdot\text{curl}\,\mathbf{F}(P).
\end{aligned}$$

**32. (a)** By Exercise 31, $\mathbf{e}_z\cdot\text{curl}\,\mathbf{F}(P) = \lim_{A\to 0}\dfrac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s}$. Here $A \approx r\Delta r\Delta\theta$.

$$\begin{aligned}
\oint_{C_A}\mathbf{F}\cdot d\mathbf{s} \approx{}& -F_r\left(r,\theta+\frac{\Delta\theta}{2},z\right)\Delta r - F_\theta\left(r-\frac{\Delta r}{2},\theta,z\right)\left(r-\frac{\Delta r}{2}\right)\Delta\theta\\
&+ F_r\left(r,\theta-\frac{\Delta\theta}{2},z\right)\Delta r + F_\theta\left(r+\frac{\Delta r}{2},\theta,z\right)\left(r+\frac{\Delta r}{2}\right)\Delta\theta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{e}_z\cdot\text{curl}\,\mathbf{F}(P) &= \lim_{A\to 0}\frac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s}\\
&= \lim_{\Delta\theta\to 0}\left[-\frac{F_r\left(r,\theta+\frac{\Delta\theta}{2},z\right) - F_r\left(r,\theta-\frac{\Delta\theta}{2},z\right)}{r\Delta\theta}\right]\\
&\quad + \lim_{\Delta r\to 0}\left[\frac{F_\theta\left(r+\frac{\Delta r}{2},\theta,z\right)\left(r+\frac{\Delta r}{2}\right) - F_\theta\left(r-\frac{\Delta r}{2},\theta,z\right)\left(r-\frac{\Delta r}{2}\right)}{r\Delta r}\right]\\
&= -\frac{1}{r}\frac{\partial F_r}{\partial\theta} + \frac{1}{r}\frac{\partial}{\partial r}(rF_\theta).
\end{aligned}$$

**(b)** Again by Exercise 31, $\mathbf{e}_r\cdot\text{curl}\,\mathbf{F}(P) = \lim_{A\to 0}\dfrac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s}$. Here $A \approx r\,\Delta z\,\Delta\theta$.

$$\begin{aligned}
\oint_{C_A}\mathbf{F}\cdot d\mathbf{s} \approx{}& F_z\left(r,\theta+\frac{\Delta\theta}{2},z\right)\Delta z - F_\theta\left(r,\theta,z+\frac{\Delta z}{2}\right)r\,\Delta\theta\\
&- F_z\left(r,\theta-\frac{\Delta\theta}{2},z\right)\Delta z + F_\theta\left(r,\theta,z-\frac{\Delta z}{2}\right)r\,\Delta\theta.
\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{e}_r \cdot \text{curl}\,\mathbf{F}(P) &= \lim_{A\to 0}\frac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s}\\ &= \lim_{\Delta\theta\to 0}\left[\frac{F_z\left(r,\theta+\frac{\Delta\theta}{2},z\right)-F_z\left(r,\theta-\frac{\Delta\theta}{2},z\right)}{r\Delta\theta}\right]\\ &\quad+\lim_{\Delta z\to 0}\left[-\frac{F_\theta\left(r,\theta,z+\frac{\Delta z}{2}\right)-F_\theta\left(r,\theta,z-\frac{\Delta z}{2}\right)}{\Delta z}\right]\\ &= \frac{1}{r}\frac{\partial F_z}{\partial\theta}-\frac{\partial F_\theta}{\partial z}.\end{aligned}$$

**(c)** Again by Exercise 31, $\mathbf{e}_\theta \cdot \text{curl}\,\mathbf{F}(P) = \lim_{A\to 0}\dfrac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s}$. Here $A = \Delta r\Delta z$.

$$\begin{aligned}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s} \approx{}& F_z\left(r-\frac{\Delta r}{2},\theta,z\right)\Delta z + F_r\left(r,\theta,z+\frac{\Delta z}{2}\right)\Delta r\\ &- F_z\left(r+\frac{\Delta r}{2},\theta,z\right)\Delta z - F_r\left(r,\theta,z-\frac{\Delta z}{2}\right)\Delta r.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{e}_\theta \cdot \text{curl}\,\mathbf{F}(P) &= \lim_{A\to 0}\frac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s}\\ &= \lim_{\Delta r\to 0}\left[-\frac{F_z\left(r+\frac{\Delta r}{2},\theta,z\right)-F_z\left(r-\frac{\Delta r}{2},\theta,z\right)}{\Delta r}\right]\\ &\quad+\lim_{\Delta z\to 0}\left[\frac{F_r\left(r,\theta,z+\frac{\Delta z}{2}\right)-F_r\left(r,\theta,z-\frac{\Delta z}{2}\right)}{\Delta z}\right]\\ &= -\frac{\partial F_z}{\partial r}+\frac{\partial F_r}{\partial z}.\end{aligned}$$

The final conclusion is just a matter of putting the three pieces together and checking that the sum agrees with the determinant given.

**33.** This is similar to Exercise 32. By Exercise 31, $\mathbf{e}_\rho \cdot \text{curl}\,\mathbf{F}(P) = \lim_{A\to 0}\dfrac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s}$. Here $A \approx \rho^2\sin\varphi\Delta\varphi\Delta\theta$.

$$\begin{aligned}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s} ={}& -F_\varphi\left(\rho,\theta+\frac{\Delta\theta}{2},\varphi\right)\rho\Delta\varphi - F_\theta\left(\rho,\theta,\varphi-\frac{\Delta\varphi}{2}\right)\rho\sin\left(\varphi-\frac{\Delta\varphi}{2}\right)\Delta\theta\\ &+ F_\varphi\left(\rho,\theta-\frac{\Delta\theta}{2},\varphi\right)\rho\Delta\varphi + F_\theta\left(\rho,\theta,\varphi+\frac{\Delta\varphi}{2}\right)\rho\sin\left(\varphi+\frac{\Delta\varphi}{2}\right)\Delta\theta.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{e}_\rho \cdot \text{curl}\,\mathbf{F}(P) &= \lim_{A\to 0}\frac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s}\\ &= \lim_{\Delta\theta\to 0}\left[-\frac{F_\varphi\left(\rho,\theta+\frac{\Delta\theta}{2},\varphi\right)-F_\varphi\left(\rho,\theta-\frac{\Delta\theta}{2},\varphi\right)}{\rho\sin\varphi\Delta\theta}\right]\\ &\quad+\lim_{\Delta\varphi\to 0}\left[\frac{F_\theta\left(\rho,\theta,\varphi+\frac{\Delta\varphi}{2}\right)\sin\left(\varphi+\frac{\Delta\varphi}{2}\right)-F_\theta\left(\rho,\theta,\varphi-\frac{\Delta\varphi}{2}\right)\sin\left(\varphi-\frac{\Delta\varphi}{2}\right)}{\rho\sin\varphi\Delta\varphi}\right]\\ &= \frac{1}{\rho\sin\varphi}\left[-\frac{\partial F_\varphi}{\partial\theta}+\frac{\partial}{\partial\varphi}(\sin\varphi F_\theta)\right].\end{aligned}$$

Again, by Exercise 31, $\mathbf{e}_\theta \cdot \text{curl}\, \mathbf{F}(P) = \lim_{A\to 0} \dfrac{1}{A}\oint_{C_A} \mathbf{F}\cdot d\mathbf{s}$. Here $A \approx \rho\Delta\varphi\Delta\rho$.

$$\oint_{C_A} \mathbf{F}\cdot d\mathbf{s} = F_\varphi\left(\rho + \frac{\Delta\rho}{2}, \theta, \varphi\right)\left(\rho + \frac{\Delta\rho}{2}\right)\Delta\varphi - F_\rho\left(\rho, \theta, \varphi + \frac{\Delta\varphi}{2}\right)\Delta\rho$$
$$- F_\varphi\left(\rho - \frac{\Delta\rho}{2}, \theta, \varphi\right)\left(\rho - \frac{\Delta\rho}{2}\right)\Delta\varphi + F_\rho\left(\rho, \theta, \varphi - \frac{\Delta\varphi}{2}\right)\Delta\rho.$$

Therefore,

$$\begin{aligned}
\mathbf{e}_\theta \cdot \text{curl}\, \mathbf{F}(P) &= \lim_{A\to 0} \frac{1}{A}\oint_{C_A} \mathbf{F}\cdot d\mathbf{s} \\
&= \lim_{\Delta\rho\to 0}\left[\frac{F_\varphi\left(\rho + \frac{\Delta\rho}{2}, \theta, \varphi\right)\left(\rho + \frac{\Delta\rho}{2}\right) - F_\varphi\left(\rho - \frac{\Delta\rho}{2}, \theta, \varphi\right)\left(\rho - \frac{\Delta\rho}{2}\right)}{\rho\Delta\rho}\right] \\
&\quad + \lim_{\Delta\varphi\to 0}\left[-\frac{F_\rho\left(\rho, \theta, \varphi + \frac{\Delta\varphi}{2}\right) - F_\rho\left(\rho, \theta, \varphi - \frac{\Delta\varphi}{2}\right)}{\rho\Delta\varphi}\right] \\
&= \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho F_\varphi) - \frac{\partial F_\rho}{\partial\varphi}\right].
\end{aligned}$$

Again, by Exercise 31, $\mathbf{e}_\varphi \cdot \text{curl}\, \mathbf{F}(P) = \lim_{A\to 0} \dfrac{1}{A}\oint_{C_A} \mathbf{F}\cdot d\mathbf{s}$. Here $A \approx \rho\sin\varphi\Delta\rho\Delta\theta$.

$$\oint_{C_A} \mathbf{F}\cdot d\mathbf{s} = F_\theta\left(\rho - \frac{\Delta\rho}{2}, \theta, \varphi\right)\left(\rho - \frac{\Delta\rho}{2}\right)\sin\varphi\Delta\theta + F_\rho\left(\rho, \theta + \frac{\Delta\theta}{2}, \varphi\right)\Delta\rho$$
$$- F_\theta\left(\rho + \frac{\Delta\rho}{2}, \theta, \varphi\right)\left(\rho + \frac{\Delta\rho}{2}\right)\sin\varphi\Delta\theta - F_\rho\left(\rho, \theta - \frac{\Delta\theta}{2}, \varphi\right)\Delta\rho.$$

Therefore,

$$\begin{aligned}
\mathbf{e}_\varphi \cdot \text{curl}\, \mathbf{F}(P) &= \lim_{A\to 0} \frac{1}{A}\oint_{C_A} \mathbf{F}\cdot d\mathbf{s} \\
&= \lim_{\Delta\rho\to 0}\left[-\frac{-F_\theta\left(\rho + \frac{\Delta\rho}{2}, \theta, \varphi\right)\left(\rho + \frac{\Delta\rho}{2}\right) + F_\theta\left(\rho - \frac{\Delta\rho}{2}, \theta, \varphi\right)\left(\rho - \frac{\Delta\rho}{2}\right)}{\rho\Delta\rho}\right] \\
&\quad + \lim_{\Delta\theta\to 0}\left[\frac{F_\rho\left(\rho, \theta + \frac{\Delta\theta}{2}, \varphi\right) - F_\rho\left(\rho, \theta - \frac{\Delta\theta}{2}, \varphi\right)}{\rho\sin\varphi\Delta\theta}\right] \\
&= \frac{1}{\rho}\left[-\frac{\partial}{\partial\rho}(\rho F_\theta) + \frac{1}{\sin\varphi}\frac{\partial F_\rho}{\partial\theta}\right].
\end{aligned}$$

Again, the final conclusion is just a matter of assembling the pieces above and checking that the sum agrees with the determinant.

**34.** We use the results of Exercises 27 and 31:

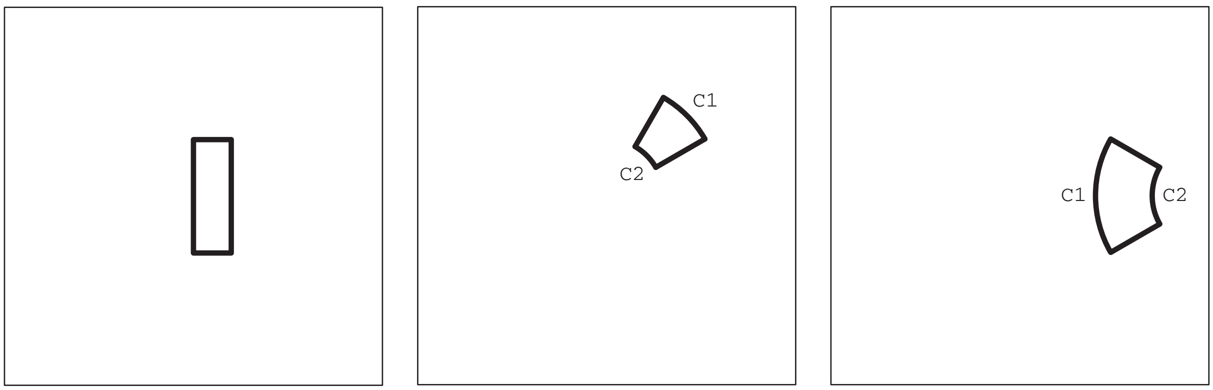
$$\text{div}\, \mathbf{F}(P) = \lim_{V\to 0} \frac{1}{V}\oiint_S \mathbf{F}\cdot d\mathbf{S}$$
$$\mathbf{n}\cdot \text{curl}\, \mathbf{F}(P) = \lim_{A\to 0} \frac{1}{A}\oint_C \mathbf{F}\cdot d\mathbf{s}.$$

The vector fields to be considered are planar, so the divergence results should actually be interpreted as

$$\text{div}\, \mathbf{F}(P) = \lim_{A\to 0} \frac{1}{A}\oint_C (\mathbf{F}\cdot\mathbf{n})\, ds.$$

(See the discussion regarding two-dimensional flux in Section 6.2.) Here **n** is the outward unit normal to $C$ that lies in the plane. We need to find the four fields for which the divergence is identically zero. Intuitively, you can see in figures (b) and (e) by looking at symmetric neighborhoods of the center point that at the center point the divergence is not zero. We will be more precise than that. For the curl result, we need only take **n** to be the unit vector pointing up out of the plane of the vector field. Using these results, we may categorize the vector fields by drawing appropriate paths.

**(a)** Draw a rectangular path $C$ with sides parallel to the $x$- and $y$-axes (see below left). Along such a path, $\oint_C \mathbf{F}\cdot d\mathbf{s} \neq 0$, since the path is tangent to the vector field along vertical segments and **F** has different magnitudes along these segments. The integrals along the horizontal segments will be equal and opposite. This will be true in the limit, so curl $\mathbf{F} \neq \mathbf{0}$. On the other hand, $\oint_C (\mathbf{F}\cdot\mathbf{n})ds = 0$ because $\mathbf{F}\cdot\mathbf{n}$ vanishes on vertical parts of $C$ and has opposite sign on the two horizontal segments. Therefore, div $\mathbf{F} = 0$.

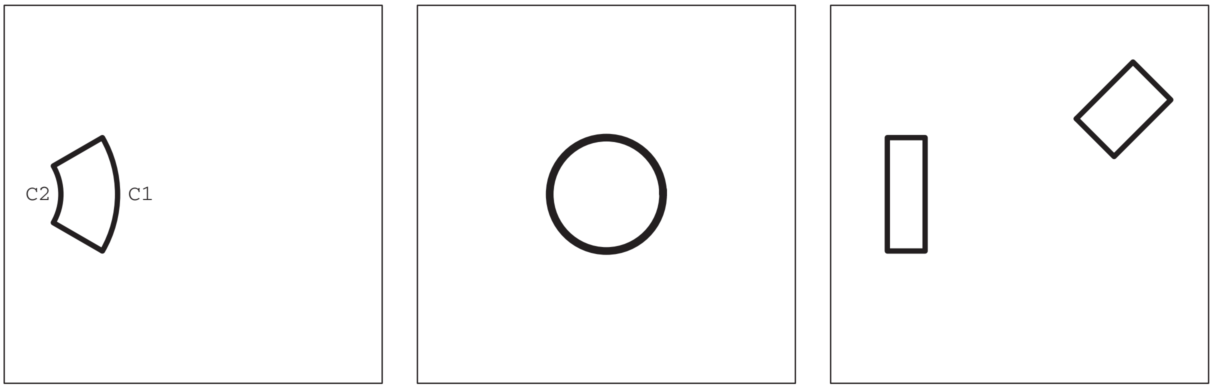



**(b)** Draw a path contained in the upper right quarter of the diagram that is a "polar rectangle" (see above center). In other words, we draw the path so that two of the sides are tangent to the vector field (one in the same direction, one in the opposite direction) and the remaining two sides are sides each of whose distance to the center of the figure is constant. Note that once the path is oriented, the segments labelled $C_1$ and $C_2$ will receive "opposite" orientations. Here $\left|\int_{C_1}(\mathbf{F}\cdot\mathbf{n})\,ds\right| > \left|\int_{C_2}(\mathbf{F}\cdot\mathbf{n})\,ds\right|$ and $\left|\int(\mathbf{F}\cdot\mathbf{n})\,ds\right| = 0$ along the radial segments. Therefore, div $\mathbf{F} \neq 0$. On the other hand, $\int_{C_1}\mathbf{F}\cdot d\mathbf{s} = \int_{C_2}\mathbf{F}\cdot d\mathbf{s} = 0$, since **F** is perpendicular to $C_1$ and $C_2$. However, $\mathbf{F}\cdot\mathbf{T}$ has opposite signs on the radial pieces so $\oint_C \mathbf{F}\cdot d\mathbf{s} = \oint_C (\mathbf{F}\cdot\mathbf{T})ds = 0$. Hence curl $\mathbf{F} = 0$.

**(c)** Again our path will be a polar rectangle (see above right). This time orient the path clockwise and picture the center of the coordinate system to be at the center of the right border of the figure. Denote the left-most, "vertical" side $C_1$ and the right-most, "vertical" side $C_2$. Orient the path either way. $C_1$ and $C_2$ will receive "opposite" orientations. The idea here is that $\int_{C_1}\mathbf{F}\cdot d\mathbf{s}$ is cancelled by $\int_{C_2}\mathbf{F}\cdot d\mathbf{s}$ because the integral of the smaller magnitude of **F** along the longer segment $C_1$ is balanced by the integral of the larger magnitude of **F** along the shorter segment $C_2$. Integrals along the other segments are 0 because **F** is perpendicular to those segments. Hence, curl $\mathbf{F} = \mathbf{0}$. The path is also arranged so $\oint_C (\mathbf{F}\cdot\mathbf{n})\,ds = 0$. It is zero along $C_1$ and $C_2$ and cancels on the other segments. Hence, div $\mathbf{F} = 0$.

**(d)** Again choose a polar rectangle for our path (see below left). This time picture the center of the coordinate system to be at the center of the left border of the figure. What makes this different from the vector field in (c) is that here $\|\mathbf{F}\|$ is constant. For this reason, $\oint_C \mathbf{F}\cdot d\mathbf{s} \neq 0$ and, therefore, curl $\mathbf{F} \neq \mathbf{0}$. On the other hand, div $\mathbf{F} = 0$ for the same reasons as in part (c).

**(e)** Let our path be an oriented circle centered at the center of the figure (see below center). It is clear that $\oint_C (\mathbf{F}\cdot\mathbf{n})ds \neq 0$ and therefore div $\mathbf{F} \neq 0$. Likewise, $\oint_C \mathbf{F}\cdot d\mathbf{s} \neq 0$, so curl $\mathbf{F} \neq \mathbf{0}$.




**(f)** Well, by elimination we must have div $\mathbf{F} = 0$ and curl $\mathbf{F} = \mathbf{0}$. For the divergence argument, choose a rectangular path in the upper right quarter of the diagram with two sides parallel to and symmetric about the diagonal from the lower left corner to the upper right corner of the diagonal. For the curl argument, use a rectangular path with sides parallel to the coordinate axes (see above right).

## 7.4 Further Vector Analysis; Maxwell's Equations

**1.** Notice the similarities between this exercise and Exercise 28 in the Miscellaneous Exercises for Chapter 6. By Gauss's theorem (Theorem 3.3),

$$\oiint_{\partial D} f\nabla g \cdot d\mathbf{S} = \iiint_D \nabla \cdot (f\nabla g)\, dV.$$

By the product rule,

$$\iiint_D \nabla \cdot (f\nabla g)\, dV = \iiint_D (\nabla f \cdot \nabla g + f\nabla^2 g)\, dV = \iiint_D (\nabla f \cdot \nabla g)\, dV + \iiint_D (f\nabla^2 g)\, dV.$$

**2.** Let $f \equiv 1$ in Green's first formula. Then $\nabla f = \mathbf{0}$ so the first term in Green's first formula is 0, so

$$\iiint_D \nabla^2 g\, dV = \oiint_S \nabla g \cdot d\mathbf{S}.$$

We assumed that $g$ is harmonic so $\nabla^2 g = 0$. Also we know that $S = \partial D$. Therefore, by the definition of the normal derivative,

$$0 = \oiint_{\partial D} \nabla g \cdot d\mathbf{S} = \oiint_{\partial D} (\nabla g \cdot \mathbf{n})\, dS = \oiint_{\partial D} \frac{\partial g}{\partial n}\, dS.$$

**3. (a)** Using Green's first formula with $f = g$, we obtain

$$\iiint_D \nabla f \cdot \nabla f\, dV + \iiint_D f\nabla^2 f\, dV = \oiint_S f\nabla f \cdot d\mathbf{S}.$$

We are assuming that $f$ is harmonic, so the second integral on the left side is 0. Therefore,

$$\iiint_D \nabla f \cdot \nabla f\, dV = \oiint_{\partial D} f\nabla f \cdot d\mathbf{S} = \oiint_{\partial D} f(\nabla f \cdot \mathbf{n})\, dS = \oiint_{\partial D} f\frac{\partial f}{\partial n}\, dS.$$

**(b)** If $f = 0$ on the boundary of $D$, then part (a) implies that

$$0 = \oiint_{\partial D} f\frac{\partial f}{\partial n}\, dS = \iiint_D \nabla f \cdot \nabla f\, dV.$$

But $\nabla f \cdot \nabla f = \|\nabla f\|^2 \geq 0$. So the right-hand integral was of a non-negative, continuous integrand. For this to be zero, the integrand must have been identically zero. In other words, $\nabla f \cdot \nabla f$ is zero on $D$. We conclude that $\nabla f$ is zero on $D$ and so $f$ is constant on $D$. Since $f(x, y, z) = 0$ on $\partial D$ and $f$ is constant on $D$, we must have that $f \equiv 0$ on $D$.

**4.** Use the hint and consider $f = f_1 - f_2$. Then, since $f_1 = f_2$ on $\partial D$, we have that $f = 0$ on $\partial D$. Note that if $f_1$ and $f_2$ are harmonic on $D$, then $f$ is harmonic on $D$. Therefore, by Exercise 3(b), $f \equiv 0$ on all of $D$ so $f_1 = f_2$ on $D$.

**5. (a)** Using the hint we see that the rate of fluid flowing into $W$ is $\iiint_W \frac{\partial\rho}{\partial t} dV$ and the rate of fluid flowing out of $W$ is $\oiint_S \rho\mathbf{F}\cdot d\mathbf{S}$. Hence we have $\oiint_S \rho\mathbf{F}\cdot d\mathbf{S} = -\iiint_W \frac{\partial\rho}{\partial t} dV$. Also, by Gauss's theorem, we have $\iiint_W \nabla\cdot(\rho\mathbf{F})dV = \oiint_S \rho\mathbf{F}\cdot d\mathbf{S}$; therefore $\iiint_W \nabla\cdot(\rho\mathbf{F})dV = -\iiint_W \frac{\partial\rho}{\partial t} dV$. Finally, as in the arguments in the text, we point out that the equation

$$\iiint_R \nabla\cdot(\rho\mathbf{F})\,dV = -\iiint_R \frac{\partial\rho}{\partial t}\,dV$$

holds for *any* solid region $R \subseteq W$ by the same argument. Thus, by shrinking $R$ to a point, we can conclude $\nabla\cdot(\rho\mathbf{F}) = -\frac{\partial\rho}{\partial t}$.

**(b)** From (14) in the text, the current density field $\mathbf{J}$ is $\rho\mathbf{v}$. Therefore, $\iiint_W \frac{\partial\rho}{\partial t} dV$ represents the current flowing into $W$ and $\oiint_S \mathbf{J}\cdot d\mathbf{S}$ represents the current flowing out of $W$ (across $S$). Hence, the same argument as that given in part (a) shows that $\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$.

**6.** We are given that the total heat leaving $D$ per unit time is $-\iiint_D \sigma\rho\frac{\partial T}{\partial t} dV$. This is equal to the flux $\oiint_S \mathbf{H}\cdot d\mathbf{S}$ which, by the definition of $\mathbf{H}$, is the same as $\oiint_S -k\nabla T\cdot d\mathbf{S}$. By Gauss's theorem, we have $\oiint_S -k\nabla T\cdot d\mathbf{S} = \iiint_D -k\nabla\cdot\nabla T dV$. Therefore, $-\iiint_D \sigma\rho\frac{\partial T}{\partial t} dV = \iiint_D -k\nabla\cdot\nabla T dV$. Since $D$ is arbitrary, shrink it to a point to conclude that $-\sigma\rho\frac{\partial T}{\partial t} = -k\nabla\cdot\nabla T$ or $\sigma\rho\frac{\partial T}{\partial t} = k\nabla\cdot\nabla T$.

**7.** We know from the argument in Exercise 6 that $-\iiint_D \sigma\rho\frac{\partial T}{\partial t} dV = \iiint_D -\nabla\cdot(k\nabla T)dV$. Use the product rule to conclude that this equals $\iiint_D -(\nabla k\cdot\nabla T + k\nabla^2 T)dV$. As before, shrink to a point to conclude $\sigma\rho\frac{\partial T}{\partial t} = k\nabla^2 T + \nabla k\cdot\nabla T$.

**8.** This is immediate from the heat equation since $\partial T/\partial t = 0$ and $\sigma, \rho, k$ are constants.

**9. (a)**

$$\oiint_{\partial D} \mathbf{H}\cdot d\mathbf{S} = \oiint_{\partial D} -k\nabla T\cdot d\mathbf{S} = \iiint_D \nabla\cdot(-k\nabla T)\,dV \quad \text{by Gauss's theorem}$$
$$= -k\iiint_D \nabla^2 T\,dV = 0 \quad \text{by Exercise 8.}$$

**(b)** By part (a), there can be no net inflow or outflow of heat. Thus, heat must be flowing into $D$ from the inner (hotter) sphere and out of $D$ through the outer sphere at the same rate.

**10. (a)** Since $w = T_1 - T_2$, $\nabla^2 w = \nabla^2(T_1 - T_2)$. But $T_1$ and $T_2$ each satisfy the heat equation given in the exercise, so

$$\nabla^2 w = \nabla^2(T_1 - T_2) = \frac{\partial T_1}{\partial t} - \frac{\partial T_2}{\partial t} = \frac{\partial}{\partial t}(T_1 - T_2) = \frac{\partial w}{\partial t}.$$

So $w$ satisfies the heat equation. Now for $(x,y,z) \in D$ we have

$$w(x,y,z,0) = T_1(x,y,z,0) - T_2(x,y,z,0) = \alpha(x,y,z) - \alpha(x,y,z) = 0.$$

So the first condition holds. Also for all $(x,y,z) \in \partial D$ and $t \geq 0$ we see

$$w(x,y,z,t) = T_1(x,y,z,t) - T_2(x,y,z,t) = \phi(x,y,z,t) - \phi(x,y,z,t) = 0.$$

So $w$ satisfies the second condition.

