7.2 | Sum and Difference Identities

Learning Objectives

In this section, you will:

7.2.1 Use sum and difference formulas for cosine.
7.2.2 Use sum and difference formulas for sine.
7.2.3 Use sum and difference formulas for tangent.
7.2.4 Use sum and difference formulas for cofunctions.
7.2.5 Use sum and difference formulas to verify identities.

Figure 7.6 Mount McKinley, in Denali National Park, Alaska, rises 20,237 feet (6,168 m) above sea level. It is the highest peak in North America. (credit: Daniel A. Leifheit, Flickr)

How can the height of a mountain be measured? What about the distance from Earth to the sun? Like many seemingly impossible problems, we rely on mathematical formulas to find the answers. The trigonometric identities, commonly used in mathematical proofs, have had real-world applications for centuries, including their use in calculating long distances.

The trigonometric identities we will examine in this section can be traced to a Persian astronomer who lived around 950 AD, but the ancient Greeks discovered these same formulas much earlier and stated them in terms of chords. These are special equations or postulates, true for all values input to the equations, and with innumerable applications.

In this section, we will learn techniques that will enable us to solve problems such as the ones presented above. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term *formula* is used synonymously with the word *identity*.

Using the Sum and Difference Formulas for Cosine

Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. We can use the special angles, which we can review in the unit circle shown in **Figure 7.7**.

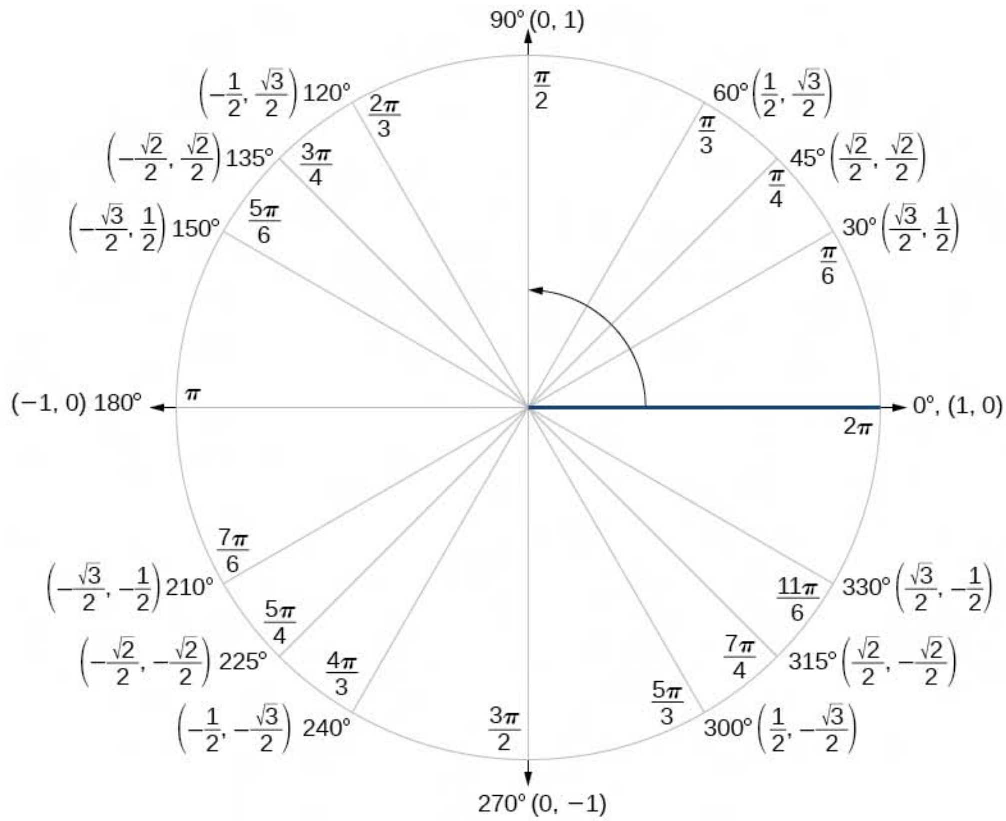

Figure 7.7 The Unit Circle

We will begin with the sum and difference formulas for cosine, so that we can find the cosine of a given angle if we can break it up into the sum or difference of two of the special angles. See **Table 7.5**.

Sum formula for cosine	$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
Difference formula for cosine	$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Table 7.5

First, we will prove the difference formula for cosines. Let's consider two points on the unit circle. See **Figure 7.8**. Point P is at an angle α from the positive x-axis with coordinates $(\cos\alpha, \sin\alpha)$ and point Q is at an angle of β from the positive x-axis with coordinates $(\cos\beta, \sin\beta)$. Note the measure of angle POQ is $\alpha - \beta$.

Label two more points: A at an angle of $(\alpha - \beta)$ from the positive x-axis with coordinates $(\cos(\alpha - \beta), \sin(\alpha - \beta))$; and point B with coordinates $(1, 0)$. Triangle POQ is a rotation of triangle AOB and thus the distance from P to Q is the same as the distance from A to B.

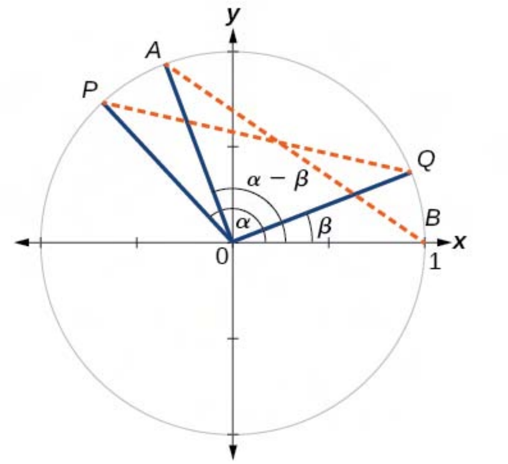

Figure 7.8

We can find the distance from P to Q using the distance formula.

$$\begin{aligned} d_{PQ} &= \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} \\ &= \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta} \end{aligned}$$

Then we apply the Pythagorean identity and simplify.

$$\begin{aligned} &= \sqrt{(\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \\ &= \sqrt{1 + 1 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \\ &= \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \end{aligned}$$

Similarly, using the distance formula we can find the distance from A to B.

$$\begin{aligned} d_{AB} &= \sqrt{(\cos(\alpha - \beta) - 1)^2 + (\sin(\alpha - \beta) - 0)^2} \\ &= \sqrt{\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)} \end{aligned}$$

Applying the Pythagorean identity and simplifying we get:

$$\begin{aligned} &= \sqrt{(\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)) - 2\cos(\alpha - \beta) + 1} \\ &= \sqrt{1 - 2\cos(\alpha - \beta) + 1} \\ &= \sqrt{2 - 2\cos(\alpha - \beta)} \end{aligned}$$

Because the two distances are the same, we set them equal to each other and simplify.

$$\begin{aligned} \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} &= \sqrt{2 - 2\cos(\alpha - \beta)} \\ 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= 2 - 2\cos(\alpha - \beta) \end{aligned}$$

Finally we subtract 2 from both sides and divide both sides by -2.

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$$

Thus, we have the difference formula for cosine. We can use similar methods to derive the cosine of the sum of two angles.

Sum and Difference Formulas for Cosine

These formulas can be used to calculate the cosine of sums and differences of angles.

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{7.18}$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad (7.19)$$

How To: Given two angles, find the cosine of the difference between the angles.

1. Write the difference formula for cosine.
2. Substitute the values of the given angles into the formula.
3. Simplify.

Example 7.11

Finding the Exact Value Using the Formula for the Cosine of the Difference of Two Angles

Using the formula for the cosine of the difference of two angles, find the exact value of $\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right)$.

