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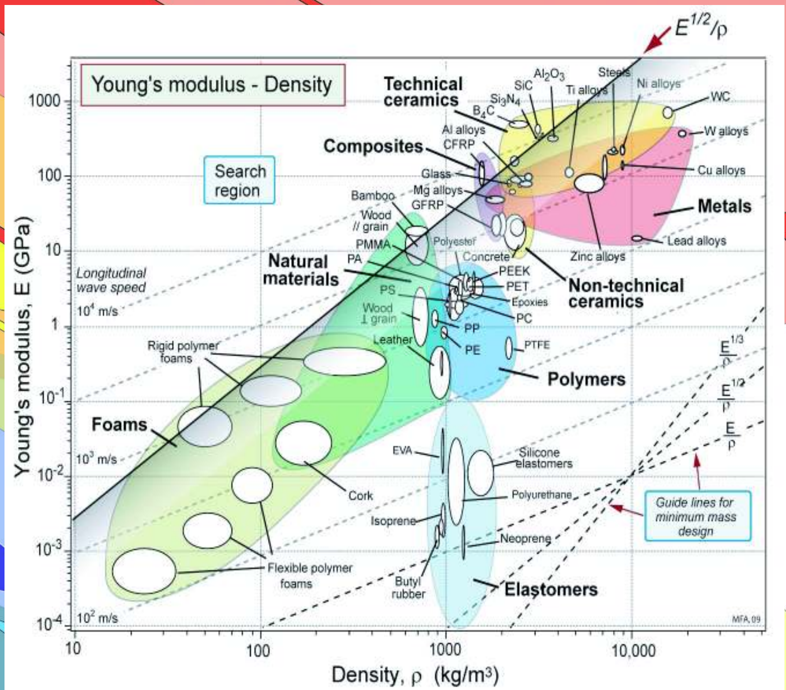
MATERIAL INSPIRATION




# CES 2010 EDUPACK

# Solution Manual:

## Material Selection for Mechanical Design 4th Edition

Cambridge University

Version MFA 10

*Materials selection in mechanical design, 4th edition*

# Exercises with worked solutions

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**Contents**

## E1 Introduction to exercises

These exercises are designed to develop facility in selecting materials, processes and shape, and in devising hybrid materials when no monolithic material completely meets the design requirements. Each exercise is accompanied by a worked solution. They are organized into the twelve sections listed on the first page.

The early exercises are easy. Those that follow lead the reader through the use of *material properties* and *simple solutions to mechanics problems*, drawing on data and results contained in Appendices A and B; the use of *material property charts*; techniques for the translation of design requirement to identify *constraints* and *objectives;* the derivation of *indices*, *screening* and *ranking*, *multi-objective optimization*; coupled choice of *material* and *shape*; devising *hybrids*; and the choice of materials to meet *environmental criteria*.

Three important points.

1. Selection problems are open-ended and, generally, under-specified; there is seldom a single, correct answer. The proper answer is sensible translation of the design requirements into material constraints and objectives, applied to give a short-list of potential candidates with commentary suggesting what supporting information would be needed to narrow the choice further.
2. The positioning of selection-lines on charts is a matter of judgement. The goal is to place the lines such that they leave an adequately large "short list" of candidates (aim for 4 or so), drawn, if possible, from more than one class of material.
3. A request for a selection based on one material index alone (such as $M = E^{1/2} / \rho$) is correctly answered by listing the subset of materials that maximize this index. But a request for a selection of materials for a component – a wing spar, for instance (which is a light, stiff beam, for which the index is $M = E^{1/2} / \rho$) – requires more: some materials with high $E^{1/2} / \rho$ such as silicon carbide, are unsuitable for obvious reasons. It is a poor answer that ignores common sense and experience and fails to add further constraints to incorporate them. Students should be encouraged to discuss the implications of their selection and to suggest further selection stages.

The best way to use the charts that are a feature of the book is to make clean copies (or down-load them from http://www.grantadesign.com ) on which you can draw, try out alternative selection criteria, write comments and so forth. Although the book itself is copyrighted, the reader is authorized to make copies of the charts and to reproduce these, with proper reference to their source, as he or she wishes.

All the materials selection problems can be solved using the *CES EduPack* software, which is particularly effective when multiple criteria and unusual indices are involved.

## E2 Material evolution in products (Chapter 1)

**E 2.1.** Use Google to research the history and uses of one of the following materials

- Tin
- Glass
- Cement
- Titanium
- Carbon fiber

Present the result as a short report of about 100 - 200 words (roughly half a page).

***Specimen answer: tin.*** Tin (symbol Sn), a silver-white metal, has a long history. It was traded in the civilisations of the Mediterranean as early as 1500 BC (the Old Testament of the Christian bible contains many references to it). Its importance at that time lay in its ability to harden copper to give *bronze* (copper containing about 10% tin), the key material for weapons, tools and statuary of the Bronze age (1500 BC – 500 BC). Today tin is still used to make bronze, for solders and as a corrosion resistant coating on steel sheet ("tin plate" ) for food and drink containers – a "tinnie", to an Australian, is a can of beer. Plate glass is made by floating molten glass on a bed of liquid tin (the Pilkington process). Thin deposits of tin compounds on glass give transparent, electrically conducting coatings used for frost-free windshields and for panel lighting.

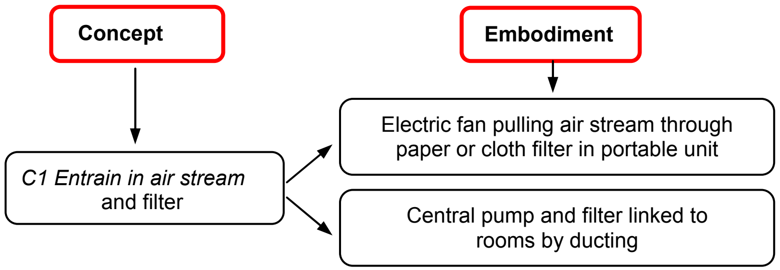
**E2.2** Research, at the level of the mini case studies in this chapter, the evolution of material use in

- Writing implements (charcoal, "lead" (graphite), quill pens, steel nib pens, gold plus osmium pens, ball points..)
- Watering cans (wood – galvanized iron – polypropylene)
- Bicycles (wood – bamboo – steel, aluminum, magnesium, titanium – CFRP)
- Small boat building (wood – aluminum – GFRP)
- Book binding (Wood – leather – cardboard – vinyl)

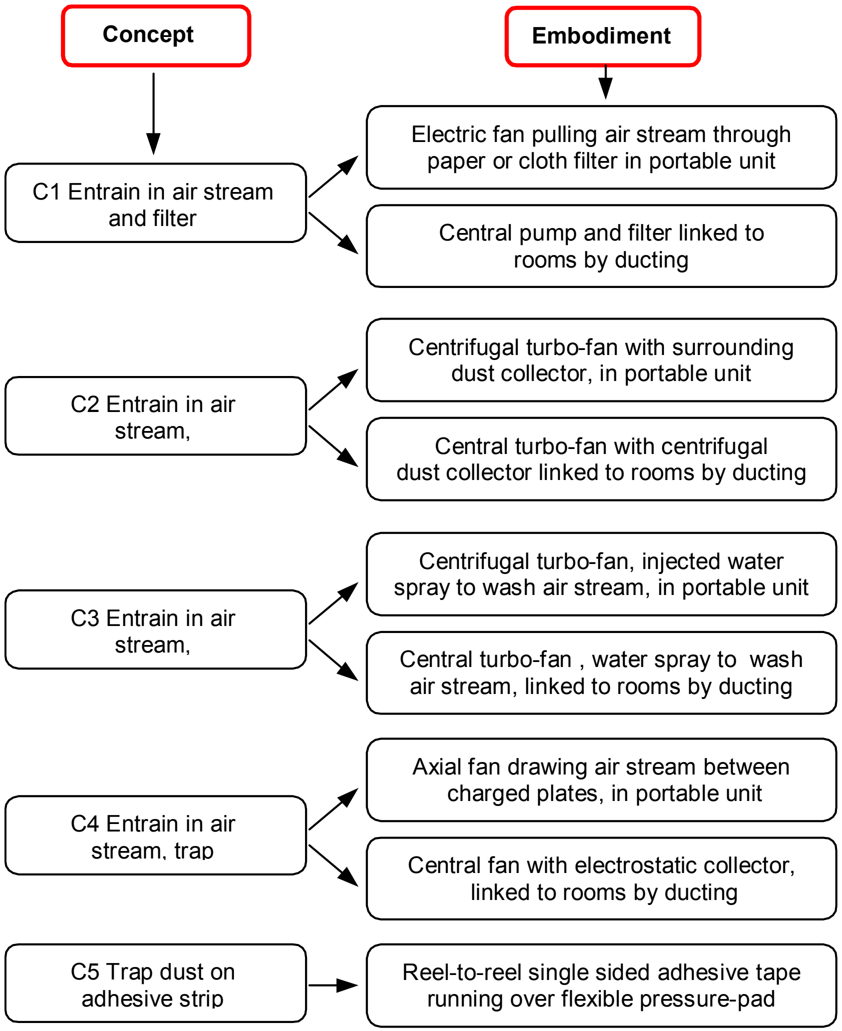
## E3 Devising concepts (Chapter 2)

*These two examples illustrate the way in which concepts are generated. The left-hand part of each diagram describes a physical principle by which the need might be met; the right-hand part elaborates, suggesting how the principle might be used.*

**E3.1 Concepts and embodiments for dust removers**. We met the need for a "device to remove household dust" in Chapter 1, with examples of established solutions. Now it is time for more creative thinking. Devise as many concepts to meet this need as you can. Nothing, at the concept stage, is too far-fetched; decisions about practicality and cost come later, at the detailed stage. So think along the lines of Figure 2.2 of the main text and list concepts and outline embodiments as block diagrams like this:



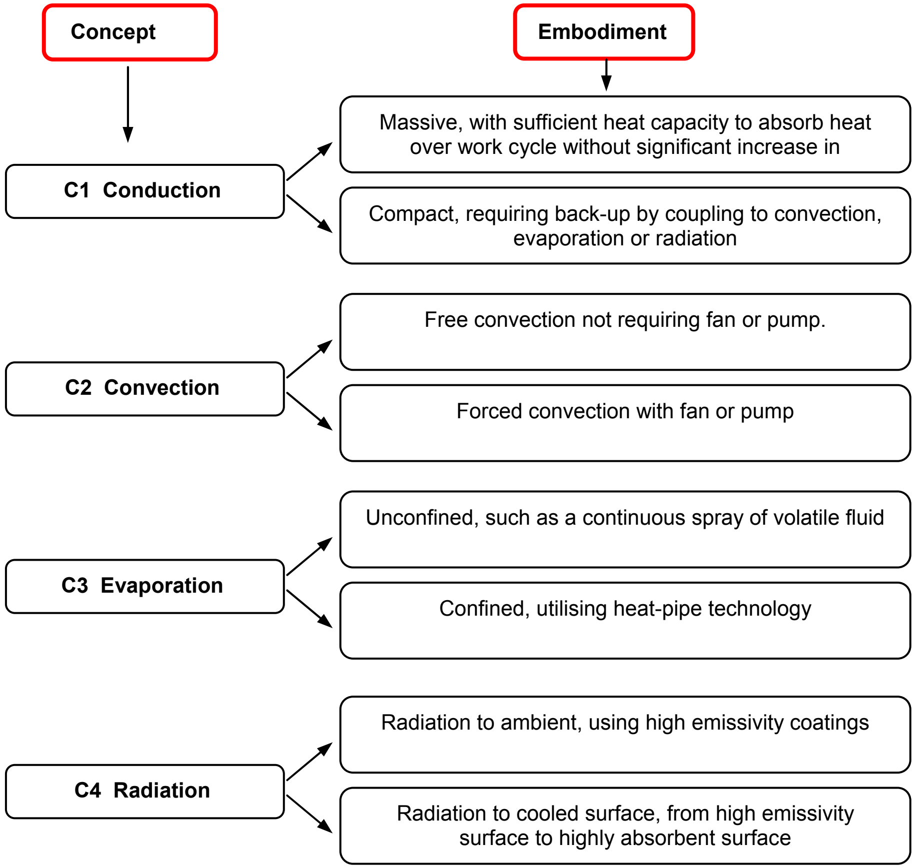

***Answer.*** Design problems are open-ended; there are always alternative solutions. Here are some for dust removers.



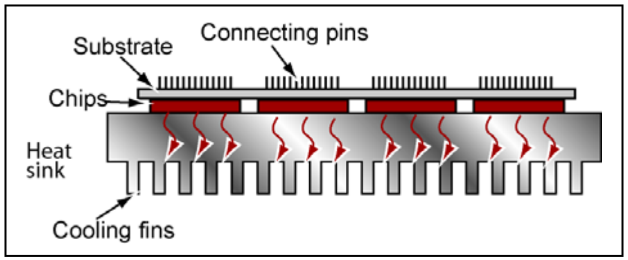
**E3.2 Cooling power electronics.** Microchips, particularly those for power electronics, get hot. If they get too hot they cease to function. The need: a scheme for removing heat from power microchips. Devise concepts to meet the need and sketch an embodiment of one of them, laying out your ideas in the way suggested in exercise E3.1.

***Answer.*** Four working principles are listed below: thermal conduction, convection by heat transfer to a fluid medium, evaporation exploiting the latent heat of evaporation of a fluid, and radiation, best achieved with a surface with high emissivity.

The best solutions may be found by combining two of these: conduction coupled with convection (as in the sketch – an often-used combination) or radiation coupled with evaporation (a possibility for short-life space structures).




Sketch.



## E4 Using material properties (Chapter 3)

*These exercises introduce the reader to 2 useful resources: the data sheets of Appendix A and the Solutions to Standard Problems of Appendix B.*

**E4.1** A cantilever beam has a length *L* = 50 mm, a width *b* = 5 mm and a thickness *t* = 1 mm. It is made of an aluminum alloy. By how much will the end deflect under and end-load of 5 N? Use data from Appendix A4 for the (mean) value of Young's modulus of aluminum alloys, the equation for the elastic deflection of a cantilever from Appendix B3 and for the second moment of a beam from Appendix B2 to find out.

***Answer.*** The deflection $\delta$ of a cantilever under an end load $F$ is, from Appendix B,

$$\delta = \frac{F L^3}{3 E I} \qquad \text{with} \qquad I = \frac{b t^3}{12}$$

The mean Young's modulus $E$ for aluminum, from Appendix A4, is 75 GPa. Inserting the data from the question results in an end deflection $\delta$ = 6.7 mm.

**E4.2** A spring, wound from stainless steel wire with a wire diameter $d$ = 1mm, has $n$ = 20 turns of radius $R$ = 10 mm. How much will it extend when loaded with a mass $P$ of 1 kg? Assume the shear modulus $G$ of stainless steel to be 3/8 $E$ where $E$ is Young's modulus, retrieve this from Appendix A4, and use the expression for the extension of springs from Appendix B6 to find out.

***Answer.*** The extension $u$ of a spring under a force $F = Pg$ = 9.81 N (here $g$ is the acceleration due to gravity) is

$$u = \frac{64 F R^3 n}{G d^4}$$

Young's modulus for stainless steel is 200 GPa, so shear modulus $G \approx$ 76 GPa. Inserting the data gives a deflection $u$ = 10.4 mm.

**E4.3** A thick-walled tube has an inner radius $r_i$ = 10 mm and an outer radius $r_o$ = 15 mm. It is made from polycarbonate, PC. What is the maximum torque that the tube can carry without the onset of yield? Retrieve the (mean) yield strength $\sigma_y$ of PC from Appendix A5, the expression for the torque at onset of yield from Appendix B6 and that for the polar moment of a thick walled tube from Appendix B2 to find out.

***Answer.*** The torque at the onset of yield for a thick walled tube is

$$T_f = \frac{K \sigma_y}{2 r_o} \qquad \text{with} \qquad K = \frac{\pi}{2}(r_o^4 - r_i^4)$$

The mean yield strength of $\sigma_y$ of PC from Appendix A5 is 65 MPa. Inserting the data from the question gives a torque at the onset of yield of $T_f$ = 138 N.m.

**E4.4** A round bar, 20 mm in diameter, has a shallow circumferential notch with a depth $c$ = 1 mm with a root radius $r$ = 10 microns. The bar is made of a low carbon steel with a yield strength of $\sigma_y$ = 250 MPa. It is loaded axially with a nominal stress, $\sigma_{nom}$ (the axial load divided by the un-notched area). At what value of $\sigma_{nom}$ will yield first commence at the root of the notch? Use the stress concentration estimate of Appendix B9 to find out.

***Answer.*** The stress concentration caused by notch of depth $c$ and root radius $r$ is

$$\frac{\sigma_{max}}{\sigma_{nom}} = 1 + \alpha \left( \frac{c}{r} \right)^{1/2} \qquad \text{with} \qquad \alpha \approx 2 \quad \text{for tension}$$

Yield first starts when $\sigma_{max} = \sigma_y$. Inserting the data from the question gives a nominal stress for first yield of 11.9 MPa. Stress concentrations can be very damaging – in this example, a cyclic stress of only ± 12 MPa will ultimately initiate a fatigue crack at the notch root.

**E4.5** An acrylic (PMMA) window is clamped in a low carbon steel frame at $T$ = 20 C. The temperature falls to $T$ = -20 C, putting the window under tension because the thermal expansion coefficient of PMMA is larger than that of steel. If the window was stress-free at 20C, what stress does it carry at -20 C? Use the result that the bi-axial stress caused by a bi-axial strain difference $\Delta\varepsilon$ is

$$\sigma = \frac{E\Delta\varepsilon}{1-\nu}$$

where $E$ is Young's modulus for PMMA and Poisson's ratio $\nu = 0.33$. You will find data for expansion coefficients in Table A7, and for moduli in Table B5. Use mean values.

***Answer.*** The strain difference caused by difference in thermal expansion, $\alpha$, when the temperature changes by $\Delta T$ is

$$\Delta\varepsilon = \left(\alpha_{PMMA} - \alpha_{Low\ C\ steel}\right)\Delta T$$

From Appendix A7,

$\alpha_{PMMA}$ = 117x $10^{-6}$/C and $\alpha_{Low\,C\,steel}$ = 12.3 x $10^{-6}$/C

giving $\Delta\varepsilon =$ 4.2 x $10^{-3}$. The modulus of PMMA, from Appendix A5, is $E =$ 3.0 GPa. The equation given in the question then predicts a tensile stress in the window of $\sigma$ = 19 MPa.

**E4.6** The PMMA window described in Exercise 4.5 has a contained crack of length $2a$ = 0.5 mm. If the maximum tensile stress that is anticipated in the window is $\sigma$ = 20 MPa, will the crack propagate? Choose an appropriate equation for crack propagation from Appendix B10 and data for the fracture toughness $K_{1c}$ of PMMA from Appendix A6 to calculated the length of crack that is just unstable under this tensile stress.

***Answer.*** The crack length is small compared with the width of the window, so the appropriate choice of equation describing crack instability is

$$C\sigma\sqrt{\pi a} \geq K_{1c} \qquad \text{with} \qquad C = 1.0$$

Inserting the data we find the length of the shortest crack that is just unstable:

$$2a = \frac{2}{\pi}\left(\frac{K_{1c}}{\sigma}\right)^2$$ = 2.1 mm, using $(K_{1c})_{PMMA}$ = 1.15 MPa.m$^{1/2}$

Thus the 0.5mm crack will not propagate.

**E4.7** A flywheel with a radius $R$ = 200 mm is designed to spin up to 8000 rpm. It is proposed to make it out of cast iron, but the casting shop can guarantee only that it will have no crack-like flaws greater than $2a$ = 2 mm in length. Use the expression for the maximum stress in a spinning disk in Appendix B7, that for the stress intensity at a small enclosed crack from Appendix B10 and data for cast iron from Appendix A3 and A6 to establish if the flywheel is safe. Take Poisson's ratio $\nu$ for cast iron to be 0.33.

***Answer.*** The maximum tensile stress in a spinning disk is

$$\sigma_{max} = \frac{(3+\nu)}{8}\rho\,\omega^2 R^2, \quad \text{and} \quad K_1 = \sigma_{max}\sqrt{\pi a} \leq K_{1c}$$

for a contained crack. Here $\omega = 2\pi W/60$ radians/sec when $W$ is the rotational velocity in rpm. Inserting the data from the question and the mean values for density $\rho$ = 7150 kg/m$^3$ and $K_{1c}$ = 38 MPa.m$^{1/2}$ from Appendix A, we find the maximum that the rotational velocity that will just cause the cracks to propagate is 4350 radians/s, or 41,600 rpm. The flywheel is safe.

**E4.8** You wish to assess, approximately, the thermal conductivity $\lambda$ of polyethylene (PE). To do so you block one end of a PE pipe with a wall thickness of $x$ = 3 mm and diameter of 30 mm and fill it with boiling water while clutching the outside with your other hand. You note that the outer surface of the pipe first becomes appreciably hot at a time $t \approx$ 18 seconds after filling the inside with water. Use this information, plus data for specific heat $C_p$ and density $\rho$ of PE from Appendices A3 and A8, to estimate $\lambda$ for PE. How does your result compare with the listed value in Table A7?

***Answer.*** The distance $x$ that heat diffuses in a time $t$ is approximately

$$x = \sqrt{2\,a\,t} \qquad \text{with} \qquad a = \frac{\lambda}{\rho\, C_p}$$

($a$ is the thermal diffusivity). Inserting the data from the question and the mean values $\rho$ = 950 kg/m$^3$ and $C_p$ = 1850 J/kg/K from the Appendix, we find $\lambda \approx$ 0.44 W/m.K. The result given in Appendix A7 for the thermal conductivity of PE is 0.40 – 0.44 W/m.K.

**E4.9** The capacitance $C$ of a condenser with two plates each of area $A$ separated by a dielectric of thickness $t$ is

$$C = \varepsilon_r \varepsilon_o \frac{A}{t}$$

where $\varepsilon_o$ is the permittivity of free space and $\varepsilon_r$ is the dielectric constant of the material between the plates. Select a dielectric by scanning data in Appendix A9 (a) to maximize $C$ and (b) to minimize it, for a given $A$ and $t$.

***Answer.*** (a) Capacitance is maximized by selecting materials with high $\varepsilon_r$ . Appendix A shows that Neoprene, Polyurethane thermoplastic, Polyurethane elastomers, and certain ceramics have values of $\varepsilon_r > 6.0$.

(b) Capacitance is minimized by materials with low $\varepsilon_r$ : Polyethylene, Polypropylene and Teflon (PTFE), and, particularly, polymer foams.

**E4.10** It is proposed to replace the cast iron casing of a power tool with one with precisely the same dimension molded from nylon. Will the material cost of the nylon casing be greater or less than that made of cast iron? Use data from Appendix A3 and A11 to find out.

***Answer.*** If the dimensions of the cast iron and nylon cases are the same, the volume of material required to make them are equal. Thus the cheaper option is the one with the lower material cost per unit volume $C_v$, where

$$C_v = \rho\, C_m$$

and $\rho$ is the material density and $C_m$ the material cost per kg. Data from Appendix A3 and A11 are assembled below, using the means of the ranges..

| | Density kg/m$^3$ | Price $/kg | Cost per unit vol $/m$^3$ |
|---|---|---|---|
| Cast iron | 7150 | 0.63 | 4500 |
| Nylon | 1130 | 3.45 | 3900 |

Surprisingly, the nylon casing has a lower material cost than that made of cast iron.

## E5. Using material selection charts (Chapter 4)

*The 20 exercises in this section involve the simple use of the charts of Chapter 4 to find materials with give property profiles. They are answered by placing selection lines on the appropriate chart and reading off the materials that lie on the appropriate side of the line. It is a good idea to present the results as a table. All can be solved by using the printed charts.*

*If the CES EduPack software is available the same exercises can be solved by its use. This involves first creating the chart, then applying the appropriate box or line selection. The results, at Level 1 or 2, are the same as those read from the hard copy charts (most of which were made using the Leve1 / 2 database). The software offers links to processes, allows a wider search by using the Level 3 database, and gives access to supporting information via the "Search Web" function.*

**E5.1** A component is at present made from a brass, a copper alloy. Use the Young's modulus – Density (*E*-□ chart of Figure 4.3 to suggest three other metals that, in the same shape, would be stiffer. "Stiffer" means a higher value of Young's modulus.

***Answer.*** Metals that are stiffer than brass are listed in the table.

| Material | Comment |
|---|---|
| Steels | The cheapest stiff, strong structural metal, widely used. |
| Nickel alloys | More expensive than steel |
| Tungsten alloys | Refractory (high-melting) and relatively expensive |

**E5.2** Use the Young's modulus – Density ($E$-$\rho$) chart of Figure 4.3 to identify materials with both a modulus $E$ > 50 GPa and a density $\rho$ < 2000 kg/m$^3$.

***Answer.*** There is only two materials on the chart with modulus $E$ > 50 GPa and density $\rho$ < 2000 kg/m$^3$.

| Material | Comment |
|---|---|
| Magnesium alloys | Magnesium is the lightest of all common structural metals – only beryllium is lighter, but it is very expensive and its oxide is toxic. |
| CFRP – carbon-fiber reinforced plastic | CFRP is both lighter and stiffer than magnesium. That is one reason it is used for competition cars and bikes. |

**E5.3** Use the Young's Modulus-Density ($E$-$\rho$) chart of Figure 4.3 to find (a) metals that are stiffer and less dense than steels and (b) materials (not just metals) that are both stiffer and less dense than steel.

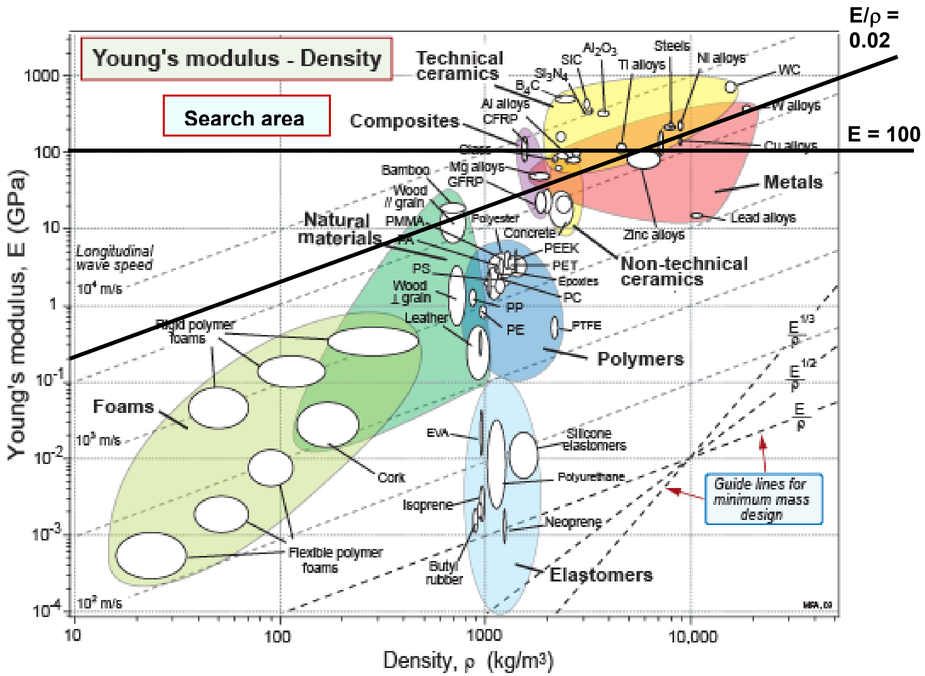
***Answer.*** (a) No metals in Figure 4.3 are both stiffer and less dense than steel, though nickel alloys come close. (b) Several ceramics qualify: Boron carbide, $B_4C$, silicon carbide, SiC, silicon nitride $Si_3N_4$ and alumina $Al_20_3$.

| Material | Comment |
|---|---|
| Alumina $Al_20_3$<br>Silicon nitride $Si_3N_4$<br>Boron carbide, $B_4C$<br>Silicon carbide, SiC | Alumina is the most widely used of all technical ceramics (spark plugs, circuit boards…) All ceramics are brittle – they have low values of fracture toughness $K_{1c}$ and toughness $G_{1c}$ . |

**E5.4** Use the $E$-$\rho$ chart of Figure 4.3 to identify metals with both $E >$ 100 GPa and $E/\rho > 0.02$ GPa/(kg/m$^3$).

***Answer.*** The chart shows the selection. The metals that lie in the search area are listed in the table.

| Material | Comment |
|---|---|
| Steels | Cheap, widely used. Stiff structural material. |
| Nickel alloys | More expensive than steel |
| Titanium alloys | Titanium alloys are very expensive. |






**E5.5** Use the E-$\rho$ chart of Figure 4.3 to identify materials with both E > 100 GPa and $E^{1/3}/\rho > 0.003$ (GPa)$^{1/3}$/(kg/m$^3$).

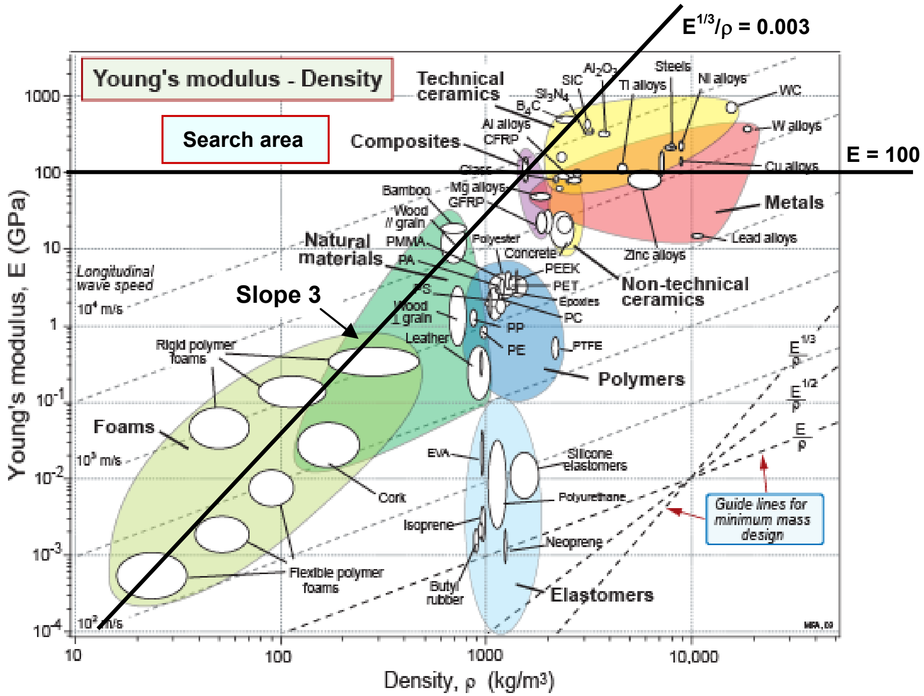
Remember that, on taking logs, the index $M = E^{1/3}/\rho$ becomes

$$Log(E) = 3\,Log(\rho) + 3\,Log(M)$$

and that this plots as a line of slope 3 on the chart, passing through the point $E = 27$ when $\rho = 1000$ in the units on the chart.

***Answer*.** The chart shows the selection. The materials that lie in the search area are listed in the table. No metals survive.

| Material | Comment |
|---|---|
| CFRP | Carbon-fiber composites excel in stiffness at low weight. |
| Boron carbide, $B_4C$ | Boron carbide is exceptionally stiff, hard and light; it is used for body armour. |



**E5.6** Use the $E$-$\rho$ chart of Figure 4.3 to establish whether woods have a higher specific stiffness $E/\rho$ than epoxies.

***Answer.*** Parallel to the grain, woods have much higher specific stiffness than epoxies. Perpendicular to the grain, woods have about the same value as epoxies.

| Material | $E/\rho$ (GPa/(kg/m$^3$) x 10$^3$ |
|---|---|
| Woods parallel to the grain<br>Woods transverse to the grain | 8 - 29<br>0.7 – 4.0 |
| Epoxies | 1.8 – 2.5 |

**E5.7** Do titanium alloys have a higher or lower specific strength (strength/density, $\sigma_f / \rho$ ) than the best steels? This is important when you want strength at low weight (landing gear of aircraft, mountain bikes). Use the $\sigma_f / \rho$ chart of Figure 4.4 to decide.