**(b)** We take the derivative

$$E'(t) = \frac{d}{dt}\left[\frac{1}{2}\iiint_D w^2\,dV\right] = \frac{1}{2}\iiint_D \frac{\partial}{\partial t}(w^2)\,dV = \iiint_D w\frac{\partial w}{\partial t}\,dV.$$

From part (a) we know that $w$ satisfies the heat equation so

$$\iiint_D w\frac{\partial w}{\partial t}\,dV = \iiint_D w\nabla^2 w\,dV.$$

Using Green's first formula with $f = g = w$, we have

$$\iiint_D w\nabla^2 w\,dV = \oiint_{\partial D} w\nabla w\cdot d\mathbf{S} - \iiint_D \nabla w\cdot\nabla w\,dV = -\iiint_D \nabla w\cdot\nabla w\,dV$$

since we showed in part (a) that $w \equiv 0$ on $\partial D$. Thus, $E'(t) = -\iiint_D \|\nabla w\|^2 dV \le 0$.

**(c)** In part (a) we showed that $w(x, y, z, 0) = 0$ on $D$. Therefore,

$$E(0) = \frac{1}{2}\iiint_D [w(x,y,z,0)]^2\,dV = 0.$$

Now $E(t)$ is the integral of a non-negative integrand so $E(t) \ge 0$. On the other hand, from part (b) we know that $E$ is nonincreasing. Therefore, $E$ is a nonincreasing, nonnegative function such that $E(0) = 0$. Hence $E(t) = 0$ for all $t \ge 0$.

**(d)** By part (c), $\iiint_D w^2 dV = 0$ for all $t \ge 0$. Since $w^2 \ge 0$, we must have $[w(x,y,z,t)]^2 = 0$ for all $(x,y,z) \in D$ and $t \ge 0$. Therefore $w(x,y,z,t) = 0$ for all $(x,y,z) \in D$ and $t \ge 0$. Hence $T_1(x,y,z,t) = T_2(x,y,z,t)$ for all $(x,y,z) \in D$ and $t \ge 0$.

**11.** From Ampère's law we have $\mathbf{J} = \dfrac{1}{\mu_0}\nabla\times\mathbf{B} - \epsilon_0\dfrac{\partial \mathbf{E}}{\partial t}$. Therefore,

$$\begin{aligned}\nabla\times\mathbf{J} &= \frac{1}{\mu_0}\nabla\cdot(\nabla\times\mathbf{B}) - \epsilon_0\nabla\cdot\frac{\partial\mathbf{E}}{\partial t} = -\epsilon_0\nabla\cdot\frac{\partial\mathbf{E}}{\partial t}\\ &= -\epsilon_0\frac{\partial}{\partial t}(\nabla\cdot\mathbf{E}) = -\epsilon_0\frac{\partial}{\partial t}\left(\frac{\rho}{\epsilon_0}\right) \quad \text{by Gauss's law,}\\ &= -\frac{\partial\rho}{\partial t}.\end{aligned}$$

**12.** We find where $\nabla\cdot\mathbf{E} = 0$.

$$\nabla\cdot\mathbf{E} = \frac{\partial}{\partial x}(x^3 - x) + \frac{\partial}{\partial y}\left(\frac{1}{4}y^3\right) + \frac{\partial}{\partial z}\left(\frac{1}{9}z^3 - 2z\right).$$

So $\nabla\cdot\mathbf{E} = 3x^2 + \frac{3}{4}y^2 + \frac{1}{3}z^2 - 3$. This is zero for points on the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$.

**13.** First we check that $\nabla\cdot\mathbf{F} = 0$ wherever $\mathbf{F}$ is defined (i.e., away from the origin):

$$\begin{aligned}\nabla\cdot\mathbf{F} &= k\left(\frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)\\ &= k\left(\frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^{3/2}} + \frac{(x^2+y^2+z^2)^{3/2} - 3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^{3/2}}\right.\\ &\qquad\left. + \frac{(x^2+y^2+z^2)^{3/2} - 3z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^{3/2}}\right).\end{aligned}$$

Multiply numerator and denominator by $(x^2+y^2+z^2)^{1/2}$:

$$\begin{aligned}\nabla\cdot\mathbf{F} &= \frac{k}{(x^2+y^2+z^2)^{7/2}}\Big(3(x^2+y^2+z^2)^2 - 3x^2(x^2+y^2+z^2) - 3y^2(x^2+y^2+z^2)\\ &\qquad - 3z^2(x^2+y^2+z^2)\Big)\\ &= \frac{k(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}}(3(x^2+y^2+z^2) - 3x^2 - 3y^2 - 3z^2) \equiv 0.\end{aligned}$$

Thus, by Gauss's theorem, $\oiint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \nabla\cdot\mathbf{F}\,dV = 0$ if $S = \partial D$ and $S$ does not enclose the origin. If $S$ does enclose the origin, let $D$ be the solid region between $S$ and a small sphere $S_b$ of radius $b$ that encloses the origin and is inside $S$ (as in Figure 7.54). Then

$$0 = \iiint_D \nabla\cdot\mathbf{F}\,dV = \oiint_S \mathbf{F}\cdot d\mathbf{S} - \oiint_{S_b}\mathbf{F}\cdot d\mathbf{S}$$

where $S$ and $S_b$ are both oriented by *outward* normals.

Hence $\displaystyle\oiint_S \mathbf{F}\cdot d\mathbf{S} = \oiint_{S_b} \mathbf{F}\cdot d\mathbf{S} = \oiint_{S_b} \frac{k\mathbf{x}}{\|\mathbf{x}\|^3}\cdot\frac{1}{b}\mathbf{x}\,dS$ (outward normal $\mathbf{n}$ to $S_b$ is $\frac{1}{b}\mathbf{x}$)

$$= \oiint_{S_b} \frac{k\|\mathbf{x}\|^2}{b\|\mathbf{x}\|^3}\,dS = \oiint_{S_b} \frac{kb^2}{b^4}\,dS \quad (\|\mathbf{x}\| = b \text{ on } S_b)$$

$$= \frac{k}{b^2}\cdot(\text{surface area of } S_b) = \frac{k}{b^2}(4\pi b^2) = 4\pi k.$$

**14. (a)** We may write $\mathbf{E}(\mathbf{x}) = E_\rho(\mathbf{x})\mathbf{e}_\rho + E_\varphi(\mathbf{x})\mathbf{e}_\varphi + E_\theta(\mathbf{x})\mathbf{e}_\theta$. The field of a point charge at the origin must be symmetric about the origin. Thus $E_\varphi = E_\theta = 0$, so $\mathbf{E}(\mathbf{x}) = E_\rho(\mathbf{x})\mathbf{e}_\rho = E(\mathbf{x})\mathbf{e}_\rho$. Once again, by symmetry, $E(\mathbf{x})$ must be constant on any sphere centered at the origin, so $E$ can only depend on $\rho$. Hence $\mathbf{E}(\mathbf{x}) = E(\rho)\mathbf{e}_\rho$.

**(b)** We have

$$\begin{aligned}\oiint_S E(\rho)\mathbf{e}_\rho\cdot d\mathbf{S} &= \oiint_S \mathbf{E}\cdot d\mathbf{S} \quad \text{from part (a),}\\ &= \iiint_D \nabla\cdot\mathbf{E}\,dV \quad \text{using Gauss's theorem,}\\ &= \iiint_D \frac{\rho}{\epsilon_0}\,dV \quad \text{using Gauss's law,}\\ &= \frac{q}{\epsilon_0} \quad \text{by definition of } \rho \text{ and } q.\end{aligned}$$

**(c)** We have $\displaystyle\oiint_S E(\rho)\mathbf{e}_\rho\cdot d\mathbf{S} = \oiint_S E(\rho)\mathbf{e}_\rho\cdot\mathbf{n}\,dS = \oiint_S E(\rho)\mathbf{e}_\rho\cdot\mathbf{e}_\rho\,dS = \oiint_S E(\rho)\,dS$. Since, by part (b), $\displaystyle\oiint_S E(\rho)\mathbf{e}_\rho\cdot d\mathbf{S} = \frac{q}{\epsilon_0}$, we have $\displaystyle\oiint_S E(\rho)\,dS = \frac{q}{\epsilon_0}$.

**(d)** By part (c), $q/\epsilon_0 = \displaystyle\oiint_S E(\rho)\,dS$. But, obviously, $\rho$ is constant on the sphere of radius $a$ and so on that sphere $q/\epsilon_0 = \displaystyle\oiint_S E(\rho)\,dS = E(a)\cdot 4\pi a^2$. Thus we see that $E(\rho) = q/(4\pi\epsilon_0\rho^2)$. Hence,

$$\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi\epsilon_0\rho^2}\mathbf{e}_\rho = \frac{q}{4\pi\epsilon_0\|\mathbf{x}\|^2}\frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{x}}{\|\mathbf{x}\|^3} \text{ as desired.}$$

**15. (a)** This is just a straightforward calculation. Write $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$. Then

$$\nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ M & N & P\end{vmatrix} = (P_y - N_z)\mathbf{i} + (M_z - P_x)\mathbf{j} + (N_x - M_y)\mathbf{k}$$

and

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{F}) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ P_y - N_z & M_z - P_x & N_x - M_y\end{vmatrix}\\ &= (N_{xy} - M_{yy} - M_{zz} + P_{xz})\mathbf{i} + (P_{yz} - N_{zz} - N_{xx} + M_{yx})\mathbf{j}\\ &\quad + (M_{zx} - P_{xx} - P_{yy} + N_{zy})\mathbf{k}.\end{aligned}$$

On the other hand,

$$\begin{aligned}
\nabla(\nabla\cdot\mathbf{F}) &= \nabla(M_x+N_y+P_z)\\
&= (M_{xx}+N_{yx}+P_{zx})\mathbf{i}+(M_{xy}+N_{yy}+P_{zy})\mathbf{j}+(M_{xz}+N_{yz}+P_{zz})\mathbf{k}\\
\text{and } (\nabla\cdot\nabla)\mathbf{F} &= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\mathbf{F}\\
&= (M_{xx}+M_{yy}+M_{zz})\mathbf{i}+(N_{xx}+N_{yy}+N_{zz})\mathbf{j}+(P_{xx}+P_{yy}+P_{zz})\mathbf{k}.\\
\text{Hence, } \nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F} &= (N_{yx}+P_{zx}-M_{yy}-M_{zz})\mathbf{i}+(M_{xy}+P_{zy}-N_{xx}-N_{zz})\mathbf{j}\\
&\quad+(M_{xz}+N_{yz}-P_{xx}-P_{yy})\mathbf{k}.
\end{aligned}$$

By assumption **F** is of class $C^2$ and so the mixed partials are equal; thus we have the result:

$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F}.$$

**(b)** First we show that **E** satisfies the wave equation.

$$\begin{aligned}
\nabla^2\mathbf{E} &= \nabla(\nabla\cdot\mathbf{E})-\nabla\times(\nabla\times\mathbf{E}) \quad \text{from part (a),}\\
&= \nabla\left(\frac{\rho}{\epsilon_0}\right)-\nabla\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right) \quad \text{using Gauss's and Faraday's laws,}\\
&= \frac{1}{\epsilon_0}\nabla\rho+\frac{\partial}{\partial t}(\nabla\times\mathbf{B})\\
&= \frac{1}{\epsilon_0}\nabla\rho+\frac{\partial}{\partial t}\left(\mu_0\mathbf{J}+\epsilon_0\,\mu_0\frac{\partial\mathbf{E}}{\partial t}\right) \quad \text{using Ampère's law}\\
&= \mathbf{0}+\epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2} \quad \text{since there are no charges or currents (so } \rho\equiv 0 \text{ and } \mathbf{J}\equiv\mathbf{0}).
\end{aligned}$$

Thus $\nabla^2\mathbf{E} = k\dfrac{\partial^2\mathbf{E}}{\partial t^2}$ where $k=\epsilon_0\,\mu_0$.

Next we show that **B** satisfies the wave equation.

$$\begin{aligned}
\nabla^2\mathbf{B} &= \nabla(\nabla\cdot\mathbf{B})-\nabla\times(\nabla\times\mathbf{B}) \quad \text{from part (a),}\\
&= \mathbf{0}-\nabla\times\left(\mu_0\,\mathbf{J}+\epsilon_0\,\mu_0\frac{\partial\mathbf{E}}{\partial t}\right) \quad \text{using Maxwell's equations,}\\
&= -\epsilon_0\,\mu_0\nabla\times\frac{\partial\mathbf{E}}{\partial t} \text{ since } \mathbf{J}\equiv\mathbf{0} \quad \text{(no currents),}\\
&= -\epsilon_0\,\mu_0\frac{\partial}{\partial t}(\nabla\times\mathbf{E})\\
&= -\epsilon_0\,\mu_0\frac{\partial}{\partial t}\left(-\frac{\partial\mathbf{B}}{\partial t}\right) \quad \text{from Faraday's law,}\\
&= \epsilon_0\,\mu_0\frac{\partial^2\mathbf{B}}{\partial t^2}.
\end{aligned}$$

So $\nabla^2\mathbf{B} = k\dfrac{\partial^2 B}{\partial t^2}$ where $k=\epsilon_0\,\mu_0$.

**(c)** By part (a),

$$\begin{aligned}\nabla(\nabla\cdot\mathbf{E})-(\nabla\cdot\nabla)\mathbf{E} &= \nabla\times(\nabla\times\mathbf{E})\\ &= \nabla\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right) \quad \text{by Faraday's law,}\\ &= -\frac{\partial}{\partial t}(\nabla\times\mathbf{B})\\ &= -\frac{\partial}{\partial t}\left[\mu_0\mathbf{J}+\epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}\right] \quad \text{by Ampère's law,}\\ &= -\mu_0\frac{\partial}{\partial t}\left[\mathbf{J}+\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right].\end{aligned}$$

**(d)** Again from part (a),

$$\begin{aligned}\nabla^2\mathbf{E} &= \nabla(\nabla\cdot\mathbf{E})-\nabla\times(\nabla\times\mathbf{E})\\ &= \mathbf{0}-\nabla\times(\nabla\times\mathbf{E}) \quad \text{by Gauss's law and the fact that } \rho=0,\\ &= \mu_0\frac{\partial\mathbf{J}}{\partial t}+\epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2} \quad \text{from the argument in part (c).}\end{aligned}$$

**16.** Start with the non-static version of Ampère's law.

$$\begin{aligned}\nabla\cdot(\nabla\times\mathbf{B}) = \nabla\cdot\left(\mu_0\,\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) &= \nabla\cdot(\mu_0\mathbf{J})+\nabla\cdot\left(\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\\ &= -\mu_0\frac{\partial\rho}{\partial t}+\mu_0\epsilon_0\nabla\cdot\frac{\partial\mathbf{E}}{\partial t} \quad \text{from the continuity equation}\\ &= -\mu_0\frac{\partial\rho}{\partial t}+\mu_0\frac{\partial\rho}{\partial t} \quad \text{from Gauss's law}\\ &= 0.\end{aligned}$$

**17. (a)** From Ampère's law in the static case, $\nabla\times\mathbf{B}-\mu_0\mathbf{J}$ must be $\mathbf{0}$ when $\mathbf{J}$ does not depend on time. Otherwise, the difference must depend on time. If $\mathbf{F}_1$ is a time-varying vector field then $\partial\mathbf{F}_1/\partial t\neq\mathbf{0}$. If, on the other hand, $\mathbf{F}_1$ does not depend on time, then $\partial\mathbf{F}_1/\partial t=\mathbf{0}$. Hence, if we take $\nabla\times\mathbf{B}-\mu_0\mathbf{J}=\partial\mathbf{F}_1/\partial t$, then we will have an equation that is valid in both the static and the non-static cases.

**(b)** This is similar to our calculation in Exercise 16.

$$\begin{aligned}\nabla\cdot(\nabla\times\mathbf{B}) = \nabla\cdot(\mu_0\mathbf{J})+\nabla\cdot\frac{\partial\mathbf{F}_1}{\partial t} &= -\mu_0\frac{\partial\rho}{\partial t}+\nabla\cdot\frac{\partial\mathbf{F}_1}{\partial t} \quad \text{from the continuity equation}\\ &= -\mu_0\epsilon_0\nabla\cdot\frac{\partial\mathbf{E}}{\partial t}+\nabla\cdot\frac{\partial\mathbf{F}_1}{\partial t} \quad \text{from Gauss's law.}\end{aligned}$$

So to have $\nabla\cdot(\nabla\times\mathbf{B})=0$ we conclude that $\mu_0\epsilon_0\nabla\cdot\dfrac{\partial\mathbf{E}}{\partial t}=\nabla\cdot\dfrac{\partial\mathbf{F}_1}{\partial t}$.

**(c)** If $\nabla\cdot\dfrac{\partial\mathbf{F}_1}{\partial t}=\mu_0\,\epsilon_0\nabla\dfrac{\partial\mathbf{E}}{\partial t}$, then by part (b),

$$\frac{\partial\mathbf{F}_1}{\partial t}=\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}+\mathbf{F}_2 \quad \text{where} \quad \nabla\cdot\mathbf{F}_2=0.$$

Therefore the most general formulation is

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}+\mathbf{F}_2.$$

**18.** We first show that $\mathbf{E}$ satisfies the telegrapher's equation. From Exercise 15(d) we know that

$$\nabla^2\mathbf{E}=\mu_0\frac{\partial\mathbf{J}}{\partial t}+\epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2},$$

but here $\mathbf{J} = \sigma\mathbf{E}$, so

$$\nabla^2\mathbf{E} = \mu_0\sigma\frac{\partial \mathbf{E}}{\partial t} + \epsilon_0\mu_0\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Next we show that $\mathbf{B}$ satisfies the telegrapher's equation. Now,

$$\begin{aligned}
\nabla^2\mathbf{B} &= \nabla(\nabla\cdot\mathbf{B}) - \nabla\times(\nabla\times\mathbf{B}) \quad \text{from Exercise 15(a)}\\
&= \mathbf{0} - \nabla\times\left(\mu_0\,\mathbf{J} + \epsilon_0\,\mu_0\frac{\partial \mathbf{E}}{\partial t}\right) \quad \text{by Maxwell's equations,}\\
&= -\nabla\times\left(\mu_0\sigma\mathbf{E} + \epsilon_0\,\mu_0\frac{\partial \mathbf{E}}{\partial t}\right)\\
&= -\mu_0\sigma(\nabla\times\mathbf{E}) - \epsilon_0\mu_0\frac{\partial}{\partial t}(\nabla\times\mathbf{E})\\
&= -\mu_0\sigma\left(-\frac{\partial \mathbf{B}}{\partial t}\right) - \epsilon_0\mu_0\frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{B}}{\partial t}\right) \quad \text{by Faraday's law,}\\
&= \mu_0\sigma\frac{\partial \mathbf{B}}{\partial t} + \epsilon_0\mu_0\frac{\partial^2 \mathbf{B}}{\partial t^2}.
\end{aligned}$$

**19.** Since $\mathbf{P} = \mathbf{E}\times\mathbf{B}$,

$$\begin{aligned}
\iint_S \mathbf{P}\cdot d\mathbf{S} &= \iint_S (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{S}\\
&= \iiint_D \nabla\cdot(\mathbf{E}\times\mathbf{B})\,dV \quad \text{by Gauss's theorem,}\\
&= \iiint_D (\mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B}))\,dV\\
&= \iiint_D \left[-\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t} - \mathbf{E}\cdot\left(\mu_0\mathbf{J} + \epsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t}\right)\right]dV \quad \text{from Faraday and Ampère's laws.}
\end{aligned}$$

Since $\mathbf{B}$ and $\mathbf{E}$ are both assumed to be constant in time, $\dfrac{\partial \mathbf{B}}{\partial t} = \dfrac{\partial \mathbf{E}}{\partial t} = \mathbf{0}$. Therefore, we get the desired result:

$$\iint_S \mathbf{P}\cdot d\mathbf{S} = \iiint_D -\mu_0\mathbf{E}\cdot\mathbf{J}\,dV.$$

**20. (a)** If $\mathbf{r} = (r_1, r_2, r_3)$ and $\mathbf{x} = (x, y, z)$,

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\iiint_D \rho(\mathbf{x})\frac{\mathbf{r}-\mathbf{x}}{\|\mathbf{r}-\mathbf{x}\|^3}\,dV\\
&= \frac{1}{4\pi\epsilon_0}\left(\iiint_D \rho(x,y,z)\frac{r_1 - x}{\|\mathbf{r}-\mathbf{x}\|^3}\,dV, \iiint_D \rho(x,y,z)\frac{r_2 - y}{\|\mathbf{r}-\mathbf{x}\|^3}\,dV, \iiint_D \rho(x,y,z)\frac{r_3 - z}{\|\mathbf{r}-\mathbf{x}\|^3}\,dV\right).
\end{aligned}$$

**(b)** Look at the first component of $\mathbf{E}$. (The arguments for the other two components are similar.) We have

$$\left|\frac{\rho(x,y,z)}{4\pi\epsilon_0}\frac{r_1 - x}{\|\mathbf{r}-\mathbf{x}\|^3}\right| \le \frac{|\rho(x,y,z)|}{4\pi\epsilon_0}\frac{\|\mathbf{r}-\mathbf{x}\|}{\|\mathbf{r}-\mathbf{x}\|^3} \le \frac{K}{\|\mathbf{r}-\mathbf{x}\|^2}$$

where $K$ may be taken to be the maximum value of $|\rho|$ on $D$ divided by $4\pi\epsilon_0$. Thus,

$$\left|\frac{1}{4\pi\epsilon_0}\iiint_D \rho(x,y,z)\frac{r_1 - x}{\|\mathbf{r}-\mathbf{x}\|^3}\,dV\right| \le \iiint_D \frac{K}{\|\mathbf{r}-\mathbf{x}\|^2}\,dV.$$

**(c)** Use spherical coordinates with $\mathbf{r}$ as the origin so that the spherical coordinate $\rho$ is $\|\mathbf{r}-\mathbf{x}\|$. Then

$$\iiint_D \frac{K}{\|\mathbf{r}-\mathbf{x}\|^2}\,dV = \iiint_D \frac{K}{\rho^2}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \iiint_D K\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

Note that $K\sin\varphi$ is a bounded, continuous integrand. Since $D$ is a bounded region, this last integral must converge. Hence, by the remarks in the exercise, the original triple integral must converge.

**21.** We are given $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\iiint_D \mathbf{J} \times \dfrac{\mathbf{r}-\mathbf{x}}{\|\mathbf{r}-\mathbf{x}\|^3}\, dV$. Now,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ J_1 & J_2 & J_3 \\ r_1 - x & r_2 - y & r_3 - z \end{vmatrix} = [(r_3 - z)J_2 - (r_2 - y)J_3]\,\mathbf{i} + [(r_1 - x)J_3 - (r_3 - z)J_1]\,\mathbf{j}$$

$$+ [(r_2 - y)J_1 - (r_1 - x)J_2]\,\mathbf{k}.$$

Hence the first component of the triple integral for **B** is

$$\frac{\mu_0}{4\pi}\iiint_D \left( J_2 \frac{r_3 - z}{\|\mathbf{r}-\mathbf{x}\|^3} - J_3 \frac{r_2 - y}{\|\mathbf{r}-\mathbf{x}\|^3} \right) dV.$$

(The other components are of the same form.) Note that each term in the integrand is of the form described in Exercise 20. Thus, using the arguments in Exercise 20, each component integral of **B** must converge.

## True/False Exercises for Chapter 7

**1.** True.
**2.** False. (Note that the parametrization only gives $y \geq 3$.)
**3.** True. (Let $u = s^3$ and $v = \tan t$.)
**4.** False. (The standard normal vanishes when $s$ or $t$ is zero.)
**5.** False. (The limits of integration are not correct.)
**6.** True. (Use symmetry.)
**7.** False. (The value of the integral is 24.)
**8.** True. (Use symmetry.)
**9.** True.
**10.** True. ($\mathbf{F} \cdot \mathbf{n} = 0$.)
**11.** False. (The integral has value $32\pi$.)
**12.** True.
**13.** False. (The value is $0$.)
**14.** True.
**15.** False. (The surface must be connected.)
**16.** False. (Consider the Möbius strip.)
**17.** True. (The result follows from Stokes's theorem.)
**18.** False. (The value is the same only up to sign.)
**19.** True. (Use Gauss's theorem.)
**20.** True. (Apply Gauss's theorem.)
**21.** False. (Gauss's theorem implies that the integral is *at most* twice the surface area.)
**22.** False.
**23.** True.
**24.** True.
**25.** False. (Should be the flux of the *curl* of **F**.)
**26.** True. (This is what Gauss's theorem says.)
**27.** True. (Apply Green's first formula.)
**28.** False. (The negative sign is incorrect.)
**29.** False. ($f$ is determined up to addition of a harmonic function.)
**30.** False. (Only if $S$ doesn't enclose the origin.)

## Miscellaneous Exercises for Chapter 7

**1.** Here are the matches:

(a) C (b) E (c) A

(d) D (e) F (f) B

Brief reasons:

**(c)** The projection of **X** into the *xy*-plane, for fixed $s$, is a circle centered at the origin of radius $2 + \cos\ s$.
**(b)** Note that $x^2 + y^2 = z^2$, so we have a conical surface.
**(a)** Since $y = s$, the intersection of the surface with the plane $y = 0$ is the parametrized curve $x = -t^3, y = 0, z = -t^2$ or $z = -x^{2/3}, y = 0$, which is a cuspidal curve.
**(d)** Let $t = \pi/2$. Then $x = 0, y = s, z = \sin s$. So the intersection of the surface by the $x = 0$ plane is a sinusoidal curve.
**(f)** For constant values of $s$ we have a helix, so the surface should be a helicoid.
**(e)** By elimination, this must correspond to F.

**2. (a)** Consider all the lines through (0, 0, 1). Either such a line is tangent to the sphere, or else it passes through another point of the sphere $S$. The lines tangent to $S$ at (0, 0, 1) fill out the tangent plane $z = 1$. All the other lines therefore have "slope vectors" with nonzero **k**-components. Hence they intersect the $z = 0$ plane somewhere. Thus any line joining (0, 0, 1) and ($s$, $t$, 0) intersects $S$ at a point other than (0, 0, 1).

