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On connection between values of Riemann zeta function at rationals and generalized harmonic numbers

Paweł J. Szabłowski

Department of Mathematics and Information Sciences,
Warsaw University of Technology
ul. Koszykowa 75,
00-662 Warsaw, Poland

Abstract

Using Euler transformation of series, we relate values of Hurwitz zeta function $\zeta(s,t)$ at integer and rational values of arguments to certain rapidly converging series, where some generalized harmonic numbers appear. Most of the results of the paper can be derived from the recent, more advanced results, on the properties of Arakawa-Kaneko zeta functions. We derive our results directly, by solving simple recursions. The form of mentioned above generalized harmonic numbers carries information, about the values of the arguments of Hurwitz function. In particular we prove: $\forall k\in\mathbb{N}:\zeta(k,1)=\zeta(k)=\frac{2^{k-1}}{2^{k-1}-1}\sum_{n=1}^{\infty}\frac{H_n^{(k-1)}}{n2^n}$, where $H_n^{(k)}$ are defined below generalized harmonic numbers, or that $\mathbb{K}=\sum_{n=0}^{\infty}\frac{n!(H_{2n+1}-H_n/2)}{2(2n+1)!!}$, where $\mathbb{K}$ denotes Calatan constant and H_n denotes $n-$th (ordinary) harmonic number. Further we show that generating function of the numbers $\hat{\zeta}(k)=\sum_{j=1}^{\infty}(-1)^{j-1}/j^k$, $k\in\mathbb{N}$ and $\hat{\xi}(0)$ $=1/2$ is equal to $B(1/2,1-y,1+y)$ where $B(x,a,b)$ denotes incomplete beta function.

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1 Introduction

First let us recall basic notions and definitions that we will work with. By the Hurwitz function $\zeta(s,\alpha)$ we will mean:

$$\zeta(s,\alpha)=\sum_{j=0}^{\infty}\frac{1}{(j+\alpha)^s},$$

considered for $\operatorname{Re} s>1$, $\operatorname{Re}\alpha\in(0,1]$. Function $\zeta(s,1)$ is called Riemann zeta function. We will denote it also by $\zeta(s)$, if it will not cause misunderstanding. It turns out that both these functions can be extended to holomorphic functions of s on the whole complex plane except $s=1$ where a single pole exists. Of great help in doing so is the formula

$$\zeta(s)=\frac{2^{s-1}}{2^{s-1}-1}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j^s}, \tag{1}$$

that enables to extend Riemann zeta function to the whole half plane $\operatorname{Re} s>0$.

We will consider numbers:

$$\mathbb{M}_k^{(m,i)}=\sum_{j=0}^{\infty}\frac{(-1)^j}{(mj+i)^k},$$

for $m\in\mathbb{N}$ and $i\in\{1,\ldots,m-1\}$. Notice that $\mathbb{M}_k^{(1,1)}=\sum_{j=1}^{\infty}(-1)^{j-1}/j^k$ and $\mathbb{M}_1^{(2,1)}=\pi/4$. The number $\mathbb{M}_2^{(2,1)}=\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^2}$ is called Catalan constant $\mathbb{K}$.

It is elementary to notice that

$$\mathbb{M}_k^{(m,i)}=\frac{1}{(2m)^k}(\zeta(k,i/(2m))-\zeta(k,1/2+i/(2m)).$$

The main idea of this paper is to apply the so called Euler transformation, that was nicely recalled by Sondow in [12]. As pointed out there we have:

$$\sum_{k=1}^{\infty}(-1)^{k-1}a_k=\sum_{n=0}^{\infty}\Delta^n a_1/2^{n+1},$$

where $\{a_k\}_{k\geq1}$ is a sequence of complex numbers and the sequence $\Delta^n a_k$ is defined recursively: $\Delta^0 a_k=a_k$, $\Delta^n a_k=\Delta^{n-1}a_k-\Delta^{n-1}a_{k+1}=\sum_{m=0}^{n}(-1)^m\binom{n}{m}a_{m+k}$.

Sondow in [12] presented general idea of applying Euler transformation to Riemann function. He however stopped half way in the sense that he calculated finite differences Δ^n applied to $(j+1)^{-s}$ only for s being negative integers. We are going to make a few steps further and calculate these differences pointing out the rôle of the generalized harmonic numbers in those calculations.

As stated in the abstract most of the results of this paper can be derived from recent more advanced results concerning Lerch and Arakawa-Koneko zeta functions that were presented in the series of papers [3], [4], [5], [6], [7].