Solution

Use the formula for the cosine of the difference of two angles. We have

$$\begin{aligned}
\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right) &= \cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\
&= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
&= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \frac{-\sqrt{6} - \sqrt{2}}{4}
\end{aligned}$$

Try It 7.6 Find the exact value of $\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$.

Example 7.12

Finding the Exact Value Using the Formula for the Sum of Two Angles for Cosine

Find the exact value of $\cos(75^\circ)$.

Solution

As $75^\circ = 45^\circ + 30^\circ$, we can evaluate $\cos(75^\circ)$ as $\cos(45^\circ + 30^\circ)$. Thus,

$$\begin{aligned}\cos(45° + 30°) &= \cos(45°)\cos(30°) - \sin(45°)\sin(30°)\\ &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

Try It **7.7** Find the exact value of $\cos(105°)$.

Using the Sum and Difference Formulas for Sine

The sum and difference formulas for sine can be derived in the same manner as those for cosine, and they resemble the cosine formulas.

Sum and Difference Formulas for Sine

These formulas can be used to calculate the sines of sums and differences of angles.

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad \textbf{(7.20)}$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \quad \textbf{(7.21)}$$

How To: **Given two angles, find the sine of the difference between the angles.**

1. Write the difference formula for sine.
2. Substitute the given angles into the formula.
3. Simplify.

Example 7.13

Using Sum and Difference Identities to Evaluate the Difference of Angles

Use the sum and difference identities to evaluate the difference of the angles and show that part *a* equals part *b*.

a. $\sin(45° - 30°)$

b. $\sin(135° - 120°)$

Solution

a. Let's begin by writing the formula and substitute the given angles.

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\sin(45° - 30°) = \sin(45°)\cos(30°) - \cos(45°)\sin(30°)$$

Next, we need to find the values of the trigonometric expressions.

$$\sin(45°) = \frac{\sqrt{2}}{2},\ \cos(30°) = \frac{\sqrt{3}}{2},\ \cos(45°) = \frac{\sqrt{2}}{2},\ \sin(30°) = \frac{1}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(45° - 30°) &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

b. Again, we write the formula and substitute the given angles.

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\sin(135° - 120°) = \sin(135°)\cos(120°) - \cos(135°)\sin(120°)$$

Next, we find the values of the trigonometric expressions.

$$\sin(135°) = \frac{\sqrt{2}}{2},\ \cos(120°) = -\frac{1}{2},\ \cos(135°) = \frac{\sqrt{2}}{2},\ \sin(120°) = \frac{\sqrt{3}}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(135° - 120°) &= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{-\sqrt{2} + \sqrt{6}}{4} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\sin(135° - 120°) &= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{-\sqrt{2} + \sqrt{6}}{4} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

Example 7.14

Finding the Exact Value of an Expression Involving an Inverse Trigonometric Function

Find the exact value of $\sin\left(\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{3}{5}\right)$.

Solution

The pattern displayed in this problem is $\sin(\alpha + \beta)$. Let $\alpha = \cos^{-1}\frac{1}{2}$ and $\beta = \sin^{-1}\frac{3}{5}$. Then we can write

$$\begin{aligned}\cos\alpha &= \frac{1}{2},\ 0 \le \alpha \le \pi \\ \sin\beta &= \frac{3}{5},\ -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}\end{aligned}$$

We will use the Pythagorean identities to find $\sin\alpha$ and $\cos\beta$.

$$\begin{aligned}\sin\alpha &= \sqrt{1-\cos^2\alpha}\\ &= \sqrt{1-\frac{1}{4}}\\ &= \sqrt{\frac{3}{4}}\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

$$\begin{aligned}\cos\beta &= \sqrt{1-\sin^2\beta}\\ &= \sqrt{1-\frac{9}{25}}\\ &= \sqrt{\frac{16}{25}}\\ &= \frac{4}{5}\end{aligned}$$

Using the sum formula for sine,

$$\begin{aligned}\sin\left(\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{3}{5}\right) &= \sin(\alpha+\beta)\\ &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ &= \frac{\sqrt{3}}{2}\cdot\frac{4}{5}+\frac{1}{2}\cdot\frac{3}{5}\\ &= \frac{4\sqrt{3}+3}{10}\end{aligned}$$

Using the Sum and Difference Formulas for Tangent

Finding exact values for the tangent of the sum or difference of two angles is a little more complicated, but again, it is a matter of recognizing the pattern.

Finding the sum of two angles formula for tangent involves taking quotient of the sum formulas for sine and cosine and simplifying. Recall, $\tan x = \frac{\sin x}{\cos x},\ \cos x \neq 0$.

Let's derive the sum formula for tangent.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}\\ &= \frac{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} && \text{Divide the numerator and denominator by } \cos\alpha\cos\beta\\ &= \frac{\frac{\sin\alpha\,\cancel{\cos\beta}}{\cos\alpha\,\cancel{\cos\beta}}+\frac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\frac{\cancel{\cos\alpha}\,\cancel{\cos\beta}}{\cancel{\cos\alpha}\,\cancel{\cos\beta}}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}{1-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\end{aligned}$$

We can derive the difference formula for tangent in a similar way.

Sum and Difference Formulas for Tangent

The sum and difference formulas for tangent are:

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{7.22}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \tag{7.23}$$

How To:

Given two angles, find the tangent of the sum of the angles.

1. Write the sum formula for tangent.
2. Substitute the given angles into the formula.
3. Simplify.

Example 7.15

Finding the Exact Value of an Expression Involving Tangent

Find the exact value of $\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$.

Solution

Let's first write the sum formula for tangent and substitute the given angles into the formula.

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \frac{\tan\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \left(\tan\left(\frac{\pi}{6}\right)\right)\left(\tan\left(\frac{\pi}{4}\right)\right)}$$

Next, we determine the individual tangents within the formula:

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}},\ \tan\left(\frac{\pi}{4}\right) = 1$$

So we have

$$\begin{aligned}\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) &= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \left(\frac{1}{\sqrt{3}}\right)(1)} \\ &= \frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} \\ &= \frac{1+\sqrt{3}}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}-1}\right) \\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\end{aligned}$$

Try It

7.8 Find the exact value of $\tan\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$.

Example 7.16

Finding Multiple Sums and Differences of Angles

Given $\sin\alpha = \frac{3}{5}, 0 < \alpha < \frac{\pi}{2}, \cos\beta = -\frac{5}{13}, \pi < \beta < \frac{3\pi}{2},$ find

a. $\sin(\alpha + \beta)$

b. $\cos(\alpha + \beta)$

c. $\tan(\alpha + \beta)$

d. $\tan(\alpha - \beta)$

Solution

We can use the sum and difference formulas to identify the sum or difference of angles when the ratio of sine, cosine, or tangent is provided for each of the individual angles. To do so, we construct what is called a reference triangle to help find each component of the sum and difference formulas.

a. To find $\sin(\alpha + \beta)$, we begin with $\sin\alpha = \frac{3}{5}$ and $0 < \alpha < \frac{\pi}{2}$. The side opposite α has length 3, the hypotenuse has length 5, and α is in the first quadrant. See **Figure 7.9**. Using the Pythagorean Theorem, we can find the length of side a :

$$\begin{aligned} a^2 + 3^2 &= 5^2 \\ a^2 &= 16 \\ a &= 4 \end{aligned}$$

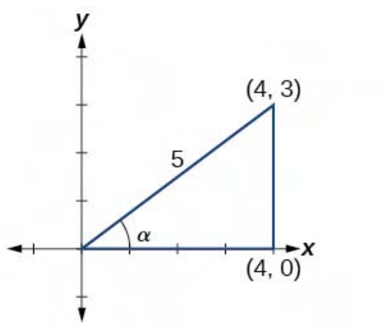

Figure 7.9

Since $\cos\beta = -\frac{5}{13}$ and $\pi < \beta < \frac{3\pi}{2}$, the side adjacent to β is -5, the hypotenuse is 13, and β is in the third quadrant. See **Figure 7.10**. Again, using the Pythagorean Theorem, we have

$$\begin{aligned} (-5)^2 + a^2 &= 13^2 \\ 25 + a^2 &= 169 \\ a^2 &= 144 \\ a &= \pm 12 \end{aligned}$$

Since β is in the third quadrant, $a = -12$.