**(b)** The line joining (0, 0, 1) and ($s$, $t$, 0) is given parametrically by

$$\mathbf{y}(u) = (1-u)(0,0,1) + u(s,t,0) = (us, ut, 1-u).$$

To see where the line intersects the sphere, we insert the parametric equations $\begin{cases} x = us \\ y = ut \\ z = 1-u \end{cases}$ into the equation for $S$ and solve for $u$. Thus:

$$\begin{aligned}(us)^2 + (ut)^2 + (1-u)^2 = 1 &\Leftrightarrow u^2(s^2+t^2+1) - 2u + 1 = 1 \\ &\Leftrightarrow u((s^2+t^2+1)u - 2) = 0.\end{aligned}$$

So either $u = 0$ (which corresponds to (0, 0, 1)) or $u = \dfrac{2}{s^2+t^2+1}$. For this second value of $u$, we may define $\mathbf{X}(s,t)$ as

$$\begin{aligned}\mathbf{X}(s,t) = \mathbf{y}\left(\frac{2}{s^2+t^2+1}\right) &= \left(\frac{2s}{s^2+t^2+1}, \frac{2t}{s^2+t^2+1}, 1 - \frac{2}{s^2+t^2+1}\right) \\ &= \left(\frac{2s}{s^2+t^2+1}, \frac{2t}{s^2+t^2+1}, \frac{s^2+t^2-1}{s^2+t^2+1}\right).\end{aligned}$$

**(c)** Check that the coordinates of **X**($s$, $t$) satisfy the equation for $S$, i.e., that

$$\begin{aligned}&\left(\frac{2s}{s^2+t^2+1}\right)^2 + \left(\frac{2t}{s^2+t^2+1}\right)^2 + \left(\frac{s^2+t^2-1}{s^2+t^2+1}\right)^2 \\ &= \frac{4s^2+4t^2+s^4+t^4+2s^2t^2-2s^2-2t^2+1}{(s^2+t^2+1)^2} \\ &= \frac{s^4+t^4+2s^2t^2+2s^2+2t^2+1}{(s^2+t^2+1)^2} = \frac{(s^2+t^2+1)^2}{(s^2+t^2+1)^2} \equiv 1.\end{aligned}$$

Note that there are no values for $s$ and $t$ so that $\mathbf{X}(s,t) = (0,0,1)$. (To see thus, look at the first two coordinates—we must have $s = t = 0$. But then $\mathbf{X}(0,0) = (0,0,-1)$). Hence the parametrization misses the north pole.

**3. (a)** If we use cylindrical coordinates $x = r\cos\theta, y = r\sin\theta, z = z$, then the equation $x^2+y^2-z^2 = 1$ becomes $r^2-z^2 = 1$ or, since $r \geq 0, r = \sqrt{z^2+1}$. Hence the desired parametrization is $\mathbf{X}(z,\theta) = (\sqrt{z^2+1}\cos\theta, \sqrt{z^2+1}\sin\theta, z)$ where $z \in \mathbf{R}$ and $0 \leq \theta \leq 2\pi$.

**(b)** Modify the cylindrical coordinate substitution by letting $x = ar\cos t, y = br\sin t, z = cs$. Substitution into the equation for the hyperboloid yields $r^2 - s^2 = 1$ so $r = \sqrt{s^2+1}$. Hence a parametrization is $\mathbf{X}(s,t) = (a\sqrt{s^2+1}\cos t,$ $b\sqrt{s^2+1}\sin t, cs)$, where $s \in \mathbf{R}$ and $0 \leq t \leq 2\pi$.

**(c)** Substitute the parametric equations for $\mathbf{l}_1$ into the left side of the equation for the hyperboloid:

$$\begin{aligned}\frac{a^2(x_0-y_0t)^2}{a^2} + \frac{b^2(x_0t+y_0)^2}{b^2} - \frac{c^2t^2}{c^2} &= x_0^2 - 2x_0y_0t + y_0^2t^2 + x_0^2t^2 + 2x_0y_0t + y_0^2 - t^2 \\ &= x_0^2 + y_0^2 + (x_0^2+y_0^2)t^2 - t^2 \\ &= 1 + t^2 - t^2 = 1,\end{aligned}$$

since $x_0^2 + y_0^2 = 1$. Thus $\mathbf{l}_1$ lies on the hyperboloid. The calculation for $\mathbf{l}_2$ is similar.

**(d)** The plane tangent to the hyperboloid at the point $(ax_0, by_0, 0)$ is given by

$$\nabla F(ax_0, by_0, 0) \cdot (x - ax_0, y - by_0, 0) = 0 \quad \text{where} \quad F = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}.$$

That is, the tangent plane is

$$(*) \quad \frac{x_0}{a}(x - ax_0) + \frac{y_0}{b}(y - by_0) = 0.$$

If we substitute the parametric equations for $\mathbf{l}_1$ into the left side of $(*)$, we find

$$\frac{x_0}{a}(a(x_0 - y_0 t) - ax_0) + \frac{y_0}{b}(b(x_0 t + y_0) - by_0) = -x_0 y_0 t + y_0 x_0 t = 0$$

for all $t$. Therefore, the line $\mathbf{l}_1$ lies in the plane. A similar calculation can be made for $\mathbf{l}_2$.

**4.** Reconsider the parametrization from Exercise 3(a), $\mathbf{X}(z, \theta) = (\sqrt{z^2+1}\cos\theta, \sqrt{z^2+1}\sin\theta, z)$ where $z \in \mathbf{R}$ and $0 \le \theta \le 2\pi$. Then we have,

$$\mathbf{T}_z = \left(\frac{z}{\sqrt{z^2+1}}\cos\theta, \frac{z}{\sqrt{z^2+1}}\sin\theta, 1\right),$$
$$\mathbf{T}_\theta = (-\sqrt{z^2+1}\sin\theta, \sqrt{z^2+1}\cos\theta, 0).$$
$$\text{Thus} \quad \mathbf{T}_z \times \mathbf{T}_\theta = (-\sqrt{z^2+1}\cos\theta, -\sqrt{z^2+1}\sin\theta, z),$$
$$\text{so that} \quad \|\mathbf{T}_z \times \mathbf{T}_\theta\| = \sqrt{(z^2+1) + z^2} = \sqrt{2z^2+1}.$$

Therefore,

$$\text{Surface area} = \int_0^{2\pi} \int_{-a}^{a} \sqrt{2z^2+1}\, dz\, d\theta = \pi(\sqrt{2}\ln(\sqrt{2a^2+1} + \sqrt{2}a) + 2a\sqrt{2a^2+1}).$$

(Let $\tan u = \sqrt{2}z$ in the $z$-integral.)

**5. (a)** This is similar to Exercise 3(b). First, consider a variant of spherical coordinates: $x = a\rho\cos\theta\sin\varphi, y = b\rho\sin\theta\sin\varphi$, and $z = c\rho\cos\varphi$. If we set $\rho = 1$, we get the desired parametrization: $x = a\sin\varphi\cos\theta, y = b\sin\theta\sin\varphi$, and $z = c\cos\varphi$ where $0 \le \varphi \le \pi$ and $0 \le \theta \le 2\pi$.

**(b)** Here we have

$$\mathbf{T}_\varphi = (a\cos\varphi\cos\theta, b\cos\varphi\sin\theta, -c\sin\varphi) \quad \text{and}$$
$$\mathbf{T}_\theta = (-a\sin\varphi\sin\theta, b\sin\varphi\cos\theta, 0).$$

Therefore,

$$\mathbf{N} = \mathbf{T}_\varphi \times \mathbf{T}_\theta = (bc\sin^2\varphi\cos\theta, ac\sin^2\varphi\sin\theta, ab\cos\varphi\sin\varphi) \quad \text{and}$$
$$\|\mathbf{N}\| = b^2c^2\sin^4\varphi\cos^2\theta + a^2c^2\sin^4\varphi\sin^2\theta + a^2b^2\cos^2\varphi\sin^2\varphi.$$

Therefore,

$$\text{Surface area} = \int_0^{2\pi} \int_0^{\pi} \sqrt{b^2c^2\sin^4\varphi\cos^2\theta + a^2c^2\sin^4\varphi\sin^2\theta + a^2b^2\cos^2\varphi\sin^2\varphi}\, d\varphi\, d\theta.$$

In the special case where $a = b = c$, we find that

$$\begin{aligned}
\text{Surface area} &= \int_0^{2\pi}\int_0^{\pi} a^2\sqrt{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta + \cos^2\varphi\sin^2\varphi}\,d\varphi\,d\theta \\
&= a^2\int_0^{2\pi}\int_0^{\pi}\sqrt{\sin^4\varphi + \cos^2\varphi\sin^2\varphi}\,d\varphi\,d\theta \\
&= a^2\int_0^{2\pi}\int_0^{\pi}\sqrt{\sin^2\varphi}\,d\varphi\,d\theta \\
&= a^2\int_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi\,d\theta \\
&= a^2\int_0^{2\pi}(-\cos\varphi)\Big|_0^{\pi}\,d\theta \\
&= a^2\int_0^{2\pi}2\,d\theta = 4\pi a^2.
\end{aligned}$$

**6. (a)** The $t$-coordinate curve is $(s_0, f(s_0)\cos t, f(s_0)\sin t)$, which is a circle of radius $|f(s_0)|$ in the $x = s_0$ plane. That is, the radius of this cross-sectional circle depends on $f(s_0)$.

**(b)** $\mathbf{T}_s = (1, f'(s)\cos t, f'(s)\sin t)$ and $\mathbf{T}_t = (0, -f(s)\sin t, f(s)\cos t)$, so $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = (f(s)f'(s), -f(s)\cos t, -f(s)\sin t)$. Thus $\|\mathbf{N}\| = \sqrt{[f(s)]^2[f'(s)]^2 + [f(s)]^2} = |f(s)|\sqrt{[f'(s)]^2 + 1}$. So

$$\begin{aligned}
\text{Surface area} &= \int_0^{2\pi}\int_a^b |f(x)|\sqrt{[f'(x)]^2+1}\,dx\,dt \\
&= \int_a^b\int_0^{2\pi} |f(x)|\sqrt{[f'(x)]^2+1}\,dt\,dx \\
&= 2\pi\int_a^b |f(x)|\sqrt{[f'(x)]^2+1}\,dx.
\end{aligned}$$

**7. (a)** This should remind students of when they were using washer and shell methods for surfaces of revolution. Of course, here we are finding a surface area and not volume. If you look at the specific value $s = s_0$ then, since we are revolving around the $y$-axis, we are sweeping out a circle of radius $s_0$ in the plane $y = f(s_0)$. Therefore, a parametrization is $\mathbf{X}(s,t) = (s\cos t, f(s), s\sin t), a \le s \le b, 0 \le t \le 2\pi$. Compare this with Exercise 6(a).

**(b)** Using the parametrization in (a), $\mathbf{T}_s = (\cos t, f'(s), \sin t)$ and $\mathbf{T}_t = (-s\sin t, 0, s\cos t)$. Therefore, $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = (sf'(s)\cos t, -s, sf'(s)\sin t)$, so $\|\mathbf{N}\| = \sqrt{s^2[f'(s)]^2 + s^2}$. Hence

$$\begin{aligned}
\text{Surface area} &= \int_a^b\int_0^{2\pi}\sqrt{s^2[f'(s)]^2+s^2}\,dt\,ds \\
&= 2\pi\int_a^b s\sqrt{[f'(s)]^2+1}\,ds \\
&= 2\pi\int_a^b x\sqrt{[f'(x)]^2+1}\,dx
\end{aligned}$$

by changing the variable of integration. Compare this result with that of Exercise 6(b).

**8. (a)** It would be helpful for you to first draw a picture. The surface is the curve $z = f(x)$ in the *xz*-plane extended so that the derivative in the $y$ direction is identically zero (i.e., we're dragging the curve in the $y$ direction). Then $S_1$, the portion of $S$ lying over $D$, may be parametrized as $\mathbf{X}(x,y) = (x, y, f(x)), (x,y) \in D$. Then $\mathbf{T}_x = (1, 0, f'(x))$ and $\mathbf{T}_y = (0,1,0)$, so that $\mathbf{N} = \mathbf{T}_x \times \mathbf{T}_y = (-f'(x), 0, 1)$, and so $\|\mathbf{N}\| = \sqrt{1 + [f'(x)]^2}$. Hence,

$$\text{Surface area} = \iint_D \sqrt{1+[f'(x)]^2}\,dx\,dy = \iint_D \sqrt{1+[f'(x)]^2}\,dA.$$

Since $s(x) = \displaystyle\int_a^x \sqrt{1+[f'(t)]^2}\,dt$, then, by the fundamental theorem of calculus, we have that $s'(x) = \sqrt{1+[f'(x)]^2}$. Thus the surface area is $\displaystyle\iint_D s'(x)\,dA$.

**(b)** From Green's theorem, $\oint_C s(x)\,dy = \iint_D s'(x)\,dA$, which, by part (a), is the surface area.

**(c)** Here we are working a specific example of what we worked out in part (a). The rectangle $D$ is $[1,3]\times[-2,2]$. Using part (a), we compute the surface area as $\iint_D s'(x)\,dA$. Now $s'(x) = \sqrt{1+[f'(x)]^2}$ where $z = f(x) = \dfrac{x^3}{3} + \dfrac{1}{4x}$, and so $f'(x) = x^2 - \dfrac{1}{4x^2}$. Therefore,

$$1 + [f'(x)]^2 = 1 + \left(x^4 - \frac{1}{2} + \frac{1}{16x^4}\right) = x^4 + \frac{1}{2} + \frac{1}{16x^4} = \left(x^2 + \frac{1}{4x^2}\right)^2.$$

Hence,

$$\begin{aligned}\text{Surface area} &= \iint_D \sqrt{1+[f'(x)]^2}\,dA = \iint_D \sqrt{\left(x^2+\frac{1}{4x^2}\right)^2}\,dA\\ &= \int_{-2}^{2}\int_1^3 \left(x^2 + \frac{1}{4x^2}\right)dx\,dy = \int_{-2}^{2}\left(\frac{1}{3}x^3 - \frac{1}{4x}\right)\Big|_{x=1}^{3} dy\\ &= \int_{-2}^{2}\left(9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4}\right) dy = \int_{-2}^{2} \frac{53}{6}\,dy = \frac{106}{3}.\end{aligned}$$

**9. (a)** The surface integral $\iint_S f\,dS$, roughly speaking, represents the "sum" of all the values of $f$ on $S$. The area of $S$ is a measure of the size of $S$. So the quotient can be thought of as the "total" amount of $f$ divided by the size of the region being sampled.

**(b)** Parametrize the sphere as $\mathbf{X}(s,t) = (7\cos s\sin t, 7\sin s\sin t, 7\cos t)$, $0 \le s \le 2\pi$, $0 \le t \le \pi$. Then, following Example 11 in Section 7.1, $\|\mathbf{T}_s \times \mathbf{T}_t\| = 49\sin t$. Note that, on the surface $S$, the temperature $T(x,y,z) = x^2+y^2-3z = 49 - z^2 - 3z$. As a result, we can calculate

$$\begin{aligned}\iint_S T(x,y,z)\,dS &= \int_0^{2\pi}\int_0^{\pi} (49 - 49\cos^2 t - 21\cos t)49\sin t\,dt\,ds\\ &= 49\int_0^{2\pi}\left(-49\cos t + \frac{49}{3}\cos^3 t + \frac{21}{2}\cos^2 t\right)\Big|_0^{\pi} ds\\ &= 49\int_0^{2\pi}\left(49(2) - \frac{49}{3}(2) + \frac{21}{2}(1-1)\right) ds = \frac{(49)^2(4)(2\pi)}{3}.\end{aligned}$$

Now, since the surface area of a sphere of radius 7 is $4\pi(49)$, we have

$$[T]_{\text{avg}} = \frac{\iint_S T(x,y,z)\,dS}{\text{surface area}} = \frac{(49)^2(4)(2\pi)}{3}\,\frac{1}{4\pi(49)} = \frac{98}{3}.$$

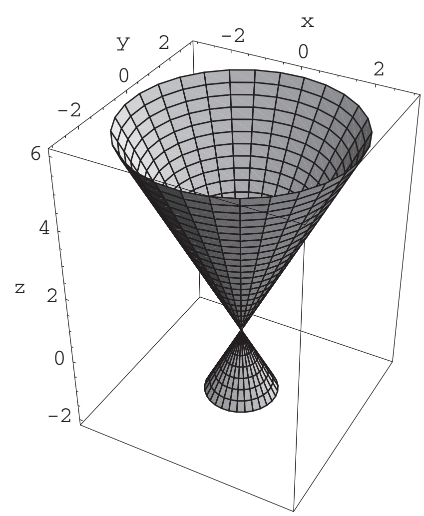
**10.** The surface area of the cylinder is $2\pi \cdot 2 \cdot 3 = 12\pi$. If we parametrize the surface as

$$\begin{cases} x = 2\cos t \\ y = 2\sin t \quad 0 \le t < 2\pi, \quad 0 \le s \le 3 \\ z = s \end{cases}$$

then $\|\mathbf{T}_s \times \mathbf{T}_t\| = 2$. Hence

$$
\begin{aligned}
[f]_{\text{avg}} &= \frac{1}{12\pi}\iint_S f\,dS = \frac{1}{12\pi}\int_0^3\int_0^{2\pi}(4e^s\cos^2 t - 4s\sin^2 t)\cdot 2\,dt\,ds \\
&= \frac{1}{3\pi}\int_0^3\int_0^{2\pi}[e^s(1+\cos 2t) - s(1-\cos 2t)]\,dt\,ds \\
&= \frac{1}{3\pi}\int_0^3 \left(e^s\left(t+\frac{1}{2}\sin 2t\right) - s\left(t - \frac{1}{2}\sin 2t\right)\right)\bigg|_{t=0}^{2\pi} ds \\
&= \frac{1}{3\pi}\int_0^3 (2\pi e^s - 2\pi s)\,ds = \frac{2}{3}\int_0^3 (e^s - s)\,ds \\
&= \frac{2}{3}\left(e^s - \frac{1}{2}s^2\right)\bigg|_0^3 = \frac{2}{3}\left(e^3 - \frac{9}{2} - 1\right) = \frac{2e^3 - 11}{3}.
\end{aligned}
$$

**11.** The cone looks as follows.




The upper nappe has a height of 6 and radius of 3; the lower nappe has a height of 2 and a radius of 1. Hence the total surface area is

$$\pi \cdot 1 \cdot \sqrt{5} + \pi \cdot 3 \cdot 3\sqrt{5} = 10\sqrt{5}\pi.$$

Next, parametrize the surface as $\begin{cases} x = s\cos t \\ y = s\sin t \\ z = 2s \end{cases}$ with $-1 \le s \le 3, 0 \le t < 2\pi$. Then

$$\|\mathbf{N}\| = \|\mathbf{T}_s \times \mathbf{T}_t\| = \|(-2s\cos t, -2s\sin t, s)\| = \sqrt{5}\,|s|.$$

Therefore,

$$
\begin{aligned}
[f]_{\text{avg}} &= \frac{1}{10\sqrt{5}\pi}\iint_S f\,dS = \frac{1}{10\sqrt{5}\pi}\int_0^{2\pi}\int_{-1}^{3}(s^2-3)\sqrt{5}|s|\,ds\,dt \\
&= \frac{1}{10\pi}\int_{-1}^{3}\int_0^{2\pi}(s^2-3)|s|\,dt\,ds \\
&= \frac{1}{10\pi}\int_{-1}^{3}2\pi(s^2-3)|s|\,ds = \frac{1}{5}\left[\int_{-1}^{0}(s^2-3)(-s)\,ds + \int_0^3(s^2-3)s\,ds\right] \\
&= \frac{1}{5}\left[\left(-\frac{1}{4}s^4+\frac{3}{2}s^2\right)\Big|_{-1}^{0} + \left(\frac{1}{4}s^4-\frac{3}{2}s^2\right)\Big|_{0}^{3}\right] \\
&= \frac{1}{5}\left(\frac{1}{4}-\frac{3}{2}+\frac{81}{4}-\frac{27}{2}\right) = \frac{11}{10}.
\end{aligned}
$$

**12.** The total mass is $\iint_{\mathbf{X}} \delta\,dS$. For the helicoid, $\mathbf{T}_s = (\cos t, \sin t, 0)$ and $\mathbf{T}_t = (-s\sin t, s\cos t, 1)$. Then $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = (\sin t, -\cos t, s)$ and $\|\mathbf{N}\| = \sqrt{1+s^2}$. Hence,

$$
\begin{aligned}
\text{Total mass} &= \iint_{\mathbf{X}} \sqrt{x^2+y^2}\,dS \\
&= \int_0^{4\pi}\int_0^1 \sqrt{(s\cos t)^2+(s\sin t)^2}\sqrt{1+s^2}\,ds\,dt \\
&= \int_0^{4\pi}\int_0^1 s\sqrt{1+s^2}\,ds\,dt \\
&= 4\pi\left(\frac{1}{2}\cdot\frac{2}{3}(1+s^2)^{3/2}\right)\Big|_{s=0}^{1} = \frac{4\pi}{3}(2\sqrt{2}-1).
\end{aligned}
$$

**13.** By the symmetry of the surface, we must have $\bar{x} = \bar{y} = \bar{z}$. We compute $\bar{z}$ explicitly. Since $\delta$ is constant, it will cancel from the center of mass integrals:

$$
\bar{z} = \frac{\iint_S z\delta\,dS}{\iint_S \delta\,dS} = \frac{\delta\iint_S z\,dS}{\delta\iint_S dS} = \frac{\iint_S z\,dS}{\text{surface area of } S}.
$$

The surface area of the first octant portion of a sphere of radius $a$ is $\frac{1}{8}(4\pi a^2) = \frac{1}{2}\pi a^2$. Therefore, $\bar{z} = \frac{2}{\pi a^2}\iint_S z\,dS$. We may parametrize the first octant portion of the sphere as $\mathbf{X}(\varphi,\theta) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$, $0 \le \varphi \le \pi/2$, $0 \le \theta \le \pi/2$. Hence,

$$
\begin{aligned}
\mathbf{T}_\varphi &= (a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, -a\sin\varphi), \\
\mathbf{T}_\theta &= (-a\sin\varphi\sin\theta, a\sin\varphi\cos\theta, 0).
\end{aligned}
$$

Therefore,

$$
\mathbf{N} = (a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\sin\varphi\cos\theta) \quad \text{and} \quad \|\mathbf{N}\| = a^2\sin\varphi.
$$

Thus,

$$
\begin{aligned}
\bar{z} &= \frac{2}{\pi a^2}\int_0^{\pi/2}\int_0^{\pi/2}(a\cos\varphi)a^2\sin\varphi\,d\varphi\,d\theta \\
&= \frac{2a^3}{\pi a^2}\int_0^{\pi/2}\int_0^{\pi/2}\cos\varphi\sin\varphi\,d\varphi\,d\theta \\
&= \frac{2a}{\pi}\int_0^{\pi/2}\left(\frac{1}{2}\sin^2\varphi\right)\Big|_{\varphi=0}^{\pi/2}d\theta \\
&= \frac{2a}{\pi}\int_0^{\pi/2}\frac{1}{2}\,d\theta = \frac{a}{\pi}\cdot\frac{\pi}{2} = \frac{a}{2}.
\end{aligned}
$$

**14.** A quick sketch should convince you that, by symmetry, $\bar{x} = 0$ and $\bar{y} = \dfrac{a}{2}$. The equation for the surface may be written as $z = \sqrt{a^2 - x^2}$, so that $\dfrac{\partial z}{\partial x} = -\dfrac{x}{\sqrt{a^2 - x^2}}$ and $\dfrac{\partial z}{\partial y} = 0$. Then

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy = \sqrt{1 + \frac{x^2}{a^2 - x^2}}\, dx\, dy = \sqrt{\frac{a^2}{a^2 - x^2}}\, dx\, dy.$$

Hence,

$$\bar{z} = \frac{\iint_S z\delta\, dS}{\iint_S \delta\, dS} = \frac{\iint_S z\, dS}{\iint_S dS} = \frac{\iint_S z\, dS}{\pi a^2},$$

since the surface area of half a cylinder is $\pi a^2$. Now we calculate

$$\begin{aligned}\bar{z} &= \frac{1}{\pi a^2}\iint_S z\, dS = \frac{1}{\pi a^2}\int_0^a \int_{-a}^a \sqrt{a^2 - x^2}\sqrt{\frac{a^2}{a^2 - x^2}}\, dx\, dy \\ &= \frac{1}{\pi a^2}\int_0^a \int_{-a}^a a\, dx\, dy = \frac{a}{\pi a^2}(2a^2) = \frac{2a}{\pi}.\end{aligned}$$

**15.** By symmetry $\bar{x} = \bar{y} = 0$, so we only need to calculate $\bar{z} = \frac{\iint_S z\delta\, dS}{\iint_S \delta\, dS}$. Now

$$\delta(x, y, z) = x^2 + y^2 + (z + a)^2.$$

If we parametrize the sphere: $\begin{cases} x = a\cos s \sin t & 0 \le s < 2\pi \\ y = a \sin s \sin t & 0 \le t \le \pi \\ z = a\cos t \end{cases}$, then $\|\mathbf{N}\| = a^2 \sin t$ (see Example 1 of §7.2). We therefore have

$$\begin{aligned}\iint_S \delta\, dS &= \iint_S \left(x^2 + y^2 + (z + a)^2\right)\, dS \\ &= \int_0^\pi \int_0^{2\pi} \left(a^2 \sin^2 t + (a\cos t + a)^2\right) a^2 \sin t\, ds\, dt \\ &= 2\pi a^2 \int_0^\pi \left(a^2 \sin^2 t + a^2 \cos^2 t + 2a^2 \cos t + a^2\right) \sin t\, dt \\ &= 4\pi a^4 \int_0^\pi (1 + \cos t) \sin t\, dt \\ &= 4\pi a^4 \left(-\cos t + \tfrac{1}{2}\sin^2 t\right)\Big|_0^\pi = 8\pi a^4\end{aligned}$$

$$\begin{aligned}\iint_S z\delta\, dS &= \iint_S z\left(x^2 + y^2 + (z + a)^2\right)\, dS \\ &= \int_0^\pi \int_0^{2\pi} a\cos t\left(a^2 \sin^2 t + (a\cos t + a)^2\right) a^2 \sin t\, ds\, dt \\ &= 4\pi a^5 \int_0^\pi (1 + \cos t)\cos t \sin t\, dt \\ &= 4\pi a^5 \left(\tfrac{1}{2}\sin^2 t - \tfrac{1}{3}\cos^3 t\right)\Big|_0^\pi = \frac{8\pi a^5}{3}.\end{aligned}$$

Hence,

$$\bar{z} = \frac{8\pi a^5/3}{8\pi a^4} = \frac{a}{3}.$$

**16.** Parametrize the cylinder as $\begin{cases} x = a\cos s \\ y = t \\ z = a \sin s \end{cases} \quad 0 \le s < 2\pi, 0 \le t \le 2.$

Then $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = (-a\sin s, 0, a\cos s) \times (0,1,0) = (-a\cos s, 0, -a\sin s)$ so $\|\mathbf{N}\| = a$. Hence

$$\begin{aligned} M &= \iint_S \delta\, dS = \int_0^2 \int_0^{2\pi} (a^2\cos^2 s + t)\cdot a\, ds\, dt \\ &= \int_0^2 \int_0^{2\pi} \left( at + \frac{a^3}{2}(1+\cos^2 s)\right) ds\, dt = \int_0^2 (2\pi a t + \pi a^3)\, dt \\ &= (\pi a t^2 + \pi a^3 t)\Big|_0^2 = 4\pi a + 2\pi a^3 = 2\pi a(a^2+2). \end{aligned}$$

Symmetry implies $\bar{z} = 0$, so we calculate

$$\begin{aligned} \bar{x} &= \frac{1}{2\pi a(a^2+2)} \int_0^2 \int_0^{2\pi} a\cos s\,(a(t + a^2\cos^2 s))\, ds\, dt \\ &= \frac{a}{2\pi(a^2+2)} \int_0^2 \int_0^{2\pi} (t\cos s + a^2\cos^3 s)\, ds\, dt \\ &= \frac{a}{2\pi(a^2+2)} \int_0^2 \int_0^{2\pi} (t\cos s + a^2(1-\sin^2 s)\cos s)\, ds\, dt \\ &= \frac{a}{2\pi(a^2+2)} \int_0^2 \left( t\sin s + a^2\sin s - \frac{a^2}{3}\sin^3 s\right)\Big|_0^{2\pi} dt = 0. \end{aligned}$$

(Actually, you can really see this from symmetry.)

$$\begin{aligned} \bar{y} &= \frac{1}{2\pi a(a^2+2)} \iint_S y(x^2+y)\, dS = \frac{1}{2\pi a(a^2+2)} \int_0^2 \int_0^{2\pi} t\,(t + a^2\cos^2 s)\cdot a\, ds\, dt \\ &= \frac{1}{2\pi(a^2+2)} \int_0^2 \int_0^{2\pi} \left[ t^2 s + a^2 t\left(\frac{1}{2}s + \frac{1}{4}\sin 2s\right)\Big|_{s=0}^{2\pi}\right] dt \\ &= \frac{1}{2\pi(a^2+2)} \int_0^2 (2\pi t^2 + \pi a^2 t)\, dt = \frac{1}{2(a^2+2)} \int_0^2 (2t^2 + a^2 t)\, dt \\ &= \frac{1}{2(a^2+2)} \left(\frac{2}{3}t^3 + \frac{a^2}{2}t^2\right)\Big|_0^2 = \frac{1}{2(a^2+2)}\left(\frac{16}{3} + 2a^2\right) \\ &= \frac{1}{a^2+2}\left(\frac{8}{3} + a^2\right) = \frac{3a^2+8}{3a^2+6}. \end{aligned}$$

So $(\bar{x}, \bar{y}, \bar{z}) = \left(0, \dfrac{3a^2+8}{3a^2+6}, 0\right)$.