***Answer.*** The guide line for $\sigma_f / \rho$ on the strength – density chart of Figure 4.4, if drawn through titanium alloys shows that they have a much the higher specific strength than any steel, even though the best steel are as strong..

| Material | $\sigma_f / \rho$ (MPa/(kg/m$^3$)) x 10$^3$ |
|---|---|
| Titanium alloys | 54 - 270 |
| Steels | 32 - 190 |

**E 5.8** Use the modulus-strength $E-\sigma_f$ chart of Figure 4.5 to find materials that have $E$ > 10 GPa and $\sigma_f \geq$ 1000 MPa.

***Answer.*** The strongest steels, titanium alloys (Ti-alloys) and carbon fiber reinforced polymers (CFRP) meet these limits.

| Material | Comment |
|---|---|
| High strength steels | All have $\sigma_f$ above 1000 MPa, a very large value |
| High strength nickel alloys | |
| Titanium alloys | |
| CFRP | |

**E5.9** Are the fracture toughness, $K_{1c}$, of the common polymers polycarbonate, ABS, or polystyrene larger or smaller than the engineering ceramic alumina, $Al_2O_3$? Are their toughness $G_{1c} = K_{1c}^2 / E$ larger or smaller? The $K_{1c}$ –$E$ chart, Figure 4.7, will help.

***Answer.*** Most polymers have a lower fracture toughness, $K_{1c}$, than alumina. Their toughness, $G_{1c} = K_{1c}^2 / E$, however, are much larger. Even the most brittle of polymers, polystyrene, has a toughness $G_{1c}$ that is nearly ten times greater than that of alumina. The values in the table are read from Chart 4.7, using the $K_{1c}$ axis to read off values of fracture toughness, and the $G_{1c} = K_{1c}^2 / E$ contours to read off values of toughness.

| Material | $K_{1c}$ (MPa.m$^{1/2}$) | $G_{1c}$ (kJ/m$^2$) |
|---|---|---|
| Polycarbonate | 2.1 – 4.6 | 0.5 - 8 |
| ABS | 1.2 – 4.3 | 1 - 8 |
| Polystyrene | 0.7 – 1.1 | 0.25 – 0.7 |
| Alumina $Al_2O_3$ | 3.3 – 4.8 | 0.04 – 0.07 |

**E 5.10** Use the fracture toughness-modulus chart (Figure 4.7) to find materials that have a fracture toughness $K_{1c}$ greater than 100 MPa.m$^{1/2}$ and a toughness $G_{1c} = K_{1c}^2 / E$ (shown as contours on Figure 4.7) greater than 10 kJ/m$^3$.

***Answer.*** The table lists the results. Only metals have both high fracture toughness $K_{1c}$ *and* high toughness $G_{1c}$. That is one reason that they are used for pressure vessels (boilers, submarine hulls, gas containers etc).

| Material | Comment |
|---|---|
| High strength steels | All have high $K_{1c}$ and high $G_{1c}$ |
| High strength nickel alloys | |
| Titanium alloys | |

**E5.11** The elastic deflection at fracture (the "resilience") of an elastic-brittle solid is proportional to the failure strain, $\varepsilon_{fr} = \sigma_{fr} / E$, where $\sigma_{fr}$ is the stress that will cause a crack to propagate:

$$\sigma_{fr} = \frac{K_{1c}}{\sqrt{\pi c}}$$

Here $K_{1c}$ is the fracture toughness and $c$ is the length of the longest crack the materials may contain. Thus

$$\varepsilon_{fr} = \frac{1}{\sqrt{\pi c}}\left(\frac{K_{1c}}{E}\right)$$

Materials that can deflect elastically without fracturing are therefore those with large values of $K_{1c} / E$. Use the $K_{1c} - E$ chart of Figure 4.7 to identify the class of materials with $K_{1c}$ > 1 MPa.m$^{1/2}$ and high values of $K_{1c} / E$.

***Answer.*** Polymers and foams both have large $K_{1c} / E$ allowing them to flex without fracturing. They have much higher values $K_{1c} / E$ than metals or ceramics, and thus large fracture strains. Only polymers have, additionally, $K_{1c}$ > 1 MPa.m$^{1/2}$. This makes them attractive for snap-together parts that must flex without failing. Leather has a particularly high value of resilience, $K_{1c} / E$.

**E 5.12** One criterion for design of a safe pressure vessel is that it should leak before it breaks: the leak can be detected and the pressure released. This is achieved by designing the vessel to tolerate a crack of length equal to the thickness $t$ of the pressure vessel wall, without failing by fast fracture. The safe pressure $p$ is then

$$p \le \frac{4}{\pi}\frac{1}{R}\left(\frac{K_{1c}^2}{\sigma_f}\right)$$

where $\sigma_f$ is the elastic limit, $K_{1c}$ is the fracture toughness, $R$ is the vessel radius. The pressure is maximized by choosing the material with the greatest value of

$$M = \frac{K_{1c}^2}{\sigma_y}$$

Use the $K_{1c} - \sigma_f$ chart of Figure 4.8 to identify three alloys that have particularly high values of *M.*

***Answer.*** Alloys with high values of $K_{1c}^2 / \sigma_y$, read from the chart, are listed below

| Material | Comment |
|---|---|
| Low alloy steels | Traditional material for pressure vessels. |
| Stainless steels | Used for nuclear pressure vessels. |
| Copper alloys | Small boilers are made of copper. |
| Nickel alloys | Reactors for chemical engineering and turbine combustion chambers are made of nickel based alloys. |

**E5.13** A material is required for the blade of a rotary lawn-mower. Cost is a consideration. For safety reasons, the designer specified a minimum fracture toughness for the blade: it is $K_{1c}$ > 30 MPa m$^{1/2}$. The other mechanical requirement is for high hardness, *H*, to minimize blade wear. Hardness, in applications like this one, is related to strength:

$$H \approx 3\sigma_y$$

where $\sigma_f$ is the strength (Chapter 4 gives a fuller definition). Use the $K_{1c} - \sigma_f$ chart of Figure 4.8 to identify three materials that have $K_{1c}$ > 30 MPa m$^{1/2}$ and the highest possible strength. To do this, position a "$K_{1c}$" selection line at 30 MPa m$^{1/2}$ and then adjust a "strength" selection line such that it just admits three candidates. Use the Cost chart of Figure 4.19 to rank your selection by material cost, hence making a final selection.

***Answer.*** Applying the property limit $K_{1c}$ > 30 MPam$^{1/2}$, and reading of materials with $\sigma_f$ > 800 MPa, gives three groups of materials. Chart 4.19 identifies the price of each of these per unit volume. The steels are far cheaper than the other two.

| Material | Comment |
|---|---|
| High-strength steels | Traditional material for blades |
| Nickel alloys | Meets the requirements, but more expensive than steel |
| Titanium alloys | Meets the requirements, but MUCH more expensive than steel. |

**E5.14** Bells ring because they have a low loss (or damping) coefficient, $\eta$; a high damping gives a dead sound. Use the Loss coefficient – Modulus ($\eta - E$) chart of Figure 4.9 to identify material that should make good bells.

***Answer.*** Materials with low loss coefficient, any of which could be used to make a bell, are listed below.

| Material | Comment |
|---|---|
| Copper alloys | The traditional material for bells: bronzes and brasses |
| Glass, silica, $SiO_2$ | Glass makes excellent bells |
| Ceramics: $Al_2O_3$, SiC | Unusual choice, expensive, but should work. |

**E5.15.** Use the Loss coefficient-Modulus ($\eta - E$) chart (Figure 4.9) to find metals with the highest possible damping.

***Answer.*** Lead alloys have very high damping – they are used to clad buildings to deaden sound and vibration. Magnesium alloys also have high damping: they are used to dampen vibration in machine tools.

| Material | Comment |
|---|---|
| Lead alloys | Used to clad buildings to damp sound and vibration |
| Magnesium alloys | Used to damp vibration in machine tools |

**E5.16** Use the Thermal conductivity-Electrical resistivity ( $\lambda - \rho_e$ ) chart (Figure 4.10) to find three materials with high thermal conductivity, $\lambda$ , and high electrical resistivity, $\rho_e$ .

***Answer.*** Aluminum nitride is the best choice. The next best are alumina and silicon carbide.

| Material | Comment |
|---|---|
| Aluminum nitride, AlN | Favored material for heat sinks requiring this combination of properties |
| Alumina, $Al_2O_3$ | Both meet the requirements |
| Silicon carbide, SiC | |

**E5.17** The window through which the beam emerges from a high-powered laser must obviously be transparent to light. Even then, some of the energy of the beam is absorbed in the window and can cause it to heat and crack. This problem is minimized by choosing a window material with a high thermal conductivity $\lambda$ (to conduct the heat away) and a low expansion coefficient $\alpha$ (to reduce thermal strains), that is, by seeking a window material with a high value of

$$M = \lambda / \alpha$$

Use the $\alpha - \lambda$ chart of Figure 4.12 to identify the best material for an ultra-high powered laser window.

***Answer.*** The chart shows three transparent materials with high $\lambda / \alpha$ : soda glass, silica glass and alumina, which, in single crystal form (sapphire) or ultra-fine grained form ("Lucalox") is transparent and hard. Diamond, not shown on the chart, has an exceptionally high value: it has been used for ultra high-powered laser windows.

| Material | Comment |
|---|---|
| Soda glass | $M$ = 3 x $10^5$ W/m; poor resistance to thermal pulse |
| Silica glass | $M$ =2 x $10^6$ W/m; much better than soda glass |
| Alumina (sapphire) | $M$ = 3 x $10^6$ W/m; a little better than silica glass |
| (Diamond) | $M$ = 3 x $10^8$ W/m; outstanding |

**E5.18** Use the Maximum service temperature ($T_{max}$) chart (Figure 4.14) to find polymers that can be used above $200^0$ C.

***Answer.*** The chart shows just two classes of polymer with maximum service temperatures greater than $200^0$ C. They are listed below.

| Material | Comment |
|---|---|
| Polytetrafluorethylene, PTFE | PTFE (Teflon) is used as non-stick coatings for cooking ware, easily surviving the temperatures of baking and frying. |
| Silicone elastomers | Silicones are polymers with a Si-O-Si chain structure instead of the C-C-C chain of polyolefins. They are more stable than carbon-based polymers, but expensive. |

**E5.19** (a) Use the Young's modulus-Relative cost ($E - C_{v,R}$) chart (Figure 4.18) to find the cheapest materials with a modulus, *E,* greater than 100 GPa.

(b) Use the Strength-Relative cost ($\sigma_f - C_R$) chart (Figure 4.19) to find the cheapest materials with a strength, $\sigma_f$, above 100MPa.

***Answer.*** (a) The two cheap classes of material that meet the constraints are cast irons and carbon steels.

(b) Cast irons and steels are again the best choice. It is because of their high stiffness and strength at low cost that they are so widely used.

**E5.20** Use the Friction coefficient chart, (Figure 4.15) to find two materials with exceptionally low coefficient of friction.

***Answer.*** PTFE and polyethylene (PE) have low coefficient of friction when sliding on steel (and on most other materials). Both are used for the sliding surface of skis.

## E6. Translation: constraints and objectives (Chapters 5 and 6)

*Translation is the task of re-expressing design requirements in terms that enable material and process selection. Tackle the exercises by formulating the answers to the questions in this table. Don't try to model the behavior at this point (that comes in later exercises). Just think out what the component does, and list the constraints that this imposes on material choice, including processing requirements.*

| | |
|---|---|
| **Function** | • *What does component do?* |
| **Constraints** | • *What essential conditions must be met?* |
| **Objective** | • *What is to be maximized or minimized?* |
| **Free variables** | • *What parameters of the problem is the designer free to change?* |

*Here it is important to recognize the distinction between constraints and objectives. As the table says, a constraint is an essential condition that must be met, usually expressed as a **limit** on a material or process attribute. An objective is an quantity for which an **extremum** (a maximum or minimum) is sought, frequently cost, mass or volume, but there are others, several of which appear in the exercises below. Take the example of a bicycle frame. It must have a certain stiffness and strength. If it is not stiff and strong enough it will not work, but it is never required to have infinite stiffness or strength. Stiffness and strength are therefore constraints that become limits on modulus, elastic limit and shape. If the bicycle is for sprint racing, it should be as light as possible – if you could make it infinitely light, that would be best of all. Minimizing mass, here, is the objective, perhaps with an upper limit (a constraint) on cost. If instead it is a shopping bike to be sold through supermarkets it should be as cheap as possible – the cheaper it is, the more will be sold. This time minimizing cost is the objective, possible with an upper limit (a constraint) on mass. For most bikes, of course, minimizing mass and cost are both objectives, and then trade-off methods are needed.*

*They come later. For now use judgement to choose the single most important objective and make all others into constraints.*

*Two rules-of-thumb, useful in many "translation" exercises. Many applications require sufficient fracture toughness for the component can survive mishandling and accidental impact during service; a totally brittle material (like un-toughened glass) is unsuitable. Then a necessary constraint is that of "adequate toughness". This is achieved by requiring that the fracture toughness $K_{1c} > 15 MPa.m^{1/2}$. Other applications require some ductility, sufficient to allow stress redistribution under loading points, and some ability to bend or shape the material plastically. This is achieved by requiring that the (tensile) ductility $\varepsilon_f > 2\%$.*

*(If the CES EduPack software is available it can be used to impose the constraints and to rank the survivors using the objective.)*

**E6.1** A material is required for the windings of an electric air-furnace capable of temperatures up to 1000°C. Think out what attributes a material must have if it is to be made into windings and function properly in a furnace. List the function and the constraints; set the objective to "minimize cost" and the free variables to "choice of material".

***Answer.*** If the material is to be used as windings it must be able to be drawn to wire and wound to a coil. It must conduct electricity and be able to operate at 1000°C in air. The constraints are tabulated below.

| | |
|---|---|
| **Function** | • *High temperature furnace winding* |
| **Constraints** | • *Maximum service temperature, $T_{max}$ > 1000 C*<br>• *Able to be rolled or drawn to wire*<br>• *Good electrical conductor*<br>• *Good resistance to oxidation at elevated temperature* |
| **Objective** | • *Minimize material cost* |
| **Free variables** | • *Choice of material* |

**E6.2** A material is required to manufacture office scissors. Paper is an abrasive material, and scissors sometimes encounter hard obstacles like staples. List function and constraints; set the objective to "minimize cost" and the free variables to "choice of material".

***Answer.*** To resist abrasive wear the scissors must have blades of high hardness. In cutting, they will sooner or later encounter a staple or other hard obstruction that would chip a brittle blade – some toughness is required. These two parameters help reduce wear, but there are other factors that influence it, so it is sensible to specify good wear resistance. Finally, the scissors must be formed – if the handles are integral with the blades, they must be forged or stamped from sheet, requiring the ability to be processed in this way.

| | |
|---|---|
| **Function** | • *Scissors* |
| **Constraints** | • *High hardness*<br>• *Adequate toughness:* $K_{1c} > 15 MPa.m^{1/2}$<br>• *Good wear resistance*<br>• *Able to be forged* |
| **Objective** | • *Minimize material cost* |
| **Free variables** | • *Choice of material* |

**E6.3** A material is required for a heat exchanger to extract heat from geo-thermally heated, saline, water at 120°C (and thus under pressure). List function and constraints; set the objective to "minimize cost" and the free variables to "choice of material".

***Answer.*** The obvious constraints here are those on service-temperature, corrosion resistance, the ability to conduct heat well and strength. There are manufacturing constraints too: if the heat exchanger is to be made from tubes or folded sheet, the material must be available in these forms, and have sufficient ductility to allow manufacture.

| | |
|---|---|
| **Function** | • *Heat exchanger* |
| **Constraints** | • *Maximum service temperature, $T_{max}$ > 120 C*<br>• *Good thermal conductor*<br>• *Resistance to corrosion in salt water: good/excellent*<br>• *Sufficient strength to support the pressure of the super-heated water*<br>• *Ability to be rolled to sheet or tube*<br>• *Adequate ductility to allow shaping,* $\varepsilon_f > 2\%$ |
| **Objective** | • *Minimize material cost* |
| **Free variables** | • *Choice of material* |

**E6.4** A C-clamp is required for the processing of electronic components at temperatures up to 450 °C. It is essential that the clamp has as low a thermal inertia as possible so that it reaches that temperature quickly, and it must not charge-up when exposed to an electron beam. The time *t* it takes a component of thickness *x* to reach thermal equilibrium when the temperature is suddenly changed (a transient heat flow problem) is

$$t \approx \frac{x^2}{2a}$$

where the thermal diffusivity $a = \lambda / \rho C_p$ and $\lambda$ is the thermal conductivity, $\rho$ the density and $C_p$ the specific heat.

List function, constraints and objective; set the free variables to "choice of material".

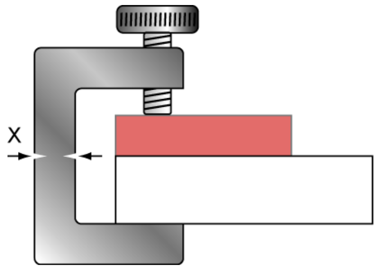



Figure E1

***Answer.*** If the C-clamp is to reach temperature quickly it must, according to the equation given in the question, be made of a material with as high a thermal diffusivity, *a* as possible. It carries loads, so it must have adequate strength, and it must not charge up, so it must be an electrical conductor.

| | |
|---|---|
| **Function** | • *C-clamp of low thermal inertia* |
| **Constraints** | • *Maximum service temperature, $T_{max}$ > 450 C*<br>• *Sufficient strength to carry clamping loads without failure*<br>• *Electrical conductor (to prevent charging)* |
| **Objective** | • *Maximise the thermal diffusivity, $a$ of the material* |
| **Free variables** | • *Choice of material* |

**E6.5** A furnace is required to sinter powder-metal parts. It operates continuously at 650 °C while the parts are fed through on a moving belt. You are asked to select a material for furnace insulation to minimize heat loss and thus to make the furnace as energy-efficient as possible. For reasons of space the insulation is limited to a maximum thickness of $x = 0.2$ m. List the function, constraints, objective and free variable.

***Answer.*** This is a problem involving steady-state heat flow. The heat lost by conduction per unit area of insulation per second, *q*, is

$$q = \lambda \frac{\Delta T}{x}$$

where $\lambda$ is the thermal conductivity, *x* the insulation thickness and $\Delta T$ the temperature difference between the interior of the furnace and its surroundings. The aim is to minimize *q*, leading (via the equation) to the objective of minimizing the thermal conductivity of the insulation material. There are two constraints: one on thickness, the other on service temperature.

| | |
|---|---|
| **Function** | • *Insulation for energy-efficient furnace* |
| **Constraints** | • *Maximum service temperature, $T_{max}$ > 650 C*<br>• *Insulation thickness $x \leq$ 0.2 m* |
| **Objective** | • *Minimize thermal conductivity $\lambda$* |
| **Free variables** | • *Choice of material* |

**E6.6** Ultra-precise bearings that allow a rocking motion make use of knife-edges or pivots. As the bearing rocks, it rolls, translating sideways by a distance that depends on the radius of contact. The further it rolls, the less precise is its positioning, so the smaller the radius of contact R the better. But the smaller the radius of contact, the greater is the contact pressure (*F/A*). If this exceeds the hardness *H* of either face of the bearing, it will be damaged. Elastic deformation is bad too: it flattens the contact, increasing the contact area and the roll.

A rocking bearing is required to operate in a micro-chip fabrication unit using fluorine gas at 100$^o$C, followed by e-beam processing requiring that all structural parts of the equipment can be earthed to prevent stray charges. Translate the requirements into material selection criteria, listing function, constraints, objective and free variable.

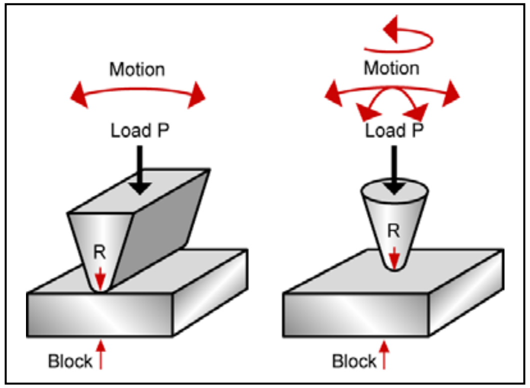



Figure E2

***Answer.*** The objective is to minimize the radius of curvature at the contact. This is limited by the hardness *H* and modulus *E* of the materials. The objective is met by seeking materials with: maximum hardness *H* to enable the smallest possible area of contact without damage, and maximum modulus *E* to minimize elastic flattening of the contact. In a later exercise we return to this problem of two objectives, treating it by trade-off methods. Here we choose maximizing hardness as the primary objective, since deficiency here results in a damaged bearing. We then treat modulus as a constraint, together with the other obvious constraints suggested by the design requirements.

| | |
|---|---|
| **Function** | • *Rocking bearing* |
| **Constraints** | • *Maximum service temperature, $T_{max}$ > 100 C*<br>• *Good electrical conductor*<br>• *Good resistance to fluorine gas*<br>• *High modulus, E* |
| **Objective** | • *Maximize hardness H of bearing faces* |
| **Free variables** | • *Choice of material* |

**E6.7** The standard CD ("Jewel" case) cracks easily and, if broken, can scratch the CD. Jewel cases are made of injection molded polystyrene, chosen because it is transparent, cheap and easy to mold. A material is sought to make CD cases that do not crack so easily. The case must still be transparent, able to be injection molded, and able to compete with polystyrene in cost.

***Answer.*** The question expresses constraints on transparency, moldablity and fracture toughness (it must be greater than that of polystyrene). Given these, the cheapest material is the best choice.

| | |
|---|---|
| **Function** | • *Improved CD case* |
| **Constraints** | • *Optically transparent*<br>• *Fracture toughness greater than that of polystyrene*<br>• *Able to be injection molded* |
| **Objective** | • *Minimize material cost* |
| **Free variables** | • *Choice of material* |

**E6.8** A storage heater captures heat over a period of time, then releases it, usually to an air stream, when required. Those for domestic heating store solar energy or energy from cheap off-peak electricity and release it slowly during the cold part of the day. Those for research release the heat to a supersonic air stream to test system behavior in supersonic flight. What is a good material for the core of a compact storage material capable of temperatures up to 120°C?

***Answer.*** When a material is heated from room temperature $T_o$ to a working temperature $T$, it absorbs heat $Q$ per unit volume where

$$Q = C_p \rho (T - T_o)$$

and $C_p$ is the specific heat of the material of the core (in J/kg.C) and $\rho$ is its density (in kg/m$^3$). Thus the most compact storage heater is one made from a material with a high $C_p \rho$. Even a small storage heater contains a considerable quantity of core (that is why they are heavy), so it is probable that an objective will be to minimize its cost per unit volume. If, however, space were critical, maximizing $C_p \rho$ might become the objective.

| | |
|---|---|
| **Function** | • *Core for compact storage heater* |
| **Constraints** | • *Maximum service temperature, $T_{max}$ > 120 C*<br>• *High heat capacity per unit volume $C_p \rho$* |
| **Objective** | • *Minimize material cost per unit volume* |
| **Free variables** | • *Choice of material* |

## E7. Deriving and using material indices (Chapters 5 and 6)

*The exercises in this section give practice in deriving indices.*

(a) *Start each by listing function, constraints, objectives and free variables; without having those straight, you will get in a mess. Then write down an equation for the objective. Consider whether it contains a free variable other than material choice; if it does, identify the constraint that limits it, substitute, and read off the material index.*

(b) *If the CES EduPack software is available, use it to apply the constraints and rank the survivors using the index (start with the Level 2 database). Are the results sensible? If not, what constraint has been overlooked or incorrectly formulated?*

**E7.1 Aperture grills for cathode ray tubes** (Figure E3). There are two types of cathode ray tube (CRT). In the older technology, colour separation is achieved by using a *shadow mask*: a thin metal plate with a grid of holes that allow only the correct beam to strike a red, green or blue phosphor. A shadow mask can heat up and distort at high brightness levels ('doming'), causing the beams to miss their targets, and giving a blotchy image.

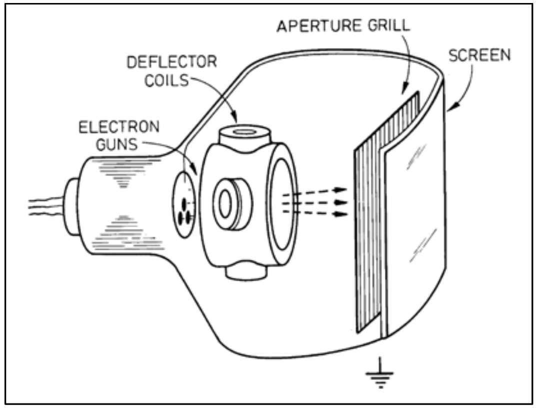



Figure E3

To avoid this, shadow masks are made of Invar, a nickel alloy with a near-zero expansion coefficient between room temperature and 150˚C. It is a consequence of shadow-mask technology that the glass screen of the CRT curves inward on all four edges, increasing the probability of reflected glare.

Sony's 'Trinitron' technology overcame this problem and allowed greater brightness by replacing the shadow mask by an *aperture grill* of fine vertical wires, each about 200 μm in thickness, that allows the intended beam to strike either the red, the green or the blue phosphor to create the image. The glass face of the Trinitron tube was curved in one plane only, reducing glare.

The wires of the aperture grill are tightly stretched, so that they remain taut even when hot – it is this tension that allows the greater brightness. What index guides the choice of material to make them? The table summarises the requirements.

| | |
|---|---|
| **Function** | • *Aperture grill for CRT* |
| **Constraints** | • *Wire thickness and spacing specified*<br>• *Must carry pre-tension without failure*<br>• *Electrically conducting to prevent charging*<br>• *Able to be drawn to wire* |
| **Objective** | • *Maximize permitted temperature rise without loss of tension* |
| **Free variables** | • *Choice of material* |

***Answer.***

**The model.** A thin, taut wire slackens and sags when the strain due to thermal expansion,

$$\varepsilon_{th} = \alpha \Delta T$$

exceeds the elastic strain caused by the pre-tension,

$$\varepsilon_{pt} = \frac{\sigma}{E}.$$

Here $\alpha$ is the thermal expansion coefficient of the wire, $\Delta T$ the temperature rise caused by the electron beams that strike it, $\sigma$ the tensile pre-stress in the wire and $E$ its modulus. We wish to maximize the brightness, and thus $\Delta T$.

The tension is limited by the elastic limit of the wire, $\sigma_f$. Inserting this and writing

$$\varepsilon_{pt} \geq \varepsilon_{th}$$

as the condition that the wire remains taut gives

$$\Delta T = \frac{\sigma_f}{E\alpha}.$$

The result could hardly be simpler. To maximize the brightness, maximize

$$\boxed{M = \frac{\sigma_f}{E\alpha}.}$$

There is a second requirement. The wires must conduct, otherwise they would charge up, distorting the image. We therefore require, also, that the material be a good electrical conductor and that it is capable of being rolled or drawn to wire.

**The selection.** Applying the constraints listed in the table above to the CES EduPack Level 1 or 2 database and ranking the survivors by the index *M* leads to the selection listed below.

| Material | Comment |
|---|---|
| Carbon steel | Carbon steel is ferro-magnetic, so will interact with the scanning magnetic fields – reject. |
| Commercially pure Titanium | Extracting titanium from its oxide is difficult making it an expensive option |
| Tungsten | A logical choice – tungsten has a high melting point and is routinely produced as fine wire |
| Nickel-based alloys | Many nickel alloys are weakly ferromagnetic – reject for the same reason as carbon steel |

The final choice, using this database, is stainless steel or tungsten. If the selection is repeated using the more detailed Level 3 database, carbon fiber emerges as a potential candidate. Carbon fibre of the desired diameter is available; it conducts well, both electrically and thermally, it has high strength and – best of all – it has almost zero thermal expansion along the fiber direction.

**E7.2 Material indices for elastic beams with differing constraints** (Figure E4). *Start each of the four parts of this problem by listing the function, the objective and the constraints. You will need the equations for the deflection of a cantilever beam with a square cross-section t x t, given in Appendix B, Section B3. The two that matter are that for the deflection $\delta$ of a beam of length L under an end load F:*

$$\delta = \frac{F L^3}{3EI}$$

*and that for the deflection of a beam under a distributed load f per unit length:*

$$\delta = \frac{1}{8}\frac{f L^4}{E I}$$

*where $I = t^4/12$. For a self-loaded beam $f = \rho A g$ where $\rho$ is the density of the material of the beam, A its cross-sectional area and g the acceleration due to gravity.*

(a) Show that the best material for a cantilever beam of given length *L* and *given* (i.e. fixed) square cross-section (*t x t*) that will deflect least under a given end load *F* is that with the largest value of the index *M* = *E*, where *E* is Young's modulus (neglect self-weight). (Figure E4a.)

(b) Show that the best material choice for a cantilever beam of given length *L* and with a given section (*t x t*) that will deflect least under its own self-weight is that with the largest value of *M* = *E/ρ*, where $\rho$ is the density. (Figure E4b.)

(c) Show that the material index for the lightest cantilever beam of length *L* and square section (not given, i.e., the area is a free variable) that will not deflect by more than $\delta$ under its own weight is $M = E/\rho^2$. (Figure E4c.)

(d) Show that the lightest cantilever beam of length *L* and square section (area free) that will not deflect by more than $\delta$ under an end load *F* is that made of the material with the largest value of $M = E^{1/2}/\rho$ (neglect self weight). (Figure E4d.)

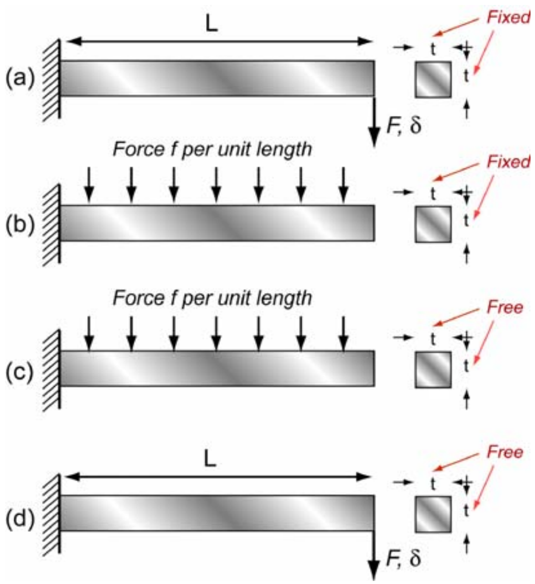



Figure E4

***Answer.***

**The model.** The point of this problem is that the material index depends on the mode of loading, on the geometric constraints and on the design objective.