We present here an alternative, simple way of obtaining them by solving simple recursion.

The paper is organized as follows. In the next section 2 we present an auxiliary result that enables application of Euler transformation to the analyzed series. Further we present transformed series approximating numbers $\mathbb{M}_k^{(m,i)}$. In Section 3 we calculate generating functions of certain series of numbers and functions. More precisely we calculate generating functions of the generalized harmonic numbers that we have defined in the previous section. We also calculate generating function of the series of the generating functions that were defined previously. It turns out that this calculation enables to obtain the generating function of the series sums that appear on the right hand side of (1). Finally in the last Section 4 there are collected cases when exact values of numbers $\mathbb{M}$ are known.

2 Euler transformation

To proceed further we need the following result.

Proposition 1 *Let us denote* $A_{n,k}^{(m,i)} = \sum_{j=0}^{n}(-1)^j\binom{n}{j}/(mj+i)^k$, $n = 0, 1, \ldots,$ *and the family of sequences defined recursively:* $B_{n,0}^{(m,i)} = 1$, $B_{0,k} = \frac{1}{i^{k-1}}$, $k \geq 1$, $\forall n, k \geq 0 : B_{n,k}^{(m,i)} = \sum_{j=0}^{n}\frac{1}{(mj+i)}B_{j,k-1}^{(m,i)}$. *We have then:*
$\forall\ m \in \mathbb{N} :\ A_{0,0}^{(m,i)} = 1$, $A_{n,0}^{(m,i)} = 0$, $A_{n,1}^{(m,i)} = \frac{n!}{m(i/m)_{n+1}}$, *where*
$(a)_n = a(a+1)\ldots(a+n-1)$ *is the so called 'rising factorial'.*
$\forall\ n \geq 0, k \geq 1$ *we get:*

$$A_{n,k}^{(m,i)} = \frac{n!}{m(i/m)_{n+1}}B_{n,k-1}^{(m,i)}.$$

Proof. i) The fact that $A_{n,0} = 0$ follows immediately properties of binomial coefficients. Notice that we have

$$A_{n+1,k}^{(m,i)} - \frac{m(n+1)}{m(n+1)+i}A_{n,k}^{(m,i)} = \sum_{j=0}^{n}(-1)^j\binom{n+1}{j}/(mj+i)^k + \frac{(-1)^{n+1}}{(m(n+1)+i)^k}$$
$$-\frac{m(n+1)}{m(n++1)+i}\sum_{j=0}^{n}(-1)^j\binom{n}{j}/(mj+i)^k =$$

$$\frac{(-1)^{n+1}}{(m(n+1)+i)^k}+\sum_{j=0}^{n}(-1)^j(\binom{n+1}{j}-\frac{m(n+1)!}{j!(n-j)!(m(n+1)+i)})/(mj+i)^k$$
$$=\frac{(-1)^{n+1}}{(m(n+1)+i)^k}+\frac{1}{m(n+1)+j}\sum_{j=0}^{n}(-1)^j\binom{n+1}{j}/(mj+i)^{k-1}$$
$$=\frac{1}{m(n+1)+i}\sum_{j=0}^{n+1}(-1)^j\binom{n+1}{j}/(mj+i)^{k-1}=\frac{1}{m(n+1)+i}A_{n+1,k-1}^{(m,i)}$$

since $(1-\frac{m(n+1-j)}{(mn+m+i)})=\frac{jm+i}{m+mn+i}$. Now notice that we have $A_{n+1,1}^{(m.i)}-\frac{m(n+1)}{m(n+1)+i}A_{n,1}^{(m,i)}=0$ from which immediately follows that $A_{n,1}^{(m,i)}=\frac{n!}{m(i/m)_{n+1}}$ since $A_{0,1}^{(m,i)}=\frac{1}{i}$. Now divide both sides of the identity $A_{n+1,k}^{(m,i)}-\frac{m(n+1)}{mn+m+i}A_{n,k}^{(m,i)}=\frac{1}{mn+m+i}A_{n+1,k-1}^{(m,i)}$ by $A_{n+1,1}^{(m,i)}$ and denote $B_{n,k}^{(m,i)}=A_{n,k}^{(m,i)}/A_{n,1}^{(m,i)}$. We get $B_{n+1,k}^{(m,i)}-B_{n,k}^{(m,i)}=\frac{1}{m(n+1)+i}B_{n+1,k-1}^{(m,i)}$ since $A_{n+1,1}^{(m.i)}=\frac{m(n+1)}{mn+m+i}A_{n,1}^{(m,i)}$. Hence $B_{n,k}^{(m,i)}=\sum_{j=0}^{n}\frac{1}{mj+i}B_{j,k-1}^{(m,i)}$ since $\forall k\geq 1:$ $B_{0,k}^{(n,i)}=1/i^{k-1}$. ■