**17. (a)** Parametrize the frustum $z^2 = 4x^2 + 4y^2$, $2 \le z \le 4$, as $\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, 2r)$, $0 \le \theta \le 2\pi$, $1 \le r \le 2$. Then

$$\begin{aligned} \frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \\ \frac{\partial(x,z)}{\partial(r,\theta)} &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ 2 & 0 \end{vmatrix} = 2r\sin\theta \\ \frac{\partial(y,z)}{\partial(r,\theta)} &= \begin{vmatrix} \sin\theta & r\cos\theta \\ 2 & 0 \end{vmatrix} = -2r\cos\theta. \end{aligned}$$

Therefore,

$$\begin{aligned} I_z &= \iint_S (x^2+y^2)\, dS = \int_0^{2\pi} \int_1^2 r^2\sqrt{r^2+4r^2}\, dr\, d\theta \\ &= \int_0^{2\pi} \int_1^2 \sqrt{5}r^3\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{5}}{4}(15)\, d\theta = \frac{15\sqrt{5}\pi}{2}. \end{aligned}$$

**(b)** The radius of gyration is given by $r_z = \sqrt{\dfrac{I_z}{M}}$. Assuming, as in part (a), that the density is 1, the total mass is just the surface area of the frustum. This can be computed from the surface area of the cone without much trouble. We view the frustum as a large cone (of height 4) with the tip (a similar cone of height 2) removed and note that the surface area of a cone is $\pi$(radius)(slant height). Then

$$\text{Surface area of frustum} = \pi(2)(2\sqrt{5}) - \pi(1)(\sqrt{5}) = 3\sqrt{5}\pi.$$

Hence

$$r_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{15\sqrt{5}\pi}{2}\frac{1}{3\sqrt{5}\pi}} = \sqrt{\frac{5}{2}}.$$

(Note: you can also compute the surface area as $\displaystyle\int_0^{2\pi}\int_1^2 \sqrt{5}r\,dr\,d\theta$.)

**(c)** We recompute the integral for $I_z$ with $\delta = kr$. Thus

$$\begin{aligned}
I_z &= \iint_S (x^2+y^2)\delta\,dS \\
&= \int_0^{2\pi}\int_1^2 r^2 kr\sqrt{5r^2}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^2 \sqrt{5}kr^4\,dr\,d\theta \\
&= \int_0^{2\pi} \frac{\sqrt{5}k}{5}(2^5-1)\,d\theta = \frac{62\sqrt{5}\pi k}{5}.
\end{aligned}$$

The total mass of the frustum is

$$\begin{aligned}
M &= \iint_S \delta\,dS = \int_0^{2\pi}\int_1^2 kr\sqrt{5}r\,dr\,d\theta \\
&= \int_0^{2\pi}\frac{\sqrt{5}}{3}(2^3-1)\,d\theta = \frac{14\sqrt{5}\pi k}{3}.
\end{aligned}$$

Hence

$$r_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{62\sqrt{5}\pi k}{5}\frac{3}{14\sqrt{5}\pi k}} = \sqrt{\frac{93}{35}}.$$

**18. (a)**

$$\begin{aligned}
I_z = \iint_S (x^2+y^2)\,\delta\,dS = \delta\iint_S a^2\,dS &= \delta a^2\cdot\text{surface area} \\
&= \delta a^2\cdot 2\pi a\cdot 2b = 4\pi\delta a^3 b
\end{aligned}$$

**(b)** $M = \displaystyle\iint_S \delta\,dS = \delta\cdot 4\pi ab$, so $r_z = \sqrt{\dfrac{I_z}{M}} = \sqrt{\dfrac{4\pi\delta a^3 b}{4\pi\delta ab}} = a.$

**19. (a)** $I_x = \iint_S (y^2+z^2)\delta\,dS$. If we parametrize $S$ by $\begin{cases} x = a\cos t \\ y = a\sin t \\ z = s \end{cases}$ $-b\le s\le b, 0\le t<2\pi$, then $\|\mathbf{N}\| = \|\mathbf{T}_s\times\mathbf{T}_t\| = a$ and so

$$\begin{aligned}
I_x &= \int_{-b}^b\int_0^{2\pi}(a^2\sin^2 t + s^2)\delta a\,dt\,ds = \delta a\int_{-b}^b\left(\frac{a^2}{2}\left(t-\frac{1}{2}\sin 2t\right)+s^2t\bigg|_{t=0}^{2\pi}\right)ds \\
&= \delta a\int_{-b}^b(\pi a^2 + 2\pi s^2)\,ds = \pi\delta a\left(2a^2b+\frac{4}{3}b^3\right) = \frac{2\pi ab\delta}{3}(3a^2+2b^2) \\
I_y &= \iint_S (x^2+z^2)\delta\,dS = \int_{-b}^b\int_0^{2\pi}(a^2\cos^2 t+s^2)\delta a\,dt\,ds \\
&= \delta a\int_{-b}^b\left(\frac{a^2}{2}\left(t+\frac{1}{2}\sin 2t\right)+s^2t\right)\bigg|_{t=0}^{2\pi} ds = \pi\delta a\left(2a^2b+\frac{4}{3}b^3\right) \quad \text{as before.}
\end{aligned}$$

**(b)** From Exercise 18, $M = 4\pi ab\delta$, so

$$r_x = r_y = \sqrt{\frac{\pi\delta a\left(2a^2b + \frac{4}{3}b^3\right)}{4\pi ab\delta}} = \sqrt{\frac{2a^2 + \frac{4}{3}b^2}{4}} = \sqrt{\frac{a^2 + \frac{2}{3}b^2}{2}}$$
$$= \sqrt{\frac{3a^2 + 2b^2}{6}}.$$

**20. (a)** Let $M$ be the maximum value of $f$ on $D$ and $m$ the minimum value. (The numbers $M$ and $m$ must exist since $D$ is compact.) Then

$$m = \frac{\iint_D mg\,dA}{\iint_D g\,dA} \le \frac{\iint_D fg\,dA}{\iint_D g\,dA} \le \frac{\iint_D Mg\,dA}{\iint_D g\,dA} = M.$$

Hence by the intermediate value theorem, there must be some point $P$ in $D$ such that

$$f(P) = \frac{\iint_D fg\,dA}{\iint_D g\,dA},$$

which gives the result, provided $\iint_D g\,dA \neq 0$.
If $\iint_D g\,dA = 0$ then we have

$$0 = m\iint_D g\,dA = \iint_D mg\,dA \le \iint_D fg\,dA \le \iint_D Mg\,dA = M\iint_D g\,dA = 0,$$

so $\iint_D fg\,dA = 0$ and any $P$ in $D$ gives the desired result.

**(b)** Assume that $S$ may be parametrized by a single function $\mathbf{X}$. Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D \mathbf{F}(\mathbf{X}(s,t))\cdot\mathbf{n}(s,t)\|\mathbf{N}(s,t)\|\,ds\,dt$$
$$= \mathbf{F}(\mathbf{X}(s_0,t_0))\cdot\mathbf{n}(s_0,t_0)\iint_D \|\mathbf{N}(s,t)\|\,ds\,dt \quad \text{by part (a),}$$
$$= \mathbf{F}(P)\cdot\mathbf{n}(P)(\text{area of } S)$$

where $P = \mathbf{X}(s_0, t_0)$.

**21. (a)** Let $\mathbf{a} = (a_1, a_2, a_3)$ and assume $\mathbf{x}(t) = (x(t), y(t), z(t))$ parametrizes $C$. Then

$$\oint_C \mathbf{a}\cdot d\mathbf{s} = \int_a^b \mathbf{a}\cdot\mathbf{x}'(t)\,dt$$
$$= \int_a^b (a_1x'(t) + a_2y'(t) + a_3z'(t))\,dt$$
$$= (a_1x(t) + a_2y(t) + a_3z(t))\Big|_a^b$$
$$= \mathbf{a}\cdot\mathbf{x}(t)\Big|_a^b = \mathbf{a}\cdot\mathbf{x}(b) - \mathbf{a}\cdot\mathbf{x}(a)$$
$$= 0$$

since $\mathbf{x}(a) = \mathbf{x}(b)$ because $C$ is a closed curve.

**(b)** Let $S$ be any smooth, orientable surface with boundary curve $C$. If we orient $S$ appropriately and use Stokes's theorem, we have

$$\oint_C \mathbf{a}\cdot d\mathbf{s} = \iint_S \nabla\times\mathbf{a}\cdot d\mathbf{S} = \iint_S \mathbf{0}\cdot d\mathbf{S} = 0.$$

**22.** Note that $C$ lies in the surface $z = x^2 - y^2$. The line integral is

$$\oint_C \mathbf{F}\cdot d\mathbf{s}, \text{ where } \mathbf{F} = (x^2 + z^2)\,\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Therefore, Stokes's theorem implies that

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S},$$

where $S$ is the portion of $z = x^2 - y^2$ bounded by $C$. Note that $S$ lies over the unit disk in the $xy$-plane. We may take for unit normal

$$\mathbf{n} = \frac{-2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2+4y^2+1}} \quad \text{and}$$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+z^2 & y & z \end{vmatrix} = 2z\mathbf{j} = 2(x^2-y^2)\,\mathbf{j} \text{ on } S.$$

Thus $\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \iint_D 4y(x^2-y^2)\,dA$ where $D$ is the unit disk. This is

$$\begin{aligned}
&\int_0^{2\pi}\int_0^1 4r^4(\cos^2\theta\sin\theta - \sin^3\theta)\,dr\,d\theta \\
&= \int_0^{2\pi}\frac{4}{5}(\cos^2\theta\sin\theta - \sin^3\theta)\,d\theta = \frac{4}{5}\int_0^{2\pi}(\cos^2\theta\sin\theta - (1-\cos^2\theta)\sin\theta)\,d\theta \\
&= \frac{4}{5}\left(-\frac{2}{3}\cos^3\theta + \cos\theta\right)\Big|_0^{2\pi} = 0.
\end{aligned}$$

**23.** By Stokes's theorem

$$\begin{aligned}
\oint_{\partial S}(f\nabla g)\cdot d\mathbf{s} &= \iint_S \nabla\times(f\nabla g)\cdot d\mathbf{S} \\
&= \iint_S (\nabla f\times\nabla g + f\nabla\times(\nabla g))\cdot d\mathbf{S} = \iint_S (\nabla f\times\nabla g)\cdot d\mathbf{S},
\end{aligned}$$

since $\nabla\times(\nabla g) = \mathbf{0}$ (see §3.4).

**24.** Using the result of Exercise 23 (twice):

$$\oint_{\partial S}(f\nabla g + g\nabla f)\cdot d\mathbf{s} = \iint_S (\nabla f\times\nabla g + \nabla g\times\nabla f)\cdot d\mathbf{S} = \iint_S \mathbf{0}\cdot d\mathbf{S} = 0$$

because $\nabla f\times\nabla g = -\nabla g\times\nabla f$.

**25.**

$$\begin{aligned}
\oint_{\partial S}(f\nabla f)\cdot d\mathbf{s} &= \oint_{\partial S}\frac{1}{2}(f\nabla f + f\nabla f)\cdot d\mathbf{s} \\
&= 0 \quad \text{by Exercise 24.}
\end{aligned}$$

**26. (a)** First apply Stokes's theorem:

$$\begin{aligned}
&\frac{1}{2}\oint_C (bz-cy)\,dx + (cx-az)\,dy + (ay-bx)\,dz \\
&= \frac{1}{2}\iint_D \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ bz-cy & cx-az & ay-bx \end{vmatrix}\cdot d\mathbf{S} \quad (D \text{ is the region enclosed by } C) \\
&= \frac{1}{2}\iint_D (2a,2b,2c)\cdot d\mathbf{S} = \iint_D (a,b,c)\cdot d\mathbf{S} \\
&= \iint_D \mathbf{n}\cdot\mathbf{n}\,dS = \iint_D dS \quad \text{since } \mathbf{n} \text{ is a unit vector,} \\
&= \text{area enclosed by } C.
\end{aligned}$$

**(b)** If $C$ is contained in the $xy$-plane, then $\mathbf{n} = \mathbf{k}$, so $a = b = 0$ and $c = 1$ in the notation above. Hence the result reduces to

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \text{area enclosed by } C.$$

**27.** By Faraday's law

$$\iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\frac{d}{dt}\iint_S \mathbf{B} \cdot d\mathbf{S}.$$

On the other hand, using Stokes's theorem,

$$\iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{E} \cdot d\mathbf{s} = \int_{\partial S} (\mathbf{E} \cdot \mathbf{T})\,ds = 0,$$

since $\mathbf{E}$ is everywhere perpendicular to $\partial S$. Thus $\frac{d}{dt}\iint_S \mathbf{B} \cdot d\mathbf{S} = 0$, so the magnetic flux does not vary with time.

**28.** For Gauss's theorem to apply to the situation, $S$ must be closed. Hence $\partial S$ is empty. But then there really is no line integral $\int_{\partial S} \mathbf{G} \cdot d\mathbf{s}$. If we try to apply Stokes's theorem in general (i.e., to surfaces with nonempty boundary) then we cannot also apply Gauss's theorem.

**29.** Note that the boundary $\partial W$ of $W$ consists of three parts: $S$, $\tilde{S}_a$ and the lateral surfaces $L$ of $\partial W$. With $\partial W$ oriented by outward normal, and if we take $S$ and $\tilde{S}_a$ to be oriented in the same way,

$$\oiint_{\partial W} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \pm\left(\iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} - \iint_{\tilde{S}_a} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S}\right) + \iint_L \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S}$$

(The $\pm$ sign depends on how $S$, $\tilde{S}_a$ are oriented with respect to the orientation of $\partial W$.) Now $L$ consists of a collection of segments of the rays defining $\Omega(S, O)$. Thus $L$ is *tangent* to $\mathbf{x}$. Hence $\mathbf{x} \cdot \mathbf{n} = 0$ where $\mathbf{n}$ is the appropriate unit normal to $L$. Thus $\iint_L \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = 0$. Thus

$$\oiint_{\partial W} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \pm\left(\iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} - \iint_{\tilde{S}_a} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S}\right).$$

Gauss's theorem implies

$$\oiint_{\partial W} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \iiint_W \left(\nabla \cdot \frac{\mathbf{x}}{\|\mathbf{x}\|^3}\right) dV = \iiint_W 0\,dV.$$

Hence $\iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \iint_{\tilde{S}_a} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S}$. On $\tilde{S}_a$, $\mathbf{n} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$, so

$$\begin{aligned}\Omega(S, O) &= \iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \iint_{\tilde{S}_a} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot \frac{\mathbf{x}}{\|\mathbf{x}\|}\,dS \\ &= \iint_{\tilde{S}_a} \frac{\|\mathbf{x}\|^2}{\|\mathbf{x}\|^4}\,dS = \iint_{\tilde{S}_a} \frac{1}{\|\mathbf{x}\|^2}\,dS.\end{aligned}$$

But on $\tilde{S}_a$, $\|\mathbf{x}\| = a$, so

$$\Omega(S, O) = \iint_{\tilde{S}_a} \frac{1}{a^2}\,dS = \frac{1}{a^2}\,(\text{surface area of } \tilde{S}_a).$$

**30.** From the definition of $\Omega(S, O)$, we calculate $\Omega(S, O) = \iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S}$. Now $\mathbf{x} = (x(s,t), y(s,t), z(s,t))$, so that $\|\mathbf{x}\| = \sqrt{x^2 + y^2 + z^2}$. Moreover, the standard normal $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t$ is

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} & \dfrac{\partial z}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} & \dfrac{\partial z}{\partial t} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial y}{\partial s} & \dfrac{\partial z}{\partial s} \\ \dfrac{\partial y}{\partial t} & \dfrac{\partial z}{\partial t} \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial z}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial z}{\partial t} \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{vmatrix}\mathbf{k}.$$

Hence

$$\iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \iint_D \frac{1}{(x^2+y^2+z^2)^{3/2}} \mathbf{x} \cdot (\mathbf{T}_s \times \mathbf{T}_t)\, ds\, dt$$

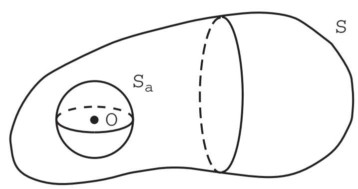
$$= \iint_D \frac{1}{(x^2+y^2+z^2)^{3/2}} \begin{vmatrix} x & y & z \\ \partial x/\partial s & \partial y/\partial s & \partial z/\partial s \\ \partial x/\partial t & \partial y/\partial t & \partial z/\partial t \end{vmatrix} ds\, dt \quad \text{as desired.}$$

**31.** First, if $S$ does not enclose the origin then, by Gauss's theorem

$$\Omega(S,O) = \iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \pm \iiint_W \nabla \cdot \left( \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \right) dV = \iiint_W 0\, dV = 0.$$

Here the $\pm$ sign depends on the orientation of $S$ and $W$ is the region enclosed by $S$.

Next, if $S$ does enclose the origin, let $S_a$ be the sphere of radius $a$ centered at $O$ and contained inside $S$. Let $D$ be the solid region in $\mathbf{R}^3$ between $S_a$ and $S$.




Note that $\nabla \cdot \left( \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \right)$ throughout $D$ since $D$ doesn't contain $O$. If $S_a$ is oriented by *inward* normal (which points away from $D$), then, by Gauss's theorem, we have:

$$0 = \iiint_D \nabla \cdot \left( \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \right) dV = \oiint_{\partial D} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \pm \iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} + \iint_{S_a} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S}.$$

Hence $\Omega(S,O) = \pm \iint_{S_a} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S}$. On $S_a$, $\mathbf{n} = -\frac{\mathbf{x}}{\|\mathbf{x}\|} = -\frac{1}{a}$ so

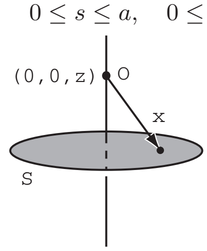
$$\Omega(S,O) = \pm \iint_{S_a} \frac{\mathbf{x}}{a^3} \cdot \left( -\frac{1}{a}\mathbf{x} \right) dS = \pm \iint_{S_a} -\frac{1}{a^4} (\mathbf{x} \cdot \mathbf{x})\, dS$$

$$= \pm \iint_{S_a} -\frac{a^2}{a^4}\, dS = \pm \frac{1}{a^2} \text{ (surface area of } S_a)$$

$$= \pm \frac{1}{a^2} (4\pi a^2) = \pm 4\pi.$$

**32.** We may parametrize $S$ as

$$\begin{cases} x = s\cos t \\ y = s \sin t \quad 0 \le s \le a, \quad 0 \le t < 2\pi \ . \\ z = 0 \end{cases}$$




Then one way to orient $S$ is with unit normal $\mathbf{n} = \mathbf{k}$. Also, we have the vector $\mathbf{x}$ from $O$ to a point of $S$ given by

$$\mathbf{x} = (s \cos t, s \sin t, -z) \Rightarrow \|\mathbf{x}\| = \sqrt{s^2 + z^2}.$$

Hence

$$\begin{aligned}\Omega(S,O) &= \iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^a \frac{-z}{(s^2+z^2)^{3/2}}\,s\,ds\,dt \\ &= -z\int_0^a\int_0^{2\pi}\frac{s}{(s^2+z^2)^{3/2}}\,dt\,ds \\ &= -\pi z\int_0^a \frac{2s}{(s^2+z^2)^{3/2}}\,ds = -\pi z(s^2+z^2)^{-1/2}(-2)\Big|_0^a \\ &= 2\pi z\left((a^2+z^2)^{-1/2}-\frac{1}{|z|}\right) = 2\pi z\left(\frac{1}{\sqrt{a^2+z^2}}-\frac{1}{|z|}\right) \\ &= 2\pi z\left(\frac{1}{\sqrt{a^2+z^2}}-\frac{1}{\sqrt{z^2}}\right) = 2\pi z\left(\frac{\sqrt{z^2}-\sqrt{a^2+z^2}}{\sqrt{z^2}\sqrt{a^2+z^2}}\right).\end{aligned}$$

Now

$$\frac{z}{\sqrt{z^2}} = \begin{cases} +1 & \text{if } z>0 \\ -1 & \text{if } z<0\end{cases}$$

and

$$\sqrt{z^2} = \begin{cases} z & \text{if } z\ge 0 \\ -z & \text{if } z<0\end{cases}.$$

Thus

$$\Omega(S,O) = \begin{cases} 2\pi\left(\dfrac{z-\sqrt{a^2+z^2}}{\sqrt{a^2+z^2}}\right) & \text{if } z>0 \\ 2\pi\left(\dfrac{z+\sqrt{a^2+z^2}}{\sqrt{a^2+z^2}}\right) & \text{if } z<0.\end{cases}$$

($z\neq 0$ because $O$ should not be a point of $S$.)

Note that if $z>0, z-\sqrt{a^2+z^2}<0$ and $|z-\sqrt{a^2+z^2}|<\sqrt{a^2+z^2}$. Hence $0>\Omega(S,O)>-2\pi$. If $z<0$, then $z+\sqrt{a^2+z^2}>0$ and $z+\sqrt{a^2+z^2}<\sqrt{a^2+z^2}$. Hence $0<\Omega(S,O)<2\pi$. Either way $-2\pi<\Omega(S,O)<2\pi$. Now as $z\to 0^+, \Omega(S,O)\to -2\pi$ and as $z\to 0^-, \Omega(S,O)\to 2\pi$. Hence as $O$ passes through $S$, there is a jump of $4\pi$.

**33.** We have

$$\begin{aligned}\nabla\times\mathbf{G} &= \nabla\times\int_0^1 t\mathbf{F}(t\mathbf{r})\times\mathbf{r}\,dt \quad \text{where } \mathbf{r}=(x,y,z), \\ &= \int_0^1 \nabla\times(t\mathbf{F}(t\mathbf{r})\times\mathbf{r})\,dt \\ &= \int_0^1 t\nabla\times(\mathbf{F}(t\mathbf{r})\times\mathbf{r})\,dt \quad \text{since } t \text{ behaves as a constant with respect to } \nabla, \\ &= \int_0^1 t\{\mathbf{F}(t\mathbf{r})\nabla\cdot\mathbf{r}-\mathbf{r}\nabla\cdot\mathbf{F}(t\mathbf{r})+(\mathbf{r}\cdot\nabla)\mathbf{F}(t\mathbf{r})-(\mathbf{F}(t\mathbf{r})\cdot\nabla)\mathbf{r}\}\,dt \quad \text{by the first identity,} \\ &= \int_0^1 t\{3\mathbf{F}(t\mathbf{r})-\mathbf{r}\nabla\cdot\mathbf{F}(t\mathbf{r})+(\mathbf{r}\cdot\nabla)\mathbf{F}(t\mathbf{r})-\mathbf{F}(t\mathbf{r})\}\,dt \\ &= \int_0^1 t\{2\mathbf{F}(t\mathbf{r})-\mathbf{r}\nabla\cdot\mathbf{F}(t\mathbf{r})+(\mathbf{r}\cdot\nabla)\mathbf{F}(t\mathbf{r})\}\,dt.\end{aligned}$$

To compute $\nabla\cdot\mathbf{F}(t\mathbf{r})$, note that $\dfrac{\partial}{\partial x}\mathbf{F}(t\mathbf{r}) = t\dfrac{\partial\mathbf{F}}{\partial\mathbf{X}}$ by the hint. This implies that $\nabla\cdot\mathbf{F}(t\mathbf{r}) = t\nabla_{\mathbf{X,Y,Z}}\mathbf{F}(\mathbf{X,Y,Z})$ where $\nabla_{\mathbf{X,Y,Z}}$ signifies that all partials are to be taken with respect to $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ where $\mathbf{X}=tx, \mathbf{Y}=ty$, and $\mathbf{Z}=tz$. Thus $\nabla\cdot\mathbf{F}(t\mathbf{r})=0$ since $\mathbf{F}$ is assumed to be divergenceless. By the second identity given in the hint,

$$(\mathbf{r}\cdot\nabla)\mathbf{F}(t\mathbf{r}) = \frac{d}{dt}[t\mathbf{F}(t\mathbf{r})]-\mathbf{F}(t\mathbf{r}).$$

Hence,

$$\begin{aligned}\nabla \times \mathbf{G} &= \int_0^1 t\left\{2\mathbf{F}(t\mathbf{r}) + \frac{d}{dt}[t\mathbf{F}(t\mathbf{r})] - \mathbf{F}(t\mathbf{r})\right\} dt \\ &= \int_0^1 t\left\{\mathbf{F}(t\mathbf{r}) + \frac{d}{dt}[t\mathbf{F}(t\mathbf{r})]\right\} dt \\ &= \int_0^1 \frac{d}{dt}[t^2\mathbf{F}(t\mathbf{r})]\, dt \quad \text{by the last identity in the hint,} \\ &= t^2\mathbf{F}(t\mathbf{r})\Big|_{t=0}^{1} = \mathbf{F}(\mathbf{r}).\end{aligned}$$

**34.** Note $\nabla \cdot \mathbf{F} = 2 - 1 - 1 = 0$ so, by the result of Exercise 33, a vector potential for $\mathbf{F}$ must exist. We can compute it by

$$\begin{aligned}\mathbf{G} &= \int_0^1 t(2xt, -yt, -zt) \times (x, y, z)\, dt = \int_0^1 t(0, -3xzt, 3xyt)\, dt \\ &= \int_0^1 (0, -3xzt^2, 3xyt^2)\, dt = (0, -xzt^3, xyt^3)\Big|_{t=0}^{1} \\ &= (0, -xz, xy).\end{aligned}$$

**35.** $\nabla \cdot \mathbf{F} = 1 + 1 + 1 = 3 \neq 0$, so, by the result of Exercise 33, $\mathbf{F}$ has no vector potential.