(a) The table lists the design requirements for part (a) of the problem.

| | |
|---|---|
| **Function** | • *End-loaded cantilever beam* |
| **Constraints** | • *Length L specified*<br>• *Section t x t specified*<br>• *End load F specified* |
| **Objective** | • *Minimize the deflection, $\delta$* |
| **Free variables** | • *Choice of material only* |

The objective function is an equation for the deflection of the beam. An end-load *F* produces a deflection $\delta$ of

$$\delta = \frac{F L^3}{3EI}$$

where *E* is the modulus of the beam material and $I = t^4/12$ is the second moment of the area, so that the deflection becomes

$$\delta = 4\frac{F L^3}{t^4}\left(\frac{1}{E}\right)$$

The magnitude of the load *F* and the dimensions *L* and *t* are all given. The deflection $\delta$ is minimized by maximizing

$$M_1 = E$$

(b) The design requirements for part (b) are listed below

| | |
|---|---|
| **Function** | • *Self-loaded cantilever beam* |
| **Constraints** | • *Length L specified*<br>• *Section t x t specified* |
| **Objective** | • *Minimize the deflection, $\delta$* |
| **Free variables** | • *Choice of material only* |

The beam carries a distributed load, *f* per unit length, where

$$f = \rho g t^2$$

where $\rho$ is the density of the beam material and $g$ is the acceleration due to gravity. Such a load produces a deflection (Appendix B, Section B3)

$$\delta = \frac{3}{2}\frac{f L^4}{E t^4} = \frac{3 g L^4}{2 t^2}\left(\frac{\rho}{E}\right)$$

(the objective function). As before, $t$ and $L$ are given. The deflection is minimized by maximizing

$$\boxed{M_2 = \frac{E}{\rho}}$$

(c) The design requirements for part (c) are listed below

| | |
|---|---|
| **Function** | • *Self-loaded cantilever beam* |
| **Constraints** | • *Length L specified*<br>• *Maximum deflection, $\delta$, specified* |
| **Objective** | • *Minimize the mass, m* |
| **Free variables** | • *Choice of material*<br>• *Section area $A = t^2$* |

The beam deflects under its own weight but now the section can be varied to reduce the weight provided the deflection does not exceed $\delta$, as in the figure. The objective function (the quantity to be minimized) is the mass $m$ of the beam

$$m = t^2 L \rho$$

Substituting for $t$ (the free variable) from the second equation into the first, gives

$$m = \frac{3 g L^5}{2 \delta}\left(\frac{\rho^2}{E}\right)$$

The quantities *L* and $\delta$ are given. The mass is minimized by maximizing

$$\boxed{M_3 = \frac{E}{\rho^2}}$$

(d) The design requirements for part (d) are listed below

| | |
|---|---|
| **Function** | • *End-loaded cantilever beam* |
| **Constraints** | • *Length L specified*<br>• *End-load F specified*<br>• *Maximum deflection, $\delta$, specified* |
| **Objective** | • *Minimize the mass, m* |
| **Free variables** | • *Choice of material*<br>• *Section area $A = t^2$* |

The section is square, but the dimension *t* is free. The objective function is

$$m = t^2 L \rho$$

The deflection is, as in part (a)

$$\delta = 4\frac{F L^3}{t^4}\left(\frac{1}{E}\right)$$

Using this to eliminate the free variable, t, gives

$$m = 2\left(\frac{F L^5}{\delta}\right)^{1/2}\left(\frac{\rho}{E^{1/2}}\right)$$

The quantities F , $\delta$ and L are given. The mass is minimized by maximizing

$$M_4 = \frac{E^{1/2}}{\rho}$$

From a selection standpoint, $M_3$ and $M_4$ are equivalent.

**The selection.** Applying the three indices to the CES EduPack Level 1 or 2 database gives the top-ranked candidates listed below.

| Index | Material choice |
|---|---|
| *High* $M_1 = E$ | Metals: tungsten alloys, nickel alloys, steels.<br>Ceramics: SiC, $Si_3N_4$, $B_4C$, $Al_2O_3$ and AlN, but of course all of these are brittle. |
| *High* $M_2 = \frac{E}{\rho}$ | Metals: aluminum, magnesium, nickel and titanium alloys and steels all have almost the same value of $E/\rho$<br>Composites: CFRP<br>Ceramics SiC, $Si_3N_4$, $B_4C$ and AlN |
| *High* $M_3 = \frac{E}{\rho^2}$<br>*High* $M_4 = \frac{E^{1/2}}{\rho}$ | Metals: aluminum and magnesium alloys superior to all other metals.<br>Composites: CFRP excels<br>Ceramics: SiC, $Si_3N_4$, $B_4C$, $Al_2O_3$ and AlN |

**E7.3 Material index for a light, strong beam** (Figure E5). In stiffness-limited applications, it is elastic deflection that is the active constraint: it limits performance. In strength-limited applications, deflection is acceptable provided the component does not fail; strength is the active constraint. Derive the material index for selecting materials for a beam of length *L*, specified strength and minimum weight. For simplicity, assume the beam to have a solid square cross-section *t x t.* You will need the equation for the failure load of a beam (Appendix B, Section B4). It is

$$F_f = \frac{I\sigma_f}{y_m L}$$

where $y_m$ is the distance between the neutral axis of the beam and its outer filament and $I = t^4/12 = A^2/12$ is the second moment of the cross-section. The table itemizes the design requirements.

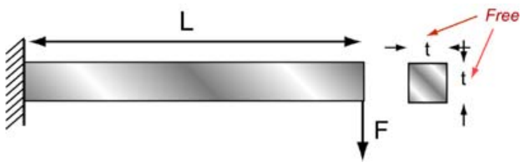



Figure E5

| | |
|---|---|
| **Function** | • *Beam* |
| **Constraints** | • *Length L is specified*<br>• *Beam must support a bending load F without yield or fracture* |
| **Objective** | • *Minimize the mass of the beam* |
| **Free variables** | • *Cross-section area, A*<br>• *Choice of material* |

***Answer.***

**The model.** The objective is to minimize the mass, giving the objective function

$$m = A L \rho$$

Inverting the equation given in the question leads to an expression for the area *A* that will support the design load *F*:

$$A = \left( 6 \frac{F L}{\sigma_f} \right)^{2/3}$$

Substituting *A* into the objective function gives the mass of the beam that will just support the load $F_f$ :

$$m = (6 F)^{2/3} L^{5/3} \left[ \frac{\rho}{\sigma_f^{2/3}} \right]$$

The mass is minimized by selecting materials with the largest values of the index

$$M = \frac{\sigma_f^{2/3}}{\rho}$$

If the cantilever is part of a mechanical system it is important that it have sufficient fracture toughness to survive accidental impact loads. For this we add the requirement of adequate toughness:

$$K_{1c} > 15 MPa.m^{1/2}$$

**The selection.** Applying the constraint on $K_{1c}$ and ranking by the index *M* using the CES EduPack Level 1 or 2 database gives the top-ranked candidates listed below.

| Material | Comment |
|---|---|
| CFRP | Exceptionally good, mainly because of its very low density. |
| Metals: titanium, aluminum and magnesium alloys | Here the light alloys out-perform steel |

## E7.4 Material index for a cheap, stiff column (Figure E6).

*In the last two exercises the objective has been that of minimizing weight. There are many others. In the selection of a material for a spring, the objective is that of maximizing the elastic energy it can store. In seeking materials for thermal-efficient insulation for a furnace, the best are those with the lowest thermal conductivity and heat capacity. And most common of all is the wish to minimize cost. So here is an example involving cost.*

Columns support compressive loads: the legs of a table; the pillars of the Parthenon. Derive the index for selecting materials for the cheapest cylindrical column of specified height, *H,* that will safely support a load *F* without buckling elastically. You will need the equation for the load $F_{crit}$ at which a slender column buckles. It is

$$F_{crit} = \frac{n^2 \pi^2 E I}{H^2}$$

where *n* is a constant that depends on the end constraints and $I = \pi r^4 / 4 = A^2 / 4\pi$ is the second moment of area of the column (see Appendix B for both). The table lists the requirements.

| | |
|---|---|
| **Function** | • *Cylindrical column* |
| **Constraints** | • *Length L is specified*<br>• *Column must support a compressive load F without buckling* |
| **Objective** | • *Minimize the material cost of the column* |
| **Free variables** | • *Cross-section area, A*<br>• *Choice of material* |

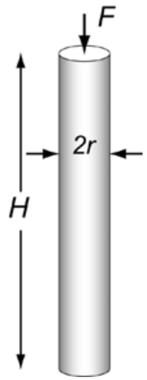



Figure E6

## *Answer.*

**The model.** A slender column uses less material than a fat one, and thus is cheaper; but it must not be so slender that it will buckle under the design load, *F*. The objective function is the cost

$$C = A\,H\,\rho\,C_m$$

Where $C_m$ is the cost/kg of the material* of the column. It will buckle elastically if *F* exceeds the Euler load, $F_{crit}$, given in the question. Eliminating *A* between the two equations, using the definition of *I*, gives:

$$C \geq \left(\frac{4\,F}{n\pi}\right)^{1/2} H^2 \left(\frac{C_m\,\rho}{E^{1/2}}\right)$$

The material cost of the column is minimized by choosing materials with the largest value of the index

$$M = \frac{E^{1/2}}{C_m\,\rho}$$

* $C_m$ is the cost/kg of the *processed* material, here, the material in the form of a circular rod or column.

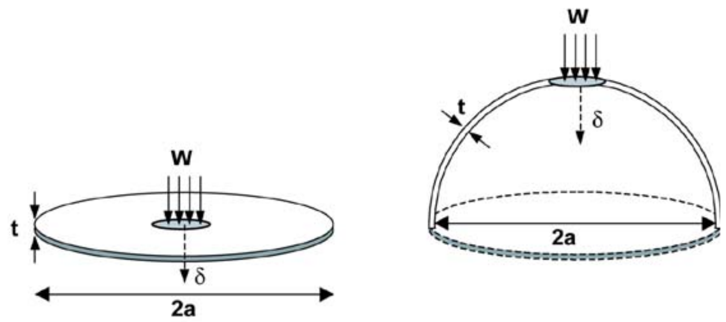
**The selection.** The loading here is compressive, so brittle materials are viable candidates. Since some are also very cheap, they dominate the selection. Applying the index *M* to the CES EduPack Level 1 or 2 database gives the top-ranked candidates listed below.

| Material | Comment |
|---|---|
| Ceramics: brick, cement, concrete and stone | The low cost and fairly high modulus makes these the top-ranked candidates. |
| Wood | Exceptional stiffness parallel to the grain, and cheap. |
| Carbon steel, cast iron | Steel out-performs all other metals when strength at low cost is sought. |

**E7.5 Indices for stiff plates and shells** (Figure E7). Aircraft and space structures make use of plates and shells. The index depends on the configuration. Here you are asked to derive the material index for

(a) a circular plate of radius *a* carrying a central load *W* with a prescribed stiffness $S = W/\delta$ and of minimum mass, and

(b) a hemispherical shell of radius *a* carrying a central load *W* with a prescribed stiffness $S = W/\delta$ and of minimum mass, as shown in the figures.



Use the two results listed below for the mid-point deflection $\delta$ of a plate or spherical shell under a load $W$ applied over a small central, circular area.

Circular plate: $$\delta = \frac{3}{4\pi}\frac{W a^2}{E t^3}(1-\nu^2)\left(\frac{3+\nu}{1+\nu}\right)$$

Hemispherical shell $$\delta = A\frac{W a}{E t^2}(1-\nu^2)$$

in which $A \approx 0.35$ is a constant. Here *E* is Young's modulus, *t* is the thickness of the plate or shell and $\nu$ is Poisson's ratio. Poisson's ratio is almost the same for all structural materials and can be treated as a constant. The table summarizes the requirements.

| | |
|---|---|
| **Function** | • *Stiff circular plate, or*<br>• *Stiff hemispherical shell* |
| **Constraints** | • *Stiffness S under central load W specified*<br>• *Radius a of plate or shell specified* |
| **Objective** | • *Minimize the mass of the plate or shell* |
| **Free variables** | • *Plate or shell thickness, t*<br>• *Choice of material* |

***Answer.***

(a) The plate. The objective is to minimize the mass, $m$

$$m = \pi a^2 t \rho$$

where $\rho$ is the density of the material of which the plate is made. The thickness *t* is free, but must be sufficient to meet the constraint on stiffness. Inverting the first equation in the question gives, for the plate,

$$t = \left(\frac{3 S a^2}{4\pi E}\right)^{1/3} f_1(\nu)$$

where $f_1(\nu)$ is simply a function of $\nu$, and thus a constant.

Inserting this into the equation for the mass gives

$$m = \pi a^2 \left(\frac{3 S a^2}{4\pi}\right)^{1/3}\left[\frac{\rho}{E^{1/3}}\right] f(\nu)$$

The lightest plate is that made from a material with a large value of the index

$$\boxed{M_1 = \frac{E^{1/3}}{\rho}}$$

(a) The hemispherical shell The objective again is to minimize the mass, $m$

$$m = 2\pi a^2 t \rho$$

Inverting the second equation in the question gives, for the shell,

$$t = \left(A\frac{S a}{E}\right)^{1/2} f_2(\nu)$$

where $f_2(\nu)$ as before is a function of $\nu$, and thus a constant. Inserting this into the equation for the mass gives

$$m = 2\pi a^2 (A S a)^{1/2}\left[\frac{\rho}{E^{1/2}}\right] f(\nu)$$

The lightest shell is that made from a material with a large value of the index

$$\boxed{M_2 = \frac{E^{1/2}}{\rho}}$$

The index for the shell differs from that for the plate, requiring a different choice of material. This is because the flat plate, when loaded, deforms by bending. The hemispherical shell, by contrast, carries membrane stresses (tension and compression in the plane of the shell wall), and because of this is much stiffer. Singly-curved shells behave like the plate, doubly-curved shells like the hemisphere.

**The selection.** Applying the three indices to the CES EduPack Level 1 or 2 database gives the top-ranked candidates listed below.

| Index | Material choice |
|---|---|
| High $M_1 = \frac{E^{1/3}}{\rho}$ | Natural materials: wood and plywood<br>Composites: CFRP |
| High $M_2 = \frac{E^{1/2}}{\rho}$ | Metals: aluminum and magnesium alloys superior to all other metals.<br>Composites: CFRP<br>Ceramics: SiC, $Si_3N_4$, $B_4C$ and AlN |

**E7.6 The C-clamp in more detail** (Figure E8). Exercise E4.4 introduced the C-clamp for processing of electronic components. The clamp has a square cross-section of width *x* and given depth *b*. It is essential that the clamp have low thermal inertia so that it reaches temperature quickly. The time *t* it takes a component of thickness *x* to reach thermal equilibrium when the temperature is suddenly changed (a transient heat flow problem) is

$$t \approx \frac{x^2}{2a}$$

where the thermal diffusivity $a = \lambda / \rho C_p$ and $\lambda$ is the thermal conductivity, $\rho$ the density and $C_p$ the specific heat.

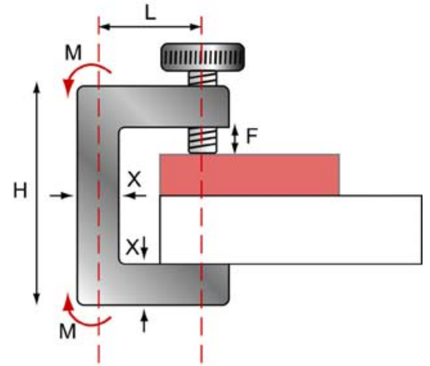



Figure E8

The time to reach thermal equilibrium is reduced by making the section *x* thinner, but it must not be so thin that it fails in service. Use this constraint to eliminate *x* in the equation above, thereby deriving a material index for the clamp. Use the fact that the clamping force *F* creates a moment on the body of the clamp of $M = F L$, and that the peak stress in the body is given by

$$\sigma = \frac{x}{2}\frac{M}{I}$$

where $I = b x^3 / 12$ is the second moment of area of the body.

The table summarizes the requirements.

| | |
|---|---|
| **Function** | • *C-clamp of low thermal inertia* |
| **Constraints** | • *Depth b specified*<br>• *Must carry clamping load F without failure* |
| **Objective** | • *Minimize time to reach thermal equilibrium* |
| **Free variables** | • *Width of clamp body, x*<br>• *Choice of material* |

***Answer.*** The clamp will fail if the stress in it exceeds its elastic limit $\sigma_f$. Equating the peak stress to $\sigma_f$ and solving for *x* gives

$$x = \left(6 \frac{F\,L}{b\,\sigma_f}\right)^{1/2}$$

Inserting this into the equation for the time to reach equilibrium gives

$$t = 3\frac{F\,L}{b}\left(\frac{1}{a\,\sigma_f}\right)$$

The time is minimized by choosing materials with large values of the index

$$\boxed{M = a\,\sigma_f\,.}$$

Additional constraints on modulus *E* > 50 GPa (to ensure that the clamp is sufficiently stiff) on fracture toughness $K_{1c} > 18 MPa.m^{1/2}$ (to guard against accidental impact) and on formability will, in practice, be needed.

**The selection.** Applying the constraint on $K_{1c}$ and formability, and ranking by the index *M* using the CES EduPack Level 1 or 2 database gives the top-ranked candidates listed below.

| Material | Comment |
|---|---|
| Aluminum alloys | The obvious candidate – good thermal conductor, adequately stiff and strong, and easy to work. |
| Copper alloys | Here the high thermal diffusivity of copper is dominating the selection. |

**E7.7 Springs for trucks.** In vehicle suspension design it is desirable to minimize the mass of all components. You have been asked to select a material and dimensions for a light spring to replace the steel leaf-spring of an existing truck suspension. The existing leaf-spring is a beam, shown schematically in the figure. The new spring must have the same length *L* and stiffness *S* as the existing one, and must deflect through a maximum safe displacement $\delta_{max}$ without failure. The width *b* and thickness *t* are free variables.

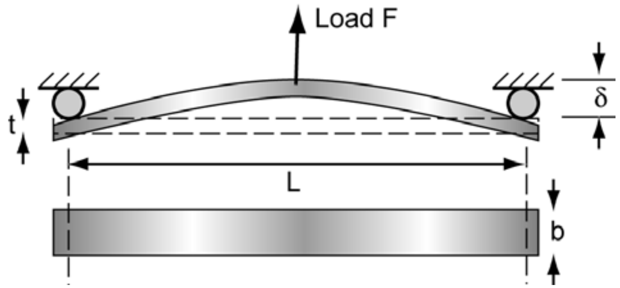



Figure E9

Derive a material index for the selection of a material for this application. Note that this is a problem with two free variables: *b* and *t*; and there are two constraints, one on safe deflection $\delta_{max}$ and the other on stiffness *S*. Use the two constraints to fix free variables. The table catalogs the requirements.

| | |
|---|---|
| **Function** | • *Leaf spring for truck* |
| **Constraints** | • *Length L specified*<br>• *Stiffness S specified*<br>• *Maximum displacement $\delta_{max}$ specified* |
| **Objective** | • *Minimize the mass* |
| **Free variables** | • *Spring thickness t*<br>• *Spring width b*<br>• *Choice of material* |

You will need the equation for the mid-point deflection of an elastic beam of length *L* loaded in three-point bending by a central load F:

$$\delta = \frac{1}{48}\frac{F L^3}{E I}$$

and that for the deflection at which failure occurs

$$\delta_{max} = \frac{1}{6}\frac{\sigma_f L^2}{t E}$$

where *I* is the second moment of area; for a beam of rectangular section, $I = b t^3 / 12$ and *E* and $\sigma_f$ are the modulus and failure stress of the material of the beam. (See Appendix B.)

***Answer.***

**The model.** The objective function – the quantity to be minimized – is the mass *m* of the spring:

$$m = b t L \rho \qquad (1)$$

where $\rho$ is its density. The length *L* is fixed. The dimensions *b* and *t* are free. There are two constraints. The first is a required stiffness, *S*. From the first equation given in the question

$$S = \frac{F}{\delta} = 48\frac{EI}{L^3} = 4\frac{E b t^3}{L^3} \qquad (2)$$

The second constraint is that of a maximum allowable displacement $\delta_{max}$ without damage to the spring, given in the question as

$$\delta_{max} = \frac{1}{6}\frac{\sigma_f L^2}{t E} \qquad (3)$$

Equations (2) and (3) can now be solved for *t* and *b*, and these substituted back into (1). The result is

$$m = 9 S \delta_{max}^2 L \left(\frac{\rho E}{\sigma_f^2}\right)$$

The mass of the spring is minimized by maximizing the index

$$M = \frac{\sigma_f^2}{\rho E}$$

Additional constraints on fracture toughness $K_{1c} > 15 MPa.m^{1/2}$ (to guard against accidental impact) and on formability will, in practice, be needed.

**The selection.** Applying the constraint on $K_{1c}$ and formability, and ranking by the index *M* using the CES EduPack Level 1 or 2 database gives the top-ranked candidates listed below.

| Material | Comment |
|---|---|
| Elastomers (rubber) | Oops! we have missed a constraint here. Elastomers excel as light springs, but the constraint on thickness *t* and depth *b* in this application translates via equation (2) into an additional constraint on modulus: $E > SL^3 / 4bt^3$ . |
| Titanium alloys | An expensive solution, but one that is lighter than steel. |
| CFRP | CFRP makes exceptionally good light springs. |
| High carbon steel | The standard solution, but one that is heavier than the others above. |

**E7.8 Fin for a rocket** (Figure E10). A tube-launched rocket has stabilizing fins at its rear. During launch the fins experience hot gas at $T_g = 1700^oC$ for a time $t = 0.3$ seconds. It is important that the fins survive launch without surface melting. Suggest a material index for selecting a material for the fins. The table summarizes the requirements.

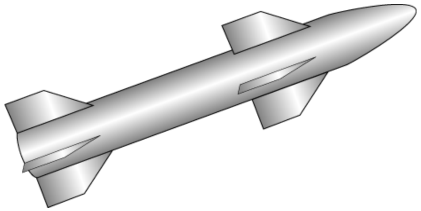


Figure E10

| | |
|---|---|
| **Function** | • *High heat-transfer rocket fins* |
| **Constraints** | • *All dimensions specified*<br>• *Must not suffer surface melting during exposure to gas at 1700 C for 0.3 sec.* |
| **Objective** | • *Minimize the surface temperature rise during firing*<br>• *Maximize the melting point of the material* |
| **Free variables** | • *Choice of material* |

This is tricky. Heat enters the surface of the fin by transfer from the gas. If the heat transfer coefficient is *h,* the heat flux per unit area is

$$q = h(T_g - T_s)$$

If the heating time is small compared with the characteristic time for heat to diffuse through the fin, a quasi steady-state exists in which the surface temperature adjusts itself such that the heat entering from the gas is equal to that diffusing inwards by conduction. This second is equal to

$$q = \lambda \frac{(T_s - T_i)}{x}$$

where $\lambda$ is the thermal conductivity, $T_i$ is the temperature of the (cold) interior of the fin, and $x$ is a characteristic heat-diffusion length. When the heating time is short (as here) the thermal front, after a time *t*, has penetrated a distance

$$x \approx (2at)^{1/2}$$

where $a = \lambda / \rho C_p$ is the thermal diffusivity. Substituting this value of $x$ in the previous equation gives

$$q = (\lambda \rho C_p)^{1/2} \frac{(T_s - T_i)}{\sqrt{2t}}$$

where $\rho$ is the density and $C_p$ the specific heat of the material of the fin.

Proceed by equating the two equations for *q,* solving for the surface temperature $T_s$ to give the objective function. Read off the combination of properties that minimizes $T_s$; it is the index for the problem.

The selection is made by seeking materials with large values of the index and with a high melting point, $T_m$. If the CES software is available, make a chart with these two as axes and identify materials with high values of the index that also have high melting points.

where $T_s$ is the surface temperature of the fin – the critical quantity we wish to minimize. Heat is diffuses into the fin surface by thermal conduction.

***Answer.*** Equating the equations and solving for $T_s$ gives

$$T_s = \frac{h T_g \sqrt{t} + \sqrt{\lambda \rho C_p}\, T_i}{h \sqrt{t} + \sqrt{\lambda \rho C_p}}$$

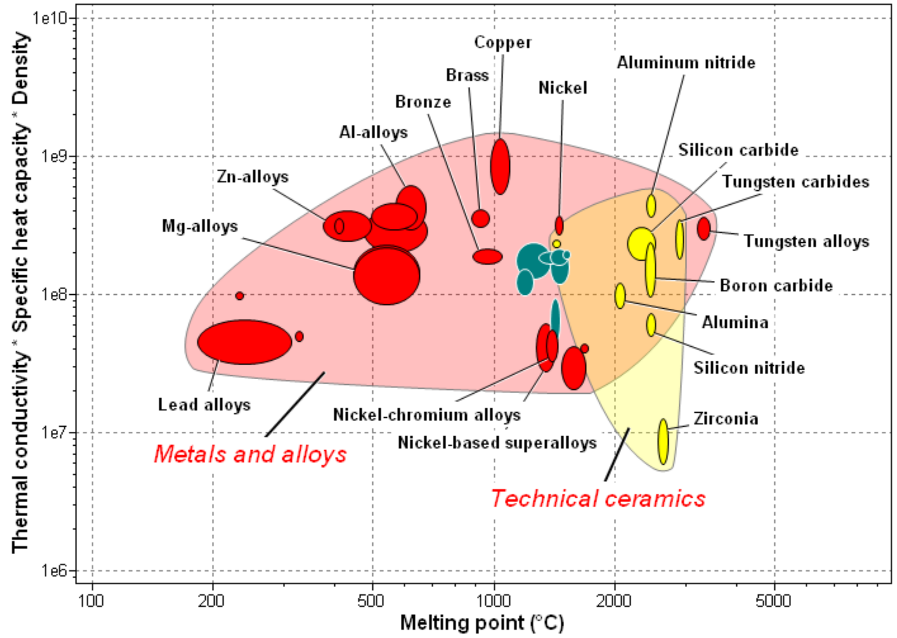
When *t* = 0 the surface temperature $T_s = T_i$ and the fin is completely cold. When *t* is large, $T_s = T_g$ and the surface temperature is equal to the gas temperature. For a given *t*, $T_s$ is minimized by maximizing $\lambda \rho C_p$. The first index is therefore

$$M_1 = \lambda \rho C_p$$

and index that often appears in problems involving transient heat flow. Melting is also made less likely by choosing a material with a high melting point $T_m$. The second index is therefore

$$M_2 = T_m$$

**The selection.** The figure shows the two indices, created using the CES EduPack Level 2 database. The top-ranked candidates are listed below.




| Material | Comment |
|---|---|
| Silicon carbide, SiC | Silicon carbide out-performs all metals except tungsten, and is much lighter. If used, its brittleness would have to be reckoned with.. |
| Copper alloys | The exceptional thermal conductivity of copper is dominating here – it is able to conduct heat away from the surface quickly, limiting the surface heating. |
| Aluminum alloys | An attractive choice, since Al-alloys are also light |
| Aluminum – Silicon carbide MMC | This metal matrix composite has almost the thermal conductivity of aluminum and is stiffer and stronger. |

## E8 Multiple constraints and objectives (Chapters 7 and 8)

*Over-constrained problems are normal in materials selection. Often it is just a case of applying each constraint in turn, retaining only those solutions that meet them all. But when constraints are used to eliminate free variables in an objective function, (as discussed in Section 9.2 of the text) the "active constraint" method must be used. The first three exercises in this section illustrate problems with multiple constraints.*

*The remaining two concern multiple objectives and trade-off methods. When a problem has two objectives – minimizing both mass $m$ and cost $C$ of a component, for instance – a conflict arises: the cheapest solution is not the lightest and vice versa. The best combination is sought by constructing a trade-off plot using mass as one axis, and cost as the other. The lower envelope of the points on this plot defines the trade-off surface. The solutions that offer the best compromise lie on this surface. To get further we need a penalty function. Define the penalty function*

$$Z = C + \alpha m$$

*where $\alpha$ is an exchange constant describing the penalty associated with unit increase in mass, or, equivalently, the value associated with a unit decrease. The best solutions are found where the line defined by this equation is tangential to the trade-off surface. (Remember that objectives must be expressed in a form such that a minimum is sought; then a low value of Z is desirable, a high one is not.)*
*When a substitute is sought for an existing material it is better to work with ratios. Then the penalty function becomes*

$$Z^* = \frac{C}{C_o} + \alpha^* \frac{m}{m_o}$$

*in which the subscript "o" means properties of the existing material and the asterisk * on Z* and $\alpha$* is a reminder that both are now dimensionless. The relative exchange constant $\alpha^*$ measures the fractional gain in value for a given fractional gain in performance.*

**E8.1 Multiple constraints: a light, stiff, strong tie** (Figure E11).

A tie, of length *L* loaded in tension, is to support a load *F*, at minimum weight without failing (implying a constraint on strength) or extending elastically by more than $\delta$ (implying a constraint on stiffness, $F/\delta$ ). The table summarises the requirements.

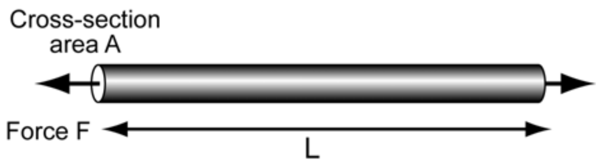


Figure E11

| | |
|---|---|
| **Function** | • *Tie rod* |
| **Constraints** | • *Must not fail by yielding under force F*<br>• *Must have specified stiffness, F/δ*<br>• *Length L and axial load F specified* |
| **Objective** | • *Minimize mass m* |
| **Free variables** | • *Section area A*<br>• *Choice of material* |

(a) Follow the method of Chapter 7 to establish two performance equations for the mass, one for each constraint, from which two material indices and one coupling equation linking them are derived. Show that the two indices are

$$M_1 = \frac{\rho}{E} \quad \text{and} \quad M_2 = \frac{\rho}{\sigma_y}$$

and that a minimum is sought for both.

(b) Use these and the material chart of Figure E12, which has the indices as axes, to identify candidate materials for the tie (i) when $\delta/L = 10^{-3}$ and (ii) when $\delta/L = 10^{-2}$.

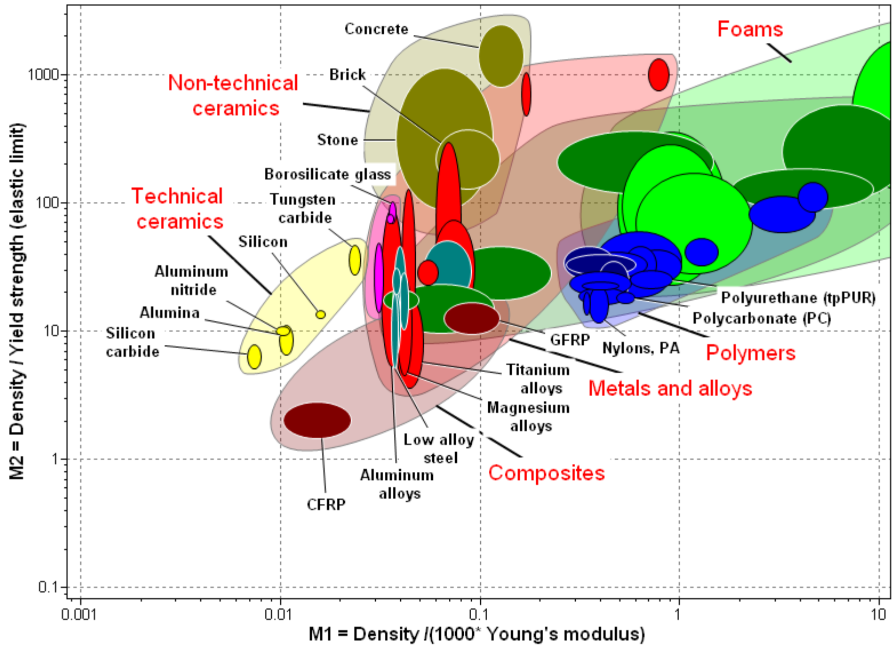



Figure E12

***Answer.*** The derivation of performance equations and the indices they contain is laid out here:

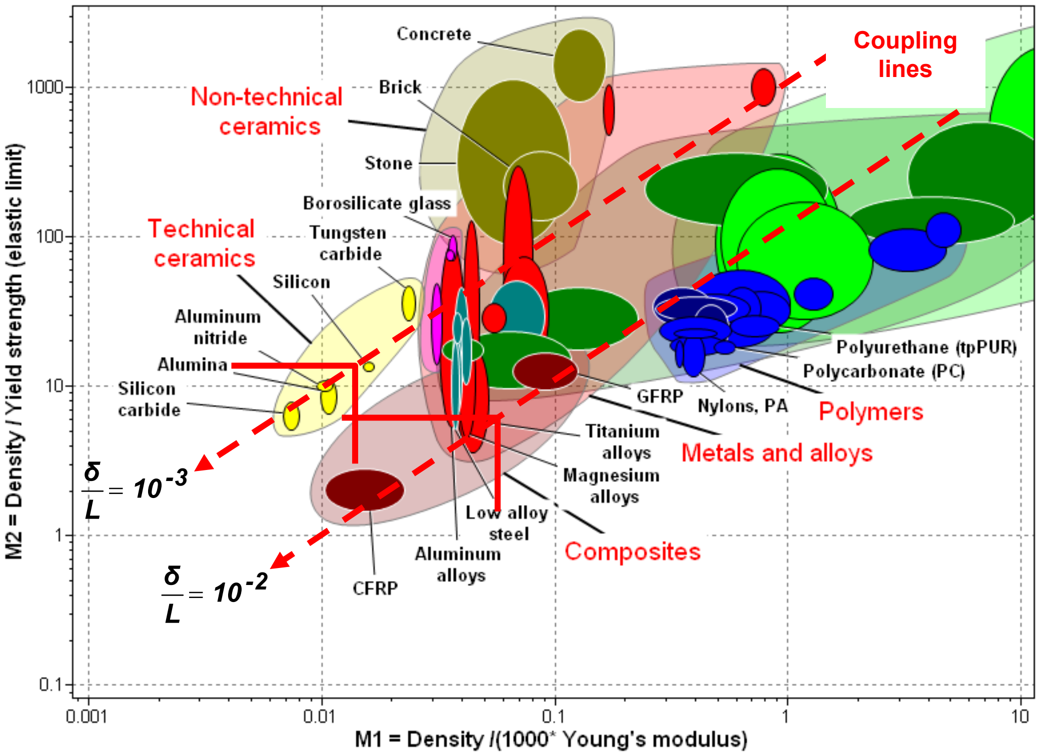
| ***Objective*** | ***Constraints*** | ***Performance equation*** | ***Index*** | |
|---|---|---|---|---|
| $m = A\,L\,\rho$ | *Substitute for A* — ***Stiffness constraint*** $\dfrac{F}{\delta} = \dfrac{E\,A}{L}$ | $m_1 = L^2 \dfrac{F}{\delta}\left(\dfrac{\rho}{E}\right)$ | $M_1 = \dfrac{\rho}{E}$ | (1) |
| | *Substitute for A* — ***Strength constraint*** $F = \sigma_y A$ | $m_2 = L\,F\left(\dfrac{\rho}{\sigma_y}\right)$ | $M_2 = \dfrac{\rho}{\sigma_y}$ | (2) |

(The symbols have their usual meanings: $A$ = area, $L$ = length, $\rho$ = density, $F/\delta$ = stiffness, $E$ = Young's modulus, $\sigma_y$ = yield strength or elastic limit.)