Remark 1 *In the literature (compare e.g. [2], [8], [10]) there function notions of harmonic and generalized harmonic numbers defined by* $h_n^{(k)}=\sum_{j=1}^{n}1/j^k$, $n\geq 1$. *Numbers* $h_n^{(1)}$ *are called simply (ordinary) harmonic numbers. Another way to generalize the notion of harmonic numbers was presented by Coppo and Candelpergher in their papers [5], [6], [7]. There the generalized harmonic numbers were defined using Bell's polynomials.*

We are going to define differently generalized harmonic numbers.

Definition 1 *For every* $k\in\mathbb{N}$ *numbers* $\left\{H_n^{(k)}\right\}_{n\geq 1,k\geq 0}$ *defined recursively by* $H_n^{(0)}=1$, $H_n^{(k)}=\sum_{j=1}^{n}H_j^{(k-1)}/j$, $n\geq 1$ *will be called generalized harmonic numbers of order* k.

Remark 2 *It is easy to see that* $B_{n,k}^{(1,1)}=H_{n+1}^{(k)}$ *and that* $H_n^{(1)}$ *is an ordinary* $n-$*th harmonic number.*

Remark 3 *Notice that* $H_n^{(k)}$ *is a symmetric function of order* k *of the numbers* $\{1,1/2,\ldots,1/n\}$ *hence it can be expressed as a linear combination of some other symmetric functions of order less or equal* k. *For example we have:* $H_n^{(1)}=h_n^{(1)}=H_n$ *(the ordinary harmonic number),* $H_n^{(2)}=H_n^2/2+h_n^{(2)}/2$, $H_n^{(3)}=H_n^3/6+H_nh_n^{(2)}/2+h_n^{(3)}/3$ *and so on.*

Remark 4 *Notice also that recursive equation, that was obtained in the proof of Proposition 1 i.e.*

$$A_{n+1,k}^{(m,i)}-\frac{m(n+1)}{m(n+1)+i}A_{n,k}^{(m,i)}=\frac{1}{m(n+1)+i}A_{n+1,k-1},$$

is valid also for $k = 0, -1, -2, \ldots$. *Of course then we apply it in the following form:*

$$A_{n+1,k-1} = (m(n+1)+i)A_{n+1,k} - m(n+1)A_{n,k}^{(m,i)},$$

getting for example : $A_{0,-1}^{(m,i)} = 1$, $A_{1,-1}^{(m,i)} = -m$, $A_{n,-1}^{(m,i)} = 0$, $A_{0,-2}^{(m,i)} = 1$, $A_{1,-2}^{(m,i)} = -m(m+i+1)$, $A_{2,-2}^{(m,i)} = 2m^2$, $A_{n,-2}^{(m,i)} = 0$ *for* $n = 3, 4, \ldots$. *The fact that* $A_{n,-k}^{(m,i)} = 0$ *for* $n \geq k+1$ *was already noticed, justified and applied by Sondow in [12].*

As a corollary we have the following result:

Theorem 2

$$\mathbb{M}_k^{(m,i)} = \sum_{n=0}^{\infty} \frac{n!}{2^{n+1} m (i/m)_{n+1}} B_{n,k-1}^{(m,i)}, \tag{2}$$

where numbers $B_{n,k}^{(m,i)}$ *are defined above.*

i) In particular :

$$\mathbb{M}_{2k+1}^{(2m,m)} = \frac{1}{m^{2k+1}} M_{2k+1}^{(2,1)} = \pi^{2k+1} \frac{(-1)^k E_{2k}}{2(2m)^{2k+1}(2k)!}, \tag{3}$$

$$\mathbb{M}_2^{(2,1)} = \mathbb{K} = \sum_{n=0}^{\infty} \frac{n!(H_{2n+1} - H_n/2)}{2(2n+1)!!}, \tag{4}$$

where H_n *denotes* $n-th$ *(ordinary) harmonic number.*

ii) for $m = i = 1$, $k \in \mathbb{N}$:

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^k} = \sum_{n=1}^{\infty} \frac{H_n^{(k-1)}}{n 2^n}, \tag{5}$$

and consequently for $k = 2, 3, \ldots$

$$\zeta(k) = \frac{2^{k-1}}{2^{k-1}-1} \sum_{n=1}^{\infty} \frac{H_n^{(k-1)}}{n 2^n}. \tag{6}$$