**36.** $\nabla \cdot \mathbf{F} = 0 + 0 + 0 = 0$, so, by the result of Exercise 33, a vector potential for $\mathbf{F}$ must exist. We compute it as follows.

$$\begin{aligned}\mathbf{G} &= \int_0^1 t(3yt, 2xzt^2, -7x^2yt^3) \times (x, y, z)\, dt \\ &= \int_0^1 (2xz^2t^3 + 7x^2y^2t^4, -7x^3yt^4 - 3yzt^2, 3y^2t^2 - 2x^2zt^3)\, dt \\ &= \left(\frac{1}{2}xz^2 + \frac{7}{5}x^2y^2, -\frac{7}{5}x^3y - yz, y^2 - \frac{1}{2}x^2z\right).\end{aligned}$$

**37.** Since $\nabla \times (\nabla\phi) = \mathbf{0}$ for any $C^2$ function, we have

$$\nabla \times (\mathbf{G} + \nabla\phi) = \nabla \times \mathbf{G} + \nabla \times (\nabla\phi) = \nabla \times \mathbf{G} + \mathbf{0} = \mathbf{F}.$$

Thus $\mathbf{G} + \nabla\phi$ is a vector potential for $\mathbf{F}$.

**38. (a)** Write $\mathbf{F} = -\dfrac{GMm}{(x^2+y^2+z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$. Then

$$\begin{aligned}\frac{\partial F_1}{\partial x} &= \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} \\ &= \frac{(x^2+y^2+z^2)^2 - 3x^2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial F_2}{\partial y} &= \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{(x^2+y^2+z^2)^2 - 3y^2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}} \\ \frac{\partial F_3}{\partial z} &= \frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{(x^2+y^2+z^2)^2 - 3z^2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}}.\end{aligned}$$

Thus

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{3(x^2+y^2+z^2)^2 - (3x^2+3y^2+3z^2)(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}} = 0.$$

**(b)** Let $S$ be a sphere of radius $a$ enclosing the origin. Consider $S$ to be the union of hemispheres $S_1$ and $S_2$, each oriented so that the normal vector points away from the center of the sphere. If $\mathbf{F} = \nabla \times \mathbf{G}$, then

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \nabla \times \mathbf{G} \cdot d\mathbf{S} &= \iint_{S_1} \nabla \times \mathbf{G} \cdot d\mathbf{S} + \iint_{S_2} \nabla \times \mathbf{G} \cdot d\mathbf{S} \\ &= \oint_{\partial S_1} \mathbf{G} \cdot d\mathbf{s} + \oint_{\partial S_2} \mathbf{G} \cdot d\mathbf{s} \quad \text{by Stokes's theorem} \\ &= 0,\end{aligned}$$

since $\partial S_1$ and $\partial S_2$ inherit opposite orientations from $S_1$ and $S_2$ and are equal as unoriented curves. On the other hand $\mathbf{n} = \mathbf{r}/\|\mathbf{r}\|$, so

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S -\frac{GMm}{\|\mathbf{r}\|^3}\mathbf{r} \cdot \frac{\mathbf{r}}{\|\mathbf{r}\|}\, dS = -GMm \iint_S \frac{\|\mathbf{r}\|^2}{\|\mathbf{r}\|^4}\, dS \\ &= -GMm \iint_S \frac{1}{\|\mathbf{r}\|^2}\, dS = -GMm \iint_S \frac{1}{a^2}\, dS \quad \text{since } \|\mathbf{r}\| = a \text{ on } S, \\ &= -GMm\frac{4\pi a^2}{a^2} = -4\pi GMm \neq 0.\end{aligned}$$

Hence, it cannot be that $\mathbf{F} = \nabla \times \mathbf{G}$.

**(c)** $\mathbf{F}$ is not of class $C^1$ on $\mathbf{R}^3$; $\mathbf{F}$ is undefined at the origin. The $C^1$ hypothesis is assumed in Exercise 33, so there's no contradiction.

**39.** We calculate the curl:

$$\begin{aligned}\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) &= \nabla \times \mathbf{E} + \nabla \times \frac{\partial \mathbf{A}}{\partial t} \\ &= \nabla \times \mathbf{E} + \frac{\partial}{\partial t}\nabla \times \mathbf{A} \\ &= \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}\end{aligned}$$

by Faraday's law. Since **E, B** and thus **A** are all defined on a simply-connected region, we must have that $\mathbf{E} + \partial\mathbf{A}/\partial t$ is conservative.

**40.** Substituting $\nabla \times \mathbf{A}$ for $\mathbf{B}$ in Ampère's law, we have

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}.$$

From the identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$, we have

$$\mu_0\mathbf{J} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}.$$

Since $\mathbf{E} + \partial\mathbf{A}/\partial t$ is conservative, $\mathbf{E} = \nabla f - \dfrac{\partial \mathbf{A}}{\partial t}$, so that

$$\begin{aligned}\mu_0\mathbf{J} &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial}{\partial t}\left(\nabla f - \frac{\partial \mathbf{A}}{\partial t}\right) \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} - \mu_0\epsilon_0\left(\nabla\left(\frac{\partial f}{\partial t}\right) - \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) \quad \text{since } f \text{ is of class } C^2.\end{aligned}$$

Thus

$$\mu_0\mathbf{J} = \nabla\left(\nabla \cdot \mathbf{A} - \mu_0\epsilon_0\frac{\partial f}{\partial t}\right) - \nabla^2\mathbf{A} + \mu_0\epsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2},$$

which is equivalent to the desired formula.

**41.** Again we have $\mathbf{E} = \nabla f - \dfrac{\partial \mathbf{A}}{\partial t}$ so that Gauss's law becomes $\rho/\epsilon_0 = \nabla \cdot \mathbf{E} = \nabla \cdot \left(\nabla f - \dfrac{\partial \mathbf{A}}{\partial t}\right) = \nabla^2 f - \dfrac{\partial}{\partial t}(\nabla \cdot \mathbf{A})$ or $\nabla^2 f = \rho/\epsilon_0 + \dfrac{\partial}{\partial t}(\nabla \cdot \mathbf{A})$.

**42. (a)** If $\tilde{\mathbf{A}} = \mathbf{A} + \nabla\phi$, then in order to have

$$\begin{aligned}\nabla \tilde{f} &= \mathbf{E} + \frac{\partial \tilde{\mathbf{A}}}{\partial t}\\ &= \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} + \nabla\frac{\partial \phi}{\partial t}\\ &= \nabla f + \nabla\frac{\partial \phi}{\partial t},\end{aligned}$$

we must have $\nabla \tilde{f} = \nabla\left(f + \dfrac{\partial \phi}{\partial t}\right)$. Thus, up to addition of a constant, $\tilde{f} = f + \dfrac{\partial \phi}{\partial t}$.

**(b)** The condition that $\nabla \cdot \tilde{\mathbf{A}} = \mu_0\epsilon_0 \dfrac{\partial \tilde{f}}{\partial t}$ is equivalent to

$$\begin{aligned}\nabla \cdot (\mathbf{A} + \nabla\phi) &= \mu_0\epsilon_0\frac{\partial}{\partial t}\left(f + \frac{\partial \phi}{\partial t}\right) \qquad \text{or}\\ \nabla \cdot \mathbf{A} + \nabla^2\phi &= \mu_0\epsilon_0\left(\frac{\partial f}{\partial t} + \frac{\partial^2 \phi}{\partial t^2}\right) \quad \Leftrightarrow\\ \nabla^2\phi - \mu_0\epsilon_0\frac{\partial^2\phi}{\partial t^2} &= -\nabla \cdot \mathbf{A} + \mu_0\epsilon_0\frac{\partial f}{\partial t}.\end{aligned}$$

**43.** If the final equation in part (b) above can be solved for $\phi$, then we may arrange things so that $\nabla \cdot \mathbf{A} = \mu_0\epsilon_0\dfrac{\partial f}{\partial t}$. Then the equation in Exercise 40 is

$$\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \nabla\overbrace{\left(\nabla\cdot\mathbf{A} - \mu_0\epsilon_0\frac{\partial f}{\partial t}\right)}^{0} = -\mu_0\mathbf{J}$$

and the equation in Exercise 41 is

$$\nabla^2 f = \frac{\rho}{\epsilon_0} + \mu_0\epsilon_0\frac{\partial^2 f}{\partial t} \quad \text{or} \quad \nabla^2 f - \mu_0\epsilon_0\frac{\partial^2 f}{\partial t^2} = \frac{\rho}{\epsilon_0}.$$

**44.** We check all the equations, given the assumptions.

$$\begin{aligned}\nabla\cdot\mathbf{E} = \nabla\cdot\underbrace{\left(-\frac{\partial \mathbf{A}}{\partial t} + \nabla f\right)}_{\mathbf{E}} &= -\frac{\partial}{\partial t}\nabla\cdot\mathbf{A} + \nabla^2 f = -\frac{\partial}{\partial t}\left(\mu_0\epsilon_0\frac{\partial f}{\partial t}\right) + \nabla^2 f\\ &= -\mu_0\epsilon_0\frac{\partial^2 f}{\partial t^2} + \nabla^2 f = \frac{\rho}{\epsilon_0}\end{aligned}$$

from the second equation in Exercise 43.

$$\begin{aligned}\nabla\times\mathbf{E} &= -\nabla\times\left(\frac{\partial\mathbf{A}}{\partial t} - \nabla f\right) = -\nabla\times\frac{\partial\mathbf{A}}{\partial t} = -\frac{\partial}{\partial t}(\nabla\times\mathbf{A}) = -\frac{\partial\mathbf{B}}{\partial t}\\ \nabla\cdot\mathbf{B} &= \nabla\cdot(\nabla\times\mathbf{A}) = 0\\ \nabla\times\mathbf{B} = \nabla\times(\nabla\times\mathbf{A}) &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \quad \text{(identity)}\\ &= \nabla(\nabla\cdot\mathbf{A}) - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} + \mu_0\mathbf{J}\end{aligned}$$

by the equation in part (b) of Exercise 42

$$= \nabla\left(\mu_0\epsilon_0\frac{\partial f}{\partial t}\right) - \mu_0\epsilon_0\frac{\partial}{\partial t}(-\mathbf{E} + \nabla f) + \mu_0\mathbf{J},$$

using the condition $\nabla \cdot \mathbf{A} = \mu_0\epsilon_0\dfrac{\partial f}{\partial t}$, and that $\dfrac{\partial \mathbf{A}}{\partial t} = \nabla f - \mathbf{E}$

$$= \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} + \mu_0\mathbf{J}$$

since we may assume $f$ to be of class $C^2$.

## Chapter 8

# Vector Analysis in Higher Dimensions

### 8.1 An Introduction to Differential Forms

**1.** $(dx_1 - 3\,dx_2)(7,3) = dx_1(7,3) - 3\,dx_2(7,3) = 7 - 3(3) = -2.$

**2.**

$$\begin{aligned}&(2\,dx + 6\,dy - 5\,dz)(1,-1,-2)\\&\quad = 2\,dx(1,-1,2) + 6\,dy(1,-1,-2) - 5\,dz(1,-1,-2)\\&\quad = 2(1) + 6(-1) - 5(-2) = 6.\end{aligned}$$

**3.** $(3\,dx_1 \wedge dx_2)((4,-1),(2,0)) = 3 \text{ det} \begin{bmatrix} dx_1(4,-1) & dx_1(2,0) \\ dx_2(4,-1) & dx_2(2,0) \end{bmatrix} = 3 \text{ det} \begin{bmatrix} 4 & 2 \\ -1 & 0 \end{bmatrix} = 3(2) = 6.$

**4.**

$$\begin{aligned}&(4\,dx \wedge dy - 7\,dy \wedge dz)((0,1,-1),(1,3,2))\\&\quad = 4\,dx \wedge dy((0,1,-1),(1,3,2)) - 7\,dy \wedge dz((0,1,-1),(1,3,2))\\&\quad = 4 \text{ det} \begin{bmatrix} dx(0,1,-1) & dx(1,3,2) \\ dy(0,1,-1) & dy(1,3,2) \end{bmatrix} - 7 \text{ det} \begin{bmatrix} dy(0,1,-1) & dy(1,3,2) \\ dz(0,1,-1) & dz(1,3,2) \end{bmatrix}\\&\quad = 4 \text{ det} \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} - 7 \text{ det} \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = 4(-1) - 7(5) = -39.\end{aligned}$$

**5.** We have

$$\begin{aligned}7\,dx \wedge dy \wedge dz(\mathbf{a},\mathbf{b},\mathbf{c}) &= 7 \text{ det} \begin{bmatrix} dx(\mathbf{a}) & dx(\mathbf{b}) & dx(\mathbf{c}) \\ dy(\mathbf{a}) & dy(\mathbf{b}) & dy(\mathbf{c}) \\ dz(\mathbf{a}) & dz(\mathbf{b}) & dz(\mathbf{c}) \end{bmatrix}\\&= 7 \text{ det} \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} = 7(-1 + 12 + 15) = 182.\end{aligned}$$

**6.** We have

$$\begin{aligned}&(dx_1 \wedge dx_2 + 2\,dx_2 \wedge dx_3 + 3\,dx_3 \wedge dx_4)(\mathbf{a},\mathbf{b})\\&\quad = \text{det} \begin{bmatrix} dx_1(\mathbf{a}) & dx_1(\mathbf{b}) \\ dx_2(\mathbf{a}) & dx_2(\mathbf{b}) \end{bmatrix} + 2 \text{ det} \begin{bmatrix} dx_2(\mathbf{a}) & dx_2(\mathbf{b}) \\ dx_3(\mathbf{a}) & dx_3(\mathbf{b}) \end{bmatrix} + 3 \text{ det} \begin{bmatrix} dx_3(\mathbf{a}) & dx_3(\mathbf{b}) \\ dx_4(\mathbf{a}) & dx_4(\mathbf{b}) \end{bmatrix}\\&\quad = \text{det} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + 2 \text{ det} \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} + 3 \text{ det} \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = -5 + 2(-5) + 3(-5) = -30.\end{aligned}$$

**7.**

$$\begin{aligned}
&(2\,dx_1 \wedge dx_3 \wedge dx_4 + dx_2 \wedge dx_3 \wedge dx_5)(\mathbf{a}, \mathbf{b}, \mathbf{c}) \\
&= 2 \det \begin{bmatrix} dx_1(\mathbf{a}) & dx_1(\mathbf{b}) & dx_1(\mathbf{c}) \\ dx_3(\mathbf{a}) & dx_3(\mathbf{b}) & dx_3(\mathbf{c}) \\ dx_4(\mathbf{a}) & dx_4(\mathbf{b}) & dx_4(\mathbf{c}) \end{bmatrix} + \det \begin{bmatrix} dx_2(\mathbf{a}) & dx_2(\mathbf{b}) & dx_2(\mathbf{c}) \\ dx_3(\mathbf{a}) & dx_3(\mathbf{b}) & dx_3(\mathbf{c}) \\ dx_5(\mathbf{a}) & dx_5(\mathbf{b}) & dx_5(\mathbf{c}) \end{bmatrix} \\
&= 2 \det \begin{bmatrix} 1 & 0 & 5 \\ -1 & 9 & 0 \\ 4 & 1 & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & 0 \\ -1 & 9 & 0 \\ 2 & -1 & -2 \end{bmatrix} \\
&= 2(-185) + 0 = -370.
\end{aligned}$$

**8.** $\omega_{(3,-1,4)}(\mathbf{a}) = (-9\,dx + 4\,dy + 192\,dz)(a_1, a_2, a_3) = -9a_1 + 4a_2 + 192a_3.$

**9.**

$$\begin{aligned}
\omega_{(2,-1,-3,1)}(\mathbf{a}, \mathbf{b}) &= (-6\,dx_1 \wedge dx_3 + dx_2 \wedge dx_4)(\mathbf{a}, \mathbf{b}) \\
&= -6 \det \begin{bmatrix} dx_1(\mathbf{a}) & dx_1(\mathbf{b}) \\ dx_3(\mathbf{a}) & dx_3(\mathbf{b}) \end{bmatrix} + \det \begin{bmatrix} dx_2(\mathbf{a}) & dx_2(\mathbf{b}) \\ dx_4(\mathbf{a}) & dx_4(\mathbf{b}) \end{bmatrix} \\
&= -6(a_1b_3 - a_3b_1) + a_2b_4 - a_4b_2.
\end{aligned}$$

**10.**

$$\begin{aligned}
\omega_{(0,-1,\pi/2)}(\mathbf{a}, \mathbf{b}) &= (1\,dx \wedge dy - 1\,dy \wedge dz + 4\,dx \wedge dz)(\mathbf{a}, \mathbf{b}) \\
&= \begin{vmatrix} dx(\mathbf{a}) & dx(\mathbf{b}) \\ dy(\mathbf{a}) & dy(\mathbf{b}) \end{vmatrix} - \begin{vmatrix} dy(\mathbf{a}) & dy(\mathbf{b}) \\ dz(\mathbf{a}) & dz(\mathbf{b}) \end{vmatrix} + 4 \begin{vmatrix} dx(\mathbf{a}) & dx(\mathbf{b}) \\ dz(\mathbf{a}) & dz(\mathbf{b}) \end{vmatrix} \\
&= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} - \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + 4 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \\
&= a_1b_2 - a_2b_1 - (a_2b_3 - a_3b_2) + 4(a_1b_3 - a_3b_1)
\end{aligned}$$

**11.**

$$\begin{aligned}
\omega_{(x,y,z)}((2,0,-1),\ (1,7,5)) &= \cos x \begin{vmatrix} 2 & 1 \\ 0 & 7 \end{vmatrix} - \sin z \begin{vmatrix} 0 & 7 \\ -1 & 5 \end{vmatrix} + (y^2 + 3) \begin{vmatrix} 2 & 1 \\ -1 & 5 \end{vmatrix} \\
&= 14\ \cos x - 7 \sin z + 11(y^2 + 3)
\end{aligned}$$

**12.** We have

$$\begin{aligned}
\omega_{(0,0,0)}(\mathbf{a}, \mathbf{b}, \mathbf{c}) &= (3\,dx \wedge dy \wedge dz)(\mathbf{a}, \mathbf{b}, \mathbf{c}) \\
&= 3 \det \begin{bmatrix} dx(\mathbf{a}) & dx(\mathbf{b}) & dx(\mathbf{c}) \\ dy(\mathbf{a}) & dy(\mathbf{b}) & dy(\mathbf{c}) \\ dz(\mathbf{a}) & dz(\mathbf{b}) & dz(\mathbf{c}) \end{bmatrix} = 3 \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \\
&= 3(a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3).
\end{aligned}$$

**13.** We have

$$\begin{aligned}
\omega_{(x,y,z)}\left((1,0,0),(0,2,0),(0,0,3)\right) &= \left(e^x \cos y + (y^2 + 2)e^{2z}\right) \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\
&= 6\left(e^x \cos y + (y^2 + 2)e^{2z}\right).
\end{aligned}$$

**14.** From Definition 1.3 of exterior product,

$$\begin{aligned}
&(3\,dx + 2\,dy - x\,dz) \wedge (x^2\,dx - \cos y\,dy + 7\,dz)\\
&\quad = 3x^2\,dx\wedge dx + 2x^2\,dy\wedge dx - x^3\,dz\wedge dx - 3\cos y\,dx\wedge dy - 2\cos y\,dy\wedge dy + x\cos y\,dz\wedge dy\\
&\qquad + 21\,dx\wedge dz + 14\,dy\wedge dz - 7x\,dz\wedge dz\\
&\quad = 2x^2\,dy\wedge dx - x^3\,dz\wedge dx - 3\cos y\,dx\wedge dy + x\cos y\,dz\wedge dy\\
&\qquad + 21\,dx\wedge dz + 14\,dy\wedge dz \quad \text{using (4)},\\
&\quad = -(2x^2 + 3\cos y)\,dx\wedge dy + (x^3+21)\,dx\wedge dz\\
&\qquad + (14 - x\cos y)\,dy\wedge dz \quad \text{using (3)}.
\end{aligned}$$

**15.** Again from Definition 1.3 of exterior product,

$$\begin{aligned}
&(y\,dx - x\,dy)\wedge(z\,dx\wedge dy + y\,dx\wedge dz + x\,dy\wedge dz)\\
&\quad = yz\,dx\wedge dx\wedge dy - xz\,dy\wedge dx\wedge dy + y^2\,dx\wedge dx\wedge dz - xy\,dy\wedge dx\wedge dz\\
&\qquad + xy\,dx\wedge dy\wedge dz - x^2\,dy\wedge dy\wedge dz\\
&\quad = 2xy\,dx\wedge dy\wedge dz \quad \text{using (3) and (4)}.
\end{aligned}$$

**16.** Again from Definition 1.3 of exterior product,

$$\begin{aligned}
&(2\,dx_1\wedge dx_2 - x_3\,dx_2\wedge dx_4)\wedge(2x_4\,dx_1\wedge dx_3 + (x_3 - x_2)\,dx_3\wedge dx_4)\\
&\quad = 4x_4\,dx_1\wedge dx_2\wedge dx_1\wedge dx_3 - 2x_3x_4\,dx_2\wedge dx_4\wedge dx_1\wedge dx_3\\
&\qquad + 2(x_3-x_2)\,dx_1\wedge dx_2\wedge dx_3\wedge dx_4 - x_3(x_3-x_2)\,dx_2\wedge dx_4\wedge dx_3\wedge dx_4\\
&\quad = -2x_3x_4\,dx_2\wedge dx_4\wedge dx_1\wedge dx_3 + 2(x_3-x_2)\,dx_1\wedge dx_2\wedge dx_3\wedge dx_4 \quad \text{using (4)},\\
&\quad = 2(x_3x_4 + x_3 - x_2)\,dx_1\wedge dx_2\wedge dx_3\wedge dx_4 \quad \text{using (3)}.
\end{aligned}$$

**17.** Again from Definition 1.3 of exterior product,

$$\begin{aligned}
&(x_1\,dx_1 + 2x_2\,dx_2 + 3x_3\,dx_3)\wedge((x_1+x_2)\,dx_1\wedge dx_2\wedge dx_3 + (x_3-x_4)\,dx_1\wedge dx_2\wedge dx_4)\\
&\quad = x_1(x_1+x_2)\,dx_1\wedge dx_1\wedge dx_2\wedge dx_3 + 2x_2(x_1+x_2)\,dx_2\wedge dx_1\wedge dx_2\wedge dx_3\\
&\qquad + 3x_2(x_1+x_2)\,dx_3\wedge dx_1\wedge dx_2\wedge dx_3 + x_1(x_3-x_4)\,dx_1\wedge dx_1\wedge dx_2\wedge dx_4\\
&\quad = 2x_2(x_3-x_4)\,dx_2\wedge dx_1\wedge dx_2\wedge dx_4 + 3x_3(x_3-x_4)\,dx_3\wedge dx_1\wedge dx_2\wedge dx_4.
\end{aligned}$$

Using equation (4), this last expression is equal to

$$0+0+0+0+0+3x_3(x_3-x_4)\,dx_3\wedge dx_1\wedge dx_2\wedge dx_4 = 3x_3(x_3-x_4)\,dx_1\wedge dx_2\wedge dx_3\wedge dx_4,$$

using equation (3).

**18.** We can work everything out, or note that $\omega$ and $\eta$ in this problem are $\eta$ and $\omega$ (respectively) in Exercise 17. Thus anticommutativity (property 2 of Proposition 1.4) may thus be applied to give

$$\omega\wedge\eta = (-1)^{3\cdot 1}3x_3(x_3-x_4)\,dx_1\wedge dx_2\wedge dx_3\wedge dx_4 = -3x_3(x_3-x_4)\,dx_1\wedge dx_2\wedge dx_3\wedge dx_4.$$

**19.** Again from Definition 1.3 of exterior product,

$$\begin{aligned}
&(x_1\,dx_2\wedge dx_3 - x_2x_3\,dx_1\wedge dx_5)\wedge(e^{x_4x_5}\,dx_1\wedge dx_4\wedge dx_5 - x_1\cos x_5\,dx_2\wedge dx_3\wedge dx_4)\\
&\quad = x_1e^{x_4x_5}\,dx_2\wedge dx_3\wedge dx_1\wedge dx_4\wedge dx_5 - x_2x_3e^{x_4x_5}\,dx_1\wedge dx_5\wedge dx_1\wedge dx_4\wedge dx_5\\
&\qquad - x_1^2\cos x_5\,dx_2\wedge dx_3\wedge dx_2\wedge dx_3\wedge dx_4 + x_1x_2x_3\cos x_5\,dx_1\wedge dx_5\wedge dx_2\wedge dx_3\wedge dx_4\\
&\quad = x_1e^{x_4x_5}\,dx_2\wedge dx_3\wedge dx_1\wedge dx_4\wedge dx_5 + x_1x_2x_3\cos x_5\,dx_1\wedge dx_5\wedge dx_2\wedge dx_3\wedge dx_4 \quad \text{using (4)},\\
&\quad = (x_1e^{x_4x_5} - x_1x_2x_3\cos x_5)dx_1\wedge dx_2\wedge dx_3\wedge dx_4\wedge dx_5 \quad \text{using (3)}.
\end{aligned}$$

**20.** Using Definition 1.1,

$$
\begin{aligned}
&dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_l} \wedge \cdots \wedge dx_{i_k}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k) \\
&= \det \begin{bmatrix} dx_{i_1}(\mathbf{a}_1) & dx_{i_1}(\mathbf{a}_2) & \ldots & dx_{i_1}(\mathbf{a}_k) \\ \vdots & \vdots & & \vdots \\ dx_{i_j}(\mathbf{a}_1) & dx_{i_j}(\mathbf{a}_2) & \ldots & dx_{i_j}(\mathbf{a}_k) \\ \vdots & \vdots & & \vdots \\ dx_{i_l}(\mathbf{a}_1) & dx_{i_l}(\mathbf{a}_2) & \ldots & dx_{i_l}(\mathbf{a}_k) \\ \vdots & \vdots & & \vdots \\ dx_{i_k}(\mathbf{a}_1) & dx_{i_k}(\mathbf{a}_2) & \ldots & dx_{i_k}(\mathbf{a}_k) \end{bmatrix} \\
&= -\det \begin{bmatrix} dx_{i_1}(\mathbf{a}_1) & dx_{i_1}(\mathbf{a}_2) & \ldots & dx_{i_1}(\mathbf{a}_k) \\ \vdots & \vdots & & \vdots \\ dx_{i_l}(\mathbf{a}_1) & dx_{i_l}(\mathbf{a}_2) & \ldots & dx_{i_l}(\mathbf{a}_k) \\ \vdots & \vdots & & \vdots \\ dx_{i_j}(\mathbf{a}_1) & dx_{i_j}(\mathbf{a}_2) & \ldots & dx_{i_j}(\mathbf{a}_k) \\ \vdots & \vdots & & \vdots \\ dx_{i_k}(\mathbf{a}_1) & dx_{i_k}(\mathbf{a}_2) & \ldots & dx_{i_k}(\mathbf{a}_k) \end{bmatrix}
\end{aligned}
$$

(since switching rows $l$ and $j$ changes the sign of the determinant)

$$= -dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_k}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k).$$

**21.** This is easier to show in person, but the point is that if you switch the two identical forms then, on the one hand, nothing has changed and, on the other hand, formula (3) says that you now have the negative of what you started with. So

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_k} = -dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_k}$$

and therefore

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_k} = 0.$$

**22.** A $k$-form $\omega$ on $\mathbf{R}^n$ may be written as $\omega = \sum F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. For each summand, each of the $k\, dx_{i_j}$'s is one of $dx_1, dx_2, \ldots dx_n$. If $k > n$, then, by the pigeon hole principle, there must be at least one repeated term $dx_l$ in $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ (i.e., it will look like $dx_{i_1} \wedge \cdots \wedge dx_l \wedge \cdots \wedge dx_l \wedge \cdots \wedge dx_{i_k}$). And so, by formula (4), we have that $dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0$. Hence every term of $\omega$ is zero.