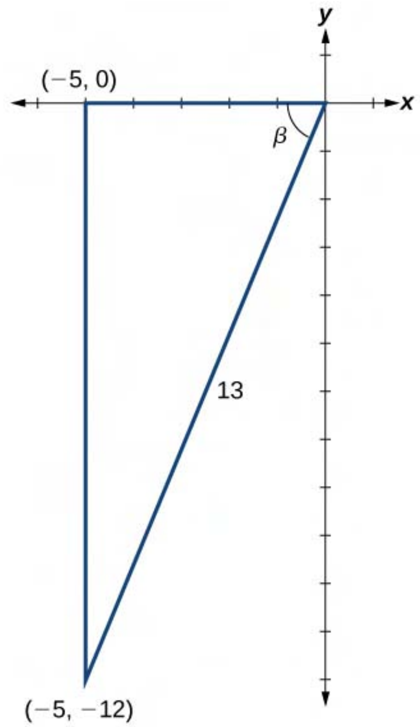

Figure 7.10

The next step is finding the cosine of α and the sine of β. The cosine of α is the adjacent side over the hypotenuse. We can find it from the triangle in Figure 7.10: $\cos\alpha = \frac{4}{5}$. We can also find the sine of β from the triangle in Figure 7.10, as opposite side over the hypotenuse: $\sin\beta = -\frac{12}{13}$. Now we are ready to evaluate $\sin(\alpha + \beta)$.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) \\ &= -\frac{15}{65} - \frac{48}{65} \\ &= -\frac{63}{65}\end{aligned}$$

b. We can find $\cos(\alpha + \beta)$ in a similar manner. We substitute the values according to the formula.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) \\ &= -\frac{20}{65} + \frac{36}{65} \\ &= \frac{16}{65}\end{aligned}$$

c. For $\tan(\alpha + \beta)$, if $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$, then

$$\tan\alpha = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

If $\sin\beta = -\frac{12}{13}$ and $\cos\beta = -\frac{5}{13}$, then

$$\tan\beta = \frac{\frac{-12}{13}}{\frac{-5}{13}} = \frac{12}{5}$$

Then,

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ &= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4}\left(\frac{12}{5}\right)} \\ &= \frac{\frac{63}{20}}{-\frac{16}{20}} \\ &= -\frac{63}{16}\end{aligned}$$

d. To find $\tan(\alpha-\beta)$, we have the values we need. We can substitute them in and evaluate.

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \\ &= \frac{\frac{3}{4} - \frac{12}{5}}{1 + \frac{3}{4}\left(\frac{12}{5}\right)} \\ &= \frac{-\frac{33}{20}}{\frac{56}{20}} \\ &= -\frac{33}{56}\end{aligned}$$

Analysis

A common mistake when addressing problems such as this one is that we may be tempted to think that α and β are angles in the same triangle, which of course, they are not. Also note that

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$

Using Sum and Difference Formulas for Cofunctions

Now that we can find the sine, cosine, and tangent functions for the sums and differences of angles, we can use them to do the same for their cofunctions. You may recall from **Right Triangle Trigonometry** that, if the sum of two positive angles is $\frac{\pi}{2}$, those two angles are complements, and the sum of the two acute angles in a right triangle is $\frac{\pi}{2}$, so they are also complements. In **Figure 7.11**, notice that if one of the acute angles is labeled as θ, then the other acute angle must be labeled $\left(\frac{\pi}{2} - \theta\right)$.

Notice also that $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$: opposite over hypotenuse. Thus, when two angles are complimentary, we can say that the sine of θ equals the cofunction of the complement of θ. Similarly, tangent and cotangent are cofunctions, and secant and cosecant are cofunctions.

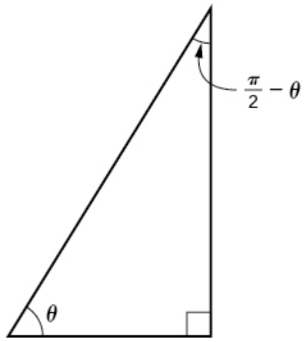

Figure 7.11

From these relationships, the cofunction identities are formed.

Cofunction Identities

The cofunction identities are summarized in Table 7.6.

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$
$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$
$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

Table 7.6

Notice that the formulas in the table may also justified algebraically using the sum and difference formulas. For example, using

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

we can write

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta \\ &= (0)\cos\theta + (1)\sin\theta \\ &= \sin\theta\end{aligned}$$

Example 7.17

Finding a Cofunction with the Same Value as the Given Expression

Write $\tan\frac{\pi}{9}$ in terms of its cofunction.

Solution

The cofunction of $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$. Thus,

$$\begin{aligned}\tan\left(\frac{\pi}{9}\right) &= \cot\left(\frac{\pi}{2} - \frac{\pi}{9}\right)\\ &= \cot\left(\frac{9\pi}{18} - \frac{2\pi}{18}\right)\\ &= \cot\left(\frac{7\pi}{18}\right)\end{aligned}$$

Try It **7.9** Write $\sin\frac{\pi}{7}$ in terms of its cofunction.

Using the Sum and Difference Formulas to Verify Identities

Verifying an identity means demonstrating that the equation holds for all values of the variable. It helps to be very familiar with the identities or to have a list of them accessible while working the problems. Reviewing the general rules from **Solving Trigonometric Equations with Identities** may help simplify the process of verifying an identity.

How To: **Given an identity, verify using sum and difference formulas.**

1. Begin with the expression on the side of the equal sign that appears most complex. Rewrite that expression until it matches the other side of the equal sign. Occasionally, we might have to alter both sides, but working on only one side is the most efficient.
2. Look for opportunities to use the sum and difference formulas.
3. Rewrite sums or differences of quotients as single quotients.
4. If the process becomes cumbersome, rewrite the expression in terms of sines and cosines.

Example 7.18

Verifying an Identity Involving Sine

Verify the identity $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$.

Solution

We see that the left side of the equation includes the sines of the sum and the difference of angles.

$$\begin{aligned}\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ \sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\end{aligned}$$

We can rewrite each using the sum and difference formulas.

$$\begin{aligned}\sin(\alpha + \beta) + \sin(\alpha - \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta\\ &= 2\sin\alpha\cos\beta\end{aligned}$$

We see that the identity is verified.

Example 7.19

Verifying an Identity Involving Tangent

Verify the following identity.

$$\frac{\sin(\alpha - \beta)}{\cos\alpha\cos\beta} = \tan\alpha - \tan\beta$$

Solution

We can begin by rewriting the numerator on the left side of the equation.

$$\begin{aligned}
\frac{\sin(\alpha - \beta)}{\cos\alpha\cos\beta} &= \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta} \\
&= \frac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} - \frac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta} && \text{Rewrite using a common denominator.} \\
&= \frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta} && \text{Cancel.} \\
&= \tan\alpha - \tan\beta && \text{Rewrite in terms of tangent.}
\end{aligned}$$

We see that the identity is verified. In many cases, verifying tangent identities can successfully be accomplished by writing the tangent in terms of sine and cosine.

Try It

7.10 Verify the identity: $\tan(\pi - \theta) = -\tan\theta.$

Example 7.20

Using Sum and Difference Formulas to Solve an Application Problem

Let L_1 and L_2 denote two non-vertical intersecting lines, and let θ denote the acute angle between L_1 and L_2. See **Figure 7.12**. Show that

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2 respectively. (**Hint:** Use the fact that $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$.)