Proof. Applying Euler transformation to the series $\mathbb{M}_{n,k}^{(m,i)}$ we have

$$\mathbb{M}_{n,k}^{(m,i)} = \sum_{n=0}^{\infty} A_{n,k}^{(m,i)}/2^{n+1}.$$

Now it remains to apply Proposition 1. i) To see that (2) reduces to (4) when $k = 2$, $m = 2$ and $i = 1$ notice that $B_{n,0}^{(2,1)} = 1$ and consequently $B_{n,1}^{(2,1)} = \sum_{j=0}^{n} 1/(2j+1) = H_{2n+1} - 2H_n$. Further we have $(1/2)_{n+1} = \prod_{j=0}^{n}(j +$

$1/2) = (2n+1)!!/2^{n+1}$. To justify (3) we have to observe that $2\mathbb{M}_{2k+1}^{(2m,m)} = \hat{S}(2k+1, 2m, m) = \sum_{j=-\infty}^{\infty} \frac{(-1)^j}{(j2m+m)^{2k+1}} = \frac{1}{m^{2k+1}} \sum_{j=-\infty}^{\infty} \frac{(-1)^j}{(j2+1)^{2k+1}}$. The fact that $\sum_{j=-\infty}^{\infty} \frac{(-1)^j}{(j2+1)^{2k+1}} = \pi^{2k+1}(-1)^k \frac{E_{2k}}{2^{2k+1}(2k)!}$ dates back to Euler and was recalled in [13].

ii) If $m = i = 1$ we have $(1)_{n+1} = (n+1)!$. Recall also that then $B_{n,k}^{(m,i)} = H_{n+1}^{(k)}$. (6) follows additionally (1). ■

Remark 5 *Notice that when $i = 1$, then the sequence $\left\{B_{n,k}^{(m,1)}\right\}$ is generated by the recursion: $B_{n,0}^{(m,1)} = 1$, $B_{n,k}^{(m,1)} = \sum_{j=0}^{n} B_{n.k-1}^{(m,1)}/(mj+1)$. Now arguing by induction we see that $\forall n \geq 0$: $B_{n,k}^{(m,1)} \geq B_{n,k-1}^{(m,1)}$. Consequently we deduce that the sequence $\left\{\mathbb{M}_k^{(m,1)}\right\}_{k\geq 1}$ is increasing, which is not so obvious when considering only definition of these numbers. It is also elementary to notice that*

$$\lim_{k\longrightarrow\infty} \mathbb{M}_k^{(m,1)} = 1.$$

In particular we deduce that the sequence $\{\zeta(k)(1-1/2^{k-1})\}_{k>1}$ is increasing.

Remark 6 *Notice that one can easily prove (by induction) that $\forall n, k \in \mathbb{N}$: $1 \leq H_n^{(k)} \leq n$. Hence, utilizing (6) we have:*

$$\frac{\ln 2 - 1/2}{2^{m+1}(m+1)} \leq \left|\zeta(k) - \frac{2^{k-1}}{2^{k-1}-1} \sum_{n=0}^{m} \frac{H_{n+1}^{(k-1)}}{2^{n+1}(n+1)}\right| \leq \frac{1}{2^{m+1}},$$

since $\frac{2^{k-1}}{2^{k-1}-1} \leq 2$ for $k \leq 2$ and further $\left|\zeta(k) - \frac{2^{k-1}}{2^{k-1}-1} \sum_{n=0}^{m} \frac{H_{n+1}^{(k-1)}}{2^{n+1}(n+1)}\right| \leq \frac{2^{k-1}}{2^{k-1}-1} \sum_{n=m+1}^{\infty} 1/2^{n+1} \leq \frac{2^{k-1}}{2^{k-1}-1}/2^{m+2}$ and $\frac{m+1}{n+1} \geq \frac{1}{n-m+1}$ and $\sum_{n=m+1}^{\infty} \frac{1}{2^{n+1}(n+1)} \geq \frac{1}{2^{m+1}(m+1)} \sum_{n=m+1}^{\infty} \frac{1}{2^{n-m+1}(n-m+1)} = \frac{\ln 2 - 1/2}{2^{m+1}(m+1)}$.