**23.** Let $\omega_1 = \sum F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, $\omega_2 = \sum G_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and $\eta = \sum H_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}$. Then

$$
\begin{aligned}
(\omega_1 + \omega_2) \wedge \eta &= \left[ \sum_{i_1, \ldots, i_k} (F_{i_1 \ldots i_k} + G_{i_1 \ldots i_k})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right] \wedge \sum_{j_1, \ldots, j_l} H_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\
&= \sum_{\substack{i_1, \ldots, i_k \\ j_1, \ldots, j_l}} (F_{i_1 \ldots i_k} + G_{i_1 \ldots i_k}) H_{j_1 \ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\
&= \sum_{\substack{i_1, \ldots, i_k \\ j_1, \ldots, j_l}} F_{i_1 \ldots i_k} H_{j_1 \ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\
&\quad + \sum_{\substack{i_1, \ldots, i_k \\ j_1, \ldots, j_l}} G_{i_1 \ldots i_k} H_{j_1 \ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\
&= \omega_1 \wedge \eta + \omega_2 \wedge \eta.
\end{aligned}
$$

**24.** Let $\omega = \sum F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and $\eta = \sum G_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}$. Then

$$\omega \wedge \eta = \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l}} F_{i_1 \ldots i_k} G_{j_1 \ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}.$$

Now move $dx_{j_1}$ to the front by switching, in reverse order, with each of the $dx_{i_p}$'s. There are $k$ switches so, by formula (3), there are $k$ sign changes and this last equation becomes

$$\omega \wedge \eta = \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l}} F_{i_1 \ldots i_k} G_{j_1 \ldots j_l} (-1)^k\, dx_{j_1} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_l}.$$

Similarly, we use $k$ more interchanges to move $dx_{j_2}$ into the second position. We repeat this for each of the $l$ $dx_{j_q}$'s. and our equation becomes

$$\omega \wedge \eta = \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l}} F_{i_1 \ldots i_k} G_{j_1 \ldots j_l} \underbrace{(-1)^k(-1)^k \cdots (-1)^k}_{l \text{ times}}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = (-1)^{kl} \eta \wedge \omega.$$

**25.** Let $\omega = \sum F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, $\eta = \sum G_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}$, and $\tau = \sum H_{u_1 \ldots u_m}\, dx_{u_1} \wedge \cdots \wedge dx_{u_m}$. Then

$$\begin{aligned}
(\omega \wedge \eta) \wedge \tau &= \left[ \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l}} F_{i_1 \ldots i_k} G_{j_1 \ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \right] \\
&\quad \wedge \sum_{u_1,\ldots,u_m} H_{u_1 \ldots u_m}\, dx_{u_1} \wedge \cdots \wedge dx_{u_m} \\
&= \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l \\ u_1,\ldots,u_m}} F_{i_1 \ldots i_k} G_{j_1 \ldots j_l} H_{u_1 \ldots u_m}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \wedge dx_{u_1} \wedge \cdots \wedge dx_{u_m}.
\end{aligned}$$

Similarly, calculate $\omega \wedge (\eta \wedge \tau)$ and you will obtain the same result.

**26.** Here $\omega = \sum F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and $\eta = \sum G_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}$ and $f$ is a function (or 0-form). First we note that

$$\begin{aligned}
(f\omega) \wedge \eta &= \left( \sum_{i_1,\ldots,i_k} f F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right) \wedge \left( \sum_{j_1,\ldots,j_l} G_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l} \right) \\
&= \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l}} f F_{i_1 \ldots i_k} G_{j_1 \ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\
&= f \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l}} F_{i_1 \ldots i_k} G_{j_1 \ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\
&= f(\omega \wedge \eta).
\end{aligned}$$

We will use this result to establish the second equality,

$$\begin{aligned}
f(\omega \wedge \eta) &= (-1)^{kl} f(\eta \wedge \omega) && \text{by property 2 of Proposition 1.4,} \\
&= (-1)^{kl} (f\eta) \wedge \omega && \text{by the result established above,} \\
&= (-1)^{kl}(-1)^{kl} \omega \wedge (f\eta) && \text{by property 2 of Proposition 1.4,} \\
&= \omega \wedge (f\eta).
\end{aligned}$$

Therefore, $(f\omega) \wedge \eta = f(\omega \wedge \eta) = \omega \wedge (f\eta)$.

## 8.2 Manifolds and Integrals of $k$-Forms

**1.** Here the map is $\mathbf{X}(\theta_1, \theta_2, \theta_3) = (3\cos\theta_1,\ 3\sin\theta_1,\ 3\cos\theta_1 + 2\cos\theta_2,\ 3\sin\theta_1 + 2\sin\theta_2,\ 3\cos\theta_1 + 2\cos\theta_2 + \cos\theta_3,\ 3\sin\theta_1 + 2\sin\theta_2 + \sin\theta_3)$.
Follow the lead of Example 2 from the text. Each component function is at least $C^1$ so the mapping is at least $C^1$. To see one-one, consider the equation $\mathbf{X}(\theta_1, \theta_2, \theta_3) = \mathbf{X}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$. The first two component equations would then have $\cos\theta_1 = \cos\hat{\theta}_1$ and $\sin\theta_1 = \sin\hat{\theta}_1$. Since $0 \le \theta_1, \hat{\theta}_1 < 2\pi$ we see that $\theta_1 = \hat{\theta}_1$. Using this information in the next two component functions, we make the same conclusion for $\theta_2$ and $\hat{\theta}_2$. Finally, use all of this information in the last set of equations to see that $\theta_3 = \hat{\theta}_3$. So $\mathbf{X}$ is one-one and $C^1$. What is left to show is that the tangent vectors $\mathbf{T}_{\theta_1}$, $\mathbf{T}_{\theta_2}$, and $\mathbf{T}_{\theta_3}$ are linearly independent.

$$\mathbf{T}_{\theta_1} = (-3\sin\theta_1, 3\cos\theta_1, -3\sin\theta_1, 3\cos\theta_1, -3\sin\theta_1, 3\cos\theta_1)$$
$$\mathbf{T}_{\theta_2} = (0, 0, -2\sin\theta_2, 2\cos\theta_2, -2\sin\theta_2, 3\cos\theta_2)$$
$$\mathbf{T}_{\theta_3} = (0, 0, 0, 0, \sin\theta_3, \cos\theta_3)$$

Because of the leading pair of zeros in $\mathbf{T}_{\theta_2}$ and $\mathbf{T}_{\theta_3}$ we can see that if $c_1\mathbf{T}_{\theta_1} + c_2\mathbf{T}_{\theta_2} + c_3\mathbf{T}_{\theta_3} = \mathbf{0}$, then $c_1 = 0$. Looking at the second pair of zeros in $\mathbf{T}_{\theta_3}$ we can then see that $c_2 = 0$. This would then force $c_3 = 0$. So $\mathbf{T}_{\theta_1}$, $\mathbf{T}_{\theta_2}$, and $\mathbf{T}_{\theta_3}$ are linearly independent. We have shown that the parametrized 3-manifold is a smooth parametrized 3-manifold.

**2.** As in Example 3, let's begin by describing the location of the point $(x_1, y_1)$. It is anywhere in the annular region described by $(l_1\cos\theta_1, l_1\sin\theta_1)$ where $1 \le l_1 \le 3$ and $0 \le \theta_1 < 2\pi$. You can now describe $(x_2, y_2)$ as being this same annular region centered at $(x_1, y_1)$. Together this means that the locus of $(x_2, y_2)$ is the interior of a disk of radius 6. Using variables $l_2$ and $\theta_2$ such that $1 \le l_2 \le 3$ and $0 \le \theta_2 < 2\pi$, the mapping is

$$\mathbf{X}(l_1, \theta_1, l_2, \theta_2) = (l_1\cos\theta_1, l_1\sin\theta_1, l_1\cos\theta_1 + l_2\cos\theta_2, l_1\sin\theta_1 + l_2\sin\theta_2).$$

As before, the component functions are at least $C^1$ so the mapping is at least $C^1$. As for one-one, consider $\mathbf{X}(l_1, \theta_1, l_2, \theta_2) = \mathbf{X}(\hat{l}_1, \hat{\theta}_1, \hat{l}_2, \hat{\theta}_2)$. From the first component functions we see that $(x_1, y_1)$ lies on a circle of radius $l_1$ and $(\hat{x}_1, \hat{y}_1)$ lies on a circle of radius $\hat{l}_1$ so $l_1 = \hat{l}_1$. Then, as in Exercise 1, $\cos\theta_1 = \cos\hat{\theta}_1$ and $\sin\theta_1 = \sin\hat{\theta}_1$. As $0 \le \theta_1, \hat{\theta}_1 < 2\pi$, we see that $\theta_1 = \hat{\theta}_1$. Now the rest of the argument follows in exactly the same way since $(x_2, y_2)$ is related to $(x_1, y_1)$ in the same way that $(x_1, y_1)$ is related to the origin. We now need to show that the four tangent vectors are linearly independent.

$$\mathbf{T}_{l_1} = (\cos\theta_1, \sin\theta_1, \cos\theta_1, \sin\theta_1)$$
$$\mathbf{T}_{\theta_1} = (-l_1\sin\theta_1, l_1\cos\theta_1, -l_1\sin\theta_1, l_1\cos\theta_1)$$
$$\mathbf{T}_{l_2} = (0, 0, \cos\theta_2, \sin\theta_2)$$
$$\mathbf{T}_{\theta_2} = (0, 0, -l_2\sin\theta_2, l_2\cos\theta_2)$$

Look at the equation $c_1\mathbf{T}_{l_1} + c_2\mathbf{T}_{\theta_1} + c_3\mathbf{T}_{l_2} + c_4\mathbf{T}_{\theta_2} = \mathbf{0}$. Because of the leading pair of zeros in $\mathbf{T}_{l_2}$ and $\mathbf{T}_{\theta_2}$ we can see that $c_1\cos\theta_1 = c_2 l_1\sin\theta_1$ and $c_1\sin\theta_1 = -c_2 l_1\cos\theta_1$. Solve for $c_1$ in the first equation and substitute into the second equation to get $c_2 l_1\sin^2\theta_1 = -c_2 l_1\cos^2\theta_1$. Because $l_1$ cannot be zero, this implies that $c_2 = 0$. This then implies that $c_1 = 0$. Given that, we can make the same argument to show $c_3 = c_4 = 0$. Therefore the four tangent vectors are linearly independent and we have described the states of the robot arm as a smooth parametrized 4-manifold in $\mathbf{R}^4$.

**3.** This is a combination of Example 3 and Exercise 2. Let's begin by describing the location of the point $(x_1, y_1)$. It is anywhere on a circle of radius 3 centered at the origin. So $(x_1, y_1) = (3\cos\theta_1, 3\sin\theta_1)$ where $0 \le \theta_1 < 2\pi$. We can then describe $(x_2, y_2)$ as being this same annular region centered at $(x_1, y_1)$. Together this means that the locus of $(x_2, y_2)$ is $(3\cos\theta_1 + l_2\cos\theta_2, 3\sin\theta_1 + l_2\sin\theta_2)$ where $1 \le l_2 \le 2$ and $0 \le \theta_2 < 2\pi$. Similarly we describe $(x_3, y_3)$ in terms of $(x_2, y_2)$ using variables $l_3$ and $\theta_3$ such that $1 \le l_3 \le 2$ and $0 \le \theta_2 < 2\pi$. The mapping is

$$\begin{aligned}\mathbf{X}(\theta_1, l_2, \theta_2, l_3, \theta_3) = (&3\cos\theta_1, 3\sin\theta_1, 3\cos\theta_1 + l_2\cos\theta_2, 3\sin\theta_1 + l_2\sin\theta_2,\\ &3\cos\theta_1 + l_2\cos\theta_2 + l_3\cos\theta_3, 3\sin\theta_1 + l_2\sin\theta_2 + l_3\sin\theta_3).\end{aligned}$$

As before, the component functions are at least $C^1$ so the mapping is at least $C^1$. As for one-one, consider $\mathbf{X}(\theta_1, l_2, \theta_2, l_3, \theta_3) = \mathbf{X}(\hat{\theta}_1, \hat{l}_2, \hat{\theta}_2, \hat{l}_3, \hat{\theta}_3)$. From the first two component functions we see that $\cos\theta_1 = \cos\hat{\theta}_1$ and $\sin\theta_1 = \sin\hat{\theta}_1$ and $0 \le \theta_1, \hat{\theta}_1 < 2\pi$ so $\theta_1 = \hat{\theta}_1$. Now, $(x_2, y_2)$ lies on a circle of radius $l_2$ and $(\hat{x}_2, \hat{y}_2)$ lies on a circle of radius $\hat{l}_2$ with each circle centered at the same point $(x_1, y_1) = (\hat{x}_1, \hat{y}_1)$. So $l_2 = \hat{l}_2$. Then, as above, $\cos\theta_2 = \cos\hat{\theta}_2$ and $\sin\theta_2 = \sin\hat{\theta}_2$. As $0 \le \theta_2, \hat{\theta}_2 < 2\pi$, we see that $\theta_2 = \hat{\theta}_2$. Now the rest of the argument follows in exactly the same way since $(x_3, y_3)$ is related to $(x_2, y_2)$ in the same way that $(x_2, y_2)$ is related to $(x_1, y_1)$.

We now need to show that the five tangent vectors are linearly independent.

$$\begin{aligned}
\mathbf{T}_{\theta_1} &= (-3\sin\theta_1, 3\cos\theta_1, -3\sin\theta_1, 3\cos\theta_1, -3\sin\theta_1, 3\cos\theta_1)\\
\mathbf{T}_{l_2} &= (0, 0, \cos\theta_2, \sin\theta_2, \cos\theta_2, \sin\theta_2)\\
\mathbf{T}_{\theta_2} &= (0, 0, -l_2\sin\theta_2, l_2\cos\theta_2, -l_2\sin\theta_2, l_2\cos\theta_2)\\
\mathbf{T}_{l_3} &= (0, 0, 0, 0, \cos\theta_3, \sin\theta_3)\\
\mathbf{T}_{\theta_3} &= (0, 0, 0, 0, -l_3\sin\theta_3, l_3\cos\theta_3)
\end{aligned}$$

Look at the equation $c_1\mathbf{T}_{\theta_1} + c_2\mathbf{T}_{l_2} + c_3\mathbf{T}_{\theta_2} + c_4\mathbf{T}_{l_3} + c_5\mathbf{T}_{\theta_3} = \mathbf{0}$. Because of the leading pair of zeros in all but the vector $\mathbf{T}_{\theta_1}$ we conclude that $c_1 = 0$. The remainder of the argument is exactly as in Exercise 2. Because the first four components of $\mathbf{T}_{l_3}$ and $\mathbf{T}_{\theta_3}$ are zero, we can see that $c_2\cos\theta_2 = c_3l_2\sin\theta_2$ and $c_2\sin\theta_2 = -c_3l_2\cos\theta_2$. Solve for $c_2$ in the first equation and substitute into the second equation to get $c_3l_2\sin^2\theta_2 = -c_3l_2\cos^2\theta_2$. Because $l_2$ cannot be zero, $c_3 = 0$. This then implies that $c_2 = 0$. Given that, we can make the same argument to show $c_4 = c_5 = 0$. Therefore the five tangent vectors are linearly independent and we have described the states of the robot arm as a smooth parametrized 5-manifold in $\mathbf{R}^6$.

**4.** We can use spherical coordinates to describe the parametrized space. The point $(x_1, y_1, z_1)$ can be written as $(2\sin\varphi_1\cos\theta_1, 2\sin\varphi_1\sin\theta_1, 2\cos\varphi_1)$ where $0 \le \varphi_1 \le \pi$ and $0 \le \theta_1 < 2\pi$. We can then write $(x_2, y_2, z_2)$ as $(x_1 + \sin\varphi_2\cos\theta_2, y_1 + \sin\varphi_2\sin\theta_2, z_1 + \cos\varphi_2)$ where $0 \le \varphi_2 \le \pi$ and $0 \le \theta_2 < 2\pi$. In other words, our mapping is

$$\begin{aligned}
\mathbf{X}(\theta_1, \varphi_1, \theta_2, \varphi_2) = (&2\sin\varphi_1\cos\theta_1, 2\sin\varphi_1\sin\theta_1, 2\cos\varphi_1,\\
&2\sin\varphi_1\cos\theta_1 + \sin\varphi_2\cos\theta_2, 2\sin\varphi_1\sin\theta_1 + \sin\varphi_2\sin\theta_2, 2\cos\varphi_1 + \cos\varphi_2).
\end{aligned}$$

As in the previous exercises, the fact that the component functions are at least $C^1$ tells us that the mapping is at least $C^1$. Checking one-one is a little more interesting than in the above exercises. Consider the implications of the equation $\mathbf{X}(\theta_1, \varphi_1, \theta_2, \varphi_2) = \mathbf{X}(\hat{\theta}_1, \hat{\varphi}_1, \hat{\theta}_2, \hat{\varphi}_2)$. By the third component functions we see that $\cos\varphi_1 = \cos\hat{\varphi}_1$. Because $0 \le \varphi_1, \hat{\varphi}_1 \le \pi$ we see that $\varphi_1 = \hat{\varphi}_1$. Substituting this into the sixth component function implies that $\varphi_2 = \hat{\varphi}_2$. Now comparing the equations from the first two component functions we see that if $\varphi_1 = 0$ or $\pi$ then $\theta_1$ need not be the same as $\hat{\theta}_1$. This is allowed—recall that the mapping might not be one-one on the boundary of the domain. Other than on the boundary, $\cos\theta_1 = \cos\hat{\theta}_1$ and $\sin\theta_1 = \sin\hat{\theta}_1$ and so, as before $\theta_1 = \hat{\theta}_1$. Again, substitute this into the equations that arise from the fourth and fifth component functions to conclude that, except when $\varphi_2$ is 0 or $\pi$, we must have $\theta_2 = \hat{\theta}_2$.

We now need to show that the four tangent vectors are linearly independent.

$$\begin{aligned}
\mathbf{T}_{\theta_1} &= (-2\sin\varphi_1\sin\theta_1, 2\sin\varphi_1\cos\theta_1, 0, -2\sin\varphi_1\sin\theta_1, 2\sin\varphi_1\cos\theta_1, 0)\\
\mathbf{T}_{\varphi_1} &= (2\cos\varphi_1\cos\theta_1, 2\cos\varphi_1\sin\theta_1, -2\sin\varphi_1, 2\cos\varphi_1\cos\theta_1, 2\cos\varphi_1\sin\theta_1, -2\sin\varphi_1)\\
\mathbf{T}_{\theta_2} &= (0, 0, 0, -\sin\varphi_2\sin\theta_2, \sin\varphi_2\cos\theta_2, 0)\\
\mathbf{T}_{\varphi_2} &= (0, 0, 0, \cos\varphi_2\cos\theta_2, \cos\varphi_2\sin\theta_2, -\sin\varphi_2)
\end{aligned}$$

Look at the equation $c_1\mathbf{T}_{\theta_1} + c_2\mathbf{T}_{\varphi_1} + c_3\mathbf{T}_{\theta_2} + c_4\mathbf{T}_{\varphi_2} = \mathbf{0}$. There is a zero in the third component of all of the tangent vectors except for $\mathbf{T}_{\varphi_1}$. This tells us that $c_2 = 0$. If that is the case, then there is a zero in the sixth component of all of the remaining tangent vectors except for $\mathbf{T}_{\varphi_2}$ so $c_4 = 0$. But then the leading trio of zeros in $\mathbf{T}_{\theta_2}$ implies that $c_1 = 0$ which in turn would mean that $c_3 = 0$. Therefore the four tangent vectors are linearly independent and we have described the states of the robot arm as a smooth parametrized 4-manifold in $\mathbf{R}^6$.

**5.** This is just an exercise in linear algebra. If $\mathbf{x} \in \mathbf{R}^n$ is orthogonal to $\mathbf{v}_i$ for $i = 1, \ldots, k$, then $\mathbf{x} \cdot \mathbf{v}_i = 0$ for $i = 1, \ldots, k$. An arbitrary vector $\mathbf{v}$ in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is of the form $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ for scalars $c_1, \ldots, c_k \in \mathbf{R}$. The calculation is straightforward:

$$\mathbf{x} \cdot \mathbf{v} = \mathbf{x} \cdot (c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) = c_1(\mathbf{x} \cdot \mathbf{v}_1) + \cdots + c_k(\mathbf{x} \cdot \mathbf{v}_k) = c_1(0) + \cdots + c_k(0) = 0.$$

In other words, $\mathbf{x}$ is orthogonal to $\mathbf{v}$.

**6.** By Definition 2.1, $\displaystyle\int_{\mathbf{x}} \omega = \int_0^{\pi} \omega_{\mathbf{x}(t)}(\mathbf{x}'(t))\,dt$. We have, $\mathbf{x}'(t) = (-a\sin t, b\cos t, c)$ and also $\omega = b\,dx - a\,dy + xy\,dz$ so that

$$\begin{aligned}\int_{\mathbf{x}} \omega &= \int_0^{\pi} [b(-a\sin t) - a(b\cos t) + (ab\cos t\sin t)c]\,dt \\ &= ab\int_0^{\pi} [-\sin t - \cos t + c\sin t\cos t]\,dt \\ &= ab\left(\cos t - \sin t + \frac{c}{2}\sin^2 t\right)\Big|_0^{\pi} = -2ab.\end{aligned}$$

**7.** Parametrize the unit circle $C$ by $\mathbf{x}(t) = (\cos t, \sin t), 0 \le t \le 2\pi$. Then

$$\begin{aligned}\int_C \omega &= \int_0^{2\pi} \omega_{\mathbf{x}(t)}(-\sin t, \cos t)\,dt = \int_0^{2\pi} (\sin t\,dx - \cos t\,dy)(-\sin t, \cos t)\,dt \\ &= \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\,dt = \int_0^{2\pi} -1\,dt = -2\pi.\end{aligned}$$

**8.** Parametrize the segment as $\mathbf{x}(t) = (t, t, \ldots, t), 0 \le t \le 3$. Then $\mathbf{x}'(t) = (1, 1, \ldots, 1)$ and so

$$\omega_{\mathbf{x}(t)}(\mathbf{x}'(t)) = (t\,dx_1 + t^2\,dx_2 + \cdots + t^n\,dx_n)(1, 1, \ldots, 1) = t + t^2 + \cdots + t^n.$$

Hence,

$$\int_C \omega = \int_0^3 (t + t^2 + \cdots + t^n)\,dt = \left(\frac{1}{2}t^2 + \frac{1}{3}t^3 + \cdots + \frac{1}{n+1}t^{n+1}\right)\Big|_0^3 = \sum_{k=2}^{n+1} \frac{3^k}{k} = \sum_{k=1}^{n} \frac{3^{k+1}}{k+1}.$$

**9.** By Definition 2.3, $\int_S \omega = \iint_D \omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t)\,ds\,dt$. For $\mathbf{X}(s,t) = (s\cos t, s\sin t, t)$, we have $\mathbf{T}_s = (\cos t, \sin t, 0)$, and $\mathbf{T}_t = (-s\sin t, s\cos t, 1)$. Then

$$\begin{aligned}\omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t) &= (t\,dx \wedge dy + 3\,dz \wedge dx - s\cos t\,dy \wedge dz)(\mathbf{T}_s, \mathbf{T}_t) \\ &= t\begin{vmatrix} \cos t & -s\sin t \\ \sin t & s\cos t \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ \cos t & -s\sin t \end{vmatrix} - s\cos t\begin{vmatrix} \sin t & s\cos t \\ 0 & 1 \end{vmatrix} \\ &= st - 3\cos t - \frac{s}{2}\sin 2t.\end{aligned}$$

Thus

$$\begin{aligned}\int_S \omega &= \int_0^{4\pi}\int_0^1 \left(st - 3\cos t - \frac{s}{2}\sin 2t\right)ds\,dt = \int_0^{4\pi}\left(\frac{1}{2}t - 3\cos t - \frac{1}{4}\sin 2t\right)dt \\ &= \left(\frac{1}{4}t^2 - 3\sin t + \frac{1}{8}\cos 2t\right)\Big|_0^{4\pi} = 4\pi^2.\end{aligned}$$

**10. (a)** First calculate the two tangent vectors for this parametrization of the helicoid. We have $\mathbf{T}_{u_1} = (\cos 3u_2, \sin 3u_2, 0)$ and $\mathbf{T}_{u_2} = (-3u_1\sin 3u_2, 3u_1\cos 3u_2, 5)$. Then

$$\Omega_{\mathbf{X}(u_1,u_2)}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}) = \det\begin{bmatrix} -5\sin 3u_2 & \cos 3u_2 & -3u_1\sin 3u_2 \\ 5\cos 3u_2 & \sin 3u_2 & 3u_1\cos 3u_2 \\ -3u_1 & 0 & 5 \end{bmatrix} = -9u_1^2 - 25 < 0$$

for all $(u_1, u_2)$. Therefore this particular parametrization is incompatible with $\Omega$.

**(b)** There is more than one solution. One possible way to do this is to switch the ordering of the variables so that the resulting determinant is positive. Try the parametrization $\mathbf{Y}(u_1, u_2) = \mathbf{X}(u_2, u_1) = (u_2\cos 3u_1, u_2\sin 3u_1, 5u_1)$ for $0 \le u_1 \le 2\pi$ and $0 \le u_2 \le 5$. Then the tangent vectors are $\mathbf{T}_{u_1} = (-3u_2\sin 3u_1, 3u_2\cos 3u_1, 5)$ and $\mathbf{T}_{u_2} = (\cos 3u_1, \sin 3u_1, 0)$. Then

$$\Omega_{\mathbf{Y}(u_1,u_2)}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}) = \det\begin{bmatrix} -5\sin 3u_1 & -3u_2\sin 3u_1 & \cos 3u_1 \\ 5\cos 3u_1 & 3u_2\cos 3u_1 & \sin 3u_1 \\ -3u_2 & 5 & 0 \end{bmatrix} = 9u_1^2 + 25 > 0$$

for all $(u_1, u_2)$. Therefore this particular parametrization is now compatible with $\Omega$.