The coupling equation is found by equating $m_1$ to $m_2$, giving

$$\left(\frac{\rho}{\sigma_y}\right) = \frac{L}{\delta}\left(\frac{\rho}{E}\right)$$

defining the coupling constant $C_c = L/\delta$. The chart below shows the positions of the coupling line when $L/\delta$ = 100 and when $L/\delta$ = $10^3$ (corresponding to the required values of $\delta/L$ in the question) and the materials that are the best choice for each.






| Coupling condition | Material choice | Comment |
| --- | --- | --- |
| $L/\delta$ = 100 | Ceramics: boron carbide, silicon carbide | These materials are available as fibers as well as bulk. |
| $L/\delta$ = 1000 | Composites: CFRP; after that, Ti, Al and Mg alloys | If ductility and toughness are also required, the metals are the best choice. |

The use of ceramics for a tie, which must carry tension, is normally ruled out by their low fracture toughness – even a small flaw can lead to brittle failure. But in the form of fibers both boron carbide and silicon carbide are used as reinforcement in composites, where they are loaded in tension, and their stiffness and strength at low weight are exploited.

*The CES EduPack software allows the construction of charts with axes that are combinations of properties, like those of* $\rho/E$ *and* $\rho/\sigma_y$ *shown here, and the application of a selection box to identify the optimum choice of material.*

**E8.2 Multiple constraints: a light, safe, pressure vessel** (Figure E13) When a pressure vessel has to be mobile; its weight becomes important. Aircraft bodies, rocket casings and liquid-natural gas containers are examples; they must be light, and at the same time they must be safe, and that means that they must not fail by yielding or by fast fracture. What are the best materials for their construction? The table summarizes the requirements.

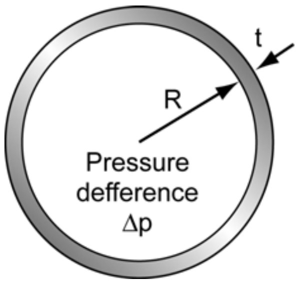



Figure E13

| | |
|---|---|
| **Function** | • *Pressure vessel* |
| **Constraints** | • *Must not fail by yielding*<br>• *Must not fail by fast fracture.*<br>• *Diameter 2R and pressure difference $\Delta p$ specified* |
| **Objective** | • *Minimize mass m* |
| **Free variables** | • *Wall thickness, t*<br>• *Choice of material* |

(a) Write, first, a performance equation for the mass *m* of the pressure vessel. Assume, for simplicity, that it is spherical, of specified radius *R*, and that the wall thickness, *t* (the free variable) is small compared with *R*. Then the tensile stress in the wall is

$$\sigma = \frac{\Delta p\, R}{2\, t}$$

where $\Delta p$, the pressure difference across this wall, is fixed by the design. The first constraint is that the vessel should not yield – that is, that the tensile stress in the wall should not exceed $\sigma_y$. The second is that it should not fail by fast fracture; this requires that the wall-stress be less than $K_{1c} / \sqrt{\pi c}$, where $K_{1c}$ is the fracture toughness of the material of which the pressure vessel is made and *c* is the length of the longest crack that the wall might contain. Use each of these in turn to eliminate *t* in the equation for *m*; use the results to identify two material indices

$$M_1 = \frac{\rho}{\sigma_y} \qquad \text{and} \qquad M_2 = \frac{\rho}{K_{1c}}$$

and a coupling relation between them. It contains the crack length, *c*.

(b) The figure shows the chart you will need with the two material indices as axes. Plot the coupling equation onto this figure for two values of *c*: one of 5 mm, the other 5 μm. Identify the lightest candidate materials for the vessel for each case.

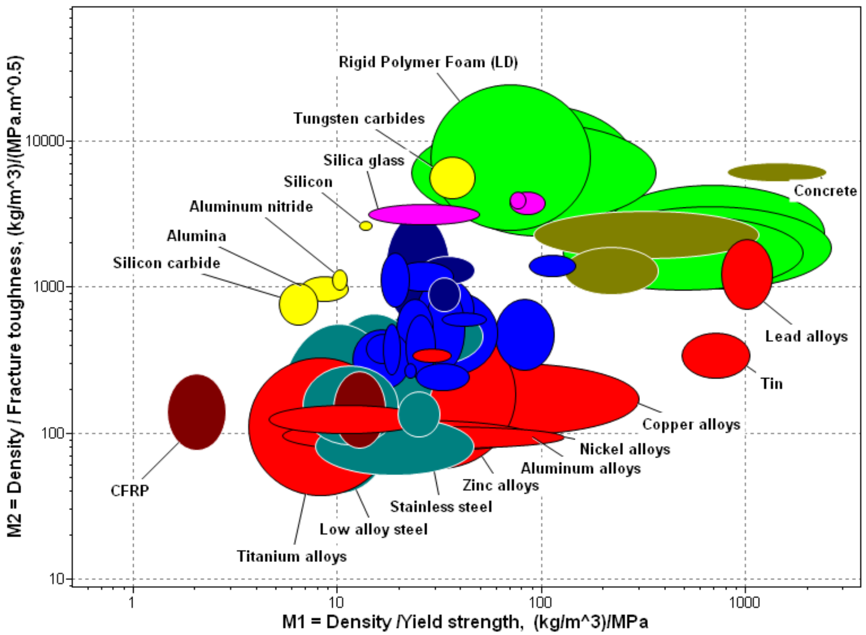



Figure E14

***Answer.*** The objective function is the mass of the pressure vessel:

$$m = 4\pi R^2 t \rho$$

The tensile stress in the wall of a thin-walled pressure vessel (Appendix B, Section B11) is

$$\sigma = \frac{\Delta p\, R}{2t}$$

Equating this first to the yield strength $\sigma_y$, then to the fracture strength $K_{1c}/\sqrt{\pi c}$ and substituting for *t* in the objective function leads to the performance equations and indices laid out below.

| ***Objective*** | ***Constraints*** | | ***Performance equation*** | ***Index*** | |
|---|---|---|---|---|---|
| $m = 4\pi R^2 t \rho$ | ***Yield constraint*** | $\sigma = \frac{\Delta p R}{2t} \le \sigma_y$ (Substitute for t) | $\Rightarrow$ $m_1 = 2\pi\, \Delta p . R^3 \cdot \left[\frac{\rho}{\sigma_f}\right]$ | $M_1 = \frac{\rho}{\sigma_y}$ | (1) |
| | ***Fracture constraint*** | $\sigma = \frac{\Delta p R}{2t} \le \frac{K_{1c}}{\sqrt{\pi c}}$ (Substitute for t) | $\Rightarrow$ $m_2 = 2\pi\, \Delta p \cdot R^3 (\pi c)^{1/2} \left[\frac{\rho}{K_{1c}}\right]$ | $M_2 = \frac{\rho}{K_{1c}}$ | (2) |

The coupling equation is found by equating $m_1$ to $m_2$, giving a relationship between $M_1$ and $M_2$:

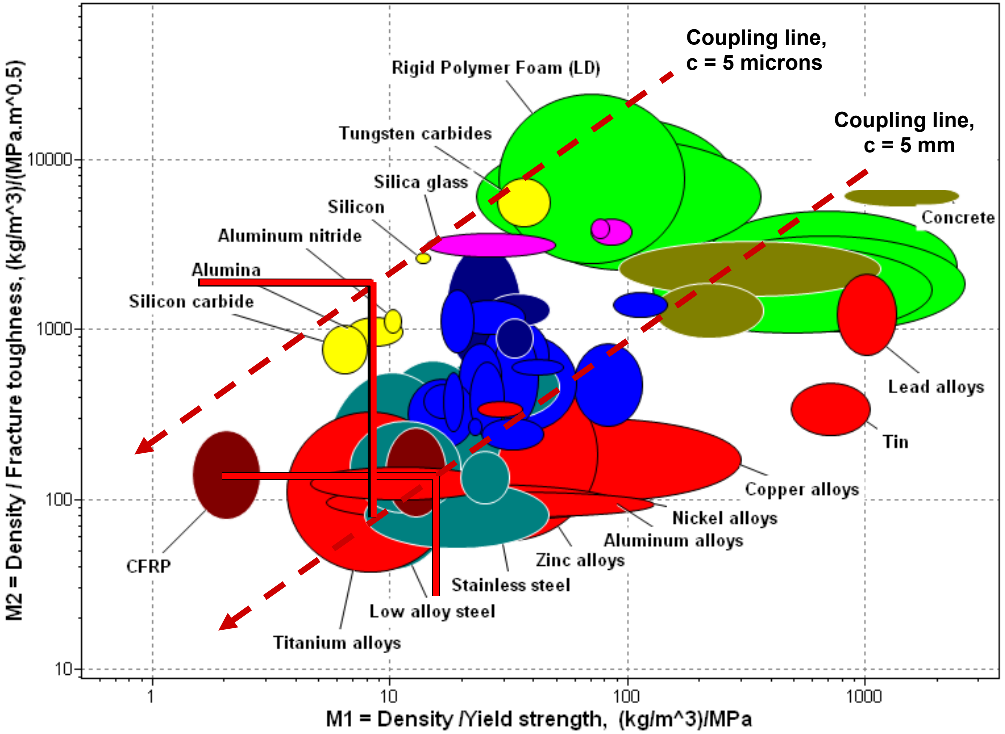
$$M_1 = (\pi c)^{1/2} M_2$$

The position of the coupling line depends on the detection limit, $c_1$ for cracks, through the term $(\pi c)^{1/2}$.

The figure shows the appropriate chart with two coupling lines, one for *c* = 5 mm and the other for *c* = 5μm.

The resulting selection is summarised in the table.

| Coupling condition | Material choice | Comment |
|---|---|---|
| Crack length $c \leq 5\,mm$ $(\sqrt{\pi c} = 0.125)$ | Titanium alloys<br>Aluminium alloys<br>Steels | These are the standard materials for pressure vessels. Steels appear, despite their high density, because their toughness and strength are so high |
| Crack length $c \leq 5\,\mu m$ $(\sqrt{\pi c} = 3.96\ x\ 10^{-3})$ | CFRP<br>Silicon carbide<br>Silicon nitride<br>Alumina | Ceramics, potentially, are attractive structural materials, but the difficulty of fabricating and maintaining them with no flaws greater than 5 μm is enormous |




In large engineering structures it is difficult to ensure that there are no cracks of length greater than 1 mm; then the tough engineering alloys based on steel, aluminum and titanium are the safe choice. In the field of MEMS (micro electro-mechanical systems), in which films of micron-thickness are deposited on substrates, etched to shapes and then loaded in various ways, it is possible – even with brittle ceramics – to make components with no flaws greater than 1 μm in size. In this regime, the second selection given above has relevance.

**E8.3 A cheap column that must not buckle or crush** (Figure E15). The best choice of material for a light strong column depends on its aspect ratio: the ratio of its height $H$ to its diameter $D$. This is because short, fat columns fail by crushing; tall slender columns buckle instead. Derive two performance equations for the material cost of a column of solid circular section and specified height $H$, designed to support a load $F$ large compared to its self-load, one using the constraints that the column must not crush, the other that it must not buckle. The table summarizes the needs.

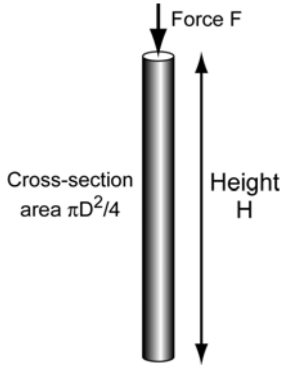



Figure E15

| | |
|---|---|
| **Function** | • *Column* |
| **Constraints** | • *Must not fail by compressive crushing*<br>• *Must not buckle*<br>• *Height H and compressive load F specified.* |
| **Objective** | • *Minimize material cost C* |
| **Free variables** | • *Diameter D*<br>• *Choice of material* |

(a) Proceed as follows

(1) Write down an expression for the material cost of the column – its mass times its cost per unit mass, $C_m$.

(2) Express the two constraints as equations, and use them to substitute for the free variable, $D$, to find the cost of the column that will just support the load without failing by either mechanism

(3) Identify the material indices $M_1$ and $M_2$ that enter the two equations for the mass, showing that they are

$$M_1 = \left(\frac{C_m \rho}{\sigma_c}\right) \quad \text{and} \quad M_2 = \left[\frac{C_m \rho}{E^{1/2}}\right]$$

where $C_m$ is the material cost per kg, $\rho$ the material density, $\sigma_c$ its crushing strength and $E$ its modulus.

(b) Data for six possible candidates for the column are listed in the Table. Use these to identify candidate materials when $F = 10^5$ N and $H = 3$m. Ceramics are admissible here, because they have high strength in compression.

**Data for candidate materials for the column**

| **Material** | Density ρ (kg/m³) | Cost/kg $C_m$ ($/kg) | Modulus E (MPa) | Compression strength $\sigma_c$ (MPa) |
|---|---|---|---|---|
| Wood (spruce) | 700 | 0.5 | 10,000 | 25 |
| Brick | 2100 | 0.35 | 22,000 | 95 |
| Granite | 2600 | 0.6 | 20,000 | 150 |
| Poured concrete | 2300 | 0.08 | 20,000 | 13 |
| Cast iron | 7150 | 0.25 | 130,000 | 200 |
| Structural steel | 7850 | 0.4 | 210,000 | 300 |
| Al-alloy 6061 | 2700 | 1.2 | 69,000 | 150 |

(c) Figure E16 shows a material chart with the two indices as axes. Identify and plot coupling lines for selecting materials for a column with $F = 10^5$ N and $H = 3$m (the same conditions as above), and for a second column with $F = 10^3$ N and $H = 20$m.

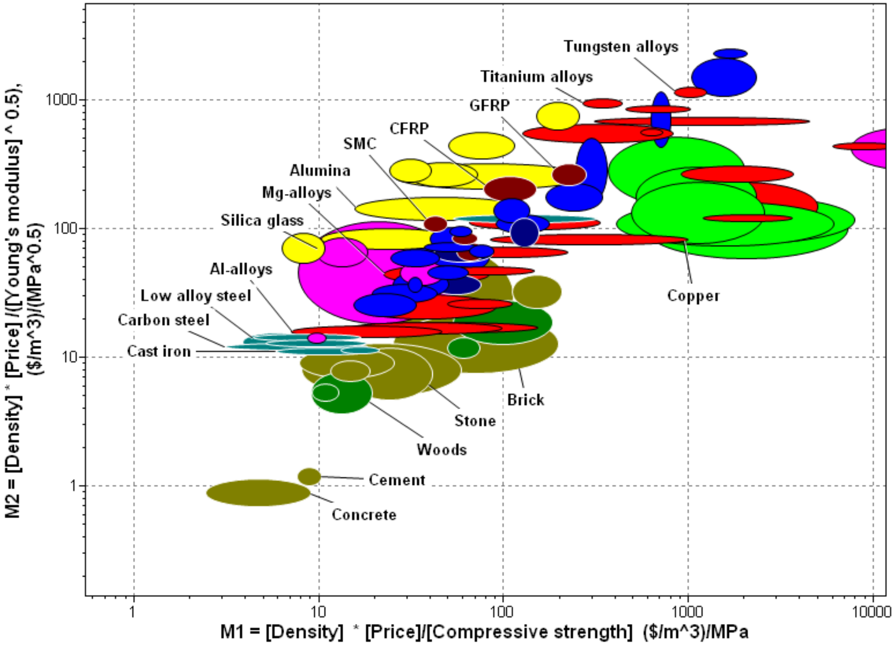



Figure E16

***Answer.*** This, and exercises E 8.1 and E 8.2 illustrate the method of solving over-constrained problems. This one concerns materials for a light column with circular section which must neither buckle nor crush under a design load *F*.

The cost, *C*, is to be minimised

$$C = \frac{\pi}{4} D^2 H\, C_m \rho$$

where *D* is the diameter (the free variable) and *H* the height of the column, $C_m$ is the cost per kg of the material and $\rho$ is its density. The column must not crush, requiring that

$$\frac{4F}{\pi D^2} \le \sigma_c$$

where $\sigma_c$ is the compressive strength. Nor must it buckle (Appendix B, Section B5):

$$F \le \frac{\pi^2 EI}{H^2}$$

The right-hand side is the Euler buckling load in which *E* is Young's modulus. The second moment of area for a circular column (Appendix B, Section B2) is

$$I = \frac{\pi}{64} D^4$$

The subsequent steps in the derivation of performance equations are laid out on the next page:

**Objective** | **Constraint** | **Performance equation**

*Substitute for D*

**Crushing constraint** $F_f \le \frac{\pi D^2}{4}\sigma_c \quad \Rightarrow \quad C_1 = F\,H\left[\frac{C_m \rho}{\sigma_c}\right]$ (1)

$$C = \frac{\pi}{4} D^2 H\, C_m \rho$$

*Substitute for D*

**Buckling constraint** $F \le \frac{\pi^2 EI}{H^2} = \frac{\pi^3 D^4 E}{64\,H^2} \quad \Rightarrow \quad C_2 = \frac{2}{\pi^{1/2}} F^{1/2} H^2 \left[\frac{C_m \rho}{E^{1/2}}\right]$ (2)

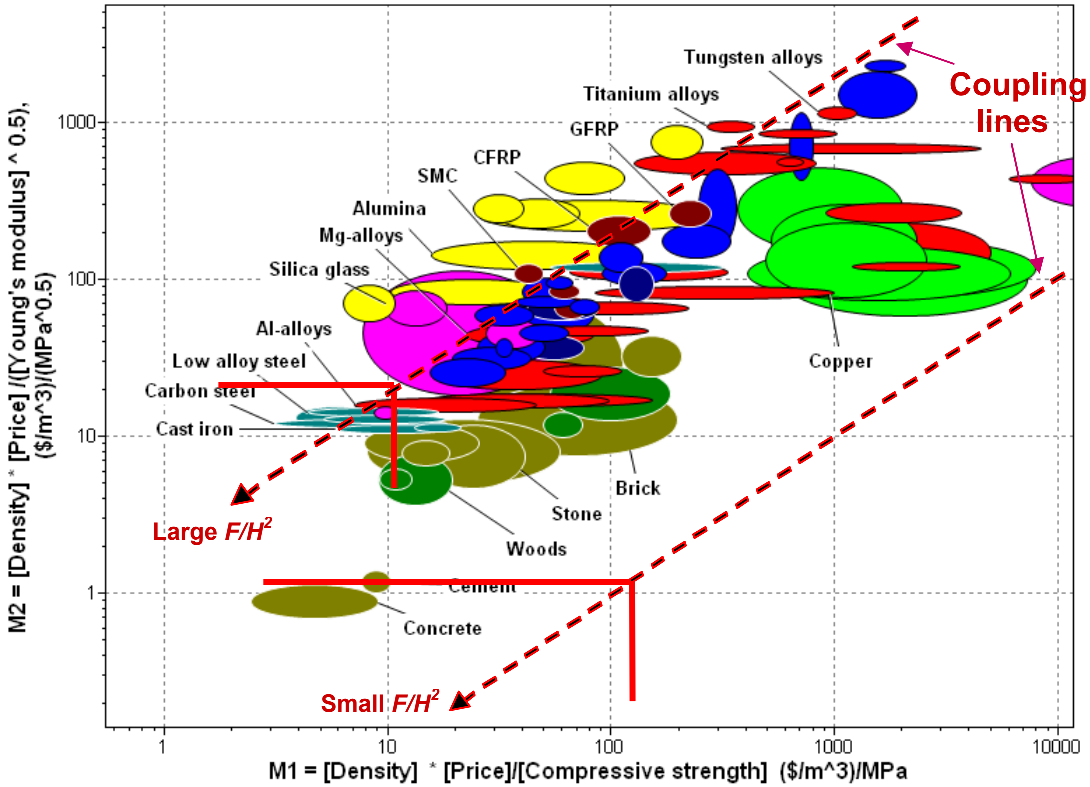
The first performance equation contains the index $M_1 = \left[\frac{C_m \rho}{\sigma_c}\right]$, the second, the index $M_2 = \left[\frac{C_m \rho}{E^{1/2}}\right]$. This is a min-max problem: we seek the material with the lowest (min) cost $\tilde{C}$ which itself is the larger (max) of $C_1$ and $C_2$. The two performance equations are evaluated in the Table, which also lists $\tilde{C} = max(C_1, C_2)$. for a column of height $H$ = 3m, carrying a load $F = 10^5$ N. The cheapest choice is concrete.

| **Material** | Density $\rho$ (kg/m$^3$) | Cost/kg $C_m$ (\$/kg) | Modulus $E$ (MPa) | Compression strength $\sigma_c$ (MPa) | $C_1$ \$ | $C_2$ \$ | $\tilde{C}$ \$ |
|---|---|---|---|---|---|---|---|
| Wood (spruce) | 700 | 0.5 | 10,000 | 25 | 4.2 | 11.2 | 11.2 |
| Brick | 2100 | 0.35 | 22,000 | 95 | 2.3 | 16.1 | 16.1 |
| Granite | 2600 | 0.6 | 20,000 | 150 | 3.1 | 35.0 | 35.0 |
| Poured concrete | 2300 | 0.08 | 20,000 | 13 | 4.3 | 4.7 | **4.7** |
| Cast iron | 7150 | 0.25 | 130,000 | 200 | 2.6 | 16.1 | 16.1 |
| Structural steel | 7850 | 0.4 | 210,000 | 300 | 3.0 | 21.8 | 21.8 |
| Al-alloy 6061 | 2700 | 1.2 | 69,000 | 150 | 6.5 | 39.5 | 39.5 |

The coupling equation is found by equating $C_1$ to $C_2$ giving

$$M_2 = \frac{\pi^{1/2}}{2} \cdot \left( \frac{F}{H^2} \right)^{1/2} \cdot M_1$$

It contains the structural loading coefficient $F/H^2$. Two positions for the coupling line are shown, one corresponding to a low value of $F/H^2$ = 0.011 MN/m$^2$ ($F$ = 10$^5$ N, $H$ = 3 m) and to a high one $F/H^2$ = 2.5 MN/m$^2$ ($F$ = 10$^7$ N, $H$ = 2 m), with associated solutions. Remember that, since $E$ and $\sigma_c$ are measured in MPa, the load $F$ must be expressed in units of MN. The selection is listed in the table.



**E8.4 An air cylinder for a truck** (Figure E17). Trucks rely on compressed air for braking and other power-actuated systems. The air is stored in one or a cluster of cylindrical pressure tanks like that shown here (length *L*, diameter 2*R*, hemispherical ends). Most are made of low-carbon steel, and they are heavy. The task: to explore the potential of alternative materials for lighter air tanks, recognizing the there must be a trade-off between mass and cost – if it is too expensive, the truck owner will not want it even if it *is* lighter. The table summarizes the design requirements.

| | |
|---|---|
| **Function** | • *Air cylinder for truck* |
| **Constraints** | • *Must not fail by yielding*<br>• *Diameter 2R and length L specified, so the ratio Q = 2R/L is fixed.* |
| **Objectives** | • *Minimize mass m*<br>• *Minimize material cost C* |
| **Free variables** | • *Wall thickness, t*<br>• *Choice of material* |

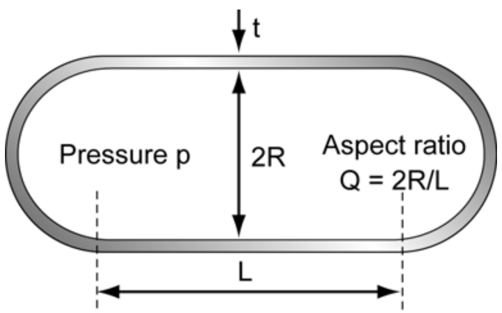



Figure E17

(a) Show that the mass and material cost of the tank relative to one made of low-carbon steel are given by

$$\frac{m}{m_o} = \left(\frac{\rho}{\sigma_y}\right)\left(\frac{\sigma_{y,o}}{\rho_o}\right) \quad \text{and} \quad \frac{C}{C_o} = \left(\frac{C_m \rho}{\sigma_y}\right)\left(\frac{\sigma_{y,o}}{C_{m,o}\rho_o}\right)$$

where $\rho$ is the density, $\sigma_y$ the yield strength and $C_m$ the cost per kg of the material, and the subscript "o" indicates values for mild steel.

(b) Explore the trade-off between relative cost and relative mass, considering the replacement of a mild steel tank with one made, first, of low alloy steel, and, second, one made of filament-wound CFRP, using the material properties in the table below Define a relative penalty function

$$Z^* = \alpha^* \frac{m}{m_o} + \frac{C}{C_o}$$

where $\alpha^*$ is a relative exchange constant, and evaluate $Z^*$ for $\alpha^*$ = 1 and for $\alpha^*$ = 100.

| **Material** | Density $\rho$ (kg/m$^3$) | Yield strength $\sigma_c$ (MPa) | Price per /kg $C_m$ ($/kg) |
|---|---|---|---|
| Mild steel | 7850 | 314 | 0.66 |
| Low alloy steel | 7850 | 775 | 0.85 |
| CFRP | 1550 | 760 | 42.1 |

(c) Figure E18, below, is a chart with axes of $m/m_o$ and $C/C_o$. Mild steel (here labelled "Low carbon steel") lies at the co-ordinates (1,1).

Sketch a trade-off surface and plot contours of $Z^*$ that are approximately tangent to the trade-off surface for $\alpha^*$ = 1 and for $\alpha^*$ = 100. What selections do these suggest?

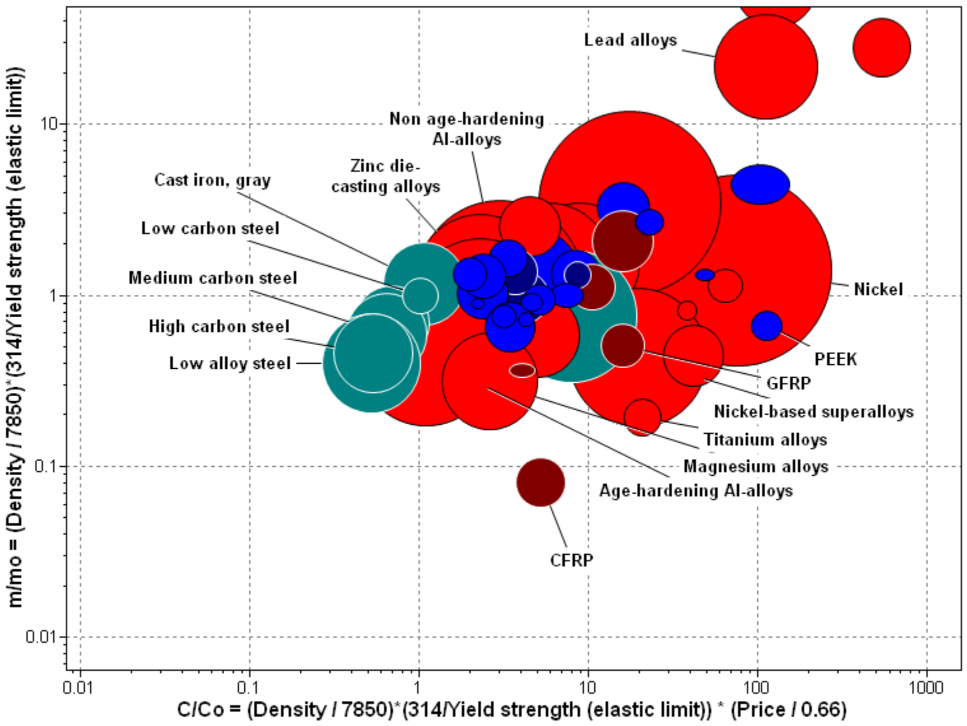



Figure E18

***Answer.*** (a) The mass $m$ of the tank is

$$m = 2\pi R\,L\,t + 4\pi\,R^2\,t = 2\pi R\,L\,t\,(1+Q)$$

where $Q$, the aspect ratio $2R/L$, is fixed by the design requirements. The stress in the wall of the tank caused by the pressure $p$ must not exceed $\sigma_y$, is the yield strength of the material of the tank wall, meaning that

$$\sigma = \frac{p\,R}{t} \le \sigma_y$$

Substituting for $t$, the free variable, gives

$$m = 2\pi R^2\,L\,p\,(1+Q)\left(\frac{\rho}{\sigma_y}\right)$$

The material cost $C$ is simply the mass $m$ times the cost per kg of the material, $C_m$, giving

$$C = C_m m = 2\pi R^2\,L\,p\,(1+Q)\left(\frac{C_m\rho}{\sigma_y}\right)$$

from which the mass and cost relative to that of a low-carbon steel (subscript "o") tank are

$$\frac{m}{m_o} = \left(\frac{\rho}{\sigma_y}\right)\left(\frac{\sigma_{y,o}}{\rho_o}\right) \quad \text{and}$$

$$\frac{C}{C_o} = \left(\frac{Cm\,\rho}{\sigma_y}\right)\left(\frac{\sigma_{y,o}}{C_{m,o}\rho_o}\right)$$

(b) To get further we need a penalty function:. The relative penalty function

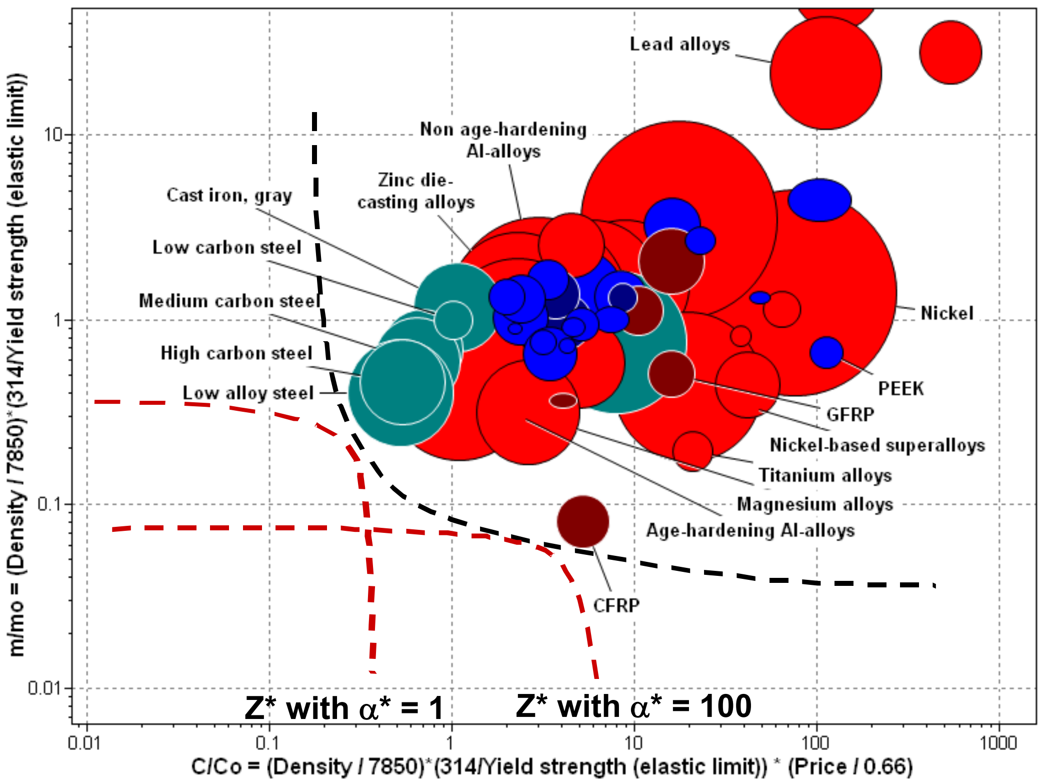
$$Z^* = \alpha^*\frac{m}{m_o} + \frac{C}{C_o}$$

This is evaluated for Low alloy steel and for CFRP in the table below, for $\alpha^*$ = 1 (meaning that weight carries a low cost premium) – Low alloy steel has by far the lowest $Z^*$. But when it is evaluated for $\alpha^*$ = 100 (meaning that weight carriers a large cost premium), CFRP has the lowest $Z^*$.