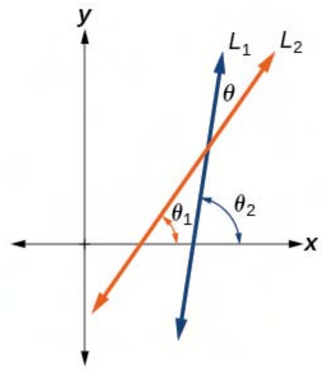

Figure 7.12

Solution

Using the difference formula for tangent, this problem does not seem as daunting as it might.

$$\begin{aligned}\tan\theta &= \tan(\theta_2 - \theta_1)\\ &= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2}\\ &= \frac{m_2 - m_1}{1 + m_1 m_2}\end{aligned}$$

Example 7.21

Investigating a Guy-wire Problem

For a climbing wall, a guy-wire R is attached 47 feet high on a vertical pole. Added support is provided by another guy-wire S attached 40 feet above ground on the same pole. If the wires are attached to the ground 50 feet from the pole, find the angle α between the wires. See Figure 7.13.

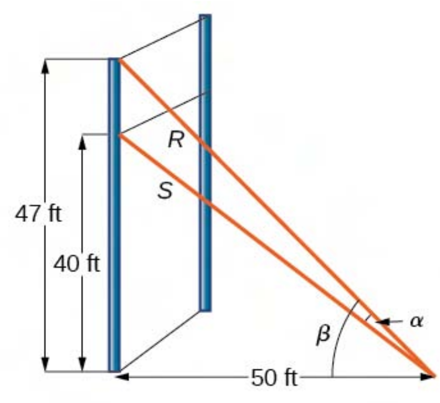

Figure 7.13

Solution

Let’s first summarize the information we can gather from the diagram. As only the sides adjacent to the right angle are known, we can use the tangent function. Notice that $\tan\beta = \frac{47}{50}$, and $\tan(\beta - \alpha) = \frac{40}{50} = \frac{4}{5}$. We can then use difference formula for tangent.

$$\tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$

Now, substituting the values we know into the formula, we have

$$\frac{4}{5} = \frac{\frac{47}{50} - \tan\alpha}{1 + \frac{47}{50}\tan\alpha}$$
$$4\left(1 + \frac{47}{50}\tan\alpha\right) = 5\left(\frac{47}{50} - \tan\alpha\right)$$

Use the distributive property, and then simplify the functions.

$$\begin{aligned}
4(1) + 4\left(\frac{47}{50}\right)\tan\alpha &= 5\left(\frac{47}{50}\right) - 5\tan\alpha \\
4 + 3.76\tan\alpha &= 4.7 - 5\tan\alpha \\
5\tan\alpha + 3.76\tan\alpha &= 0.7 \\
8.76\tan\alpha &= 0.7 \\
\tan\alpha &\approx 0.07991 \\
\tan^{-1}(0.07991) &\approx .079741
\end{aligned}$$

Now we can calculate the angle in degrees.

$$\alpha \approx 0.079741\left(\frac{180}{\pi}\right) \approx 4.57°$$

Analysis

Occasionally, when an application appears that includes a right triangle, we may think that solving is a matter of applying the Pythagorean Theorem. That may be partially true, but it depends on what the problem is asking and what information is given.

Access these online resources for additional instruction and practice with sum and difference identities.

- **Sum and Difference Identities for Cosine (http://openstaxcollege.org/l/sumdifcos)**
- **Sum and Difference Identities for Sine (http://openstaxcollege.org/l/sumdifsin)**
- **Sum and Difference Identities for Tangent (http://openstaxcollege.org/l/sumdiftan)**

7.2 EXERCISES

Verbal

43. Explain the basis for the cofunction identities and when they apply.

44. Is there only one way to evaluate $\cos\left(\frac{5\pi}{4}\right)$? Explain how to set up the solution in two different ways, and then compute to make sure they give the same answer.

45. Explain to someone who has forgotten the even-odd properties of sinusoidal functions how the addition and subtraction formulas can determine this characteristic for $f(x) = \sin(x)$ and $g(x) = \cos(x)$. (Hint: $0 - x = -x$)

Algebraic

For the following exercises, find the exact value.

46. $\cos\left(\frac{7\pi}{12}\right)$

47. $\cos\left(\frac{\pi}{12}\right)$

48. $\sin\left(\frac{5\pi}{12}\right)$

49. $\sin\left(\frac{11\pi}{12}\right)$

50. $\tan\left(-\frac{\pi}{12}\right)$

51. $\tan\left(\frac{19\pi}{12}\right)$

For the following exercises, rewrite in terms of $\sin x$ and $\cos x$.

52. $\sin\left(x + \frac{11\pi}{6}\right)$

53. $\sin\left(x - \frac{3\pi}{4}\right)$

54. $\cos\left(x - \frac{5\pi}{6}\right)$

55. $\cos\left(x + \frac{2\pi}{3}\right)$

For the following exercises, simplify the given expression.

56. $\csc\left(\frac{\pi}{2} - t\right)$

57. $\sec\left(\frac{\pi}{2} - \theta\right)$

58. $\cot\left(\frac{\pi}{2} - x\right)$

59.

$\tan\left(\frac{\pi}{2} - x\right)$

60. $\sin(2x)\cos(5x) - \sin(5x)\cos(2x)$

61. $\dfrac{\tan\left(\frac{3}{2}x\right) - \tan\left(\frac{7}{5}x\right)}{1 + \tan\left(\frac{3}{2}x\right)\tan\left(\frac{7}{5}x\right)}$

For the following exercises, find the requested information.

62. Given that $\sin a = \frac{2}{3}$ and $\cos b = -\frac{1}{4}$, with a and b both in the interval $\left[\frac{\pi}{2}, \pi\right)$, find $\sin(a + b)$ and $\cos(a - b)$.

63. Given that $\sin a = \frac{4}{5}$, and $\cos b = \frac{1}{3}$, with a and b both in the interval $\left[0, \frac{\pi}{2}\right)$, find $\sin(a - b)$ and $\cos(a + b)$.

For the following exercises, find the exact value of each expression.

64. $\sin\left(\cos^{-1}(0) - \cos^{-1}\left(\frac{1}{2}\right)\right)$

65. $\cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

66. $\tan\left(\sin^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{1}{2}\right)\right)$

Graphical

For the following exercises, simplify the expression, and then graph both expressions as functions to verify the graphs are identical.

67. $\cos\left(\frac{\pi}{2} - x\right)$

68. $\sin(\pi - x)$

69. $\tan\left(\frac{\pi}{3} + x\right)$

70. $\sin\left(\frac{\pi}{3} + x\right)$

71. $\tan\left(\frac{\pi}{4} - x\right)$

72. $\cos\left(\frac{7\pi}{6} + x\right)$

73. $\sin\left(\frac{\pi}{4} + x\right)$

74. $\cos\left(\frac{5\pi}{4} + x\right)$

For the following exercises, use a graph to determine whether the functions are the same or different. If they are the same, show why. If they are different, replace the second function with one that is identical to the first. (Hint: think $2x = x + x$.)

75. $f(x) = \sin(4x) - \sin(3x)\cos x,\ g(x) = \sin x\cos(3x)$

76. $f(x) = \cos(4x) + \sin x \sin(3x),\ g(x) = -\cos x \cos(3x)$

77. $f(x) = \sin(3x)\cos(6x),\ g(x) = -\sin(3x)\cos(6x)$

78. $f(x) = \sin(4x),\ g(x) = \sin(5x)\cos x - \cos(5x)\sin x$

79. $f(x) = \sin(2x),\ g(x) = 2\sin x \cos x$

80. $f(\theta) = \cos(2\theta),\ g(\theta) = \cos^2\theta - \sin^2\theta$

81. $f(\theta) = \tan(2\theta),\ g(\theta) = \dfrac{\tan\theta}{1 + \tan^2\theta}$

82. $f(x) = \sin(3x)\sin x,\ g(x) = \sin^2(2x)\cos^2 x - \cos^2(2x)\sin^2 x$

83. $f(x) = \tan(-x),\ g(x) = \dfrac{\tan x - \tan(2x)}{1 - \tan x \tan(2x)}$

Technology

For the following exercises, find the exact value algebraically, and then confirm the answer with a calculator to the fourth decimal point.