**(c)** Since the goal is to change the sign of the resulting determinant, we can change $\Omega$ to $\Phi$ where

$$\Phi_{\mathbf{X}(u_1,u_2)}(\mathbf{a},\mathbf{b}) = -\det\begin{bmatrix} -5\sin 3u_2 & a_1 & b_1 \\ 5\cos 3u_2 & a_2 & b_2 \\ -3u_1 & a_3 & b_3 \end{bmatrix}.$$

**(d)** The discussion following Theorem 2.11 tells us what to do if the parametrization is compatible. Since the parametrization $\mathbf{X}$ is incompatible with $\Omega$ we make the following simple adjustment: $\int_S \omega = -\int_{\mathbf{X}} \omega$. We pause to calculate

$$\begin{aligned}\omega_{\mathbf{X}(u_1,u_2)}(\mathbf{T}_{u_1},\mathbf{T}_{u_2}) &= 5u_2\begin{vmatrix} \cos 3u_2 & -3u_1\sin 3u_2 \\ \sin 3u_2 & 3u_1\cos 3u_2 \end{vmatrix} \\ &\quad - (u_1^2\cos^2 3u_2 + u_1^2\sin^2 3u_2)\begin{vmatrix} \sin 3u_2 & 3u_1\cos 3u_2 \\ 0 & 5 \end{vmatrix} \\ &= 15u_1u_2 - 5u_1^2\sin 3u_2.\end{aligned}$$

Hence,

$$\begin{aligned}\int_S \omega &= -\int_0^5\int_0^{2\pi}(15u_1u_2 - 5u_1^2\sin 3u_2)\,du_2\,du_1 = 5\int_0^5\int_0^{2\pi}(u_1^2\sin 3u_2 - 3u_1u_2)\,du_2\,du_1 \\ &= 5\int_0^5\left[u_1^2\left(\frac{-\cos 3u_2}{3}\right) - \frac{3u_1u_2^2}{2}\right]\Bigg|_{u_2=0}^{u_2=2\pi} du_1 = 5\int_0^5(-6\pi^2u_1)\,du_1 \\ &= -30\pi^2\int_0^5 u_1\,du_1 = -15\pi^2u_1^2\big|_0^5 = -375\pi^2.\end{aligned}$$

**11. (a)** For the parametrization given, we calculate the tangent vectors as $\mathbf{T}_{u_1} = (\cos u_2, \sin u_2, 0)$, $\mathbf{T}_{u_2} = (-u_1\sin u_2, u_1\cos u_2, 0)$, and $\mathbf{T}_{u_3} = (0,0,1)$. Then

$$\Omega_{\mathbf{X}(u)}(\mathbf{T}_{u_1},\mathbf{T}_{u_2},\mathbf{T}_{u_3}) = \det\begin{bmatrix} \cos u_2 & -u_1\sin u_2 & 0 \\ \sin u_2 & u_1\cos u_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = u_1.$$

As $0 \le u_1 \le \sqrt{5}$, this is positive when $u_1 \ne 0$. Note that when $u_1 = 0$ the parametrization is not one-one and also that $\mathbf{T}_{u_2} = \mathbf{0}$ so $\mathbf{T}_{u_1}$, $\mathbf{T}_{u_2}$, and $\mathbf{T}_{u_3}$ are not linearly independent. In other words, the parametrization is not smooth when $u_1 = 0$. It is, however, smooth when $u_1 \ne 0$. You can easily see that the mapping is one-one and at least $C^1$. To see that the tangent vectors are linearly independent, consider the equation $c_1\mathbf{T}_{u_1} + c_2\mathbf{T}_{u_2} + c_3\mathbf{T}_{u_3} = \mathbf{0}$. We see from the third components that $c_3 = 0$. Look at the remaining equations and we see that

$$\begin{cases} (\cos u_2)c_1 - (u_1\sin u_2)c_2 = 0 \\ (\sin u_2)c_1 + (u_1\cos u_2)c_2 = 0. \end{cases}$$

Multiply the first equation by $-\sin u_2$ and the second by $\cos u_2$ and add to obtain $u_1c_2 = 0$. Because we are assuming that $u_1 \ne 0$, this implies that $c_2 = 0$ and therefore $c_1 = 0$. This shows that the tangent vectors are linearly independent and hence the parametrization is smooth when $u_1 \ne 0$. The conclusion is then that the parametrization given is compatible with the orientation when it is smooth.

**(b)** We can read the boundary pieces right off of the original parametrization: they are paraboloids that intersect at $z = -1$ in a circle in the plane $z = -1$ of radius $\sqrt{5}$ centered at $(0,0,-1)$. The boundary is

$$\partial M = \{(x,y,z)|z = x^2 + y^2 - 6, z \le -1\} \cup \{(x,y,z)|z = 4 - x^2 - y^2, z \ge -1\}.$$

We can easily adapt the parametrization to each of these pieces. For the bottom, use

$$\mathbf{Y}_1 : [0,\sqrt{5}] \times [0,2\pi) \to \mathbf{R}^3; \quad \mathbf{Y}_1(s_1,s_2) = (s_1\cos s_2, s_1\sin s_2, s_1^2 - 6).$$

For the top, use

$$\mathbf{Y}_2 : [0,\sqrt{5}] \times [0,2\pi) \to \mathbf{R}^3; \quad \mathbf{Y}_2(s_1,s_2) = (s_1\cos s_2, s_1\sin s_2, 4 - s_1^2).$$

**(c)** On the bottom part of $\partial M$ the outward-pointing unit vector

$$\mathbf{V}_1 = \frac{(2x, 2y, -1)}{\sqrt{4x^2 + 4y^2 + 1}}. \text{ In terms of } \mathbf{Y}_1, \text{ this is } \mathbf{V}_1 = \frac{(2s_1 \cos s_2, 2s_1 \sin s_2, -1)}{\sqrt{4s_1^2 + 1}}.$$

On the top part of $\partial M$ the outward-pointing unit vector

$$\mathbf{V}_2 = \frac{(2x, 2y, 1)}{\sqrt{4x^2 + 4y^2 + 1}}. \text{ In terms of } \mathbf{Y}_2, \text{ this is } \mathbf{V}_2 = \frac{(2s_1 \cos s_2, 2s_1 \sin s_2, 1)}{\sqrt{4s_1^2 + 1}}.$$

**12.** The paraboloid can be parametrized as $\mathbf{X}(s,t) = (s, t, s^2 + t^2)$ where $0 \le s^2 + t^2 \le 4$. Therefore, $\mathbf{T}_s = (1, 0, 2s)$ and $\mathbf{T}_t = (0, 1, 2t)$. Note that this parametrization is compatible with the orientation $\Omega$ derived from the normal $\mathbf{N} = (-2x, -2y, 1)$ as

$$\Omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t) = \det[\mathbf{N} \quad \mathbf{T}_s \quad \mathbf{T}_t] = \det \begin{bmatrix} -2s & 1 & 0 \\ -2t & 0 & 1 \\ 1 & 2s & 2t \end{bmatrix} = 2s^2 + 2t^2 + 1 > 0.$$

Therefore, we may compute $\int_S \omega$ as $\int_{\mathbf{X}} \omega$. So we begin by calculating

$$\begin{aligned}
\omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t) &= (e^{s^2+t^2}\, dx \wedge dy + t\, dz \wedge dx + s\, dy \wedge dz)(\mathbf{T}_s, \mathbf{T}_t) \\
&= e^{s^2+t^2} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + t \begin{vmatrix} 2s & 2t \\ 1 & 0 \end{vmatrix} + s \begin{vmatrix} 0 & 1 \\ 2s & 2t \end{vmatrix} \\
&= e^{s^2+t^2} - 2t^2 - 2s^2.
\end{aligned}$$

Use this in the calculation:

$$\begin{aligned}
\int_S \omega &= \iint_{0 \le s^2+t^2 \le 4} [e^{s^2+t^2} - 2(s^2 + t^2)]\, ds\, dt \\
&= \int_0^{2\pi} \int_0^2 (e^{r^2} - 2r^2) r\, dr\, d\theta \quad \text{using polar coordinates,} \\
&= \int_0^{2\pi} \left( \frac{1}{2} e^{r^2} - \frac{1}{2} r^4 \right) \Big|_{r=0}^{2} d\theta \\
&= \int_0^{2\pi} \left( \frac{1}{2} e^4 - \frac{1}{2} - 8 \right) d\theta = \pi(e^4 - 17).
\end{aligned}$$

**13.** The cylinder can be parametrized as $\mathbf{X}(s,t) = (2\cos t, s, 2\sin t)$ where $-1 \le s \le 3$ and $0 \le t \le 2\pi$. Therefore, $\mathbf{T}_s = (0, 1, 0)$ and $\mathbf{T}_t = (-2\sin t, 0, 2\cos t)$. This parametrization turns out to be compatible with the orientation $\Omega$ derived from the normal $\mathbf{N} = (x, 0, z)$ as

$$\Omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t) = \det \begin{bmatrix} \mathbf{N} & \mathbf{T}_s & \mathbf{T}_t \end{bmatrix} = \begin{vmatrix} 2\cos t & 0 & -2\sin t \\ 0 & 1 & 0 \\ 2\sin t & 0 & 2\cos t \end{vmatrix} = 4\cos^2 t + 4\sin^2 t = 4 > 0.$$

Therefore, we may compute $\int_S \omega$ as $\int_{\mathbf{X}} \omega$. Hence we calculate

$$\begin{aligned}
\omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t) &= \left( 2\sin t\, dx \wedge dy + e^{s^2}\, dz \wedge dx + 2\cos t\, dy \wedge dz \right)(\mathbf{T}_s, \mathbf{T}_t) \\
&= 2\sin t \begin{vmatrix} 0 & -2\sin t \\ 1 & 0 \end{vmatrix} + e^{s^2} \begin{vmatrix} 0 & 2\cos t \\ 0 & -2\sin t \end{vmatrix} + 2\cos t \begin{vmatrix} 1 & 0 \\ 0 & 2\cos t \end{vmatrix} \\
&= 4\sin^2 t + 0 + 4\cos^2 t = 4.
\end{aligned}$$

Thus

$$\int_S \omega = \iint_{[-1,3] \times [0, 2\pi]} 4\, ds\, dt = 32\pi.$$

**14.** We have, for the given parametrization, that

$$\mathbf{T}_s = \left(\frac{\cos t}{2\sqrt{s}}, -\frac{\sin t}{2\sqrt{4-s}}, \frac{\sin t}{2\sqrt{s}}, -\frac{\cos t}{2\sqrt{4-s}}\right)$$

and

$$\mathbf{T}_t = \left(-\sqrt{s}\sin t, \sqrt{4-s}\cos t, \sqrt{s}\cos t, -\sqrt{4-s}\sin t\right).$$

Thus,

$$\begin{aligned}\omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t) &= \big(((4-s)\sin^2 t + (4-s)\cos^2 t)\, dx_1 \wedge dx_3 \\ &\quad - (2s\cos^2 t + 2s\sin^2 t)\, dx_2 \wedge dx_4\big)\,(\mathbf{T}_s, \mathbf{T}_t) \\ &= (4-s)\begin{vmatrix} \frac{\cos t}{2\sqrt{s}} & -\sqrt{s}\sin t \\ \frac{\sin t}{2\sqrt{s}} & \sqrt{s}\cos t \end{vmatrix} - 2s\begin{vmatrix} -\frac{\sin t}{2\sqrt{4-s}} & \sqrt{4-s}\cos t \\ -\frac{\cos t}{2\sqrt{4-s}} & -\sqrt{4-s}\sin t \end{vmatrix} \\ &= (4-s)\left(\tfrac{1}{2}\cos^2 t + \tfrac{1}{2}\sin^2 t\right) - 2s\left(\tfrac{1}{2}\sin^2 t + \tfrac{1}{2}\cos^2 t\right) = 2 - \tfrac{3}{2}s.\end{aligned}$$

Hence,

$$\int_{\mathbf{X}} \omega = \int_0^{2\pi}\int_1^3 \left(2 - \tfrac{3}{2}s\right)\, ds\, dt = \int_0^{2\pi} \left(2s - \tfrac{3}{4}s^2\right)\Big|_{s=1}^{s=3} dt = \int_0^{2\pi} (-2)\, dt = -4\pi.$$

**15.** We have, for the given parametrization, that $\mathbf{T}_{u_1} = (1, 0, 0, 4(2u_1 - u_3)), \mathbf{T}_{u_2} = (0, 1, 0, 0)$, and $\mathbf{T}_{u_3} = (0, 0, 1, 2(u_3 - 2u_1))$. Thus,

$$\begin{aligned}\omega_{\mathbf{X}(u_1,u_2,u_3)}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3}) &= (u_2\, dx_2 \wedge dx_3 \wedge dx_4 + 2u_1u_3\, dx_1 \wedge dx_2 \wedge dx_3)(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3}) \\ &= u_2\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4(2u_1 - u_3) & 0 & 2(u_3 - 2u_1) \end{vmatrix} + 2u_1u_3\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= u_2(8u_1 - 4u_3) + 2u_1u_3 = 8u_1u_2 - 4u_2u_3 + 2u_1u_3.\end{aligned}$$

Hence,

$$\begin{aligned}\int_{\mathbf{X}} \omega &= \int_0^1\int_0^1\int_0^1 (8u_1u_2 - 4u_2u_3 + 2u_1u_3)\, du_1\, du_2\, du_3 \\ &= \int_0^1\int_0^1 (4u_2 - 4u_2u_3 + u_3)\, du_2\, du_3 \\ &= \int_0^1 (2 - 2u_3 + u_3)\, du_3 = 2 - \frac{1}{2} = \frac{3}{2}.\end{aligned}$$

## 8.3 The Generalized Stokes's Theorem

**1.** Using Definition 3.1,

$$d(e^{xyz}) = \frac{\partial}{\partial x}(e^{xyz})\, dx + \frac{\partial}{\partial y}(e^{xyz})\, dy + \frac{\partial}{\partial z}(e^{xyz})\, dz + e^{xyz}(yz\, dx + xz\, dy + xy\, dz).$$

**2.** Using Definition 3.1,

$$d(x^3y - 2xz^2 + xy^2z) = (3x^2y - 2z^2 + y^2z)\, dx + (x^3 + 2xyz)\, dy + (xy^2 - 4xz)\, dz.$$

**3.** Again, using Definition 3.1,

$$\begin{aligned}d((x^2 + y^2)\, dx + xy\, dy) &= d(x^2 + y^2) \wedge dx + d(xy) \wedge dy \\ &= (2x\, dx + 2y\, dy) \wedge dx + (y\, dx + x\, dy) \wedge dy \\ &= 2y\, dy \wedge dx + y\, dx \wedge dy \quad \text{using (4) from Section 8.1,} \\ &= -y\, dx \wedge dy \quad \text{using (3) from Section 8.1.}\end{aligned}$$

**4.** Again, using Definition 3.1,

$$\begin{aligned}
&d(x_1\,dx_2 - x_2\,dx_1 + x_3x_4\,dx_4 - x_4x_5\,dx_5)\\
&\quad = dx_1 \wedge dx_2 - dx_2 \wedge dx_1 + (x_4\,dx_3 + x_3\,dx_4) \wedge dx_4 - (x_5\,dx_4 + x_4\,dx_5) \wedge dx_5\\
&\quad = dx_1 \wedge dx_2 - dx_2 \wedge dx_1 + x_4\,dx_3 \wedge dx_4 - x_5\,dx_4 \wedge dx_5 \quad \text{using (4) from Section 8.1,}\\
&\quad = 2\,dx_1 \wedge dx_2 + x_4\,dx_3 \wedge dx_4 - x_5\,dx_4 \wedge dx_5 \quad \text{using (3) from Section 8.1.}
\end{aligned}$$

**5.** Again, using Definition 3.1,

$$\begin{aligned}
d(xz\,dx \wedge dy - y^2 z\,dx \wedge dz) &= (z\,dx + x\,dz) \wedge dx \wedge dy - (2yz\,dy + y^2\,dz) \wedge dx \wedge dz\\
&= x\,dz \wedge dx \wedge dy - 2yz\,dy \wedge dx \wedge dz \quad \text{using (4) from Section 8.1,}\\
&= (x + 2yz)\,dx \wedge dy \wedge dz \quad \text{using (3) from Section 8.1.}
\end{aligned}$$

**6.** Again, using Definition 3.1,

$$\begin{aligned}
&d(x_1x_2x_3\,dx_2 \wedge dx_3 \wedge dx_4 + x_2x_3x_4\,dx_1 \wedge dx_2 \wedge dx_3)\\
&\quad = (x_2x_3\,dx_1 + x_1x_3\,dx_2 + x_1x_2\,dx_3) \wedge dx_2 \wedge dx_3 \wedge dx_4\\
&\qquad + (x_3x_4\,dx_2 + x_2x_4\,dx_3 + x_2x_3\,dx_4) \wedge dx_1 \wedge dx_2 \wedge dx_3\\
&\quad = x_2x_3\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + x_2x_3\,dx_4 \wedge dx_1 \wedge dx_2 \wedge dx_3 \quad \text{using (4) from Section 8.1,}\\
&\quad = 0 \quad \text{using (3) from Section 8.1.}
\end{aligned}$$

**7.** For this solution $\widehat{dx_i}$ means that the term $dx_i$ is omitted.

$$\begin{aligned}
d\omega &= \sum_{i=1}^{n} d(x_i)^2 \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\\
&= \sum_{i=1}^{n} 2x_i\,dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\\
&= \sum_{i=1}^{n} (-1)^{i-1} 2x_i\,dx_1 \wedge \cdots \wedge dx_n \quad \text{using equation (3) of Section 8.1 repeatedly}\\
&= 2(x_1 - x_2 + x_3 - \cdots + (-1)^{n-1} x_n)\,dx_1 \wedge \cdots \wedge dx_n.
\end{aligned}$$

**8.** Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$; then

$$\begin{aligned}
df_{\mathbf{x}_0}(\mathbf{u}) &= (f_{x_1}(\mathbf{x}_0)\,dx_1 + f_{x_2}(\mathbf{x}_0)\,dx_2 + \cdots + f_{x_n}(\mathbf{x}_0)\,dx_n)(\mathbf{u})\\
&= f_{x_1}(\mathbf{x}_0)u_1 + f_{x_2}(\mathbf{x}_0)u_2 + \cdots + f_{x_n}(\mathbf{x}_0)u_n\\
&= (f_{x_1}(\mathbf{x}_0), f_{x_2}(\mathbf{x}_0), \ldots, f_{x_n}(\mathbf{x}_0)) \cdot \mathbf{u}\\
&= \nabla f(\mathbf{x}_0) \cdot \mathbf{u}\\
&= D_{\mathbf{u}} f(\mathbf{x}_0) \quad \text{by Theorem 6.2 of Chapter 2.}
\end{aligned}$$

**9.** For $\omega = F(x,z)\,dy + G(x,y)\,dz$, we have $d\omega = (F_x\,dx + F_z\,dz) \wedge dy + (G_x\,dx + G_y\,dy) \wedge dz$. Expanding, this gives $d\omega = F_x\,dx \wedge dy + G_x\,dx \wedge dz + (G_y - F_z)\,dy \wedge dz$. But we are told that $d\omega = z\,dx \wedge dy + y\,dx \wedge dz$ so

$$\frac{\partial F}{\partial x} = z, \; \frac{\partial G}{\partial x} = y, \; \text{and} \; \frac{\partial G}{\partial y} - \frac{\partial F}{\partial z} = 0.$$

The first equation implies that $F(x,z) = xz + f(z)$ for some differentiable function $f$ of $z$ alone. Similarly, the second equation implies that $G(x,y) = xy + g(y)$ for some differentiable function $g$ of $y$ alone. Using these results together with the third equation we see that $x + g'(y) = x + f'(z)$ or $g'(y) = f'(z)$. This can only be true if their common value is a constant $C$. So if $g'(y) = f'(z) = C$, then $f(z) = Cz + D_1$ and $g(y) = Cy + D_2$ for arbitrary constants $C$, $D_1$, and $D_2$. We conclude that $F(x,z) = xz + Cz + D_1$ and $G(x,y) = xy + Cy + D_2$.

**10.** If $\omega = 2x\,dy \wedge dz - z\,dx \wedge dy$, then $d\omega = 2\,dx \wedge dy \wedge dz - dz \wedge dx \wedge dy = dx \wedge dy \wedge dz$. From Exercise 11 of Section 8.2, $M$ is parametrized as $\mathbf{X}\colon D \to \mathbf{R}^3; \mathbf{X}(u_1, u_2, u_3) = (u_1 \cos u_2, u_1 \sin u_2, u_3)$ where $D = \{(u_1, u_2, u_3) | u_1^2 - 6 \le u_3 \le 4 - u_1^2, 0 \le u_1 \le \sqrt{5}, 0 \le u_2 < 2\pi\}$. If we orient $M$ by the 3-form $\Omega = dx \wedge dy \wedge dz$, then

$$\Omega_{\mathbf{X(u)}}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3}) = \det \begin{bmatrix} \cos u_2 & -u_1 \sin u_2 & 0 \\ \sin u_2 & u_1 \cos u_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = u_1 \ge 0.$$

As before, this is strictly positive when the parametrization is smooth so the parametrization is compatible with the orientation. Therefore, using this orientation,

$$\int_M d\omega = \int_{\mathbf{X}} d\omega = \int_0^{2\pi} \int_0^{\sqrt{5}} \int_{u_1^2-6}^{4-u_1^2} u_1\, du_3\, du_1\, du_2 = 2\pi \int_0^{\sqrt{5}} u_1(10 - 2u_1^2)\, du_1$$

$$= 4\pi \left( \frac{5}{2} u_1^2 - \frac{1}{4} u_1^4 \right) \Big|_0^{\sqrt{5}} = 4\pi \left( \frac{25}{2} - \frac{25}{4} \right) = 25\pi.$$

On the other hand, $\partial M$ is parametrized on the bottom surface as

$$\mathbf{Y}_1 : [0, \sqrt{5}] \times [0, 2\pi) \to \mathbf{R}^3; \quad \mathbf{Y}_1(s_1, s_2) = (s_1 \cos s_2, s_1 \sin s_2, s_1^2 - 6)$$

with tangent vector normal to $\partial M$

$$\mathbf{V}_1 = \frac{(2s_1 \cos s_2, 2s_1 \sin s_2, -1)}{\sqrt{4s_1^2 + 1}}.$$

The boundary $\partial M$ is parametrized on the top surface as

$$\mathbf{Y}_2 : [0, \sqrt{5}] \times [0, 2\pi) \to \mathbf{R}^3; \quad \mathbf{Y}_2(s_1, s_2) = (s_1 \cos s_2, s_1 \sin s_2, 4 - s_1^2)$$

with tangent vector normal to $\partial M$

$$\mathbf{V}_2 = \frac{(2s_1 \cos s_2, 2s_1 \sin s_2, 1)}{\sqrt{4s_1^2 + 1}}.$$

Then we have that the induced orientation on $\partial M$ is given by $\Omega^{\partial M}(\mathbf{a}_1, \mathbf{a}_2) = \Omega(\mathbf{V}, \mathbf{a}_1, \mathbf{a}_2)$. Therefore we see that on the bottom part of $\partial M$

$$\Omega^{\partial M}_{\mathbf{Y}_1(\mathbf{s})}(\mathbf{T}_{s_1}, \mathbf{T}_{s_2}) = \Omega_{\mathbf{X}(s_1, s_2, s_1^2 - 6)}(\mathbf{V}_1, \mathbf{T}_{s_1}, \mathbf{T}_{s_2})$$

$$= \det \begin{bmatrix} \frac{2s_1 \cos s_2}{\sqrt{4s_1^2+1}} & \cos s_2 & -s_1 \sin s_2 \\ \frac{2s_1 \sin s_2}{\sqrt{4s_1^2+1}} & \sin s_2 & s_1 \cos s_2 \\ \frac{-1}{\sqrt{4s_1^2+1}} & 2s_1 & 0 \end{bmatrix}$$

$$= -\frac{4s_1^3 + s_1}{\sqrt{4s_1^2+1}} \le 0.$$

The parametrization $\mathbf{Y}_1$ is incompatible with the induced orientation on $\partial M$. Along the top part of $\partial M$

$$\Omega^{\partial M}_{\mathbf{Y}_2(\mathrm{s})}(\mathbf{T}_{s_1}, \mathbf{T}_{s_2}) = \Omega_{\mathbf{X}(s_1, s_2, 4 - s_1^2)}(\mathbf{V}_2, \mathbf{T}_{s_1}, \mathbf{T}_{s_2})$$

$$= \det \begin{bmatrix} \frac{2s_1 \cos s_2}{\sqrt{4s_1^2+1}} & \cos s_2 & -s_1 \sin s_2 \\ \frac{2s_1 \sin s_2}{\sqrt{4s_1^2+1}} & \sin s_2 & s_1 \cos s_2 \\ \frac{1}{\sqrt{4s_1^2+1}} & -2s_1 & 0 \end{bmatrix}$$

$$= \frac{4s_1^3 + s_1}{\sqrt{4s_1^2+1}} \ge 0.$$

This parametrization is compatible with the induced orientation on $\partial M$.

Therefore we set up our integral (changing signs in the first integrand because of the incompatibility of the parametrization) to obtain the following.

$$\begin{aligned}
\int_{\partial M}\omega &= -\int_{\mathbf{Y}_1}\omega + \int_{\mathbf{Y}_2}\omega \\
&= -\int_0^{2\pi}\int_0^{\sqrt{5}}\left\{2s_1\cos s_2\begin{vmatrix}\sin s_2 & s_1\cos s_2\\ 2s_1 & 0\end{vmatrix} - (s_1^2-6)\begin{vmatrix}\cos s_2 & -s_1\sin s_2\\ \sin s_2 & s_1\cos s_2\end{vmatrix}\right\} ds_1\,ds_2 \\
&\quad + \int_0^{2\pi}\int_0^{\sqrt{5}}\left\{2s_1\cos s_2\begin{vmatrix}\sin s_2 & s_1\cos s_2\\ -2s_1 & 0\end{vmatrix} - (4-s_1^2)\begin{vmatrix}\cos s_2 & -s_1\sin s_2\\ \sin s_2 & s_1\cos s_2\end{vmatrix}\right\} ds_1\,ds_2 \\
&= \int_0^{2\pi}\int_0^{\sqrt{5}}[8s_1^3\cos^2 s_2 + (2s_1^2-10)s_1]\,ds_1\,ds_2 \\
&= \int_0^{2\pi}\left[s_1^4\cos 2s_2 + \frac{3}{2}s_1^4 - 5s_1^2\right]\Bigg|_{s_1=0}^{s_1=\sqrt{5}} ds_2 \\
&= \int_0^{2\pi}[25\cos 2s_2 + 25/2]\,ds_2 = [(25/2)\sin 2s_2 + 25s_2/2]\Big|_0^{2\pi} = 25\pi.
\end{aligned}$$

**11.** One integral is easy. Since $\omega = xy\,dz\wedge dw$ and $\partial M = \{(x,y,z,w)|x=0, 8-2y^2-2z^2-2w^2=0\}$, we see that $x=0$ along $\partial M$ so $\int_{\partial M}\omega = \int_{\partial M} 0 = 0$.