(c) The figure shows the trade-off surface. Materials on or near this surface have attractive combinations of mass and cost. Several are better low-carbon steel. Two contours of $Z^*$ *that just touch the trade-off line are shown,* one for $\alpha^*$ = 1, the other for $\alpha^*$ = 100 – they are curved because of the logarithmic axes.

The first, for $\alpha^*$ = 1 identifies higher strength steels as good choices. This is because their higher strength allows a thinner tank wall. The contour for $\alpha^*$ = 100 touches near CFRP, aluminum and magnesium alloys – if weight saving is very highly valued, these become attractive solutions.

| Material | Density $\rho$ (kg/m$^3$) | Yield strength $\sigma_c$ (MPa) | Price per /kg $C_m$ ($/kg) | $Z^*$, $\alpha^* = 1$ | $Z^*$, $\alpha^* = 100$ |
|---|---|---|---|---|---|
| Mild steel | 7850 | 314 | 0.66 | 2 | 101 |
| Low alloy steel | 7850 | 775 | 0.85 | **1.03** | 45.6 |
| CFRP | 1550 | 760 | 42.1 | 5.2 | **13.4** |



**E8.5 Insulating walls for freezers** (Figure E19). Freezers and refrigerated trucks have panel-walls that provide thermal insulation, and at the same time are stiff, strong and light (stiffness to suppress vibration, strength to tolerate rough usage). To achieve this the panels are usually of sandwich construction, with two skins of steel, aluminum or GFRP (providing the strength) separated by, and bonded to, a low density insulating core. In choosing the core we seek to minimize thermal conductivity, λ, and at the same time to maximize stiffness, because this allows thinner steel faces, and thus a lighter panel, while still maintaining the overall panel stiffness. The table summarizes the design requirements.

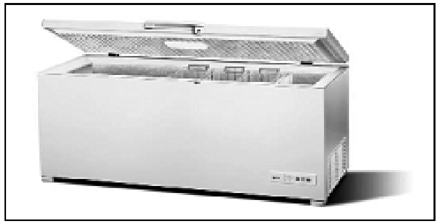


Figure E19

| | |
|---|---|
| **Function** | • *Foam for panel-wall insulation* |
| **Constraint** | • *Panel wall thickness specified.* |
| **Objectives** | • *Minimize foam thermal conductivity, λ*<br>• *Maximize foam stiffness, meaning Young's modulus, E* |
| **Free variables** | • *Choice of material* |

Figure E20 shows the thermal conductivity $\lambda$ of foams plotted against their elastic compliance $1/E$ (the reciprocal of their Young's moduli *E*, since we must express the objectives in a form that requires minimization). The numbers in brackets are the densities of the foams in $Mg/m^3$. The foams with the lowest thermal conductivity are the least stiff; the stiffest have the highest conductivity. Explain the reasoning you would use to select a foam for the truck panel using a penalty function.

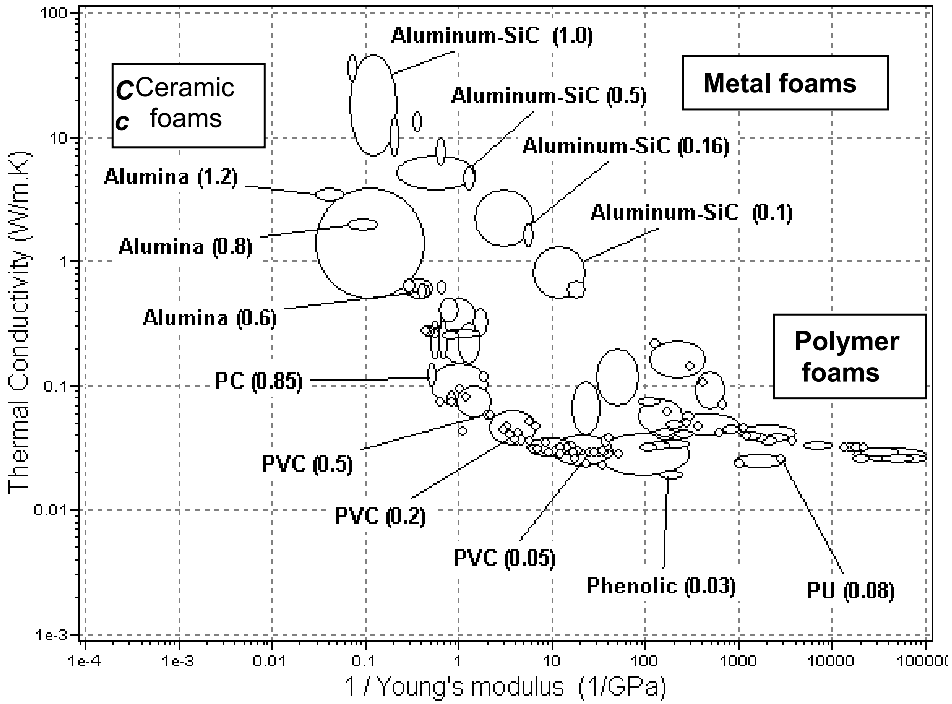



Figure E20

***Answer.*** The steps in making a reasoned choice are as follows.

(1) Sketch the trade-off surface: the low $\lambda$ vs. low *1/E* envelope of the data, as shown below. The foams that lie on or near the surface are a better choice than those far from it. This already eliminates a large number of foams and identifies the family from which a choice should be made. Note that most metal foams are not a good choice; only if the highest stiffness is wanted is the metal foam Aluminium-SiC (1.0) an attractive choice.

(2) If it is desired to go further, it is necessary to construct a penalty function:

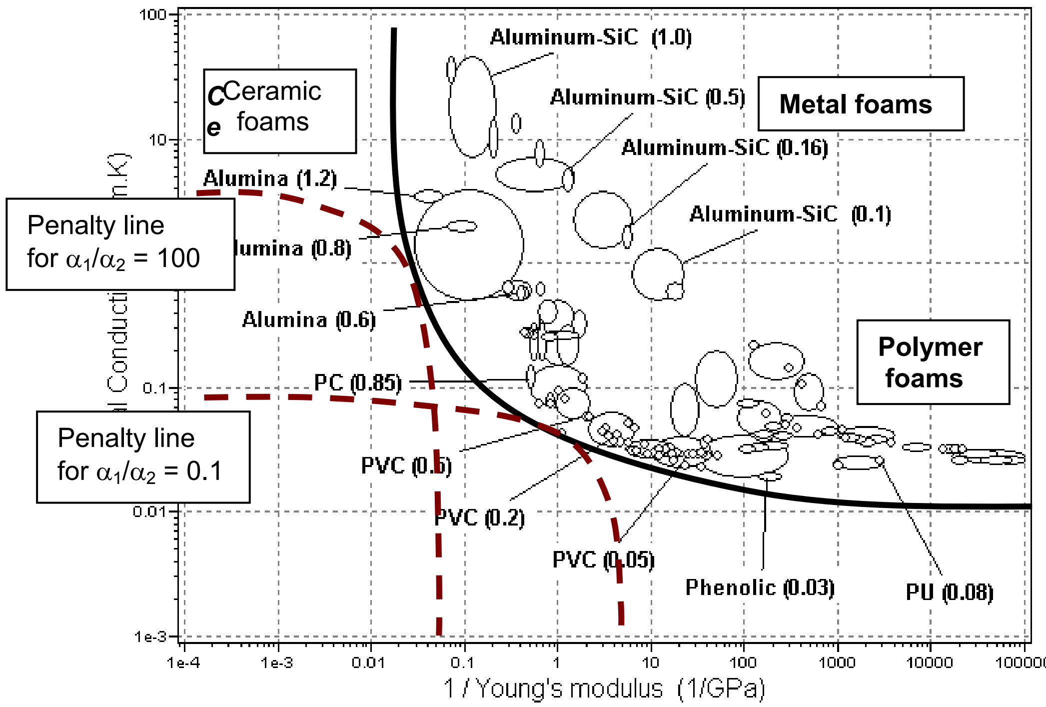
$$Z = \alpha_1 \lambda + \alpha_2 \left(\frac{1}{E}\right)$$

*Z* is to be minimized, so $\alpha_1$ is a measure of the value associated with reducing heat flow; $\alpha_2$ a measure of the value associated with reducing core compliance. Rearranging the equations gives

$$\lambda = \frac{Z}{\alpha_1} - \frac{\alpha_2}{\alpha_1}\left(\frac{1}{E}\right)$$

If the axes were linear, this equation would be that of a family of straight, parallel, lines on the $\lambda$ vs. *1/E* diagram, of slope $-\alpha_2 / \alpha_1$, each line corresponding to a value of $Z / \alpha_1$. In fact the scales are logarithmic, and that leads instead to a set of curved lines. One such line is sketched below for values $\alpha_2 / \alpha_1$ = 0.01 (meaning that thermal insulation is considered very important, and stiffness less important) and for $\alpha_2 / \alpha_1$ = 100 (meaning the opposite). The foam nearest the point at which the penalty lines are tangent to the trade-off surface is the best choice. In the first example PVC foam with a density of about 0.1 Mg/m$^3$ is the best choice, but in the second a ceramic or even a metal foam is a better choice.

Ceramic foams are brittle. This probably rules them out for the truck body because is exposed to impact loads. But in other applications ceramic foams – particularly glass foams – are viable.



## E9. Selecting material and shape (Chapters 9 and 10)

*The examples in this section relate to the analysis of material and shape of Chapters 9 and 10. They cover the derivation of shape factors, of indices that combine material and shape, and the use of the 4-quadrant chart arrays to explore material and shape combinations. For this last purpose it is useful to have clean copies of the chart arrays of Figures 9.9 and 9.12. Like the material property charts, they can be copied from the text without restriction of copyright.*

**E9.1 Shape factors for tubes** (Figure E21).

(a) Evaluate the shape factor $\phi_B^e$ for stiffness-limited design in bending of a square box section of outer edge-length $h$ = 100mm and wall thickness $t$ = 3mm. Is this shape more efficient than one made of the same material in the form of a tube of diameter $2r$ = 100mm and wall thickness $t$ = 3.82 mm (giving it the same mass per unit length, $m/L$)? Treat both as thin-walled shapes.

(b) Make the same comparison for the shape factor $\phi_B^f$ for strength-limited design.

Use the expressions given in Table 9.3 of the text for the shape factors $\phi_B^e$ and $\phi_B^f$.

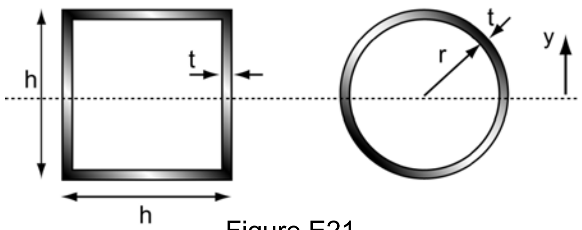



Figure E21

***Answer.***

(a) The shape-efficiency factor for elastic bending for the square box section (from Table 9.3) is

$$\phi_B^e = \frac{1}{2}\frac{h}{t} = 16.7$$

That for the tube is

$$\phi_B^e = \frac{3}{\pi}\frac{r}{t} = 12.5$$

The box is more efficient than the tube of the same $m/L$, and both are much stiffer in bending than a solid square section of the same area (and thus mass per unit length).

(b) The shape-efficiency factor for bending failure for the square box section (from Table 9.3) is

$$\phi_B^f = \sqrt{\frac{h}{t}} = 5.8$$

That for the tube is

$$\phi_B^f = \frac{3}{\sqrt{2\pi}}\sqrt{\frac{r}{t}} = 4.33$$

The box is more efficient than the tube, and both are roughly 5 times stronger in bending than a solid square section of the same area (and thus mass per unit length).

**E9.2 Deriving shape factors for stiffness-limited design** (Figure E22). (a) Derive the expression for the shape-efficiency factor $\phi_B^e$ for stiffness-limited design for a beam loaded in bending with each of the three sections listed below. Do not assume that the thin-wall approximations is valid.

(a) a closed circular tube of outer radius *5t* and wall thickness *t*,

(b) a channel section of thickness *t*, overall flange width *5t* and overall depth *10 t*, bent about its major axis; and

(c) a box section of wall thickness *t,* and height and width $h_1 = 10t$.

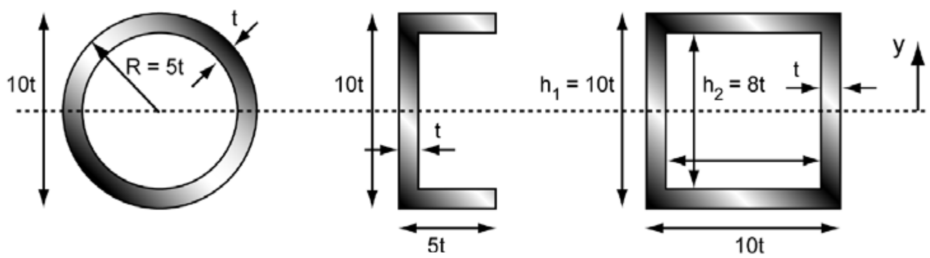



Figure E22

Answer.

(a) The shape factor for elastic bending of a tubular beam (left-hand figure) is defined by

$$\phi_B^e = \frac{12\,I}{A^2}$$

where $I$ is the second moment of area and $A$ is the area. The second moment of area, *I,* of the tube about the axis shown as a dotted line is

$$I = \frac{\pi}{4}(5\,t)^4 - \frac{\pi}{4}(4\,t)^4 = 92.25\,\pi\,t^4$$

Its cross section area is $A = \pi(5\,t)^2 - \pi(4\,t)^2 = 9\,\pi\,t^2$

from which $\phi_B^e = \frac{12\,I}{A^2} = 4.35$

(b) The second moment of area, *I*, of the channel (central figure) about the axis shown as a dotted line is

$$I = \frac{(10t)^3.5t}{12} - \frac{(8t)^3.4t}{12} = 246\,t^4$$

Its cross section area is $A = 18\,t^2$

from which

$$\phi_B^e = \frac{12\,I}{A^2} = 9.11$$

(c) The second moment of area, *I*, of the square box-section (right-hand figure) about the axis shown as a dotted line is

$$I = 2\int_0^{h_1/2} y^2 h_1\,dy - 2\int_0^{h_2/2} y^2 h_2\,dy$$

$$= \frac{1}{12}\left(h_1^4 - h_2^4\right) = \frac{2}{3}\,t h_1^3$$

and

$$A = h_1^2 - h_2^2 \approx 4\,h_1\,t$$

Assembling these results, and inserting $h_1 = 10t$ gives

$$\phi_B^e = \frac{1}{2}\frac{h_1}{t} = 5$$

**E9.3 Deriving shape factors for strength-limited design.**

(a), (b) and (c) Determine the shape-efficiency factor $\phi_B^f$ for strength limited design in bending, for the same three sections shown as Figure 29 of Exercise E9.2. You will need the results for *I* for the sections derived in E9.2.

(d) A beam of length *L*, loaded in bending, must support a specified bending moment *M* without failing and be as light as possible.. Show that to minimize the mass of the beam per unit length, *m/L*, one should select a material and a section-shape to maximize the quantity

$$M = \frac{\left(\phi_B^f \sigma_f\right)^{2/3}}{\rho}$$

where $\sigma_f$ is the failure stress and $\rho$ the density of the material of the beam, and $\phi_B^f$ is the shape-efficiency factor for failure in bending.

***Answer.***

(a) The section modulus, *Z*, of the tube about the axis shown as a dotted line is

$$Z = \frac{I}{y_{max}} = \frac{I}{r} = 18.45\,\pi\, t^3$$

(here $y_{max}$ is the distance of the outer surface from the neutral axis),from which

$$\phi_B^f = \frac{6\,Z}{A^{3/2}} = 2.31$$

(b) The second moment, *Z*, of the channel about the axis shown as a dotted line is

$$Z = \frac{I}{5\,t} = 49.2\,t^3$$

from which
$$\phi_B^f = \frac{6\,Z}{A^{3/2}} = 3.87$$

(c) The second moment, *Z*, of the box-section about the axis shown as a dotted line is

$$Z = \frac{I}{10\,t}$$

from which
$$\phi_B^f = \frac{6\,Z}{A^{3/2}} = 4.47$$

(d) The beam of length *L*, loaded in bending, must support a specified load *F* without failing and be as light as possible. Its mass *m* is

$$m = A\,L\,\rho \qquad (1)$$

where *A* is the area of its cross section. Failure occurs if the load exceeds the moment

$$M = Z\sigma_f \qquad (2)$$

where *Z* is the section modulus . Replacing *Z* by the shape-factor $\phi_B^f = 6\,Z\,/\,A^{3/2}$ gives

$$M = \frac{\sigma_f}{6}\,\phi_B^f\,A^{3/2} \qquad (3)$$

Substituting this into equation (1) for the mass gives

$$m = (6\,M)^{2/3}\,L\,\frac{\rho}{\left(\phi_B^f\,\sigma_f\right)^{2/3}} \qquad (4)$$

The best material-and-shape combination is that with the greatest value of the index

$$M = \frac{\left(\phi_B^f\,\sigma_f\right)^{2/3}}{\rho}$$

**E9.4 Determining shape factors from stiffness data.** The elastic shape factor measures the gain in stiffness by shaping, relative to a solid square section of the same area. Shape factors can be determined by experiment. Equation 9.19 of the text gives the mass $m$ of a beam of length $L$ and prescribed bending stiffness $S_B$ with a section of efficiency $\phi_B^e$ as

$$m = \left(\frac{12 S_B}{C_1}\right)^{1/2} L^{5/2} \left[\frac{\rho}{\left(\phi_B^e E\right)^{1/2}}\right]$$

where $C_1$ is a constant that depends on the distribution of load on the beam. Inverting the equation gives

$$\phi_B^e = \left(\frac{12 L^5 S_B}{C_1 m^2}\right)\left(\frac{\rho^2}{E}\right)$$

Thus if the bending stiffness $S_B$, mass $m$ and length $L$ are measured, and the modulus $E$ and density $\rho$ of the material of the beam are known, $\phi_B^e$ can be calculated.

(a) Calculate the shape factor $\phi_B^e$ from the following experimental data, measured on an aluminum alloy beam loaded in 3-point bending (for which $C_1 = 48$ – see Appendix B, Section B3) using the data shown in the table

| Attribute | Value |
|---|---|
| Beam stiffness $S_B$ | $7.2 \times 10^5$ N/m |
| Mass/unit length $m/L$ | 1 kg/m |
| Beam length $L$ | 1 m |
| Beam material | 6061 aluminum alloy |
| Material density $\rho$ | 2670 kg/m$^3$ |
| Material modulus $E$ | 69 GPa |

(b) A steel truss bridge shown in Figure E23 has a span $L$ and is simply supported at both ends. It weighs $m$ tonnes. As a rule of thumb, bridges are designed with a stiffness $S_B$ such that the central deflection $\delta$ of a span under its self-weight is less than 1/300 of the length $L$ (thus $S_B \geq 300mg / L$ where $g$ is the acceleration due to gravity, 9.81 m/s$^2$).

Use this information to calculate the minimum shape factor $\phi_B^e$ of the three steel truss bridge spans listed in the table. Take the density $\rho$ of steel to be 7900 kg/m$^3$ and its modulus $E$ to be 205 GPa. The constant $C_1 = 384/5 = 76.8$ for uniformly distributed load (Appendix B, Section B3).

| **Bridge and construction date*** | **Span *L*** (m) | **Mass *m*** (tonnes) |
|---|---|---|
| Royal Albert bridge, Tamar, Saltash UK (1857) | 139 | 1060 |
| Carquinez Strait bridge, California (1927) | 132 | 650 |
| Chesapeake Bay bridge, Maryland USA (1952) | 146 | 850 |

*Data from the Bridges Handbook.

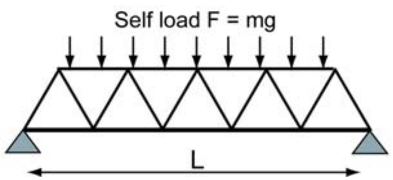



Figure E23

***Answer.***

(a) Inserting the data given in the question into the equation for $\phi_B^e$ given in the example gives

$$\phi_B^e = 18.6$$

(b) The shape factor is given in terms of *m, L* and $S_B$ by the equation given in the question. Substituting $S_B \geq 300mg / L$ gives

$$\phi_B^e = \left(\frac{3600\, L^4}{C_1} \frac{g}{m}\right)\left(\frac{\rho^2}{E}\right)$$

Inserting $C_1$ = 384/5 (Appendix B, Section B3) and the data from the table gives the results shown below.

| **Bridge and construction date*** | *F = mg* (10[6] N) | $A = m / L\rho$ ($m^2$) | $\phi_B^e$ |
|---|---|---|---|
| Royal Albert bridge, Tamar, Saltash UK (1857) | 10.6 | 0.97 | **49.3** |
| Carquinez Strait bridge, California (1927) | 6.5 | 0.62 | **65.4** |
| Chesapeake Bay bridge, Maryland USA (1952) | 8.5 | 0.74 | **74.8** |

The results show that "structured structures" (trusses, rib-stiffened structures) can offer very large shape efficiencies. Note how the efficiency has grown over time.

**E9.5 Deriving indices for bending and torsion.**

(a) A beam, loaded in bending, must support a specified bending moment *M** without failing and be as light as possible. Section shape is a variable, and "failure" here means the first onset of plasticity. Derive the material index. The table summarizes the requirements.

| | |
|---|---|
| **Function** | • *Light weight beam* |
| **Constraints** | • *Specified failure moment M**<br>• *Length L specified* |
| **Objective** | • *Minimum mass m* |
| **Free variables** | • *Choice of material*<br>• *Section shape and scale* |

(b) A shaft of length *L*, loaded in torsion, must support a specified torque *T** without failing and be as cheap as possible. Section shape is a variable and "failure" again means the first onset of plasticity. Derive the material index. The table summarizes the requirements.

| | |
|---|---|
| **Function** | • *Cheap shaft* |
| **Constraints** | • *Specified failure torque T**<br>• *Length L specified* |
| **Objective** | • *Minimum material cost C* |
| **Free variables** | • *Choice of material*<br>• *Section shape and scale* |

***Answer.***

(a) The mass $m$ of a beam of length $L$ and section area $A$ is

$$m = A L \rho \qquad (1)$$

where $\rho$ is the density of the material of which the beam is made. Failure occurs if the bending moment exceeds the value

$$M = Z \sigma_f \qquad (2)$$

Replacing $Z$ by the shape-factor $\phi_B^f$ (equation (9.10) of the text)gives

$$M = \frac{\sigma_f}{6} \phi_B^f A^{3/2} \qquad (3)$$

Substituting this into equation (1) for the mass of the beam gives

$$m = \left(6 M^*\right)^{2/3} L \left[ \frac{\rho}{\left(\phi_B^f \sigma_f\right)^{2/3}} \right] \qquad (4)$$

The best material-and-shape combination is that with the greatest value of the index

$$M_3 = \frac{\left(\phi_B^f \sigma_f\right)^{2/3}}{\rho}$$

(b) The shaft, loaded in torsion, must support a specified load $F$ without failing and be as light as possible. The material cost $C$ of a shaft of length $L$ and section area $A$ is

.

$$C = A L C_m \rho \qquad (5)$$

where $C_m$ is the cost per kg and $\rho$ is the density of the material of which the shaft is made. Failure occurs if the torque $T$ exceeds the moment

$$T = Q \tau_f \qquad (6)$$

where $\tau_f$ is the shear stress at which plasticity first starts.

Replacing $Q$ by the shape-factor $\phi_T^f$ (equation (11.13) of the text)gives

$$T = \frac{\tau_f}{4.8} \phi_T^f A^{3/2} \qquad (7)$$

Substituting this into equation (5) for the material cost of the shaft gives

$$m = \left(4.8 T^*\right)^{2/3} L \left[ \frac{C_m \rho}{\left(\phi_T^f \tau_f\right)^{2/3}} \right] \qquad (8)$$

Approximating $\tau_f$ by $\sigma_f / 2$, we find that the material-and-shape combination with the lowest associated material cost is that with the greatest value of the index

$$M_4 = \frac{\left(\phi_T^f \sigma_f\right)^{2/3}}{C_m \rho}$$

**E9.6 Use of the four segment chart for stiffness-limited design**

(a) Use the 4-segment chart for stiffness-limited design of Figure 9.9 to compare the mass per unit length, *m/L,* of a section with $EI = 10^5$ Nm$^2$ made from

(i) structural steel with a shape factor $\phi_B^e$ of 20, modulus $E$ = 210 GPa and density $\rho$ = 7900 kg/m$^3$

(ii) carbon fiber reinforced plastic with a shape factor $\phi_B^e$ of 10, modulus $E$ = 70 GPa and density $\rho$ = 1600 kg/m$^3$, and

(iii) structural timber with a shape factor $\phi_B^e$ of 2, modulus $E$ = 9 GPa and density $\rho$ = 520 kg/m$^3$.

The schematic Figure E24 illustrates the method.

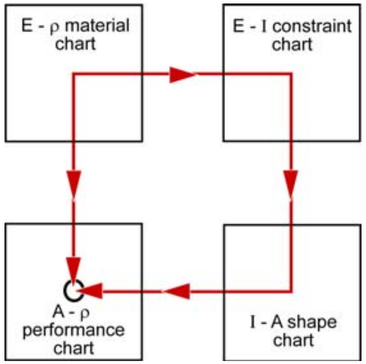



Figure E24

(b) Show, by direct calculation, that the conclusions of part (a) are consistent with the idea that to minimize mass for a given stiffness one should maximize $\sqrt{E^*}/\rho^*$ with $E^* = E/\phi_B^e$ and $\rho^* = \rho/\phi_B^e$.

***Answer.***

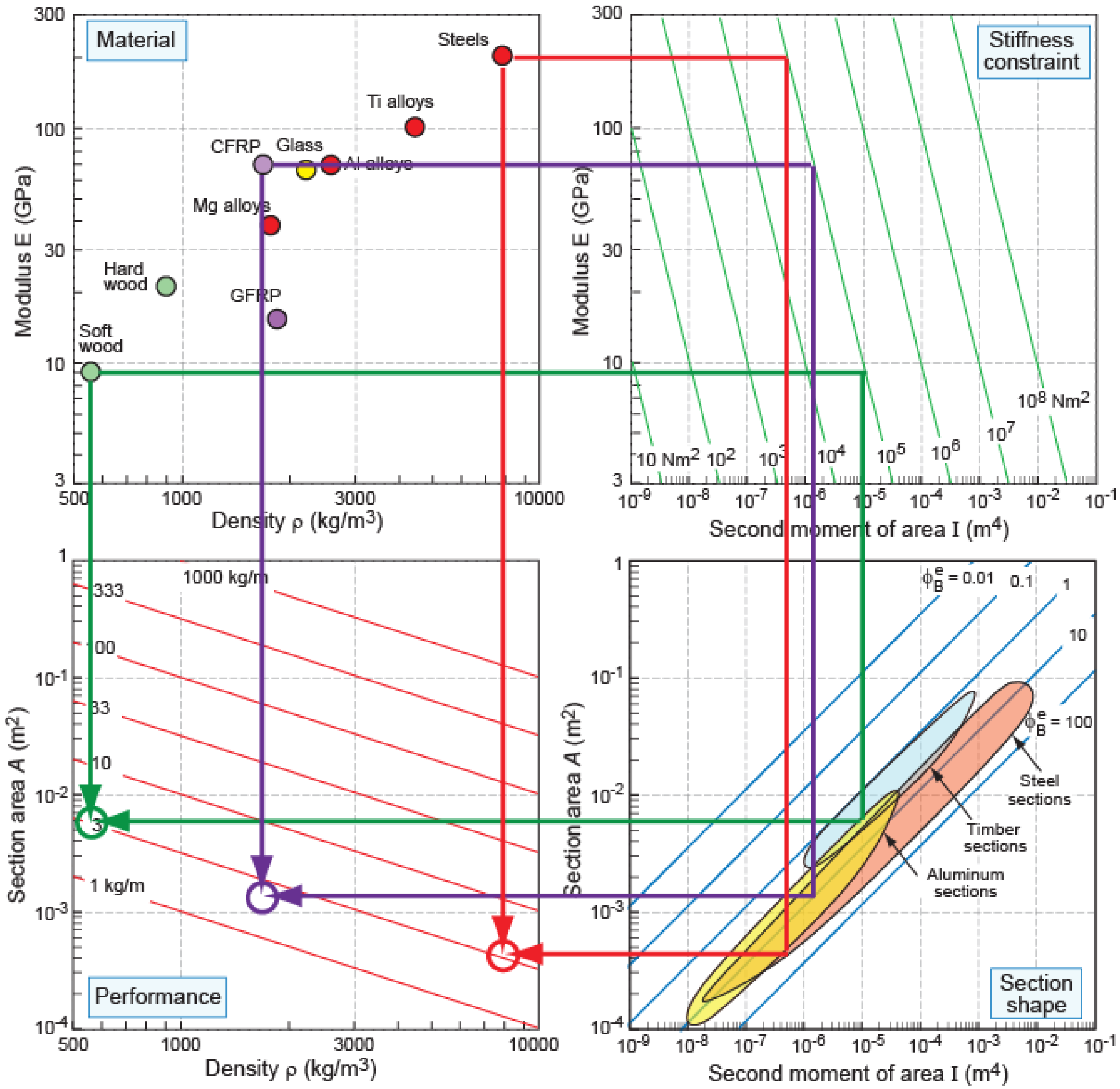
(a) The method is shown in the question. The results appear on the chart on the following page. The CFRP beam with *m/L* of about 4 kg/m is significantly (factor about 2) lighter than one made of steel or softwood.