84. $\sin(75^\circ)$

85. $\sin(195^\circ)$

86. $\cos(165^\circ)$

87. $\cos(345^\circ)$

88. $\tan(-15^\circ)$

Extensions

For the following exercises, prove the identities provided.

89. $\tan\left(x + \frac{\pi}{4}\right) = \dfrac{\tan x + 1}{1 - \tan x}$

90. $\dfrac{\tan(a+b)}{\tan(a-b)} = \dfrac{\sin a \cos a + \sin b \cos b}{\sin a \cos a - \sin b \cos b}$

91. $\dfrac{\cos(a+b)}{\cos a \cos b} = 1 - \tan a \tan b$

92. $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$

93. $\dfrac{\cos(x+h) - \cos x}{h} = \cos x \dfrac{\cos h - 1}{h} - \sin x \dfrac{\sin h}{h}$

For the following exercises, prove or disprove the statements.

94. $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$

95. $\tan(u-v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

96. $\frac{\tan(x+y)}{1+\tan x\tan x}=\frac{\tan x+\tan y}{1-\tan^2 x\tan^2 y}$

97. If α, β, and γ are angles in the same triangle, then prove or disprove $\sin(\alpha+\beta)=\sin\gamma$.

98. If α, β, and y are angles in the same triangle, then prove or disprove $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$

7.3 | Double-Angle, Half-Angle, and Reduction Formulas

Learning Objectives

In this section, you will:

7.3.1 Use double-angle formulas to find exact values.
7.3.2 Use double-angle formulas to verify identities.
7.3.3 Use reduction formulas to simplify an expression.
7.3.4 Use half-angle formulas to find exact values.

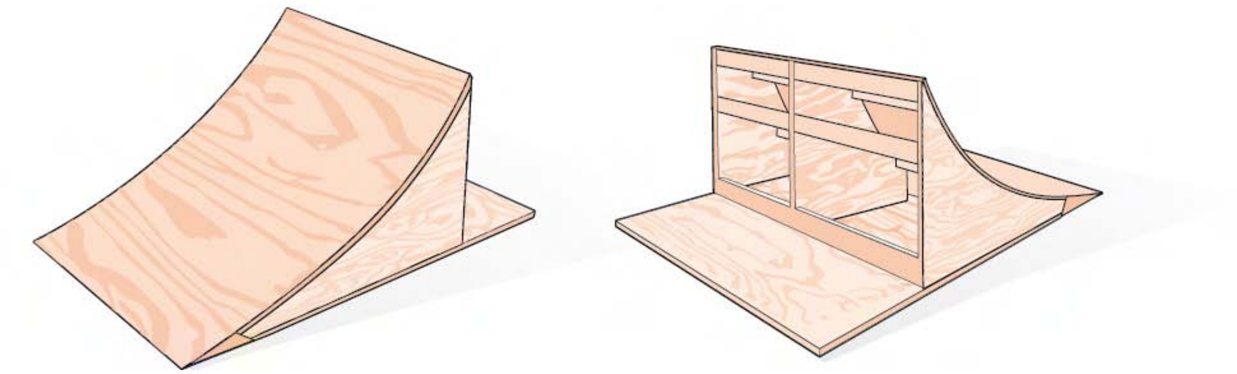

Figure 7.14 Bicycle ramps for advanced riders have a steeper incline than those designed for novices.

Bicycle ramps made for competition (see **Figure 7.14**) must vary in height depending on the skill level of the competitors. For advanced competitors, the angle formed by the ramp and the ground should be θ such that $\tan\theta = \frac{5}{3}$. The angle is divided in half for novices. What is the steepness of the ramp for novices? In this section, we will investigate three additional categories of identities that we can use to answer questions such as this one.

Using Double-Angle Formulas to Find Exact Values

In the previous section, we used addition and subtraction formulas for trigonometric functions. Now, we take another look at those same formulas. The double-angle formulas are a special case of the sum formulas, where $\alpha = \beta$. Deriving the double-angle formula for sine begins with the sum formula,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

If we let $\alpha = \beta = \theta$, then we have

$$\begin{aligned}\sin(\theta + \theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ \sin(2\theta) &= 2\sin\theta\cos\theta\end{aligned}$$

Deriving the double-angle for cosine gives us three options. First, starting from the sum formula, $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, and letting $\alpha = \beta = \theta$, we have

$$\begin{aligned}\cos(\theta + \theta) &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Using the Pythagorean properties, we can expand this double-angle formula for cosine and get two more interpretations. The first one is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta\\ &= (1 - \sin^2\theta) - \sin^2\theta\\ &= 1 - 2\sin^2\theta\end{aligned}$$

The second interpretation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \\ &= 2\cos^2\theta - 1\end{aligned}$$

Similarly, to derive the double-angle formula for tangent, replacing $\alpha = \beta = \theta$ in the sum formula gives

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ \tan(\theta + \theta) &= \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} \\ \tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta}\end{aligned}$$

Double-Angle Formulas

The **double-angle formulas** are summarized as follows:

$$\sin(2\theta) = 2\sin\theta\cos\theta \quad (7.24)$$

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \quad (7.25) \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1\end{aligned}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} \quad (7.26)$$

How To: **Given the tangent of an angle and the quadrant in which it is located, use the double-angle formulas to find the exact value.**

1. Draw a triangle to reflect the given information.
2. Determine the correct double-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 7.22

Using a Double-Angle Formula to Find the Exact Value Involving Tangent

Given that $\tan\theta = -\frac{3}{4}$ and θ is in quadrant II, find the following:

a. $\sin(2\theta)$

b. $\cos(2\theta)$

c. $\tan(2\theta)$

Solution

If we draw a triangle to reflect the information given, we can find the values needed to solve the problems on the image. We are given $\tan\theta = -\frac{3}{4},$ such that θ is in quadrant II. The tangent of an angle is equal to the opposite side over the adjacent side, and because θ is in the second quadrant, the adjacent side is on the x-axis and is negative. Use the Pythagorean Theorem to find the length of the hypotenuse:

$$\begin{aligned}(-4)^2 + (3)^2 &= c^2 \\ 16 + 9 &= c^2 \\ 25 &= c^2 \\ c &= 5\end{aligned}$$

Now we can draw a triangle similar to the one shown in **Figure 7.15**.

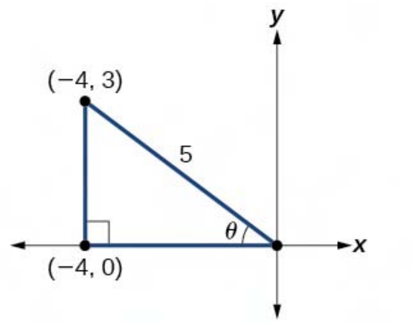

Figure 7.15

a. Let's begin by writing the double-angle formula for sine.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

We see that we to need to find $\sin\theta$ and $\cos\theta$. Based on **Figure 7.15**, we see that the hypotenuse equals 5, so $\sin\theta = \frac{3}{5}$, and $\cos\theta = -\frac{4}{5}$. Substitute these values into the equation, and simplify.