Now $d\omega = d(xy)\wedge dz\wedge dw = x\,dy\wedge dz\wedge dw + y\,dx\wedge dz\wedge dw$. We can orient $M$ any way we wish, so we won't worry about this—we'll choose the orientation to be compatible with the parametrization.

$$\mathbf{X}: D\to\mathbf{R}^4,\quad \mathbf{X}(u_1,u_2,u_3) = (8-2u_1^2-2u_2^2-2u_3^2, u_1, u_2, u_3)$$

where $D = \{(u_1,u_2,u_3)|u_1^2+u_2^2+u_3^2\le 4\}$ (i.e., the solid ball of radius 2). Then

$$\begin{aligned}
\int_M d\omega &= \int_{\mathbf{X}} d\omega = \iiint_B d\omega_{\mathbf{X}(u)}(\mathbf{T}_{u_1},\mathbf{T}_{u_2},\mathbf{T}_{u_3})\,du_1\,du_2\,du_3 \\
&= \iiint_B\left\{(8-2u_1^2-2u_2^2-2u_3^2)\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix} + u_1\begin{vmatrix}-4u_1 & -4u_2 & -4u_3\\ 0&1&0\\0&0&1\end{vmatrix}\right\} du_1\,du_2\,du_3 \\
&= \iiint_B (8-2(u_1^2+u_2^2+u_3^2)-4u_1^2)\,du_1\,du_2\,du_3.
\end{aligned}$$

At this point it is helpful to switch to spherical coordinates. The previous quantity is then

$$\begin{aligned}
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{2}(8-2\rho^2-4\rho^2\sin^2\varphi\cos^2\theta)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \\
&= 8\cdot(\text{volume of } B) - 2\int_0^{2\pi}\int_0^{\pi}\int_0^{2}\rho^4(\sin\varphi+2\sin^3\varphi\cos^2\theta)\,d\rho\,d\varphi\,d\theta \\
&= 8\cdot\left(\frac{4}{3}\pi 2^3\right) - 2\int_0^{2\pi}\int_0^{\pi}\frac{32}{5}(\sin\varphi+2\sin\varphi(1-\cos^2\varphi)\cos^2\theta)\,d\varphi\,d\theta \\
&= \frac{256\pi}{3} - \frac{64}{5}\int_0^{2\pi}\{(-\cos\varphi)\Big|_0^{\pi} + 2\cos^2\theta(-\cos\varphi+(\cos^3\varphi)/3)\Big|_{\varphi=0}^{\pi}\}\,d\theta \\
&= \frac{256\pi}{3} - \frac{64}{5}\int_0^{2\pi}\left\{2+2\cos^2\theta\cdot\left(2-\frac{2}{3}\right)\right\}d\theta = \frac{256\pi}{3} - \frac{128}{5}\int_0^{2\pi}\left(1+\frac{4}{3}\cos^2\theta\right)d\theta \\
&= \frac{256\pi}{3} - \frac{128}{5}\int_0^{2\pi}\left(\frac{5}{3}+\frac{2}{3}\cos 2\theta\right)d\theta \quad \text{(using the half angle formula)} \\
&= \frac{256\pi}{3} - \frac{128}{5}\left(\frac{5}{3}(2\pi)+\frac{1}{3}\sin 2\theta\Big|_0^{2\pi}\right) = \frac{256\pi}{3} - \frac{256\pi}{3} = 0.
\end{aligned}$$

**12. (a)** Using the generalized version of Stokes's theorem (Theorem 3.2), we have

$$\begin{aligned}
\frac{1}{3}\int_{\partial M} x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy &= \frac{1}{3}\int_M d(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy) \\
&= \frac{1}{3}\int_M dx\wedge dy\wedge dz - dy\wedge dx\wedge dz + dz\wedge dx\wedge dy \\
&= \frac{1}{3}\int_M 3\,dx\wedge dy\wedge dz \quad \text{using formula (3) of Section 8.1,} \\
&= \int_M dx\wedge dy\wedge dz = \iiint_M dx\,dy\,dz = \text{volume of } M.
\end{aligned}$$

(See Definition 2.6 and Example 6 of Section 8.2.)

**(b)** This generalizes the result demonstrated in part (a). Notice that the $k$th summand is $(-1)^{k-1}x_k$ multiplied by the $(n-1)$-form which is the wedge product of the $dx_i$'s in order with $dx_k$ missing. In other words, the $k$th summand is

$$(-1)^{k-1}x_k\,dx_1\wedge\cdots\wedge\widehat{dx_k}\wedge\cdots\wedge dx_n$$

where $\widehat{dx_k}$ means that $dx_k$ is omitted. (Make the obvious adjustments to the expression if it is the first or last term that is omitted.) Then

$$d(\text{of the } k\text{th summand}) = (-1)^{k-1}\,dx_k\wedge dx_1\wedge\cdots\wedge\widehat{dx_k}\wedge\cdots\wedge dx_n.$$

Let $\omega$ denote the $(n-1)$-form in the integrand. Then, using the generalized Stokes's theorem,

$$\begin{aligned}
\frac{1}{n}\int_{\partial M}\omega &= \frac{1}{n}\int_M d\omega \\
&= \frac{1}{n}\int_M\left(\sum_{k=1}^{n}(-1)^{k-1}\,dx_k\wedge dx_1\wedge\cdots\wedge\widehat{dx_k}\wedge\cdots\wedge dx_n\right).
\end{aligned}$$

Use formula (3) of Section 8.1 to "move" each $dx_k$ back into the slot from which it has been omitted and collect terms to obtain

$$\frac{1}{n}\int_{\partial M}\omega = \frac{1}{n}\int_M n\,dx_1\wedge\cdots\wedge dx_n = \int\cdots\int_M dx_1\cdots dx_n.$$

It is entirely reasonable to take this last $n$-dimensional integral to represent the $n$-dimensional volume of $M$.

### True/False Exercises for Chapter 8

**1.** True.
**2.** False. (There is a negative sign missing.)
**3.** True.
**4.** False.
**5.** True.
**6.** False. (A negative sign is missing.)
**7.** True.
**8.** False. (There should be no negative sign.)
**9.** True.
**10.** True.
**11.** False. ($\mathbf{X}(1, 1, -1) = \mathbf{X}(1, 1, 1)$, so $\mathbf{X}$ is not one-one on $D$.)
**12.** True. (Both manifolds are the same helicoid.)
**13.** False. (The agreement is only up to sign.)
**14.** True.
**15.** False. (This is only true if $n$ is even.)
**16.** False. (A negative sign is missing.)
**17.** True.
**18.** False. ($d\omega = 0$.)
**19.** True. ($d\omega$ would be an $(n+1)$-form, and there are no nonzero ones on $\mathbf{R}^n$.)
**20.** True. (This is the generalized Stokes's theorem, since $\partial M = \emptyset$.)

### Miscellaneous Exercises for Chapter 8

**1. (a)** First, by definition of the exterior product and derivative

$$\begin{aligned} d(f \wedge g) &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(fg)\, dx_i = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i} \right) dx_i \quad \text{by the product rule,} \\ &= g \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i + f \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\, dx_i \\ &= g \wedge df + f \wedge dg \\ &= df \wedge g + (-1)^0 f \wedge dg. \end{aligned}$$

**(b)** If $k = 0$, then write $\omega = f$ so that

$$\begin{aligned} d(\omega \wedge \eta) &= d(f \wedge \eta) = d\left( \sum f G_{j_1 \dots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l} \right) = \sum d(f G_{j_1 \dots j_l}) \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &= \sum (df \wedge G_{j_1 \dots j_l} + f \wedge dG_{j_1 \dots j_l}) \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \quad \text{from (a),} \\ &= df \wedge \sum G_{j_1 \dots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l} + f \wedge \sum dG_{j_1 \dots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &= df \wedge \eta + (-1)^0 f \wedge d\eta. \end{aligned}$$

**(c)** If $l = 0$, then write $\eta = g$ so that

$$\begin{aligned} d(\omega \wedge \eta) &= d(\omega \wedge g) = d\left(\sum gF_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) \\ &= \sum d(gF_{i_1 \ldots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum (dg \wedge F_{i_1 \ldots i_k} + g \wedge dF_{i_1 \ldots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= dg \wedge \omega + g \wedge d\omega \\ &= (-1)^k \omega \wedge dg + d\omega \wedge g \end{aligned}$$

by part 2 of Proposition 1.4 (recall $dg$ is a 1-form).

**(d)** In general,

$$\begin{aligned} d(\omega \wedge \eta) &= d\left(\sum_{1 \le i_1 < \cdots < i_k \le n} F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge \sum_{1 \le j_1 < \cdots < j_l \le n} G_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}\right) \\ &= d\left(\sum_{\substack{1 \le i_1 < \cdots < i_k \le n \\ 1 \le j_1 < \cdots < j_l \le n}} F_{i_1 \ldots i_k} G_{j_1 \ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}\right) \\ &= \sum d(F_{i_1 \ldots i_k} G_{j_1 \ldots j_l}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \quad \text{so by part (a),} \\ &= \sum (dF_{i_1 \ldots i_k} \wedge G_{j_1 \ldots j_l} + F_{i_1 \ldots i_k} \wedge dG_{j_1 \ldots j_l}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &= \sum dF_{i_1 \ldots i_k} \wedge G_{j_1 \ldots j_l} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &\quad + \sum F_{i_1 \ldots i_k} \wedge dG_{j_1 \ldots j_l} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &= \sum dF_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge G_{j_1 \ldots j_l} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &\quad + \sum F_{i_1 \ldots i_k} \wedge (-1)^k\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dG_{j_1 \ldots j_l} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \end{aligned}$$

since $G_{j_1 \ldots j_l}$ is a 0-form and $dG_{j_1 \ldots j_l}$ is a 1-form,

$$= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

**2. (a)** Define $\mathbf{X} : [0,1] \times [0,1] \times [0,1] \times [0,1] \to \mathbf{R}^5, \mathbf{X}(u_1, u_2, u_3, u_4, u_5) = (u_1, u_2, u_3, u_4, u_1 u_2 u_3 u_4)$. Then $\mathbf{T}_{u_1} = (1, 0, 0, 0, u_2 u_3 u_4), \mathbf{T}_{u_2} = (0, 1, 0, 0, u_1 u_3 u_4), \mathbf{T}_{u_3} = (0, 0, 1, 0, u_1 u_2 u_4)$, and $\mathbf{T}_{u_4} = (0, 0, 0, 1, u_1 u_2 u_3)$. From this we see that

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3}, \mathbf{T}_{u_4}) = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1.$$

**(b)** We can now calculate

$$\begin{aligned} &\int_M dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \\ &\quad = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ u_2 u_3 u_4 & u_1 u_3 u_4 & u_1 u_2 u_4 & u_1 u_2 u_3 \end{bmatrix} du_1\, du_2\, du_3\, du_4 \\ &\quad = \int_0^1 \int_0^1 \int_0^1 \int_0^1 u_1 u_2 u_3\, du_1\, du_2\, du_3\, du_4 = \frac{1}{8}. \end{aligned}$$

**3. (a)** The curve $C$ may be parametrized as $\mathbf{x}(t) = (t, f(t)), a \le t \le b$. Then $\mathbf{x}'(t) = (1, f'(t))$ and this is compatible with the orientation of $C$. By Definition 2.1, we have

$$\int_C \omega = \int_{\mathbf{x}} \omega = \int_a^b \omega_{\mathbf{x}(t)}(\mathbf{x}'(t))\, dt.$$

For $\omega = y\, dx$ this is

$$\int_a^b f(t)\cdot 1\, dt = \int_a^b f(t)\, dt = \text{area under the graph.}$$

**(b)** Parametrize $S$ by

$$\mathbf{X} : [a, b] \times [c, d] \to \mathbf{R}^3; \quad \mathbf{X}(u_1, u_2) = (u_1, u_2, f(u_1, u_2)).$$

The upward unit normal $\mathbf{N}$ is given by

$$\mathbf{N} = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

The parametrization is compatible with the orientation since

$$\mathbf{T}_{u_1} \times \mathbf{T}_{u_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_{u_1} \\ 0 & 1 & f_{u_2} \end{vmatrix} = (-f_{u_1}, -f_{u_2}, 1)$$

is parallel to $\mathbf{N}$ (when $\mathbf{N}$ is expressed in terms of the parametrization). Thus,

$$\int_S \omega = \int_{\mathbf{X}} \omega = \int_c^d \int_a^b \omega_{\mathbf{X}(u_1,u_2)}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2})\, du_1\, du_2.$$

For $\omega = z\, dx \wedge dy$, this is

$$\int_c^d \int_a^b f(u_1, u_2) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} du_1\, du_2 = \int_c^d \int_a^b f(u_1, u_2)\, du_1\, du_2 = \text{area under the graph.}$$

**(c)** Parametrize $M$ using

$$\mathbf{X} : D \to \mathbf{R}^n, \quad \mathbf{X}(u_1, \ldots, u_{n-1}) = (u_1, \ldots, u_{n-1}, f(u_1, \ldots, u_{n-1})).$$

Then, depending on how $M$ is oriented,

$$\begin{aligned} \int_M \omega &= \pm \int_{\mathbf{X}} \omega = \pm \int \cdots \int_D \omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \ldots, \mathbf{T}_{u_{n-1}})\, du_1 \cdots du_{n-1} \\ &= \pm \int \cdots \int_D f(u_1, \ldots, u_{n-1}) \det \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} du_1 \cdots du_{n-1} \\ &= \pm \int \cdots \int_D f(u_1, \ldots, u_{n-1})\, du_1 \cdots du_{n-1} = \pm(n\text{-dimensional volume under the graph}). \end{aligned}$$

If you orient $M$ with the unit normal

$$\mathbf{N} = (-1)^n \frac{(f_{x_1}, \ldots, f_{x_{n-1}}, -1)}{\sqrt{(f_{x_1})^2 + \cdots + (f_{x_{n-1}})^2 + 1}}$$

we can guarantee a $+$ sign above.

**4. (a)** Define a parametrization

$$\mathbf{X} : [0, 3] \times [0, 2\pi) \to \mathbf{R}^3; \quad \mathbf{X}(u_1, u_2) = (\cos u_2, u_1, \sin u_2).$$

Then we may define $\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{a}, \mathbf{b}) = \det[\mathbf{N}\ \mathbf{a}\ \mathbf{b}]$. Note that $\mathbf{X}$ is compatible with this orientation as $\mathbf{T}_{u_1} = (0, 1, 0)$ and $\mathbf{T}_{u_2} = (-\sin u_2, 0, \cos u_2)$ so that

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}) = \det \begin{bmatrix} \cos u_2 & 0 & -\sin u_2 \\ 0 & 1 & 0 \\ \sin u_2 & 0 & \cos u_2 \end{bmatrix} = 1 > 0.$$

(Note that the first column is the normal $\mathbf{N}$ in terms of the parametrization.)

**(b)** The boundary $\partial M$ consists of two disjoint pieces. The left piece is $\{(x,0,z)|x^2+z^2=1\}$, parametrized by $\mathbf{Y}_1 : [0,2\pi) \to \mathbf{R}^3, \mathbf{Y}_1(t) = (\cos t, 0, \sin t)$. The right piece is $\{(x,3,z)|x^2+z^2=1\}$, parametrized by $\mathbf{Y}_2 : [0,2\pi) \to \mathbf{R}^3, \mathbf{Y}_2(t) = (\cos t, 3, \sin t)$.

**(c)** We must first determine $\mathbf{V}$, a unit vector tangent to $M$, normal to $\partial M$, and pointing away from $M$. If you think about the boundary pieces we looked at in part (b), a vector corresponding to the left side is $\mathbf{V}_1 = (0,-1,0)$ and corresponding to the right side is $\mathbf{V}_2 = (0,1,0)$. Then, along the left circle of $\partial M$,

$$\Omega^{\partial M}_{\mathbf{Y}_1(t)}(\mathbf{a}) = \Omega_{\mathbf{X}(0,t)}(\mathbf{V}_1, \mathbf{a})$$

and along the right circle of $\partial M$,

$$\Omega^{\partial M}_{\mathbf{Y}_2(t)}(\mathbf{a}) = \Omega_{\mathbf{X}(3,t)}(\mathbf{V}_2, \mathbf{a}).$$

Note that

$$\Omega^{\partial M}_{\mathbf{Y}_1(t)}(\mathbf{T}_t) = \det \begin{bmatrix} \cos t & 0 & -\sin t \\ 0 & -1 & 0 \\ \sin t & 0 & \cos t \end{bmatrix} = -1.$$

So the parametrization $\mathbf{Y}_1$ is incompatible with $\Omega^{\partial M}$. However,

$$\Omega^{\partial M}_{\mathbf{Y}_2(t)}(\mathbf{T}_t) = \det \begin{bmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{bmatrix} = 1.$$

So the parametrization $\mathbf{Y}_2$ is compatible with $\Omega^{\partial M}$.

**(d)** If $\omega = z\,dx + (x+y+z)\,dy - x\,dz$, we have

$$d\omega = dz \wedge dx + (dx + dy + dz) \wedge dy - dx \wedge dz = dx \wedge dy - dy \wedge dz - 2\,dx \wedge dz.$$

Then using the orientation $\Omega$ and the parametrization $\mathbf{X}$ from part (a), we have

$$\begin{aligned} \int_M d\omega = \int_{\mathbf{X}} d\omega &= \int_0^{2\pi} \int_0^3 \left( \begin{vmatrix} 0 & -\sin u_2 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 0 & \cos u_2 \end{vmatrix} - 2 \begin{vmatrix} 0 & -\sin u_2 \\ 0 & \cos u_2 \end{vmatrix} \right) du_1\,du_2 \\ &= \int_0^{2\pi} \int_0^3 (\sin u_2 - \cos u_2)\,du_1\,du_2 = 3(-\cos u_2 - \sin u_2)\big|_0^{2\pi} = 0. \end{aligned}$$

On the other hand, using the parametrizations $\mathbf{Y}_1$ and $\mathbf{Y}_2$ for $\partial M$ in parts (b) and (c), we have (after reversing the sign for the left piece because of the incompatibility with $\Omega^{\partial M}$)

$$\begin{aligned} \int_{\partial M} \omega &= -\int_{\mathbf{Y}_1} \omega + \int_{\mathbf{Y}_2} \omega \\ &= -\int_0^{2\pi} [\sin t(-\sin t) + (\cos t + \sin t)\cdot 0 - \cos t(\cos t)]\,dt \\ &\quad + \int_0^{2\pi} [\sin t(-\sin t) + (\cos t + 3 + \sin t)\cdot 0 - \cos t(\cos t)]\,dt = 0. \end{aligned}$$

**5.** If $S^4$ is the unit 4-sphere in $\mathbf{R}^5$, then let $B$ denote the 5-dimensional unit ball

$$B = \{x_1, x_2, x_3, x_4, x_5)|x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \le 1\}.$$

Note that $\partial B = S^4$. Then using the generalized Stokes's theorem, we have

$$\int_{S^4} \omega = \int_B d\omega.$$

For $\omega = x_3\,dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_5 + x_4\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5$ we have $d\omega = dx_3 \wedge dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_5 + dx_4 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5 = dx_1 \wedge \cdots \wedge dx_5 - dx_1 \wedge \cdots \wedge dx_5 = 0$. Hence $\int_{S^4} \omega = \int_B 0 = 0$.

**6. (a)** Let $\omega = f$. Then $df = \sum_i \frac{\partial f}{\partial x_i}\, dx_i$ and

$$d(df) = \sum_i d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i = \sum_i \left(\sum_j \frac{\partial^2 f}{\partial x_j \partial x_i} \wedge dx_j\right) \wedge dx_i$$
$$= \sum_{i<j} \frac{\partial^2 f}{\partial x_j \partial x_i}\, dx_j \wedge dx_i + \sum_{i>j} \frac{\partial^2 f}{\partial x_j \partial x_i}\, dx_j \wedge dx_i,$$

since the terms where $i = j$ contain $dx_i \wedge dx_i = 0$. By exchanging the roles of $i$ and $j$ in the second sum, we find

$$d(df) = \sum_{i<j} \frac{\partial^2 f}{\partial x_j \partial x_i}\, dx_j \wedge dx_i + \sum_{i<j} \frac{\partial^2 f}{\partial x_i \partial x_j}\, dx_i \wedge dx_j$$
$$= \sum_{i<j} \left(-\frac{\partial^2 f}{\partial x_j \partial x_i} + \frac{\partial^2 f}{\partial x_j \partial x_i}\right) dx_i \wedge dx_j = 0$$

since the mixed partials are equal because $f$ is of class $C^2$.

**(b)** Now

$$d(d\omega) = d\left(d\left(\sum_{1 \le i_1 < \cdots < i_k \le n} F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right)\right)$$
$$= d\left(\sum_{1 \le i_1 < \cdots < i_k \le n} dF_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right)$$
$$= \sum_{1 \le i_1 < \cdots < i_k \le n} [d(dF_{i_1 \ldots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$
$$+ (-1)^1\, dF_{i_1 \ldots i_k} \wedge d(dx_{i_1} \wedge \cdots \wedge dx_{i_k})] \quad \text{from Exercise 1,}$$
$$= -\sum_{1 \le i_1 < \cdots < i_k \le n} dF_{i_1 \ldots i_k} \wedge d(dx_{i_1} \wedge \cdots \wedge dx_{i_k})$$

since $d(dF_{i_1 \ldots i_k}) = 0$ from part (a). But

$$d(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = d(1\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = d(1) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0.$$

Hence $d(d\omega) = 0$, as desired.

**7. (a)** If $\omega$ is a 0-form, write $\omega = f$. Then, using the first row of the chart, the 1-form $d\omega$ corresponds to the vector field $\nabla f$. Hence, from the second row of the chart, $d(d\omega)$ is the 2-form that corresponds to $\nabla \times \nabla f$. Thus $d(d\omega) = 0$ "translates" to the statement $\nabla \times (\nabla f) = \mathbf{0}$.

**(b)** If $\omega$ is a 1-form, it corresponds to the vector field $\mathbf{F}$ and, using the second row of the chart, $d\omega$ is the 2-form that corresponds to $\nabla \times \mathbf{F}$, another vector field. Then, using the third row of the chart, $d(d\omega)$ is the 3-form that corresponds to $\nabla \cdot (\nabla \times \mathbf{F})$. Hence, $d(d\omega) = 0$ "translates" to the statement that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

**8. (a)** The outward unit normal $\mathbf{N} = (x, y, z)$ gives orientation form $\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{a}_1, \mathbf{a}_2) = \det[\mathbf{N}\ \ \mathbf{a}_1\ \mathbf{a}_2]$ where $\mathbf{X}$ is a parametrization of $S$. For a specific parametrization we can use

$$\mathbf{X} : [0, \pi] \times [0, 2\pi) \to \mathbf{R}^3; \quad \mathbf{X}(u_1, u_2) = (\sin u_1 \cos u_2, \sin u_1 \sin u_2, \cos u_1).$$

Then $\mathbf{T}_{u_1} = (\cos u_1 \cos u_2, \cos u_1 \sin u_2, -\sin u_1)$ and $\mathbf{T}_{u_2} = (-\sin u_1 \sin u_2, \sin u_1 \cos u_2, 0)$, so that

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}) = \det \begin{bmatrix} \sin u_1 \cos u_2 & \cos u_1 \cos u_2 & -\sin u_1 \sin u_2 \\ \sin u_1 \sin u_2 & \cos u_1 \sin u_2 & \sin u_1 \cos u_2 \\ \cos u_1 & -\sin u_1 & 0 \end{bmatrix} = \sin u_1 \ge 0.$$

In fact, this quantity is strictly greater than 0 when the parametrization is smooth and so the parametrization is compatible with the orientation.

Next we note that on $S$ we have $\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$ as the denominators in $\omega$ are all 1 on $S$. Therefore,

$$\begin{aligned}
\int_S \omega &= \int_{\mathbf{X}} \omega \\
&= \int_0^{2\pi}\int_0^{\pi} \left\{ \sin u_1 \cos u_2 \begin{vmatrix} \cos u_1 \sin u_2 & \sin u_1 \cos u_2 \\ -\sin u_1 & 0 \end{vmatrix} \right. \\
&\quad + \sin u_1 \sin u_2 \begin{vmatrix} -\sin u_1 & 0 \\ \cos u_1 \cos u_2 & -\sin u_1 \sin u_2 \end{vmatrix} \\
&\quad \left. + \cos u_1 \begin{vmatrix} \cos u_1 \cos u_2 & -\sin u_1 \sin u_2 \\ \cos u_1 \sin u_2 & \sin u_1 \cos u_2 \end{vmatrix} \right\} du_1\, du_2 \\
&= \int_0^{2\pi}\int_0^{\pi} (\sin^3 u_1 + \cos^2 u_1 \sin u_1)\, du_1\, du_2 = \int_0^{2\pi}\int_0^{\pi} \sin u_1\, du_1\, du_2 \\
&= 2\pi(-\cos u_1)\big|_0^{\pi} = 4\pi.
\end{aligned}$$

**(b)** For $\omega$ as given we calculate

$$d\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{(y^2+z^2-2x^2)\,dx - 3xy\,dy - 3xz\,dz}{(x^2+y^2+z^2)^{5/2}}$$

$$d\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{-3xy\,dx - (x^2+z^2-2y^2)\,dy - 3yz\,dz}{(x^2+y^2+z^2)^{5/2}}$$

$$d\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{-3xz\,dx - 3yz\,dy - (x^2+y^2-2z^2)dz}{(x^2+y^2+z^2)^{5/2}}$$

Hence,

$$\begin{aligned}
d\omega = \frac{1}{(x^2+y^2+z^2)^{5/2}} &\left[(y^2+z^2-2x^2)\,dx \wedge dy \wedge dz\right. \\
&+ (x^2-2y^2+z^2)\,dy \wedge dz \wedge dx \\
&\left.+(x^2+y^2-2z^2)\,dz \wedge dx \wedge dy\right]
\end{aligned}$$

This is identically equal to 0 wherever it is defined.

**(c)** Since $M$ does not include the origin, we have $\int_M d\omega = \int_M 0 = 0$ from part (b).

$\partial M$ consists of two pieces. The outer piece $S_1$ is the unit sphere $x^2+y^2+z^2=1$, oriented by the outward unit normal $\mathbf{n}_1 = (x, y, z)$. The inner piece is the sphere $x^2+y^2+z^2=a^2$ of radius $a$, oriented by inward unit normal $\mathbf{n}_2 = (-x, -y, -z)/a$. Then, using Proposition 2.4, we have

$$\int_{\partial M} \omega = \iint_{\partial M} \mathbf{F} \cdot d\mathbf{S} \text{ where } \mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}.$$

In the following calculation we will use the fact that $x^2+y^2+z^2$ is 1 on $S_1$ and is $a^2$ on $S_2$.

$$\begin{aligned}
\int_{\partial M} \omega &= \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\, dS \\
&= \iint_{S_1} 1\, dS + \iint_{S_2} \frac{1}{a^2}\, dS \\
&= (1)(\text{surface area of } S_1) - \frac{1}{a^2}(\text{surface area of } S_2) \\
&= 4\pi - \frac{1}{a^2}(4\pi a^2) = 0.
\end{aligned}$$

This verifies Theorem 3.2.

**(d)** No—since $\omega$ is not defined at the origin, Theorem 3.2 does not apply.

**(e)** Let $M$ be the 3-manifold bounded on the outside by $S$, oriented with the outward normal, and on the inside by $S_\epsilon$, oriented by the inward normal. Then $\mathbf{0} \notin M$, so we have

$$0 = \int_M d\omega = \int_{\partial M} \omega = \int_S \omega + \int_{S_\epsilon} \omega = \int_S \omega - 4\pi.$$

The last equality follows from part (c). The conclusion is that $\int_S \omega = 4\pi$.

**9.** Because $\partial M = \emptyset$, the note following Theorem 3.2 advises us to take $\int_{\partial M} \omega \wedge \eta$ to be 0 in the equation $\int_{\partial M} \omega \wedge \eta = \int_M d(\omega \wedge \eta)$. Now substitute the results of Exercise 1 to get

$$0 = \int_M d(\omega \wedge \eta) = \int_M d\omega \wedge \eta + (-1)^k \omega \wedge d\eta = \int_M d\omega \wedge \eta + (-1)^k \int_M \omega \wedge d\eta.$$

Pull this last piece to the other side to obtain the result

$$(-1)^{k+1} \int_M \omega \wedge d\eta = \int_M d\omega \wedge \eta.$$

**10.** By the generalized Stokes's theorem,

$$\begin{aligned} \int_{\partial M} f\omega &= \int_M d(f\omega) \\ &= \int_M (df \wedge \omega + f \wedge d\omega) \quad \text{by the result of Exercise 1,} \\ &= \int_M (df \wedge \omega + f\, d\omega). \end{aligned}$$

Hence

$$\int_M f\, d\omega = \int_{\partial M} f\omega - \int_M df \wedge \omega.$$