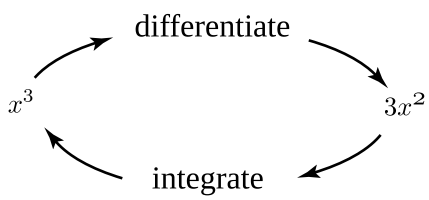
Integration as the reverse of differentiation

Introduction

Integration can be introduced in several different ways. One way is to think of it as differentiation in reverse. This approach is described in this leaflet.

1. Differentiation in reverse

Suppose we differentiate the function $y = x^3$. We obtain $\frac{dy}{dx} = 3x^2$. Integration reverses this process and we say that the integral of $3x^2$ is x^3. Pictorially we can think of this as follows:



The situation is just a little more complicated because there are lots of functions we can differentiate to give $3x^2$. Here are some of them:

$$x^3 + 14, \qquad x^3 + 7, \qquad x^3 - 0.25, \qquad x^3 - \frac{1}{2}$$

Each of these functions has the same derivative, $3x^2$, because when we differentiate the constant term we obtain zero. Consequently, when we try to reverse the process, we have no idea what the original constant term might have been. Because of this we include in our answer an unknown constant, c say, called the **constant of integration.** We state that the integral of $3x^2$ is $x^3 + c$.

The symbol for integration is $\int$, known as an **integral sign**. Formally we write

$$\int 3x^2 \, dx = x^3 + c$$

Along with the integral sign there is a term 'dx', which must always be written, and which indicates the name of the variable involved, in this case x. Technically, integrals of this sort are called **indefinite integrals**, to distinguish them from definite integrals which are dealt with on a subsequent leaflet. When asked to find an indefinite integral your answer should always contain a constant of integration.

Common integrals are usually found in a 'Table of Integrals' such as that shown here. A more complete table is available on leaflet *8.7 Table of integrals*.

Table of integrals

Function $f(x)$	Indefinite integral $\int f(x)\mathrm{d}x$
constant, k	$kx + c$
x	$\frac{x^2}{2} + c$
x^2	$\frac{x^3}{3} + c$
x^n	$\frac{x^{n+1}}{n+1} + c \quad n \neq -1$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\sin kx$	$\frac{-\cos kx}{k} + c$
$\cos kx$	$\frac{\sin kx}{k} + c$
$\tan kx$	$\frac{1}{k}\ln\lvert\sec kx\rvert$+c
e^x	$e^x + c$
e^{-x}	$-e^{-x} + c$
e^{kx}	$\frac{e^{kx}}{k} + c$
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert + c$

When dealing with the trigonometric functions the variable x must always be measured in radians.

Example

Use the table above to find a) $\displaystyle\int x^8 \mathrm{d}x$, b) $\displaystyle\int x^{-4}\mathrm{d}x$.

Solution

From the table note that

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + c$$

a) With $n = 8$ we find

$$\int x^8 \mathrm{d}x = \frac{x^{8+1}}{8+1} + c = \frac{x^9}{9} + c$$

b) With $n = -4$ we find

$$\int x^{-4}\mathrm{d}x = \frac{x^{-4+1}}{-4+1} + c = \frac{x^{-3}}{-3} + c$$

Note that the final answer can be written in a variety of equivalent ways, for example

$$-\frac{1}{3}x^{-3} + c, \qquad \text{or} \qquad -\frac{1}{3}\cdot\frac{1}{x^3} + c, \qquad \text{or} \qquad -\frac{1}{3x^3} + c$$

Exercises

1. Integrate each of the following functions:

a) x^9, b) $x^{1/2}$, c) x^{-3}, d) $\frac{1}{x^4}$, e) 4, f) $\sqrt{x}$, g) e^{4x}, h) 17, i) $\cos 5x$.

Answers

1. a) $\frac{x^{10}}{10} + c$, b) $\frac{2x^{3/2}}{3} + c$, c) $-\frac{1}{2}x^{-2} + c$, d) $-\frac{1}{3}x^{-3} + c$,

e) $4x + c$, f) same as b), g) $\frac{e^{4x}}{4} + c$, h) $17x + c$, i) $\frac{\sin 5x}{5} + c$.