Remark 7 *Formulae (2) and (6) can be considered as a series transformation to speed up its convergence. Apery for $\zeta(3)$ in his breakthrough paper and later Hessami Pilehrood et al. in [9] obtained series transformations to speedup series appearing in the definitions of Riemann or Hurwitz zeta functions. As it is remarked in [9] all these transformation give series more or less of the form $c_n/4^n$ where $c_n = O(1)$, but for different arguments of ζ one gets very different series in a very different, particular way. Apery's one is one of the simplest. Formulae (2) and (6) offer unified form of the transformed series and speed of convergence is only slightly worse. Namely of the form $c_n/2^n$.*

Remark 8 *Notice also that analyzing the proof of Proposition 1 we can formulate the following observation. Let us denote* $A_{n,s}^{(m,l)}=\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}/(mj+l)^{s}$ *for* $\mathrm{Re}(s)>0$. *Then*

$$A_{n+1,s}^{(m,l)}-\frac{m(n+1)}{m(n+1)+l}A_{n,s}^{(m,l)}=\frac{1}{m(n+1)+l}A_{n+1,s-1}^{(m,l)}.$$

Hence keeping in mind that $A_{0,s}^{(m,l)}=1/l^{s}$ *and assuming that we know numbers* $\left\{A_{n,s-1}^{(m,l)}\right\}_{n\geq 0}$ *we are able to get numbers* $\left\{A_{n,s}^{(m,l)}\right\}_{n\geq 0}$ *and consequently find* $\zeta(s,l/m)$.

In particular if $m=l=1$ *we get* $A_{n+1,s}-\frac{n+1}{n+2}A_{n,s}=\frac{1}{n+2}A_{n+1,s-1}$ *where we denoted* $A_{n,s}=A_{n,s}^{(1,1)}$ *to simplify notation. Consequently we deduce that* $A_{n,s}=\frac{1}{n+1}\sum_{j=1}^{n}A_{j,s-1}$. *Since we can iterate this relationship we see that the knowledge of functions* $A_{n,s}$, *for* $\mathrm{Re}(s)\in(0,1]$ *implies knowledge of these functions for* s *with* $\mathrm{Re}(s)>0$.

3 Generating functions and integral representation Riemann zeta functions at integer values

Let us denote by $f_{n}(x)$ the generating function of numbers $\left\{H_{j}^{(n)}\right\}_{j=0}^{\infty}$ i.e. $f_{n}(x)=\sum_{j=0}^{\infty}x^{j}H_{j+1}^{(n)}$. We have the following simple observation:

Proposition 3 *i)* $\forall x\in(-1,1):f_{-1}(x)=1,\ f_{0}(x)=1/(1-x):$

$$f_{n}(x)=\frac{1}{x(1-x)}\int_{0}^{x}f_{n-1}(y)dy, \tag{7}$$

$n\geq 1$.

ii) Let us denote $Q(x,y)$ *the generating function of function series* $\{f_{n}\}_{n\geq 0}$ *i.e.* $Q(x,y)=\sum_{j=0}^{\infty}y^{j}f_{j}(x)$, *for* $y\in(-1,1)$. *We have*

$$Q(x,y)=\frac{B(x,1-y,1+y)}{x^{1-y}(1-x)^{1+y}}, \tag{8}$$

where $B(x,a,b)$ *denotes incomplete beta function.*

Proof. i) We have $f_{n}(x)=\sum_{j=1}^{\infty}x^{j-1}H_{j}^{(n)}=\sum_{j=1}^{\infty}x^{j-1}\sum_{k=1}^{j}H_{k}^{(n-1)}/k=\sum_{k=1}^{\infty}H_{k}^{(n-1)}/k\sum_{j=k}^{\infty}x^{j-1}=\frac{1}{1-x}\sum_{k=1}x^{k-1}H_{k}^{(n-1)}/k=\frac{1}{x(1-x)}\sum_{k=1}^{\infty}H_{k}^{(n-1)}\int_{0}^{s}y^{k-1}dy=\frac{1}{x(1-x)}\int_{0}^{x}\sum_{k=1}^{\infty}y^{k-1}H_{k}^{(n-1)}dy$
$=\frac{1}{x(1-x)}\int_{0}^{x}f_{n-1}(y)dy.$

ii) We have: $(1-x)xQ(x,y) = \sum_{j=0}^{\infty} y^j(1-x)xf_j(x) = x+\sum_{j=1}^{\infty} y^j \int_0^x f_{j-1}(z)dz$ $=$
$x + \int_0^x \sum_{j=1}^{\infty} y^j f_{j-1}(z)dz) = x + y\int_0^x Q(z,y))dz$. Differentiating with respect to x we get: $(1-2x)Q(x,y) + x(1-x)Q'(x,y) = 1 + yQ(x,y)$. Now solving this differential equation we get $Q(x,y) = \frac{Beta(x,1-y,1+y)-C(y)}{x^{1-y}(1-x)^{1+y}}$. Recalling that $Q(0,y) = 1/(1-y)$ we see $C(y) = 0$. ∎