Thus,

$$\begin{aligned}\sin(2\theta) &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) \\ &= -\frac{24}{25}\end{aligned}$$

b. Write the double-angle formula for cosine.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Again, substitute the values of the sine and cosine into the equation, and simplify.

$$\begin{aligned}\cos(2\theta) &= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 \\ &= \frac{16}{25} - \frac{9}{25} \\ &= \frac{7}{25}\end{aligned}$$

c. Write the double-angle formula for tangent.

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

In this formula, we need the tangent, which we were given as $\tan\theta = -\frac{3}{4}$. Substitute this value into the equation, and simplify.

$$\begin{aligned}\tan(2\theta) &= \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2} \\ &= \frac{-\frac{3}{2}}{1-\frac{9}{16}} \\ &= -\frac{3}{2}\left(\frac{16}{7}\right) \\ &= -\frac{24}{7}\end{aligned}$$

Try It **7.11** Given $\sin\alpha = \frac{5}{8},$ with θ in quadrant I, find $\cos(2\alpha)$.

Example 7.23

Using the Double-Angle Formula for Cosine without Exact Values

Use the double-angle formula for cosine to write $\cos(6x)$ in terms of $\cos(3x)$.

Solution

$$\begin{aligned}\cos(6x) &= \cos(3x+3x) \\ &= \cos 3x\cos 3x - \sin 3x\sin 3x \\ &= \cos^2 3x - \sin^2 3x\end{aligned}$$

Analysis

This example illustrates that we can use the double-angle formula without having exact values. It emphasizes that the pattern is what we need to remember and that identities are true for all values in the domain of the trigonometric function.

Using Double-Angle Formulas to Verify Identities

Establishing identities using the double-angle formulas is performed using the same steps we used to derive the sum and difference formulas. Choose the more complicated side of the equation and rewrite it until it matches the other side.

Example 7.24

Using the Double-Angle Formulas to Establish an Identity

Establish the following identity using double-angle formulas:

$$1+\sin(2\theta) = (\sin\theta+\cos\theta)^2$$

Solution

We will work on the right side of the equal sign and rewrite the expression until it matches the left side.

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\ &= (\sin^2\theta+\cos^2\theta)+2\sin\theta\cos\theta\\ &= 1+2\sin\theta\cos\theta\\ &= 1+\sin(2\theta)\end{aligned}$$

Analysis

This process is not complicated, as long as we recall the perfect square formula from algebra:

$$(a\pm b)^2=a^2\pm 2ab+b^2$$

where $a=\sin\theta$ and $b=\cos\theta$. Part of being successful in mathematics is the ability to recognize patterns. While the terms or symbols may change, the algebra remains consistent.

Try It **7.12** Establish the identity: $\cos^4\theta-\sin^4\theta=\cos(2\theta)$.

Example 7.25

Verifying a Double-Angle Identity for Tangent

Verify the identity:

$$\tan(2\theta)=\frac{2}{\cot\theta-\tan\theta}$$

Solution

In this case, we will work with the left side of the equation and simplify or rewrite until it equals the right side of the equation.

$$\begin{aligned}\tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta} && \text{Double-angle formula}\\ &= \frac{2\tan\theta\left(\frac{1}{\tan\theta}\right)}{\left(1-\tan^2\theta\right)\left(\frac{1}{\tan\theta}\right)} && \text{Multiply by a term that results in desired numerator.}\\ &= \frac{2}{\frac{1}{\tan\theta}-\frac{\tan^2\theta}{\tan\theta}}\\ &= \frac{2}{\cot\theta-\tan\theta} && \text{Use reciprocal identity for } \frac{1}{\tan\theta}.\end{aligned}$$

Analysis

Here is a case where the more complicated side of the initial equation appeared on the right, but we chose to work the left side. However, if we had chosen the left side to rewrite, we would have been working backwards to arrive at the equivalency. For example, suppose that we wanted to show

$$\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2}{\cot\theta-\tan\theta}$$

Let's work on the right side.

$$\begin{aligned}\frac{2}{\cot\theta - \tan\theta} &= \frac{2}{\frac{1}{\tan\theta} - \tan\theta}\left(\frac{\tan\theta}{\tan\theta}\right)\\ &= \frac{2\tan\theta}{\frac{1}{\tan\theta}(\tan\theta) - \tan\theta(\tan\theta)}\\ &= \frac{2\tan\theta}{1-\tan^2\theta}\end{aligned}$$

When using the identities to simplify a trigonometric expression or solve a trigonometric equation, there are usually several paths to a desired result. There is no set rule as to what side should be manipulated. However, we should begin with the guidelines set forth earlier.

Try It **7.13** Verify the identity: $\cos(2\theta)\cos\theta = \cos^3\theta - \cos\theta\sin^2\theta.$

Use Reduction Formulas to Simplify an Expression

The double-angle formulas can be used to derive the reduction formulas, which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine. These formulas are especially important in higher-level math courses, calculus in particular. Also called the power-reducing formulas, three identities are included and are easily derived from the double-angle formulas.

We can use two of the three double-angle formulas for cosine to derive the reduction formulas for sine and cosine. Let's begin with $\cos(2\theta) = 1 - 2\sin^2\theta.$ Solve for $\sin^2\theta$:

$$\begin{aligned}\cos(2\theta) &= 1 - 2\sin^2\theta\\ 2\sin^2\theta &= 1 - \cos(2\theta)\\ \sin^2\theta &= \frac{1-\cos(2\theta)}{2}\end{aligned}$$

Next, we use the formula $\cos(2\theta) = 2\cos^2\theta - 1.$ Solve for $\cos^2\theta$:

$$\begin{aligned}\cos(2\theta) &= 2\cos^2\theta - 1\\ 1+\cos(2\theta) &= 2\cos^2\theta\\ \frac{1+\cos(2\theta)}{2} &= \cos^2\theta\end{aligned}$$

The last reduction formula is derived by writing tangent in terms of sine and cosine:

$$\begin{aligned}\tan^2\theta &= \frac{\sin^2\theta}{\cos^2\theta}\\ &= \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}} && \text{Substitute the reduction formulas.}\\ &= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{2}{1+\cos(2\theta)}\right)\\ &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)}\end{aligned}$$

Reduction Formulas

The **reduction formulas** are summarized as follows:

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2} \quad (7.27)$$

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2} \quad (7.28)$$

$$\tan^2 \theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)} \quad (7.29)$$

Example 7.26

Writing an Equivalent Expression Not Containing Powers Greater Than 1

Write an equivalent expression for $\cos^4 x$ that does not involve any powers of sine or cosine greater than 1.

Solution

We will apply the reduction formula for cosine twice.

$$\begin{aligned}
\cos^4 x &= (\cos^2 x)^2 \\
&= \left(\frac{1 + \cos(2x)}{2}\right)^2 && \text{Substitute reduction formula for } \cos^2 x. \\
&= \frac{1}{4}\left(1 + 2\cos(2x) + \cos^2(2x)\right) \\
&= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\frac{1 + \cos2(2x)}{2}\right) && \text{Substitute reduction formula for } \cos^2 x. \\
&= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{8} + \frac{1}{8}\cos(4x) \\
&= \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)
\end{aligned}$$

Analysis

The solution is found by using the reduction formula twice, as noted, and the perfect square formula from algebra.

Example 7.27

Using the Power-Reducing Formulas to Prove an Identity

Use the power-reducing formulas to prove

$$\sin^3 (2x) = \left[\frac{1}{2}\sin(2x)\right][1 - \cos(4x)]$$

Solution

We will work on simplifying the left side of the equation:

$$\begin{aligned}
\sin^3(2x) &= [\sin(2x)][\sin^2(2x)] \\
&= \sin(2x)\left[\frac{1 - \cos(4x)}{2}\right] && \text{Substitute the power-reduction formula.} \\
&= \sin(2x)\left(\frac{1}{2}\right)[1 - \cos(4x)] \\
&= \frac{1}{2}[\sin(2x)][1 - \cos(4x)]
\end{aligned}$$

Analysis

Note that in this example, we substituted

$$\frac{1-\cos(4x)}{2}$$

for $\sin^2(2x)$. The formula states

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

We let $\theta = 2x$, so $2\theta = 4x$.