(b) Direct calculation of $\sqrt{E^*}/\rho^*$ with $E^* = E/\phi_B^e$ and $\rho^* = \rho/\phi_B^e$ is shown in the table.

| Material | $E^* = E/\phi_B^e$ (N/m$^2$) | $\rho^* = \rho/\phi_B^e$ (kg/m$^3$) | $\sqrt{E^*}/\rho^*$ (N/m$^2$)$^{1/2}$/(kg/m$^3$) |
|---|---|---|---|
| Steel | $10.5 \times 10^9$ | 395 | 259 |
| CFRP | $7.0 \times 10^9$ | 160 | **523** |
| Softwood | $4.5 \times 10^9$ | 260 | 258 |

The higher the value of $\sqrt{E^*}/\rho^*$ the lighter is the beam. The ranking is the same as that arrived at in Part (a).

Stiffness-limited design in bending



**E9.7 Use of the four segment chart for strength**

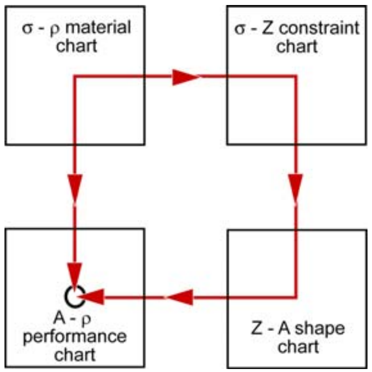
(a) Use the 4-segment chart for strength-limited design of Figure 9.12 to compare the mass per unit length, *m/L,* of a section with $Z\sigma_f$ = $10^4$ Nm (where *Z* is the section modulus) made from

(i) mild steel with a shape factor $\phi_B^f$ of 10, strength $\sigma_f$ = 200 MPa and density $\rho$ = 7900 kg/m$^3$

(ii) 6061 grade aluminum alloy with a shape factor $\phi_B^f$ of 3, strength $\sigma_f$ = 200 MPa and density $\rho$ = 2700 kg/m$^3$, and

(iii) a titanium alloy with a shape factor $\phi_B^f$ of 10, , strength $\sigma_f$ = 480 MPa and density $\rho$ = 4420 kg/m$^3$.

The schematic Figure E25 illustrates the method.




Figure

(b) Show, by direct calculation, that the conclusions of part (a) are consistent with the idea that to minimize mass for a given stiffness one should maximize $\dfrac{\left(\phi_B^f \sigma_f\right)^{2/3}}{\rho}$.

**Answer.**

(a) The method is shown in the question. The results appear on the chart on the next page. The steel section is the heaviest ($m/L \approx$ 9 kg/m). The titanium alloy is the lightest ($m/L \approx$ 2.5 kg/m).

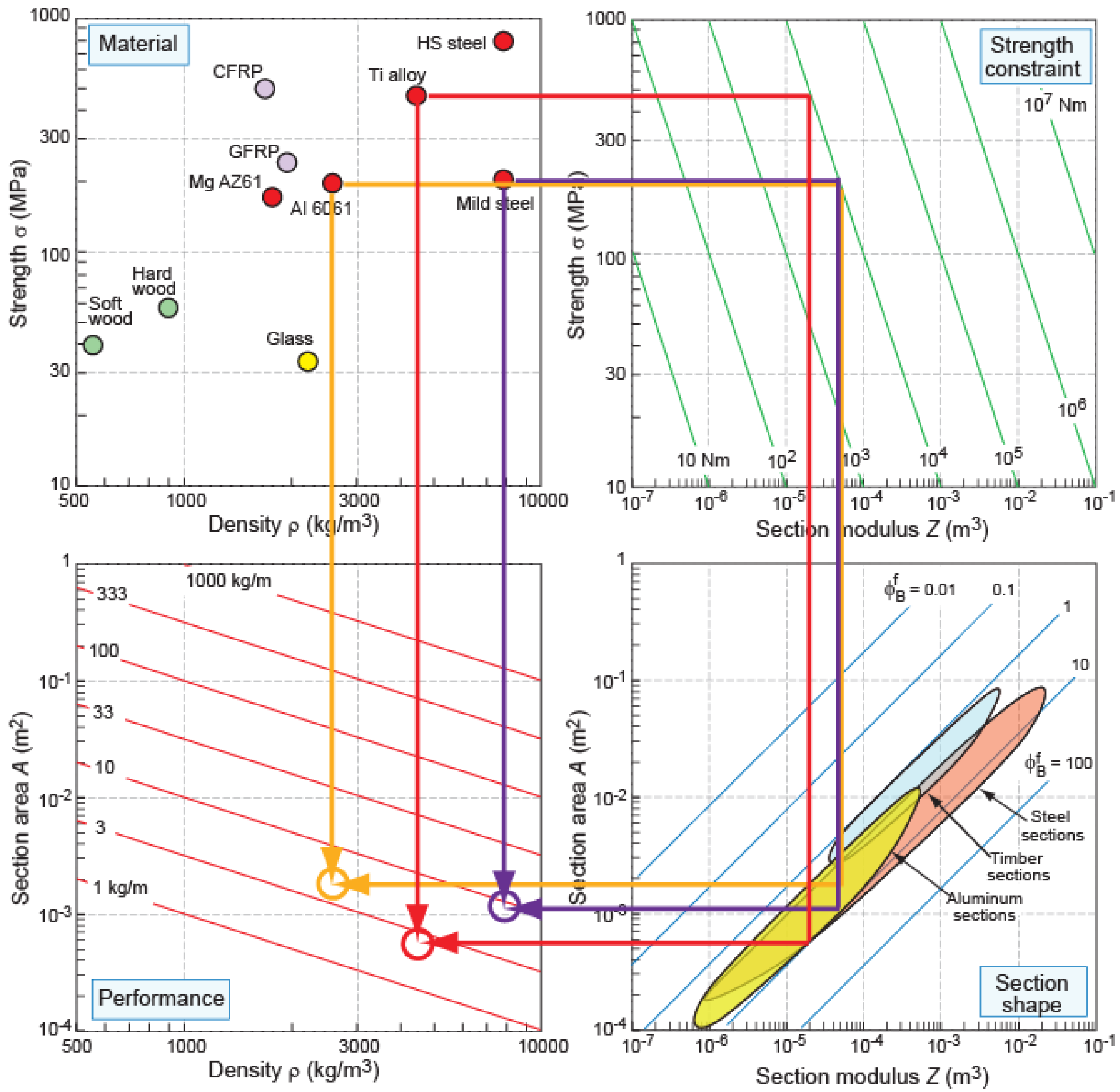
(b) Direct calculation of $\left(\phi_B^f \sigma_f\right)^{2/3} / \rho$ gives the values shown in the table.

| Material | $\sigma_f$ (MPa) | $\rho$ (kg/m$^3$) | $\phi_B^f$ | $\left(\phi_B^f \sigma_f\right)^{2/3} / \rho$ (MPa)$^{2/3}$/(kg/m$^3$) |
|---|---|---|---|---|
| Steel | 200 | 7900 | 10 | 2 x $10^{-2}$ |
| 6061 Al alloy | 200 | 2700 | 3 | 2.6 x $10^{-2}$ |
| Ti alloy | 480 | 4420 | 10 | **6.4 x $10^{-2}$** |

The higher the value of $\left(\phi_B^f \sigma_f\right)^{2/3} / \rho$ the lighter is the beam.

The ranking is the same.

## Strength-limited design in bending



**E9.8 A light weight display stand** (Figure E26). The figure shows a concept for a lightweight display stand. The stalk must support a mass *m* of 100 kg, to be placed on its upper surface at a height *h*, without failing by elastic buckling. It is to be made of stock tubing and must be as light as possible. Use the methods of Chapter 11 to derive a material index for the tubular material of the stand that meets these requirements, and that includes the shape of the section, described by the shape factor

$$\phi_B^e = \frac{12\,I}{A^2}$$

where *I* is the second moment of area and *A* is the section area. The table summarizes the requirements.

.

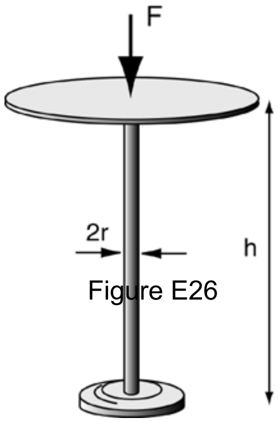



Figure E26

| | |
|---|---|
| **Function** | • *Light weight column* |
| **Constraints** | • *Specified buckling load F*<br>• *Height h specified* |
| **Objective** | • *Minimum mass m* |
| **Free variables** | • *Choice of material*<br>• *Section shape and scale* |

Cylindrical tubing is available from stock in the following materials and sizes. Use this information and the material index to identify the best stock material for the column of the stand.

| *Material* | *Modulus* E *(GPa)* | *Tube radius* r | *Wall thickness/tube radius, t/r* |
|---|---|---|---|
| Aluminum alloys | 69 | 25 mm | 0.07 to 0.25 |
| Steel | 210 | 30 mm | 0.045 to 0.1 |
| Copper alloys | 120 | 20 mm | 0.075 to 0.1 |
| Polycarbonate (PC) | 3.0 | 20 mm | 0.15 to 0.3 |
| Various woods | 7 - 12 | 40 mm | Solid circular sections only |

***Answer.*** The objective function is the mass. The mass *m* of the tubular stalk is

$$m = A\,h\,\rho \qquad (1)$$

where $\rho$ is the density and $A$ the section area of the tube. The constraint is that the stalk should not buckle. The buckling load for a slender column of height $h$ is

$$F_{cr} = \frac{n^2\pi^2 EI}{h^2} = \frac{n^2\pi^2 E}{h^2}\frac{\phi_B^e A^2}{12} \qquad (2)$$

where $E$ is Young's modulus, $I$ is the second moment of area of the tube and $n$ is a constant that depends on the end constraint ($n$ = 1/2 in this instance – Appendix B Section B5). Substituting for $A$ from equation (2) into equation (1) gives

$$m = \frac{2\sqrt{3}\,h^2}{\pi}\cdot F^{1/2}\,\frac{\rho}{(E\phi_B^e)^{1/2}} \qquad (3)$$

The best choice of tubing for the stalk is that with the greatest value of

$$M = \frac{(E\phi_B^e)^{1/2}}{\rho}$$

The shape factor for a thin-walled tube is calculated from the bending moment of inertia, $I_{xx}$ ("Appendix B, Section B2):

$$I_{xx} = \pi r^3 t \qquad (4)$$

where $r$ is the tube radius and $t$ the wall thickness; the area $A$ of the section is $2\pi rt$.

Its shape factor is

$$\phi_B^e = \frac{12\,I}{A^2} = \frac{3}{\pi}\frac{r}{t} \qquad (5)$$

(Note that, for a solid section, $\phi_B^e = 0.95 \approx 1$). The selection can be made by evaluating and comparing the quantity $M = (E\phi_B^e)^{1/2} / \rho$ for each of the materials and sections, as shown.

| Material | $E$ (GPa) | $\rho$ (kg/m$^3$) | $\phi_B^e$ | $M$ (GPa$^{1/2}$/Mg/m$^3$) |
|---|---|---|---|---|
| Aluminum alloys | 71 | 2700 | 4 - 14 | 6.2 - 11.7 |
| Steel | 210 | 7900 | 10 - 21 | 5.8 - 8.2 |
| Copper alloys | 120 | 8900 | 10 - 13 | 3.9 – 4.4 |
| Polycarbonate (PC) | 3 | 1200 | 3.3 - 6.7 | 2.6 – 3.7 |
| Various woods | 8-15 | 500 - 800 | 1 | 3.5 – 7.7 |

Aluminum offers the highest index and thus the lightest stalk. Steel is the next best choice, then wood.

**E9.9 Microscopic shape: tube arrays** (Figure E27). Calculate the gain in bending efficiency, $\psi_B^e$, when a solid is formed into small, thin-walled tubes of radius $r$ and wall thickness $t$ that are then assembled and bonded into a large array, part of which is shown in the figure. Let the solid of which the tubes are made have modulus $E_s$ and density $\rho_s$. Express the result in terms of $r$ and $t$.

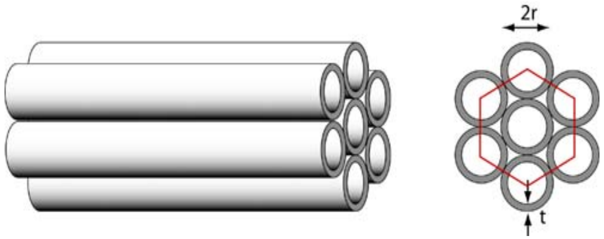



Figure E27

***Answer.*** The unit cell of the structure is shown as a hexagon on the right-hand part of the figure. The area $A_s$ of solid in the unit cell, equal to its relative density $\rho / \rho_s$, is

$$A_s = 6\pi r t$$

and the relative density of the array, $\rho / \rho_s$, is

$$\frac{\rho}{\rho_s} = \frac{A_s}{6\ \tan(60)\, r^2} = 1.8\frac{t}{r}$$

If an initially solid, square section, beam of edge length $b_s$ is formed into an array like that of the figure, with an outer, square, profile, the edge-length of this new square is

$$b = \left(\frac{\rho_s}{\rho}\right)^{1/2} b_s$$

The apparent modulus of the tube array parallel to the axis of the tubes is

$$E = \left(\frac{\rho}{\rho_s}\right) E_s$$

and the second moment of area of the tube array is

$$I = \frac{b^4}{12} = \left(\frac{\rho}{\rho_s}\right)^2 b_s$$

Thus the bending efficiency is

$$\psi_B^e = \frac{E\, I}{E_s\, I_s} = \frac{\rho_s}{\rho} = 0.56\frac{r}{t}$$

**E9.10 The structural efficiency of foamed panels** (Figure E28). Calculate the change in structural efficiency for both bending stiffness and strength when a solid flat panel of unit area and thickness $t$ is foamed to give a foam panel of unit area and thickness $h$, at constant mass. The modulus $E$ and strength $\sigma_f$ of foams scale with relative density $\rho / \rho_s$ as

$$E = \left(\frac{\rho}{\rho_s}\right)^2 E_s \quad \text{and} \quad \sigma_f = \left(\frac{\rho}{\rho_s}\right)^{3/2} \sigma_{f,s}$$

where $E$, $\sigma_f$ and $\rho$ are the modulus, strength and density of the foam and $E_s$, $\sigma_{f,s}$ and $\rho_s$ those of the solid panel.

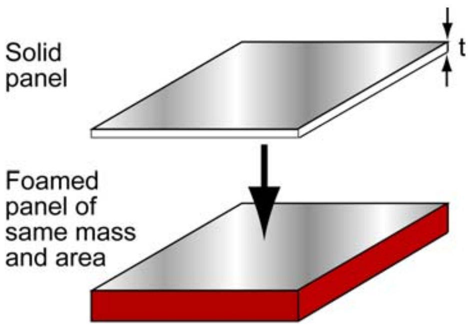



Figure E28

***Answer.*** The panel, initially solid and of thickness t, is foamed to a height $h$ at constant area. The density falls from $\rho_s$ to

$$\rho = \rho_s \frac{t}{h}$$

and the modulus and strength fall from $E_s$ to $\sigma_{f,s}$

$$E = E_s \left(\frac{t}{h}\right)^2 \quad \text{and} \quad \sigma_f = \sigma_{f,s}\left(\frac{t}{h}\right)^{3/2}$$

The bending stiffness per unit area changes from $S_s$ to $S$, so that

$$\psi_B^e = \frac{S}{S_o} = \frac{E\,I}{E_s I_s} = \frac{h}{t} = \frac{1}{(\rho / \rho_s)}$$

(using $I_s = t^3 / 12$ and $I = h^3 / 12$ per unit width). Foaming imparts a shape-efficiency factor for stiffness equal to the reciprocal of the relative density.

Following the same procedure for strength (using $Z_s = t^2 / 12$ and $I = h^2 / 6$ per unit width) gives

$$\psi_B^f = \frac{Z\,\sigma_f}{Z_s\,\sigma_{f,s}} = \sqrt{\frac{h}{t}} = \frac{1}{(\rho / \rho_s)^{1/2}}$$

The shape-efficiency factor for strength is equal to the reciprocal of the square root of the relative density.

## E10. Hybrid materials (Chapters 11 and 12)

*The examples in this section relate to the design of hybrid materials described in Chapters 11 and 12. The first three involve the use of bounds for evaluating the potential of composite systems. The fourth is an example of hybrid design to fill holes in property space. Then comes one involving a sandwich panel. The next three make use of the charts for natural materials. The last is a challenge: to explore the potential of hybridizing two very different materials.*

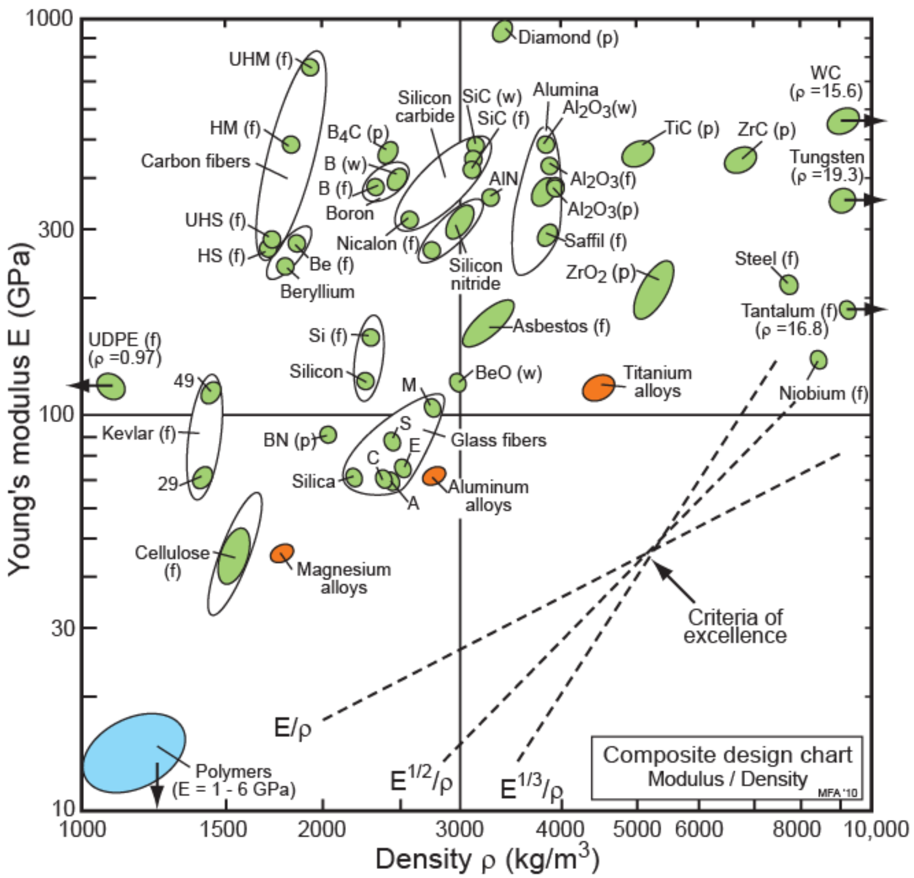
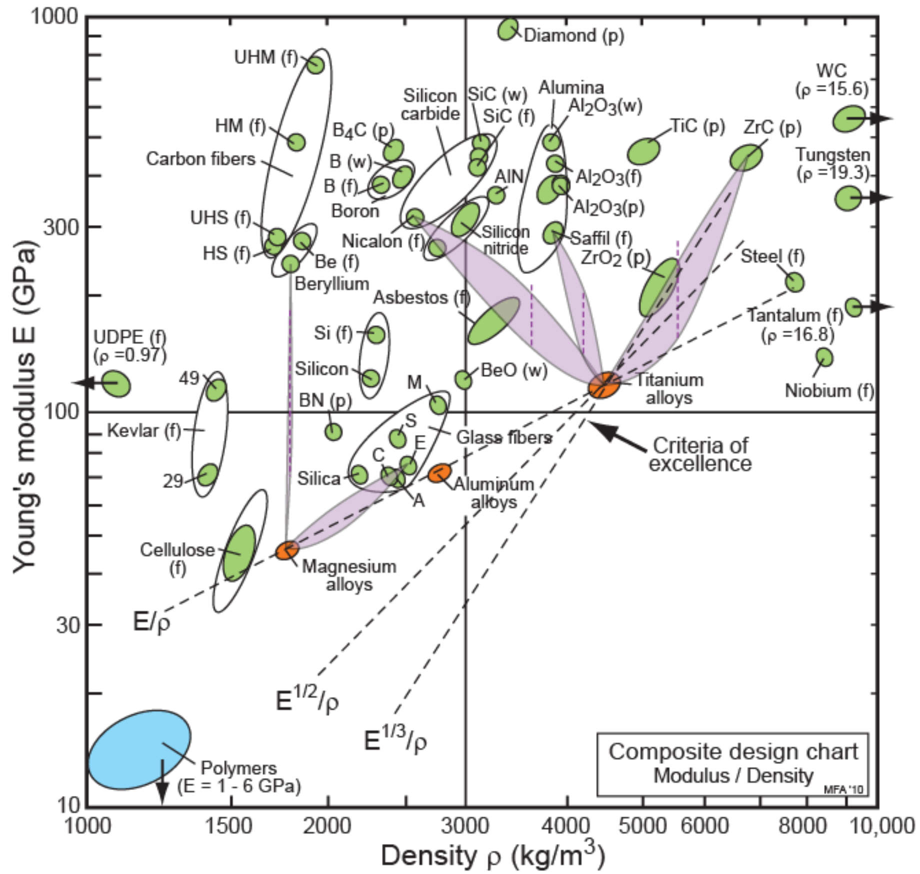



Figure E29

### E10.1 Concepts for light, stiff composites.

Figure E29 is a chart for exploring stiff composites with light alloy or polymer matrices. A construction like that of Figure 11.7 of the text allows the potential of any given matrix-reinforcement combination to be assessed. Four matrix materials are shown, highlighted in red. The materials shown in gray are available as fibers (f), whiskers (w) or particles (p). The criteria of excellence (the indices $E/\rho$, $E^{1/2}/\rho$ and $E^{1/3}/\rho$ for light, stiff structures) are shown; they increase in value towards the top left. Use the chart to compare the performance of a titanium-matrix composite reinforced with (a) zirconium carbide, ZrC, (b) Saffil alumina fibers and (c) Nicalon silicon carbide fibers. Keep it simple: use equations 11.1 – 11.2 to calculate the density and upper and lower bounds for the modulus at a volume fraction of *f* = 0.5 and plot these points. Then sketch arcs of circles from the matrix to the reinforcement to pass through them. In making your judgement, assume that *f* = 0.5 is the maximum practical reinforcement level.

***Answer.*** The adjacent figure shows the result. The guide lines have been moved to pass through titanium, allowing the performance of each composite system to be assessed. A vertical, broken, line through each of the bound-envelopes marks the *f* = 0.5 point. The Ti-ZrC composite system offers little or no gain in performance because the composite trajectory lies almost parallel to the guide-lines. By contrast, Ti-Saffil composites offer gains. Best of the lot is Ti-Nicalon. These composites have a trajectory that lies almost normal to the guide-lines, offering the greatest increase in all three criteria of excellent.

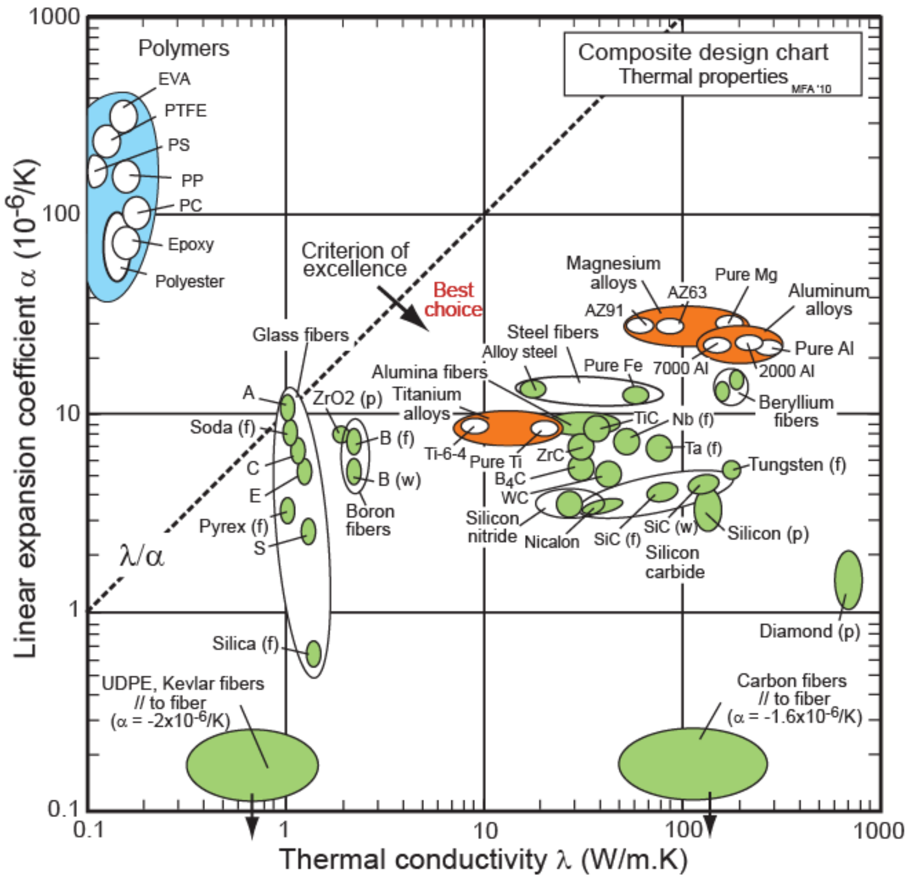



**E10.2** Use the chart of Figure E29 to explore the relative potential of Magnesium – E-glass-fiber composites and Magnesium – Beryllium composites for light, stiff structures.

***Answer.*** The adjacent figure shows the result. Mg – E-glass offers almost no gain in performance. Mg – Be, if it could be made, offers very considerable gains.

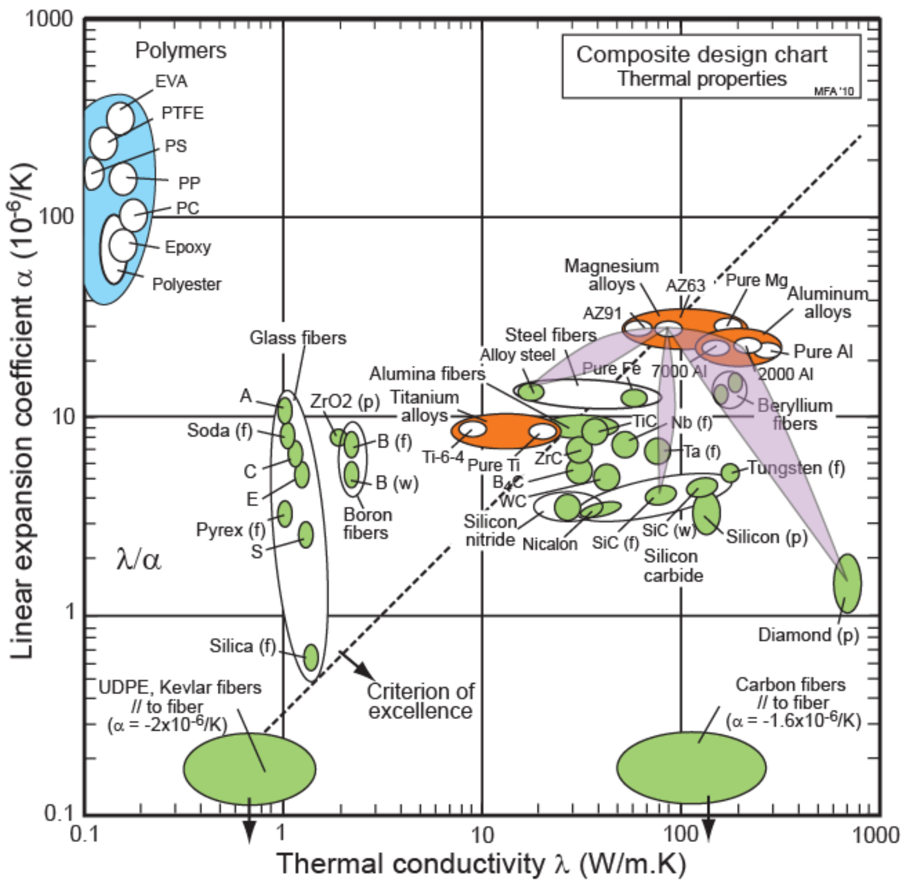
**E10.3 Concepts for composites with tailored thermal properties.**

Figure E30 is a chart for exploring the design of composites with desired combinations of thermal conductivity and expansion, using light alloy or polymer matrices. A construction like that of Figure 11.10 allows the potential of any given matrix-reinforcement combination to be assessed. One criterion of excellence (the index for materials to minimize thermal distortion, $\lambda / \alpha$ ) is shown; it increases in value towards the bottom right. Use the chart to compare the performance of a magnesium AZ63 alloy-matrix composite reinforced with (a) alloy steel fibers, (b) silicon carbide fibers, SiC (f) and (c) diamond-structured carbon particles. Keep it simple: use equation 11.7 to 11.10 to calculate the upper and lower bounds for $\alpha$ and $\lambda$ at a volume fraction of $f$ = 0.5 and plot these points. Then sketch curves linking matrix to reinforcement to pass through the outermost of the points. In making your judgement, assume that $f$ = 0.5 is the maximum practical reinforcement level.




Figure

***Answer.*** The figure shows the result. Mg-alloy steel composites perform less well than magnesium itself. By contrast, Mg-SiC composites offer gains (and are currently available). Even bigger gains lie with Mg-diamond composites. These are not as outrageous as they might sound: particulate industrial diamond is produced in large quantities for cutting tools, rock drills, grinding and polishing. Using it for a reinforcement in a composite for very high performance applications is perfectly practical.



**E10.4 Hybrids with exceptional combinations of stiffness and damping.** The Loss-coefficient – Modulus ( $\eta - E$ ) chart of Figure 4.9 is populated only along one diagonal band. (The loss coefficient $\eta$ measures the fraction of the elastic energy that is dissipated during a load-unload cycle.) Monolithic materials with low *E* have high $\eta$, those with high *E* have low $\eta$. The challenge here is to devise hybrids to fill the holes, with the following applications in mind.

(a) Sheet steel (as used in car body panels, for instance) is prone to lightly damped flexural vibration. Devise a hybrid sheet that combines the high stiffness, *E,* of steel with high loss coefficient, $\eta$.

(b) High loss coefficient means that energy is dissipated on mechanical cycling. This energy appears as heat, sometimes with undesirable consequences. Devise a hybrid with low modulus *E* and low loss coefficient, $\eta$.