Let us denote for simplicity $\hat{\zeta}(s) \overset{df}{=} \sum_{j=1}^{\infty}(-1)^j/j^s$ for $\mathrm{Re}(s) > 0$. Notice that following (5) we have

$$\hat{\zeta}(k) = \int_0^{1/2} f_{k-1}(x)dx = \frac{1}{4}f_k(1/2), \tag{9}$$

for $k = 1, 2, \ldots$.

We also have:

$$\sum_{j=0}^{\infty} y^j\hat{\zeta}(j) = B(1/2, 1-y, 1+y),$$

for $y \in (-1,1)$ following (8).

Recall that $\sum_{j=1}^{\infty}\zeta(2j)t^{2j} = 1 - \pi t\cot(\pi t)$ hence $\sum_{j=1}^{\infty}\hat{\zeta}(2j)t^{2j} = \frac{\pi t}{\sin(\pi t)} - 1$ after some algebra. Hence

$$\sum_{j=0}^{\infty} y^{2j+1}\hat{\zeta}(2j+1) = B(1/2, 1-y, 1+y) + 1 - \frac{\pi y}{\sin(\pi y)},$$

since $\hat{\zeta}(0) = 1/2$. Let us remark that there exist some expansions of incomplete beta function. Applying one of them we have for example:

$$\sum_{j=0}^{\infty} y^j\hat{\zeta}(j) = 2^{y-1}\sum_{j=0}^{\infty}\frac{(-y)_j}{j!(j+1-y)2^j},$$

for $y \in (0,1)$.

4 Remarks on particular values

In [13] the sums of the form $S(n,k,l) = \sum_{j=-\infty}^{\infty}\frac{1}{(jk+l)^n}$, $\hat{S}(n,k,l) = \sum_{j=-\infty}^{\infty}\frac{(-1)^j}{(jk+l)^n}$ were analyzed and some of them were calculated. From the results of this paper it follows that the following sums:

$$\mathbb{M}_k^{(m,i)} + (-1)^{k+1}\mathbb{M}_k^{(m,m-i)}$$

have values of the form π^k times some known, analytic number. Notice that this statement is trivial for k odd, m even and $i = m/2$.

In particular we get for $k = 2l$ we have $\mathbb{M}_{2l}^{(m,i)} - \mathbb{M}_{2l}^{(m,m-i)} = \frac{1}{m^{2l}}(\zeta(2l, l/(2m)) - \zeta(2l,(m+i)/(2m)) - \zeta(2l,(m-i)/(2m)) + \zeta(2l,(2m-i)/(2m)) = \frac{1}{m^{2l}}(\zeta(2l, l/(2m)) + \zeta(2l, 1 - i/(2m)) - \zeta(2l,(m+i)/(2m) - \zeta(2l,(m-i)/(2m))$.

Following [13] we also have for $k \geq 1$:

$$S(2k, 4, 1) = \frac{1}{4^{2k}}(\zeta(2k, 1/4) + \zeta(2k, 3/4)) = \pi^{2k}\frac{(2^{2k}-1)}{2(2k)!}(-1)^{k+1}B_{2k},$$

where B_{2k} denotes $2k - th$ Bernoulli number. In particular we have

$$16\mathbb{K} = (\zeta(2, 1/4) - \zeta(2, 3/4)); (\zeta(2, 1/4) + \zeta(2, 3/4)) = 2\pi^2.$$

Finally let us recall that $\zeta(2l, 1) = (-1)^{l+1}B_{2l}\frac{(2\pi)^{2l}}{2(2l)!}$. Using formula (6) we get:

$$(-1)^{l+1}B_{2l}\frac{(2\pi)^{2l}}{2(2l)!} = \frac{2^{2l-1}}{2^{2l-1}-1}\sum_{n=1}^{\infty}\frac{H_n^{(2l-1)}}{n2^n},$$

and consequently we obtain the following expansions of even powers of π :

$$\pi^{2l} = (-1)^{l+1}\frac{(2l)!}{(2^{2l-1}-1)B_{2l}}\sum_{n=1}^{\infty}\frac{H_n^{(2l-1)}}{n2^n}.$$

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