Try It

7.14 Use the power-reducing formulas to prove that $10\cos^4 x = \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x)$.

Using Half-Angle Formulas to Find Exact Values

The next set of identities is the set of **half-angle formulas**, which can be derived from the reduction formulas and we can use when we have an angle that is half the size of a special angle. If we replace θ with $\frac{\alpha}{2}$, the half-angle formula for sine is found by simplifying the equation and solving for $\sin\left(\frac{\alpha}{2}\right)$. Note that the half-angle formulas are preceded by a $\pm$ sign. This does not mean that both the positive and negative expressions are valid. Rather, it depends on the quadrant in which $\frac{\alpha}{2}$ terminates.

The half-angle formula for sine is derived as follows:

$$\begin{aligned}
\sin^2\theta &= \frac{1-\cos(2\theta)}{2} \\
\sin^2\left(\frac{\alpha}{2}\right) &= \frac{1-\left(\cos 2\cdot\frac{\alpha}{2}\right)}{2} \\
&= \frac{1-\cos\alpha}{2} \\
\sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos\alpha}{2}}
\end{aligned}$$

To derive the half-angle formula for cosine, we have

$$\begin{aligned}
\cos^2\theta &= \frac{1+\cos(2\theta)}{2} \\
\cos^2\left(\frac{\alpha}{2}\right) &= \frac{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}{2} \\
&= \frac{1+\cos\alpha}{2} \\
\cos\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1+\cos\alpha}{2}}
\end{aligned}$$

For the tangent identity, we have

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$
$$\tan^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}$$
$$= \frac{1-\cos\alpha}{1+\cos\alpha}$$
$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Half-Angle Formulas

The **half-angle formulas** are as follows:

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}} \quad (7.30)$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \quad (7.31)$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \quad (7.32)$$
$$= \frac{\sin\alpha}{1+\cos\alpha}$$
$$= \frac{1-\cos\alpha}{\sin\alpha}$$

Example 7.28

Using a Half-Angle Formula to Find the Exact Value of a Sine Function

Find $\sin(15^\circ)$ using a half-angle formula.

Solution

Since $15^\circ = \frac{30^\circ}{2}$, we use the half-angle formula for sine:

$$\sin\frac{30^\circ}{2} = \sqrt{\frac{1-\cos 30^\circ}{2}}$$
$$= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$
$$= \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}$$
$$= \sqrt{\frac{2-\sqrt{3}}{4}}$$
$$= \frac{\sqrt{2-\sqrt{3}}}{2}$$

Analysis

Notice that we used only the positive root because $\sin(15^o)$ is positive.

How To: Given the tangent of an angle and the quadrant in which the angle lies, find the exact values of trigonometric functions of half of the angle.

1. Draw a triangle to represent the given information.
2. Determine the correct half-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 7.29

Finding Exact Values Using Half-Angle Identities

Given that $\tan\alpha = \frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

a. $\sin\left(\frac{\alpha}{2}\right)$

b. $\cos\left(\frac{\alpha}{2}\right)$

c. $\tan\left(\frac{\alpha}{2}\right)$

Solution

Using the given information, we can draw the triangle shown in **Figure 7.16**. Using the Pythagorean Theorem, we find the hypotenuse to be 17. Therefore, we can calculate $\sin\alpha = -\frac{8}{17}$ and $\cos\alpha = -\frac{15}{17}$.

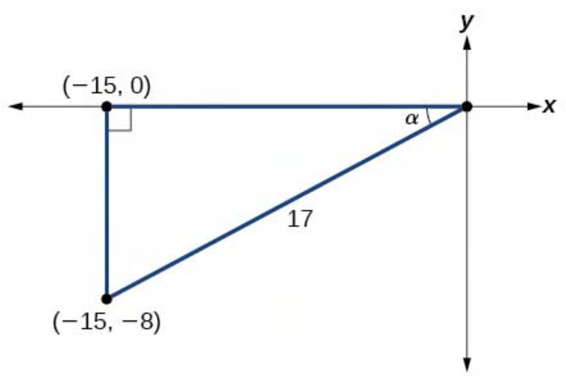

Figure 7.16

a. Before we start, we must remember that, if α is in quadrant III, then $180° < \alpha < 270°$, so $\frac{180°}{2} < \frac{\alpha}{2} < \frac{270°}{2}$. This means that the terminal side of $\frac{\alpha}{2}$ is in quadrant II, since $90° < \frac{\alpha}{2} < 135°$.

To find $\sin\frac{\alpha}{2}$, we begin by writing the half-angle formula for sine. Then we substitute the value of the cosine we found from the triangle in **Figure 7.16** and simplify.

$$\begin{aligned}
\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\
&= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{2}} \\
&= \pm\sqrt{\frac{\frac{32}{17}}{2}} \\
&= \pm\sqrt{\frac{32}{17}\cdot\frac{1}{2}} \\
&= \pm\sqrt{\frac{16}{17}} \\
&= \pm\frac{4}{\sqrt{17}} \\
&= \frac{4\sqrt{17}}{17}
\end{aligned}$$

We choose the positive value of $\sin\frac{\alpha}{2}$ because the angle terminates in quadrant II and sine is positive in quadrant II.

b. To find $\cos\frac{\alpha}{2}$, we will write the half-angle formula for cosine, substitute the value of the cosine we found from the triangle in **Figure 7.16**, and simplify.

$$\begin{aligned}
\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \\
&= \pm\sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}} \\
&= \pm\sqrt{\frac{\frac{2}{17}}{2}} \\
&= \pm\sqrt{\frac{2}{17}\cdot\frac{1}{2}} \\
&= \pm\sqrt{\frac{1}{17}} \\
&= -\frac{\sqrt{17}}{17}
\end{aligned}$$

We choose the negative value of $\cos\frac{\alpha}{2}$ because the angle is in quadrant II because cosine is negative in quadrant II.

c. To find $\tan\frac{\alpha}{2}$, we write the half-angle formula for tangent. Again, we substitute the value of the cosine we found from the triangle in **Figure 7.16** and simplify.

$$\begin{aligned} \tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \\ &= \pm\sqrt{\frac{1-(-\frac{15}{17})}{1+(-\frac{15}{17})}} \\ &= \pm\sqrt{\frac{\frac{32}{17}}{\frac{2}{17}}} \\ &= \pm\sqrt{\frac{32}{2}} \\ &= -\sqrt{16} \\ &= -4 \end{aligned}$$

We choose the negative value of $\tan\frac{\alpha}{2}$ because $\frac{\alpha}{2}$ lies in quadrant II, and tangent is negative in quadrant II.

Try It **7.15** Given that $\sin\alpha = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\cos\left(\frac{\alpha}{2}\right)$.

Example 7.30

Finding the Measurement of a Half Angle

Now, we will return to the problem posed at the beginning of the section. A bicycle ramp is constructed for high-level competition with an angle of θ formed by the ramp and the ground. Another ramp is to be constructed half as steep for novice competition. If $\tan\theta = \frac{5}{3}$ for higher-level competition, what is the measurement of the angle for novice competition?