***Answer.*** The figure shows two simple solutions.

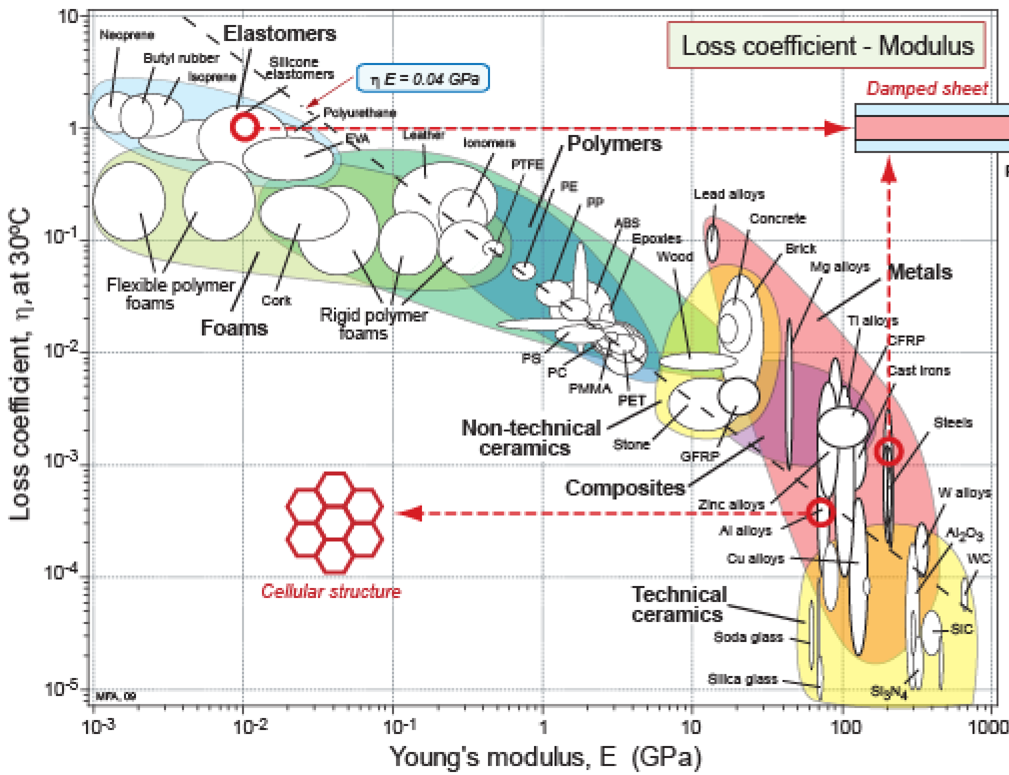
(a) High modulus with high damping is achieved by bonding the steel to a high-loss polymer or elastomeric skins to give a hybrid as suggested by the sketch at the top right. When the sheet vibrates – a bending mode – the outer high-loss layers suffer the greatest strain and dissipate energy, damping the vibration. The in-plane and flexural moduli remain those of the steel sheet.

(b) Low modulus and low damping can be achieved by internal shape, as sketched in the lower left. A bending-dominated cellular structure has a modulus *E* given be equation 11.29 of the text:

$$E = E_s \left( \frac{\rho}{\rho_s} \right)^2$$

where $\rho$ is the density of the cell-wall material, and $E_s$ and $\rho_s$ are the modulus and density of the cellular structure.

The loss coefficient of the cellular structure, $\eta_s$, is the same as that of the material from which it is made, provided deformation remains elastic. Thus a low-loss aluminum alloy with a bending-dominated cellular structure of relative density $\rho / \rho_s$ = 0.1 has a has a modulus that is 100 times lower than that of aluminum, with the same value of $\eta$. It lies at the point marked on the figure.



**E10.5 Sandwich panels.** An aircraft quality sandwich panel has the characteristics listed in the table.

(a) Use the data and equations 11.13 – 11.15 of the text to calculate the equivalent density, flexural modulus and strength.

(b) Plot these on a modulus / density and a strength / density chart. Do the flexural properties of the panel lie in a region of property space not filled by monolithic materials?

**Data for glass fiber / aluminum honeycomb sandwich panel**

| | |
|---|---|
| Face material | 0.38 mm glass fiber / epoxy |
| Core material | 3.2 mm cell, 97.8 kg/m$^3$, 5052 Alu honeycomb |
| Panel weight per unit area, $m_a$ | 2.65 kg/m$^2$ |
| Panel length, $L$ | 510 mm |
| Panel width, $b$ | 51 mm |
| Panel thickness, $d$ | 10.0 mm |
| Flexural stiffness $EI$ | 67 N.m$^2$ |
| Failure moment $M_f$ | 160 Nm |

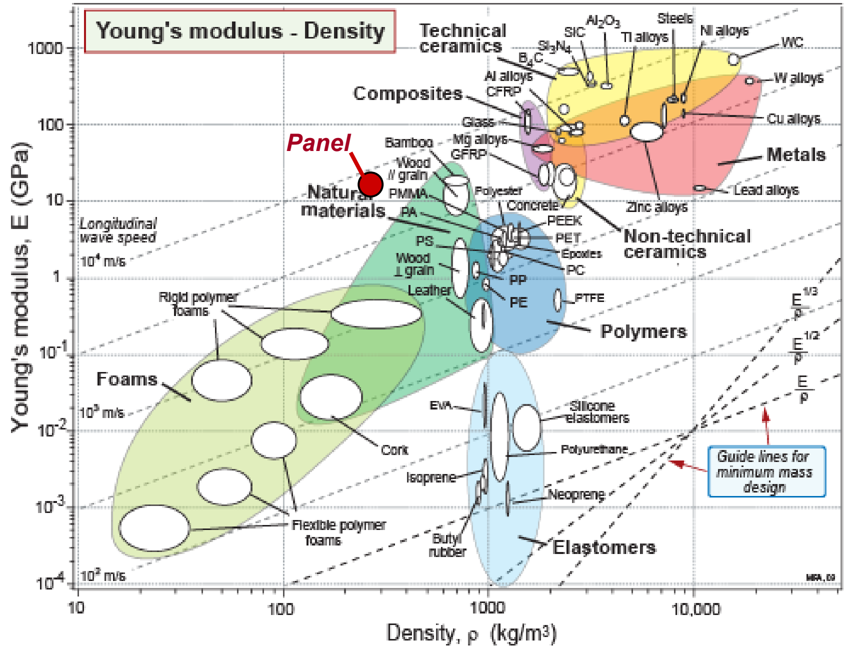
***Answer.***

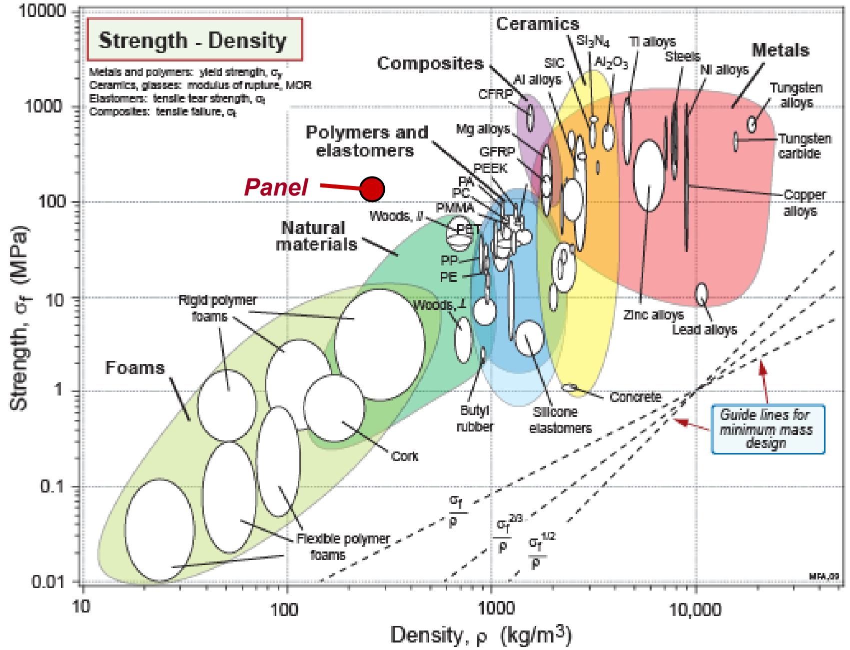
(a) The equivalent density $\tilde{\rho}$, equation (11.13) is $\tilde{\rho} = \dfrac{m_a}{d} = 265\ kg/m^3$

The equivalent modulus $\tilde{E}$, equation (11.14) is $\tilde{E} = \dfrac{12\,EI}{b\,d^3} = 15.8\,GPa$

The equivalent strength $\tilde{\sigma}_f$, equation (11.15), is $\tilde{\sigma}_f = \dfrac{4M_f}{b\,d^2} = 125\ MPa$

(b) See opposite. Yes, the flexural properties of the panel lie in an area of property space not occupied by monolithic materials.



**E10.6 Natural hybrids that are light, stiff and strong.**

(a) Plot aluminum alloys, steels, CFRP and GFRP onto a copy of the $E - \rho$ chart for natural materials (Figure 12.13 of the text), where *E* is Young's modulus and $\rho$ is the density. How do they compare, using the flexural stiffness index $E^{1/2}/\rho$ as criterion of excellence?

(b) Do the same thing for strength with a copy of Figure 12.14, using the flexural strength index $\sigma_f^{2/3}/\rho$ (where $\sigma_f$ is the failure stress) as criteria of excellence.

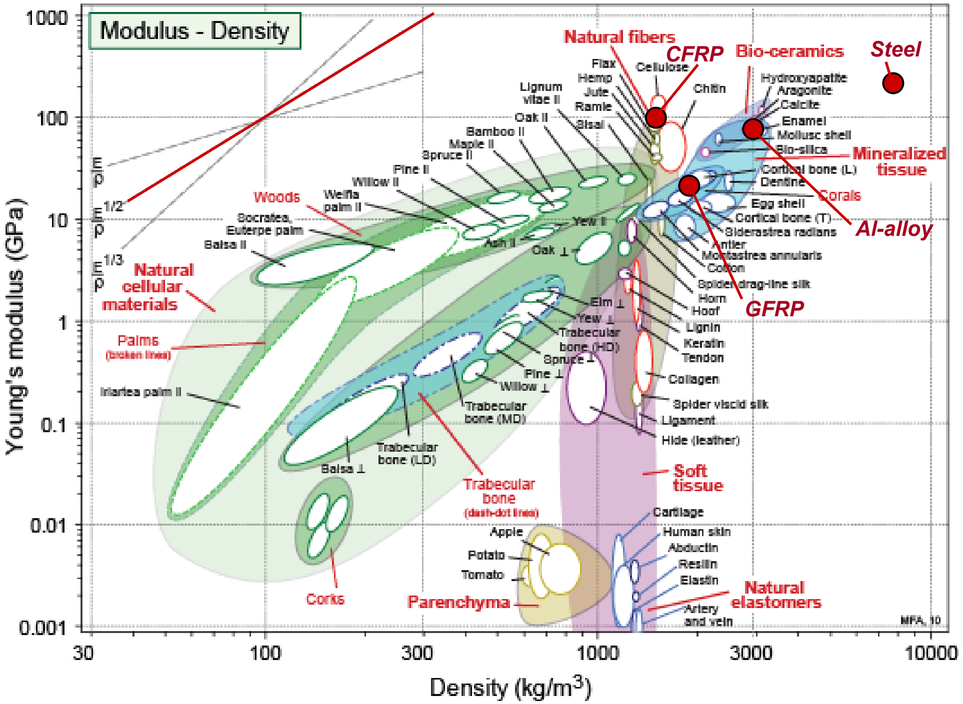
The table lists the necessary data.

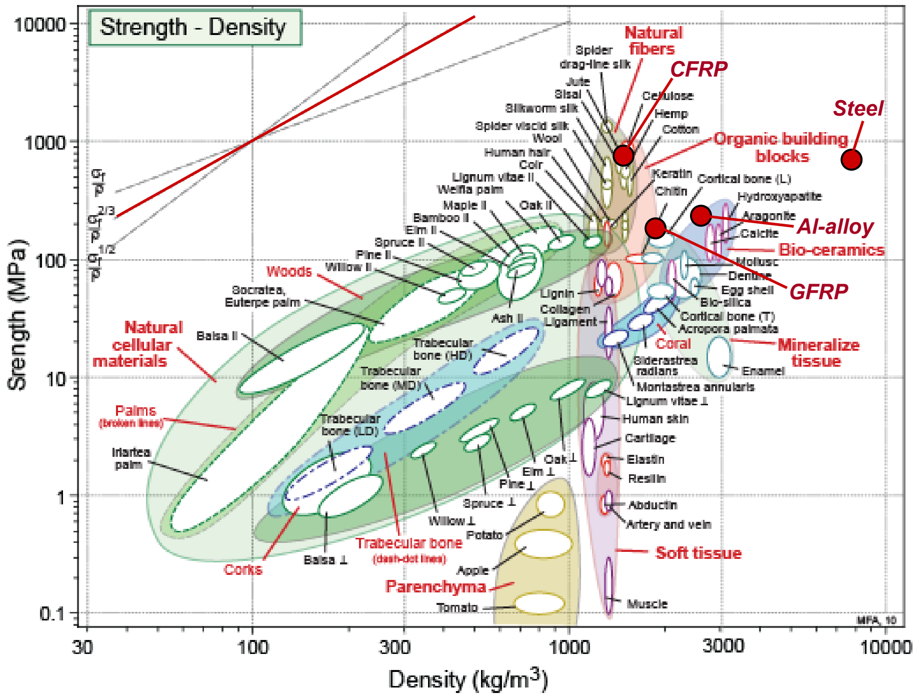
| Material | Young's modulus E (GPa) | Density $\rho$ (kg/m$^3$) | Strength, $\sigma_f$ (MPa) |
|---|---|---|---|
| Aluminum alloy | 74 | 2700 | 335 |
| Steel | 210 | 7850 | 700 |
| CFRP | 100 | 1550 | 760 |
| GFRP | 21 | 1850 | 182 |

***Answer.*** The comparisons are shown in two the figures on the next page.

(a) The comparison for stiffness, using the criterion of excellence, $E^{1/2}/\rho$, is highlighted. The nearer a material lies to this line, the better is its performance. Natural materials are extraordinarily efficient by this measure. CFRP is comparable with the best of them, but the other man-made materials compare poorly with those of nature.

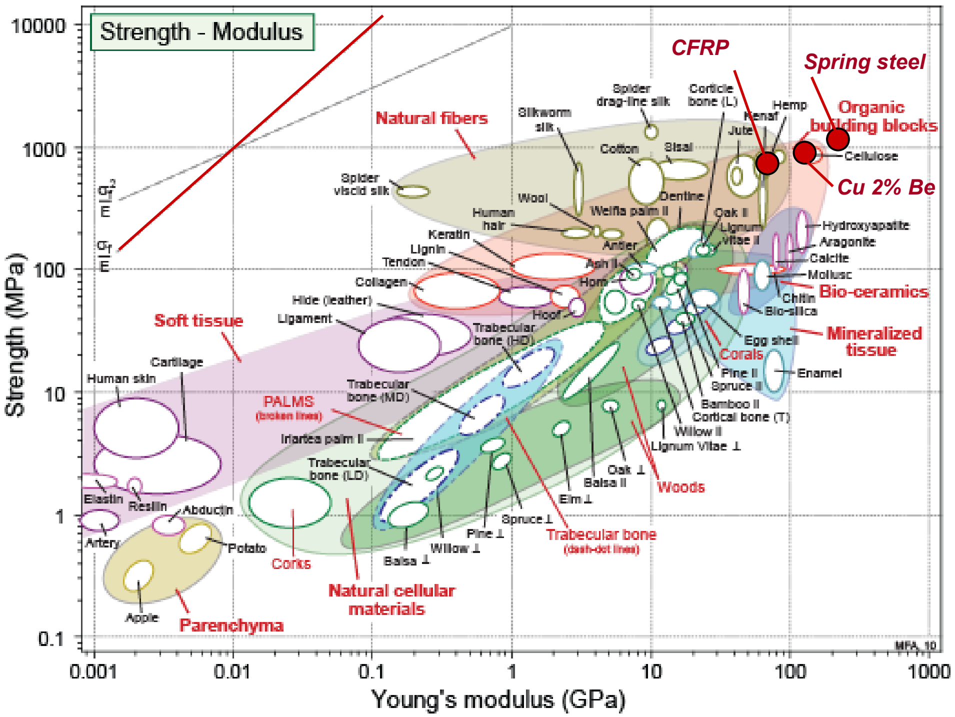
(b) The comparison for strength, using criterion of excellence, $\sigma_f^{2/3}/\rho$, is highlighted. Again, natural materials excel. CFRP alone competes with wood, bamboo and palm.


Modulus - Density
Natural fibers
CFRP
Bio-ceramics
Steel
Mineralized tissue
Al-alloy
GFRP
Woods
Natural cellular materials
Palms
Soft tissue
Trabecular bone
Corks
Parenchyma
Natural elastomers
Young's modulus (GPa)
Density (kg/m3)


Strength - Density
Natural fibers
CFRP
Steel
Organic building blocks
Al-alloy
Bio-ceramics
GFRP
Mineralize tissue
Coral
Woods
Natural cellular materials
Palms
Corks
Trabecular bone
Parenchyma
Soft tissue
Srength (MPa)
Density (kg/m3)

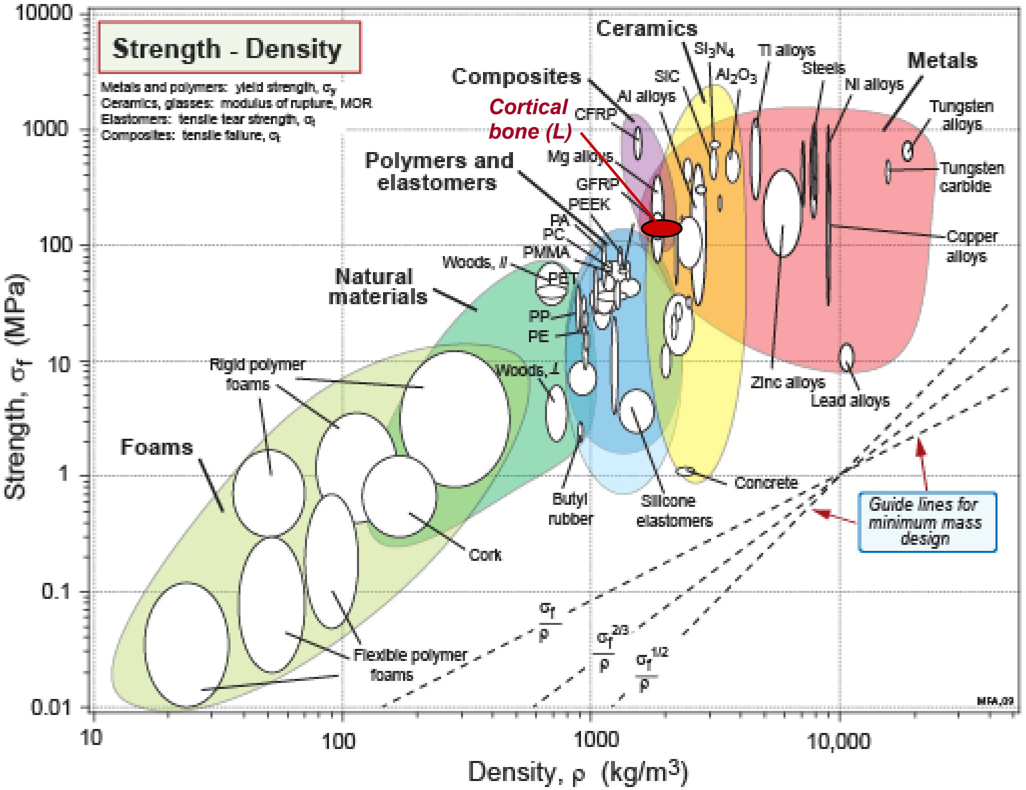
**E10.7 Natural hybrids that act as springs.** The table lists the moduli and strengths of spring materials. Plot these onto a copy of the $E-\sigma_f$ chart for natural materials of Figure 12.15, and compare their energy-storing performance with that of natural materials, using the $\sigma_f^2/E$ as the criterion of choice. Here $\sigma_f$ is the failure stress, *E* is Young's modulus and $\rho$ is the density.

| Material | Modulus (GPa) | Strength (MPa) | Density (kg/m$^3$) |
|---|---|---|---|
| Spring steel | 206 | 1100 | 7850 |
| Copper-2%Beryllium | 130 | 980 | 8250 |
| CFRP filament wound | 68 | 760 | 1580 |




***Answer.*** The comparison is shown on the adjacent figure. The criterion of excellence, $\sigma_f^2/E$, is highlighted. The nearer a material lies to this line, the better is its performance. The best man-made spring materials have a performance that is comparable with that of antler, but falls short of that offered by silks of various types. If the criterion for light springs, $\sigma_f^2/\rho E$ is used instead, the natural materials rank even more highly.

**E10.8 Finding a substitute for bone.** Find an engineering material that most closely resembles the longitudinal strength of compact bone (*cortical bone (L)*) in its strength/weight ( $\sigma_f / \rho$ ) characteristics by plotting data for this material, read from Figure 12.14, onto a copy of the $\sigma_f - \rho$ chart for engineering materials (Figure 4.4). Here $\sigma_f$ is the failure stress and $\rho$ is the density.




***Answer.*** The modulus and density of compact bone match closely those of magnesium and glass fiber reinforced epoxy, GFRP. These materials work well for external braces. For prostheses, bio-compatibility is important. This leads to the used of cobalt and titanium-based alloys even though the match of mechanical properties is very poor. Efforts are underway to develop a bio-compatible polyethylene/hydroxyapatite composites with a near perfect match of properties. Hydroxyapatite is a natural ceramic that is found in real bone – it appears on the charts for natural materials.

**E10.9 Creativity: what could you do with X?** The same 68 materials appear on all the charts of Chapter 4. These can be used as the starting point for "what if…?" exercises. As a challenge, use any chart or combination of charts to explore what might be possible by hybridizing any pair of the materials listed below, in any configuration you care to choose.

(a) Cement

(b) Wood

(c) Polypropylene

(d) Steel

(e) Copper.

## E11. Selecting processes (Chapters 13 and 14)

*The exercises of this section use the process selection charts of Chapters 13 and 14. They are useful in giving a feel for process attributes and the way in which process choice depends on material and the shape. Here the CES EduPack software offers greater advantages: what is cumbersome and of limited resolution with the charts is easy with the software, which offers much greater resolution.*

*Each exercise has two parts, labeled (a) and (b). The first involves translation. The second uses the selection charts of Chapter 13 (which you are free to copy) in the way that was illustrated in Chapter 14*

**E11.1 Elevator control quadrant** (Figure E31). The quadrant sketched here is part of the control system for the wing-elevator of a commercial aircraft. It is to be made of a light alloy (aluminum or magnesium) with the shape shown in the figure. It weighs about 5 kg. The minimum section thickness is 5 mm, and – apart from the bearing surfaces – the requirements on surface finish and precision are not strict: surface finish ≤ 10 μm and precision ≤ 0.5 mm. The bearing surfaces require a surface finish ≤ 1 μm and a precision ≤ 0.05 mm. A production run of 100 – 200 is planned.

(a) Itemize the function and constraints, leave the objective blank and enter "Choice of process" for the free variable.

(b) Use copies of the charts of Chapter 13 in succession to identify processes to shape the quadrant.

(c) If the CES EduPack software is available, apply the constraints and identify in more detail the viable processes.

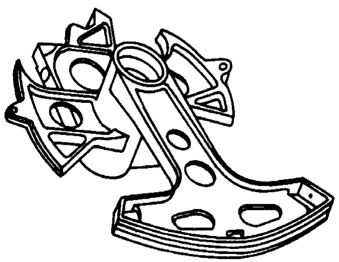


Figure E31

***Answer.*** (a) The function and constraints are listed in the table.

| | |
|---|---|
| **Function** | • *Shape an elevator quadrant* |
| **Constraints** | • *Material class: aluminum or magnesium alloy*<br>• *Shape class: 3D-solid*<br>• *Mass: 5 kg*<br>• *Minimum section: 5 mm*<br>• *Tolerance: 0.5 mm, (0.05mm on bearing surfaces)*<br>• *Surface roughness: 10 μm (1 μm on bearing surfaces)*<br>• *Planned batch size of 100 - 200* |
| **Objective** | -- |
| **Free variables** | • *Choice of process* |

(b) These constraints are applied to the selection charts, shown in the figures on the four pages following exercise E11.3. The material constraint limits the selection to those shown in the first column of the next table. The shape constraint eliminates two, leaving those in the second column. All of these can meet the constraints on size and section, which are not restrictive. The tolerance and roughness constraints on the body are not met by sand casting, though it is worth asking if they have been over-specified. Those on the bearing surfaces are much more restrictive – only machining from solid achieves them. The answer here is not to reject the others, but to add a finishing step, shown at the bottom of Mass range and Tolerance charts. The result is the list shown in the third column. Economics are important here – the Economic batch-size chart suggests that only the three listed in the last column are economic at a batch size of 100 – 200.

The conclusion: explore *investment casting, forging* (both with additional machining steps) and *numerically controlled machining from solid*.

The processes surviving the successive applications of the constraints on the quadrant:

| Processes passing the material constraint. | Processes that survive the shape constraint | Processes meeting the tolerance constraint | Processes that are economic at a batch of 100 - 200 |
|---|---|---|---|
| Sand casting | Sand casting | ***FAILS*** | |
| Die casting | Die casting | Die casting plus machining | ***FAILS*** |
| Investment casting | Investment casting | Investment casting plus machining | Investment casting plus machining |
| Low pressure casting | Low pressure casting | Low pressure casting plus machining | ***FAILS*** |
| Forging | Forging | Forging plus machining | Forging plus machining |
| Extrusion | ***FAILS*** | | |
| Sheet forming | ***FAILS*** | | |
| Powder methods | Powder methods | Powder methods | ***FAILS*** |
| Machine from solid | Machine from solid | Machine from solid | Machine from solid |

**E11.2** **Casing for an electric plug** (Figure E32). The electric plug is perhaps the commonest of electrical products. It has a number of components, each performing one or more functions. The most obvious are the casing and the pins, though there are many more (connectors, a cable clamp, fasteners, and, in some plugs, a fuse). The task is to investigate processes for shaping the two-part insulating casing, the thinnest part of which is 2 mm thick. Each part weighs about 30 grams and is to be made in a single step from a thermoplastic or thermosetting polymer with a planned batch size of 5 x $10^4$ – 2 x $10^6$. The required tolerance of 0.3 mm and surface roughness of 1 μm must be achieved without using secondary operations.

(a) Itemize the function and constraints, leave the objective blank and enter "Choice of process" for the free variable.

(b) Use the charts of Chapter 13 successively to identify possible processes to make the casing

Figure E32

***Answer.*** (a) The table lists the function and constraints.

| | |
|---|---|
| **Function** | • *Shape an electric plug casing* |
| **Constraints** | • *Material class: thermoplastic or thermosetting polymer*<br>• *Shape class: 3D-solid*<br>• *Mass: 0.03 kg*<br>• *Minimum section: 2 mm*<br>• *Tolerance: 0.3 mm*<br>• *Surface roughness: 1µm*<br>• *Planned batch size of 5 x$10^4$ – 2 x$10^6$* |
| **Objective** | -- |
| **Free variables** | • *Choice of process* |

(b) Here we seek a net-shape process – the casing must be shaped in one operation without the need for any further finishing. Applying the constraints as shown in the charts on the four pages following exercise E11.3, eliminating processes that fail a constraint in the manner of the table shown in exercise E11.1, leaves two candidates: *injection molding* and *compression molding*.

(c) Using CES EduPack to apply the constraints gives the selection shown in the below.

| Selection using CES EduPack | Comment |
|---|---|
| Injection molding<br>Compression molding | The selection is identical with that derived from the charts, but the software also delivers data files containing supporting information for each process. |

**E11.3 Ceramic valves for taps** (Figure E33). Few things are more irritating than a dripping tap. Taps drip because the rubber washer is worn or the brass seat is pitted by corrosion, or both. Ceramics have good wear resistance, and they have excellent corrosion resistance in both pure and salt water. Many household taps now use ceramic valves.

The sketch shows how they work. A ceramic valve consists of two disks mounted one above the other, spring-loaded so that their faces are in contact. Each disk has a diameter of 20 mm, a thickness of 3 mm and weighs about 10 grams. In order to seal well, the mating surfaces of the two disks must be flat and smooth, requiring high levels of precision and surface finish; typically tolerance < 0.02 mm and surface roughness < 0.1 µm. The outer face of each has a slot that registers it, and allows the upper disc to be rotated through 90$^o$ (1/4 turn). In the “off” position the holes in the upper disc are blanked off by the solid part of the lower one; in the “on” position the holes are aligned. A production run of $10^5$ –$10^6$ is envisaged.

(a) List the function and constraints, leave the objective blank and “Choice of process” for the free variable.

(b) Use the charts of Chapter 13 to identify possible processes to ı the casing.

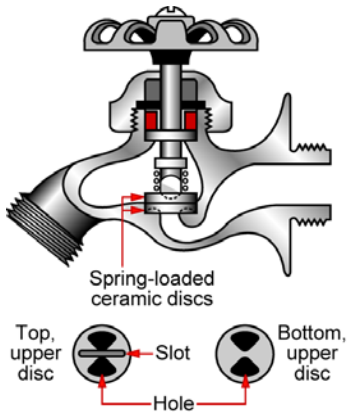
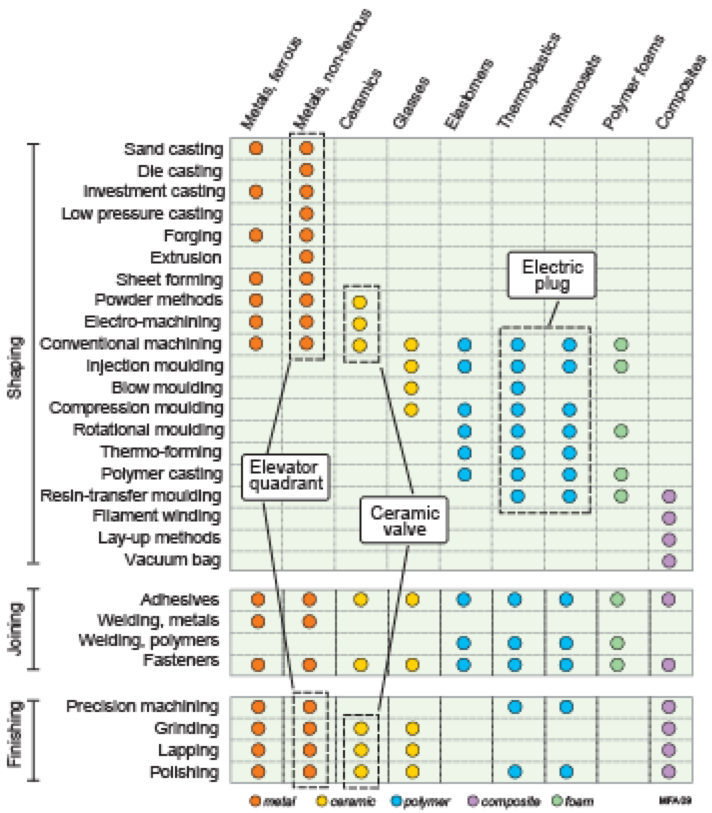



Figure E33

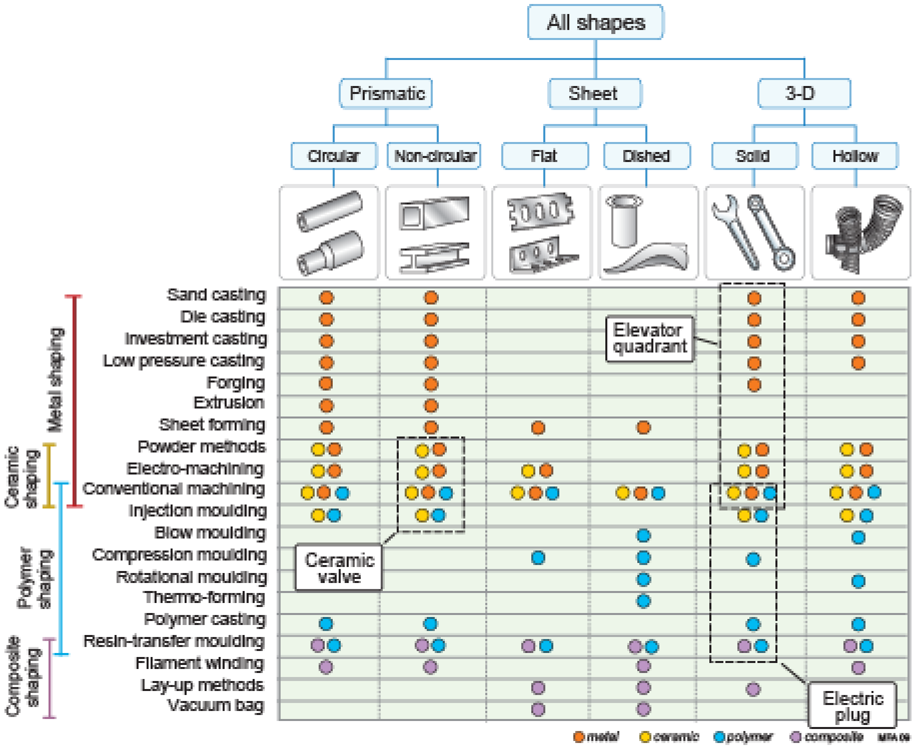
***Answer.*** (a) The table lists the function and constraints.

| | |
|---|---|
| **Function** | • *Shape a ceramic valve* |
| **Constraints** | • *Material class: technical ceramic*<br>• *Shape class: prismatic*<br>• *Mass: 0.01 kg*<br>• *Minimum section: 3 mm*<br>• *Tolerance: 0.02 mm*<br>• *Surface roughness: 0.1µm*<br>• *Planned batch size of $10^5$ –$10^6$* |
| **Objective** | -- |
| **Free variables** | • *Choice of process* |

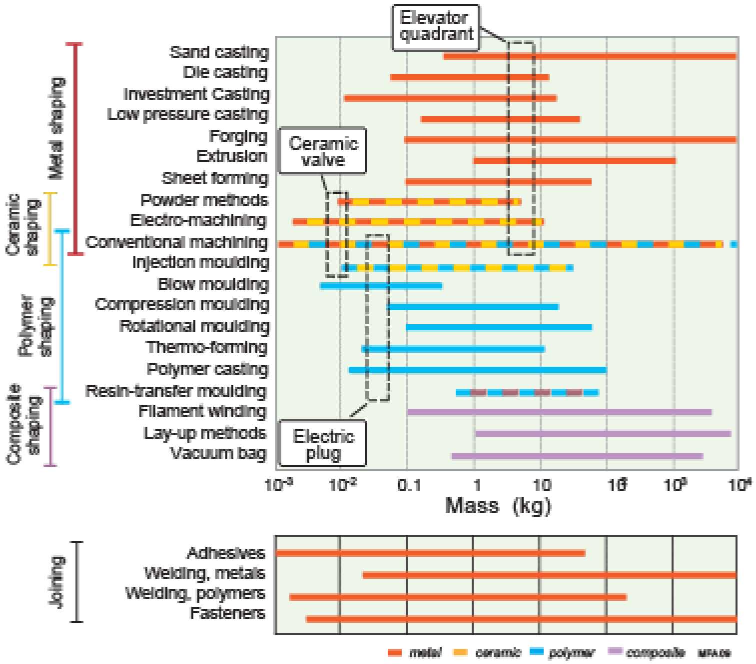
(b) Applying the constraints as shown in the charts opposite and on the next three pages, eliminating processes that fail a constraint in the manner of the table shown in exercise E11.1, leaves only one candidate: *powder methods*. The tolerance and roughness requirements for the mating faces require a subsequent *grinding, lapping* or *polishing* step.