Solution

Since the angle for novice competition measures half the steepness of the angle for the high level competition, and $\tan\theta = \frac{5}{3}$ for high competition, we can find $\cos\theta$ from the right triangle and the Pythagorean theorem so that we can use the half-angle identities. See **Figure 7.17**.

$$\begin{aligned} 3^2 + 5^2 &= 34 \\ c &= \sqrt{34} \end{aligned}$$

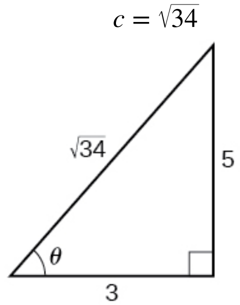

Figure 7.17

We see that $\cos\theta = \frac{3}{\sqrt{34}} = \frac{3\sqrt{34}}{34}$. We can use the half-angle formula for tangent: $\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$. Since $\tan\theta$ is in the first quadrant, so is $\tan\frac{\theta}{2}$. Thus,

$$\begin{aligned}\tan\frac{\theta}{2} &= \sqrt{\frac{1-\frac{3\sqrt{34}}{34}}{1+\frac{3\sqrt{34}}{34}}}\\ &= \sqrt{\frac{\frac{34-3\sqrt{34}}{34}}{\frac{34+3\sqrt{34}}{34}}}\\ &= \sqrt{\frac{34-3\sqrt{34}}{34+3\sqrt{34}}}\\ &\approx 0.57\end{aligned}$$

We can take the inverse tangent to find the angle: $\tan^{-1}(0.57) \approx 29.7^\circ$. So the angle of the ramp for novice competition is $\approx 29.7^\circ$.

Access these online resources for additional instruction and practice with double-angle, half-angle, and reduction formulas.

- **Double-Angle Identities (http://openstaxcollege.org/l/doubleangiden)**
- **Half-Angle Identities (http://openstaxcollege.org/l/halfangleident)**

7.3 EXERCISES

Verbal

99. Explain how to determine the reduction identities from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$.

100. Explain how to determine the double-angle formula for $\tan(2x)$ using the double-angle formulas for $\cos(2x)$ and $\sin(2x)$.

101. We can determine the half-angle formula for $\tan\left(\frac{x}{2}\right) = \frac{\sqrt{1 - \cos x}}{\sqrt{1 + \cos x}}$ by dividing the formula for $\sin\left(\frac{x}{2}\right)$ by $\cos\left(\frac{x}{2}\right)$. Explain how to determine two formulas for $\tan\left(\frac{x}{2}\right)$ that do not involve any square roots.

102. For the half-angle formula given in the previous exercise for $\tan\left(\frac{x}{2}\right)$, explain why dividing by 0 is not a concern. (Hint: examine the values of $\cos x$ necessary for the denominator to be 0.)

Algebraic

For the following exercises, find the exact values of a) $\sin(2x)$, b) $\cos(2x)$, and c) $\tan(2x)$ without solving for x.

103. If $\sin x = \frac{1}{8}$, and x is in quadrant I.

104. If $\cos x = \frac{2}{3}$, and x is in quadrant I.

105. If $\cos x = -\frac{1}{2}$, and x is in quadrant III.

106. If $\tan x = -8$, and x is in quadrant IV.

For the following exercises, find the values of the six trigonometric functions if the conditions provided hold.

107. $\cos(2\theta) = \frac{3}{5}$ and $90^\circ \leq \theta \leq 180^\circ$

108. $\cos(2\theta) = \frac{1}{\sqrt{2}}$ and $180^\circ \leq \theta \leq 270^\circ$

For the following exercises, simplify to one trigonometric expression.

109. $2\sin\left(\frac{\pi}{4}\right)2\cos\left(\frac{\pi}{4}\right)$

110. $4\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$

For the following exercises, find the exact value using half-angle formulas.

111. $\sin\left(\frac{\pi}{8}\right)$

112. $\cos\left(-\frac{11\pi}{12}\right)$

113. $\sin\left(\frac{11\pi}{12}\right)$

114. $\cos\left(\frac{7\pi}{8}\right)$

115. $\tan\left(\frac{5\pi}{12}\right)$

116. $\tan\left(-\frac{3\pi}{12}\right)$

117. $\tan\left(-\frac{3\pi}{8}\right)$

For the following exercises, find the exact values of a) $\sin\left(\frac{x}{2}\right)$, b) $\cos\left(\frac{x}{2}\right)$, and c) $\tan\left(\frac{x}{2}\right)$ without solving for x.

118. If $\tan x = -\frac{4}{3}$, and x is in quadrant IV.

119. If $\sin x = -\frac{12}{13}$, and x is in quadrant III.

120. If $\csc x = 7$, and x is in quadrant II.

121. If $\sec x = -4$, and x is in quadrant II.

For the following exercises, use **Figure 7.18** to find the requested half and double angles.

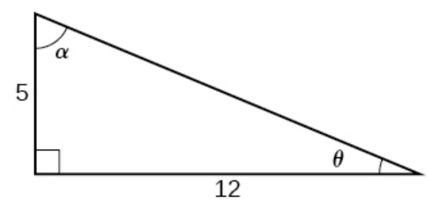

Figure 7.18

122. Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

123. Find $\sin(2\alpha)$, $\cos(2\alpha)$, and $\tan(2\alpha)$.

124. Find $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$.

125. Find $\sin\left(\frac{\alpha}{2}\right)$, $\cos\left(\frac{\alpha}{2}\right)$, and $\tan\left(\frac{\alpha}{2}\right)$.

For the following exercises, simplify each expression. Do not evaluate.

126. $\cos^2(28°) - \sin^2(28°)$

127. $2\cos^2(37°) - 1$

128. $1 - 2\sin^2(17°)$

129. $\cos^2(9x) - \sin^2(9x)$

130. $4\sin(8x)\cos(8x)$

131. $6\sin(5x)\cos(5x)$

For the following exercises, prove the identity given.

132. $(\sin t - \cos t)^2 = 1 - \sin(2t)$

133. $\sin(2x) = -2\sin(-x)\cos(-x)$

134. $\cot x - \tan x = 2\cot(2x)$

135. $\frac{\sin(2\theta)}{1+\cos(2\theta)}\tan^2\theta = \tan\theta$

For the following exercises, rewrite the expression with an exponent no higher than 1.

136. $\cos^2(5x)$

137. $\cos^2(6x)$

138. $\sin^4(8x)$

139. $\sin^4(3x)$

140. $\cos^2 x \sin^4 x$

141. $\cos^4 x \sin^2 x$

142. $\tan^2 x \sin^2 x$

Technology

For the following exercises, reduce the equations to powers of one, and then check the answer graphically.

143. $\tan^4 x$

144. $\sin^2(2x)$

145. $\sin^2 x \cos^2 x$

146. $\tan^2 x \sin x$

147. $\tan^4 x \cos^2 x$

148. $\cos^2 x \sin(2x)$

149. $\cos^2(2x)\sin x$

150. $\tan^2\left(\frac{x}{2}\right)\sin x$

For the following exercises, algebraically find an equivalent function, only in terms of $\sin x$ and/or $\cos x$, and then check the answer by graphing both equations.

151. $\sin(4x)$

152. $\cos(4x)$

Extensions

For the following exercises, prove the identities.

153. $\sin(2x) = \dfrac{2\tan x}{1 + \tan^2 x}$

154. $\cos(2\alpha) = \dfrac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha}$

155. $\tan(2x) = \dfrac{2\sin x\cos x}{2\cos^2 x - 1}$

156. $\left(\sin^2 x - 1\right)^2 = \cos(2x) + \sin^4 x$

157. $\sin(3x) = 3\sin x\cos^2 x - \sin^3 x$

158. $\cos(3x) = \cos^3 x - 3\sin^2 x\cos x$

159. $\dfrac{1 + \cos(2t)}{\sin(2t) - \cos t} = \dfrac{2\cos t}{2\sin t - 1}$

160. $\sin(16x) = 16\sin x\cos x\cos(2x)\cos(4x)\cos(8x)$

161. $\cos(16x) = \left(\cos^2(4x) - \sin^2(4x) - \sin(8x)\right)\left(\cos^2(4x) - \sin^2(4x) + \sin(8x)\right)$