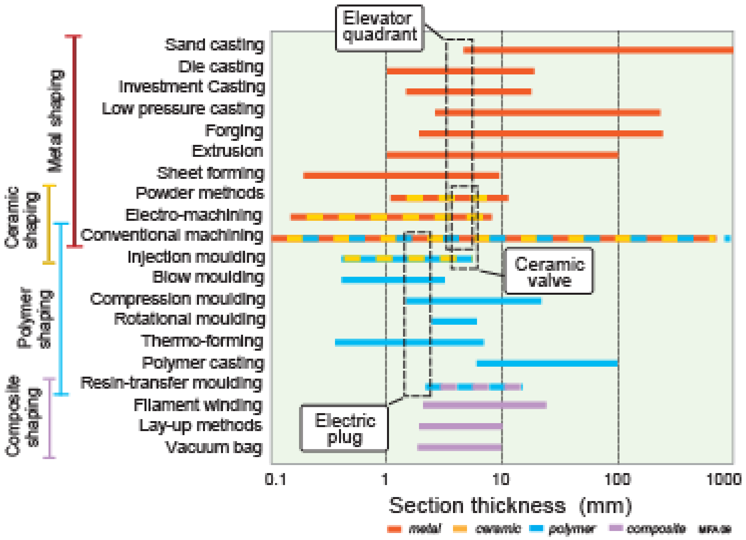

The Process-Material matrix



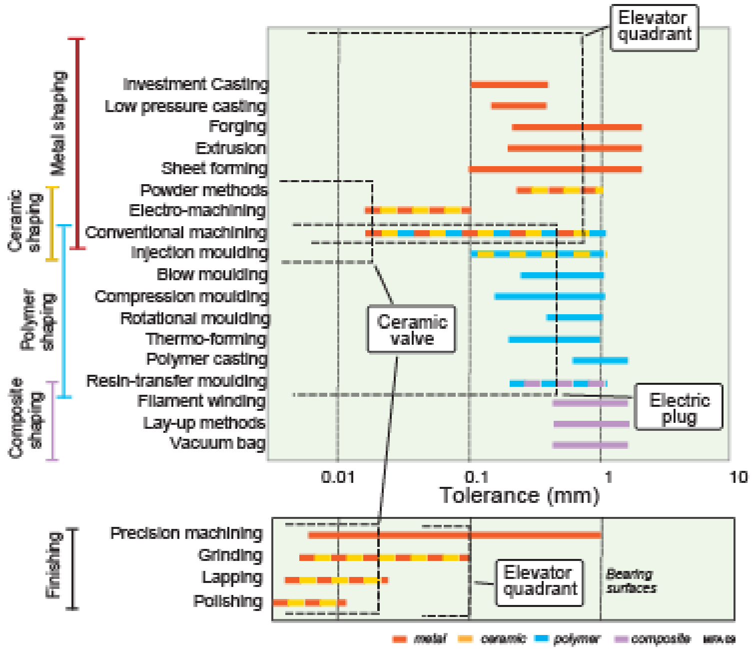

The Process-Shape matrix



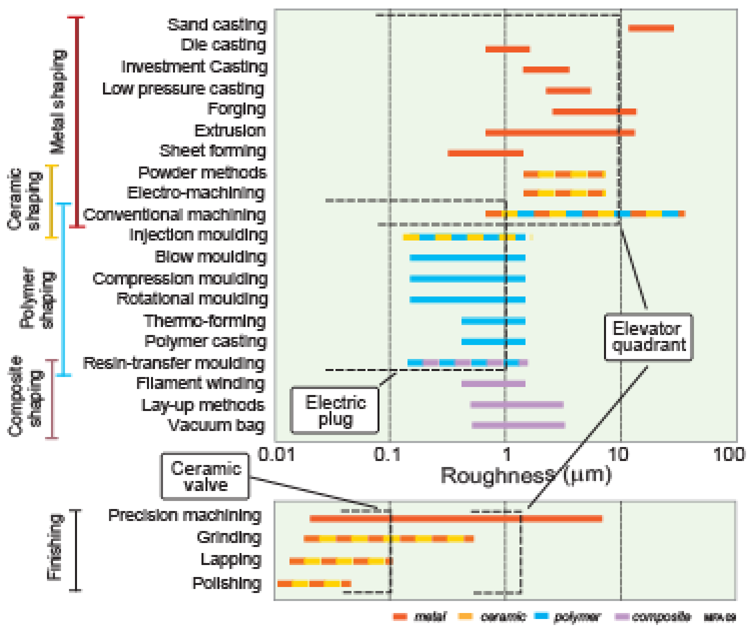

The Process-Mass range chart



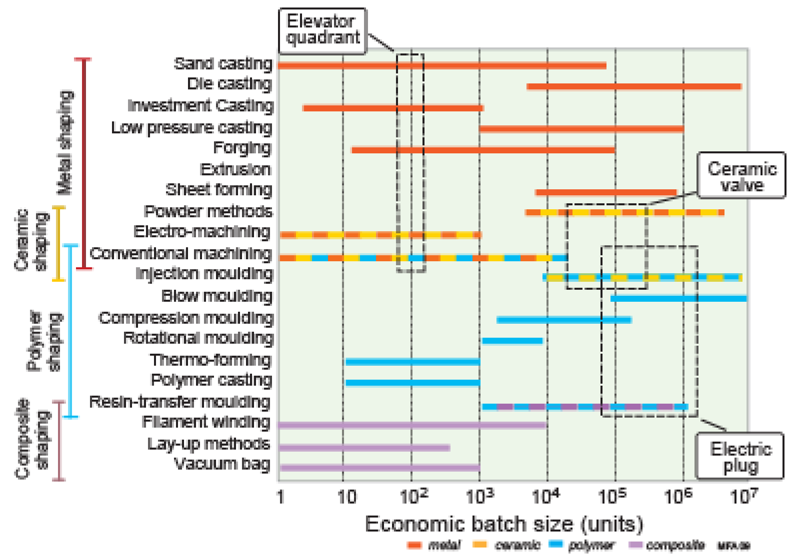

The Process-Section thickness chart




The Process-Tolerance chart




The Process-Roughness chart




The Economic batch-size chart

**E11.4** Shaping plastic bottles (Figure E34). Plastic bottles are used to contain fluids as various as milk and engine oil. A typical polyethylene bottle weighs about 30 grams and has a wall thickness of about 0.8 mm. The shape is 3-D hollow. The batch size is large (1,000,000 bottles). What process could be used to make them?

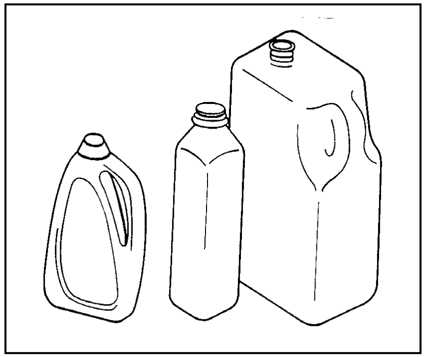


Figure E34

(a) List the function and constraints, leave the objective blank and enter "Choice of process" for the free variable.

(b) Use the charts of Chapter 13 to identify possible processes to make the casing.

***Answer.*** (a) The table lists the function and constraints.

| | |
|---|---|
| **Function** | • *Shape a polyethylene bottle* |
| **Constraints** | • *Material class: Polyethylene (or thermoplastic)*<br>• *Shape class: 3-D hollow*<br>• *Mass: 0.02 – 0.04 kg*<br>• *Minimum section: 0.7 - 1 mm*<br>• *Tolerance: 1 mm*<br>• *Surface roughness: 10 µm*<br>• *Planned batch size of > $10^6$* |
| **Objective** | -- |
| **Free variables** | • *Choice of process* |

(b) Applying the constraints as shown on the Charts eliminating processes that fail a constraint in the manner of the table shown in exercise E11.1, leaves only two candidates: *blow molding* and *injection molding.*

**E11.5 Car hood (bonnet)** (Figure E35). As weight-saving assumes greater importance in automobile design, the replacement of steel parts with polymer-composite substitutes becomes increasingly attractive. Weight can be saved by replacing a steel hood with one made from a thermosetting composites. The weight of the hood depends on the car model: a typical composite hood weighs is 8 - 10 kg. The shape is a dished-sheet and the requirements on tolerance and roughness are 1 mm and 2 μm, respectively. A production run of 100,000 is envisaged.

(a) List the function and constraints, leave the objective blank and enter "Choice of process" for the free variable.

(b) Use the charts of Chapter 13 to identify possible processes to make the casing.

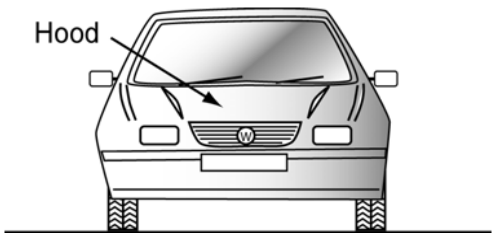



Figure E35

***Answer.*** (a) The table lists the function and constraints.

| | |
|---|---|
| **Function** | • *Shape a car hood* |
| **Constraints** | • *Material class: Thermosetting composite*<br>• *Shape class: Dished sheet*<br>• *Mass: 8 – 10 kg*<br>• *Minimum section: 3 mm*<br>• *Tolerance: 1 mm*<br>• *Surface roughness: 2 μm*<br>• *Planned batch size of 100,000* |
| **Objective** | -- |
| **Free variables** | • *Choice of process* |

(b) Applying the constraints as shown on the charts that follow exercise E11.6, eliminating processes that fail a constraint in the manner of the table shown in exercise E11.1, leaves *resin transfer molding* as the prime choice.

**E11.6 Complex structural channels** (Figure E36). Channel sections for window frames, for slide-together sections for versatile assembly and for ducting for electrical wiring can be complex in shape. The figure shows an example. The order is for 10,000 such sections, each 1 m in length and weighing 1.2 kg, with a minimum section of 4 mm. A tolerance of 0.2 mm and a surface roughness of less than 1 μm must be achieved without any additional finishing operation.

(a) List the function and constraints, leave the objective blank and enter "Choice of process" for the free variable.

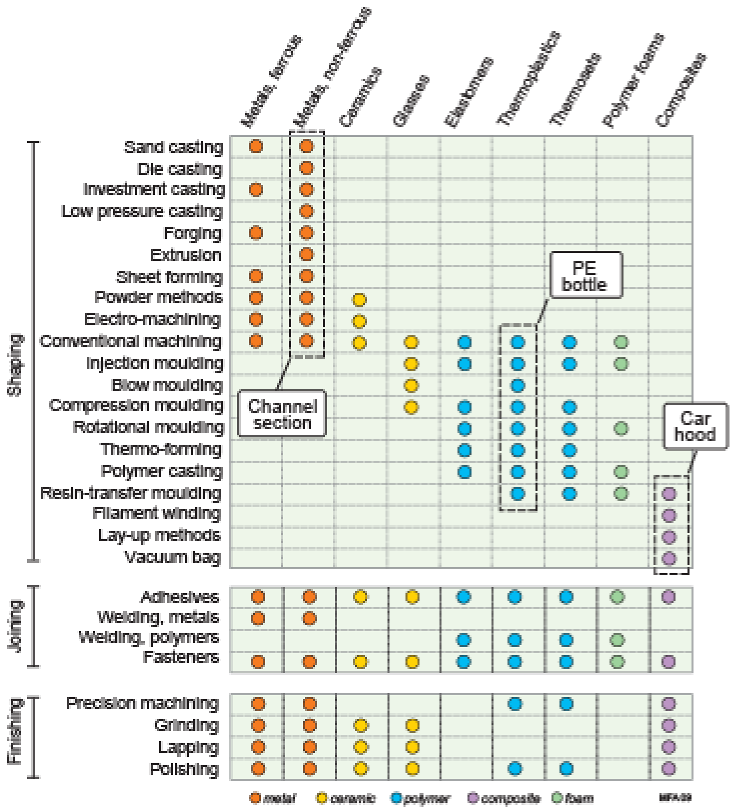
(b) Use the charts of Chapter 11 to identify possible processes to make the casing.

Figure E36

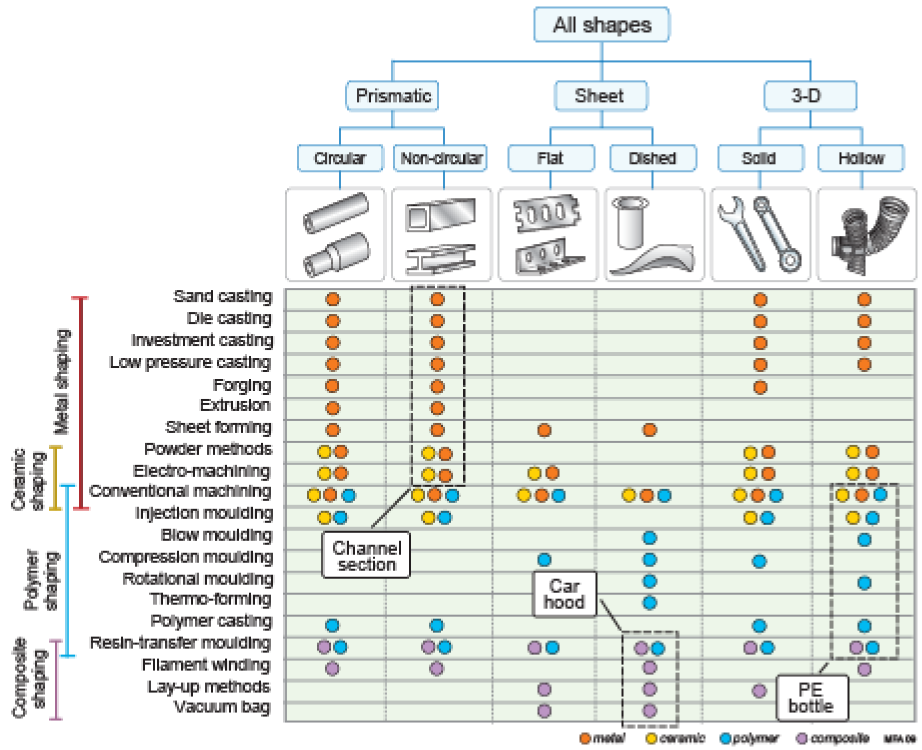
***Answer.*** (a) The table lists the function and constraints.

| | |
|---|---|
| **Function** | • *Shape an aluminum section with a complex profile* |
| **Constraints** | • *Material class: aluminum alloy*<br>• *Shape class: prismatic, non-circular*<br>• *Mass: 1.2 kg*<br>• *Minimum section: 4 mm*<br>• *Tolerance: 0.2 mm*<br>• *Surface roughness: 1 μm*<br>• *Planned batch size of 10,000* |
| **Objective** | -- |
| **Free variables** | • *Choice of process* |

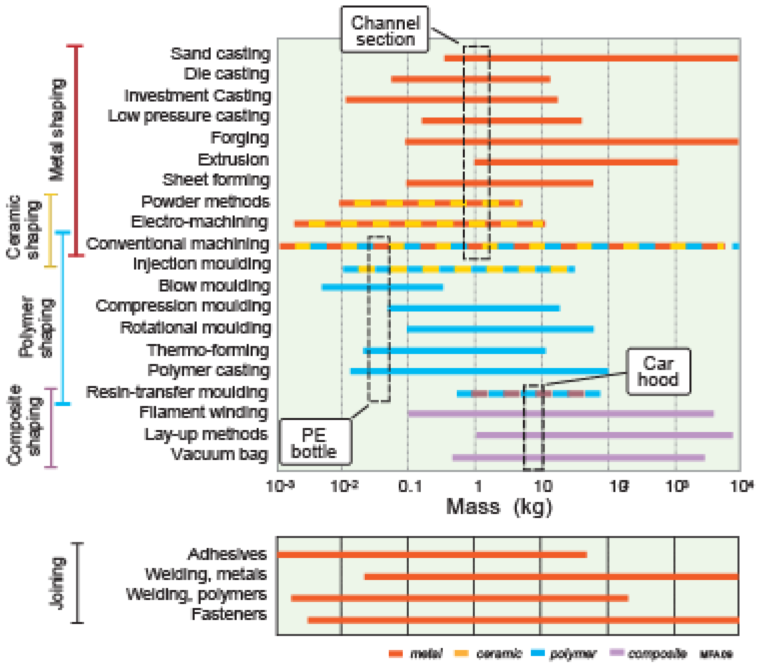
(b) Applying the constraints as shown on the Charts, below, eliminating processes that fail a constraint in the manner of the table shown in exercise E11.1, leaves *extrusion* and *die-casting* as viable candidates. The re-entrant profile makes die casting extremely difficult, leaving extrusion as the best choice.



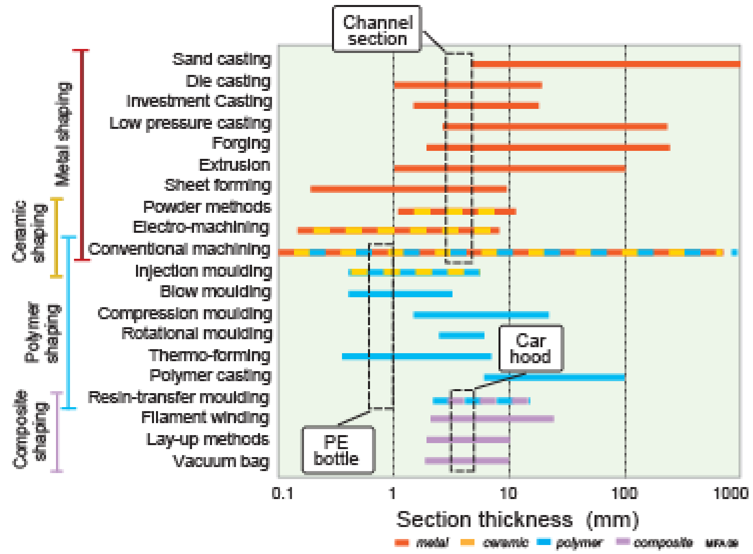

The Process-Material chart



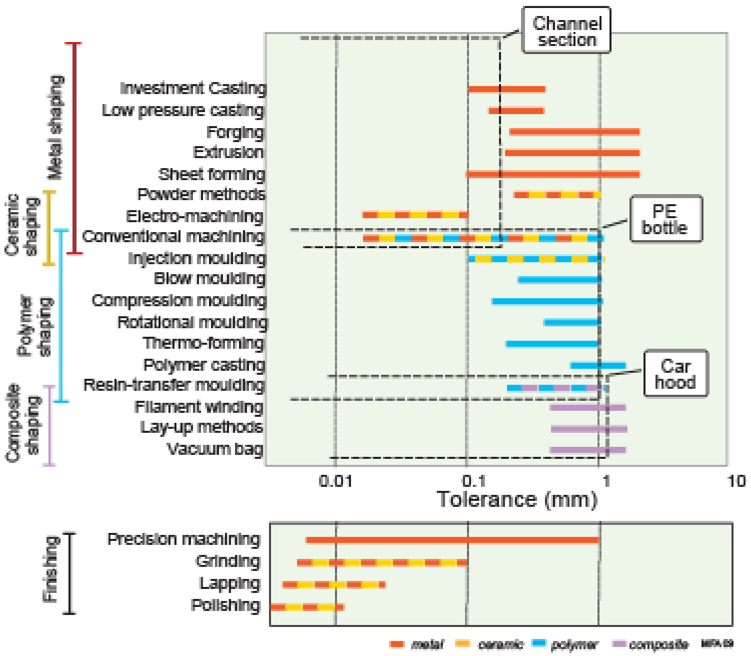

The Process-Shape chart



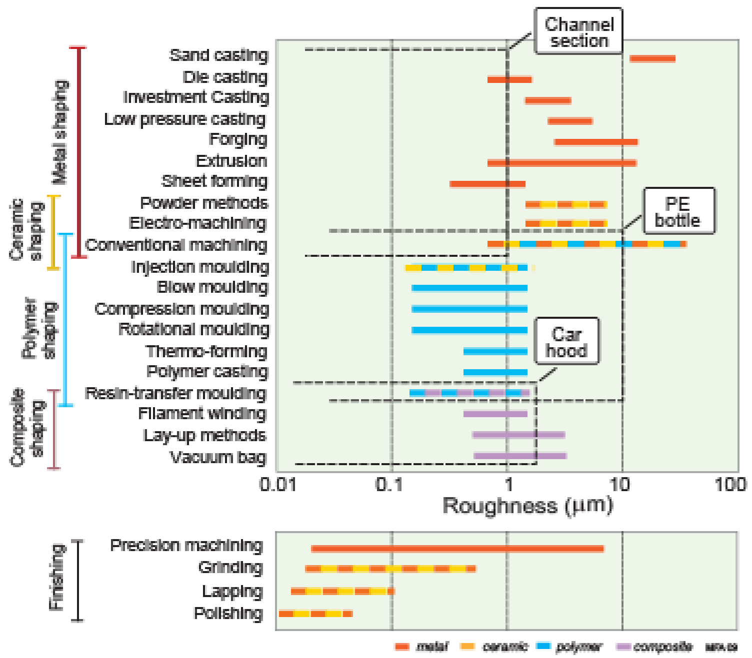

The Process-Mass range chart



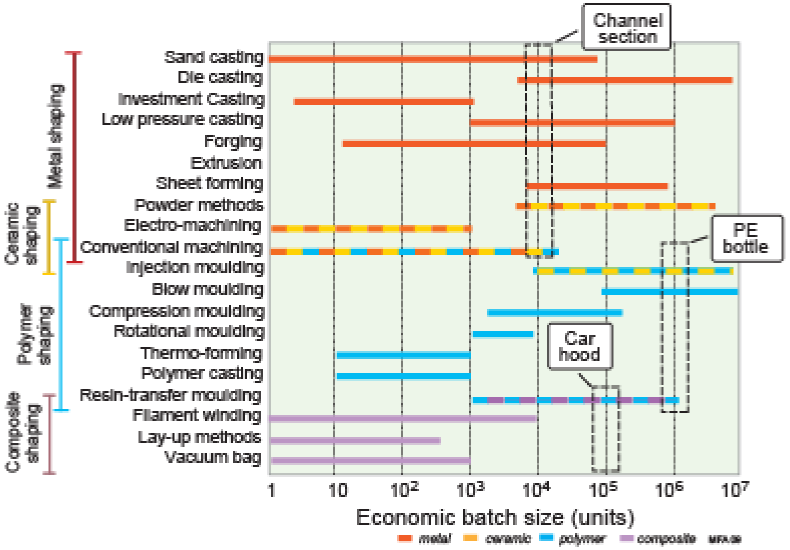

The Process-Section thickness chart




The Process-Tolerance chart




The Process-Roughness chart




The Economic batch-size chart

**E11.7 Selecting joining processes.** *This exercise and the next require the use of the CES EduPack software.*

(a) Use CES EduPack to select a joining process to meet the following requirements.

| | |
|---|---|
| **Function** | • *Create a permanent butt joint between steel plates* |
| **Constraints** | • *Material class: carbon steel*<br>• *Joint geometry: butt joint*<br>• *Section thickness: 8 mm*<br>• *Permanent*<br>• *Watertight* |
| **Objective** | -- |
| **Free variables** | • *Choice of process* |

(b) Use CES EduPack to select a joining process to meet the following requirements.

| | |
|---|---|
| **Function** | • *Create a watertight, demountable lap joint between glass and polymer* |
| **Constraints** | • *Material class: glass and polymers*<br>• *Joint geometry: lap joint*<br>• *Section thickness: 4 mm*<br>• *Demountable*<br>• *Watertight* |
| **Objective** | -- |
| **Free variables** | • *Choice of process* |

***Answer.*** (a) Applying CES EduPack to the problem gives the selection shown in the below.

| Selection using CES EduPack Level 1 or 2 | Comment |
|---|---|
| Brazing<br>Friction welding<br>Friction-stir welding<br>Gas – metal arc (MIG) welding<br>Gas-tungsten arc (TIG) welding<br>Manual metal arc (MMA) welding<br>Power (electron, laser) beam welding | There are many ways to make a permanent watertight joint in steel. |

(b) Applying CES EduPack to the problem gives the selection shown in the below.

| Selection using CES EduPack Level 1 or 2 | Comment |
|---|---|
| Snap fit<br>Threaded fasteners<br>Flexible adhesives<br>Rigid adhesives | All are practical |

**E11.8 Selecting surface treatment processes.** *This exercise, like the last, requires the use of the CES EduPack software.*

(a) Use CES EduPack to select a surface treatment process to meet the following requirements.

| | |
|---|---|
| **Function** | • *Increase the surface hardness and wear resistance of a high carbon steel component* |
| **Constraints** | • *Material class: carbon steel*<br>• *Purpose of treatment: increase surface hardness and wear resistance* |
| **Objective** | -- |
| **Free variables** | • *Choice of process* |

(b) Use CES EduPack to select a surface treatment process to meet the following requirements.

| | |
|---|---|
| **Function** | • *Apply color and pattern to the curved surface of a polymer molding* |
| **Constraints** | • *Material class: thermoplastic*<br>• *Purpose of treatment: aesthetics, color*<br>• *Curved surface coverage: good or very good* |
| **Objective** | -- |
| **Free variables** | • *Choice of process* |

***Answer.*** (a) Applying CES EduPack to the problem gives the selection shown in the below.

| Selection using CES EduPack Level 1 or 2 | Comment |
|---|---|
| Carburizing and carbonitriding<br>Nitriding<br>Induction and flame hardening<br>Laser surface hardening<br>Electro plating<br>Vapor metallizing | There are many ways to increase the surface hardness of carbon steel. |

(b) Applying CES EduPack to the problem gives the selection shown in the below.

| Selection using CES EduPack Level 1 or 2 | Comment |
|---|---|
| Cubic printing<br>Pad printing<br>Electroless plating<br>Organic-solvent based painting<br>Water-based painting | All are practical |

## E12. Materials and the environment (Chapter 15)

**E12.1.** Use the chart $E - H_p \rho$ chart of Figure 15.9 to find the polymer with a modulus $E$ greater than 1 GPa and the lowest embodied energy per unit volume.

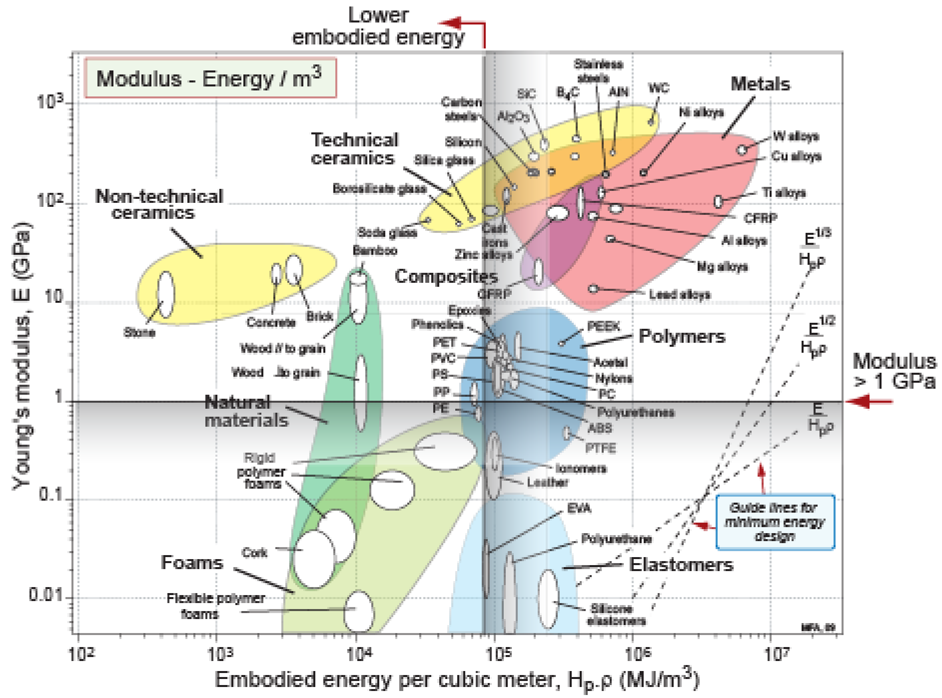
***Answer.*** The construction is shown opposite. The polymer with a modulus greater than 1 GPa and the lowest embodied energy is polypropylene, PP.

**E12.2.** A maker of polypropylene (PP) garden furniture is concerned that the competition is stealing part of his market by claiming that the "traditional" material for garden furniture, cast iron, is much less energy and $CO_2$ intensive than the PP. Atypical PP chair weighs 1.6 kg; one made of cast iron weighs 8.5 kg. Use the data for these two materials in Appendix A, Table A10, to find out who is right – are the differences significant if the data for embodied energy are only accurate to +-20%? If the PP chair lasts 5 years and the cast iron chair lasts 25 years, does the conclusion change?

***Answer.*** The table lists mean values of embodied energy and carbon footprint, per kg, for the two materials. The last two columns show the values per chair. If the difference in lifetime is ignored, the two chairs do not differ significantly in embodied energy, but they do in carbon release. If the longer life of the cast iron chair is recognized by dividing the values by the life in years (to give energy and carbon per chair-year) the cast iron chair wins easily.

| Material | Embodied energy* MJ/kg | $CO_2$* kg/kg | Embodied energy MJ/chair | $CO_2$ kg/chair |
|---|---|---|---|---|
| Cast iron | 17 | 1.05 | 145 | 8.9 |
| Polypropylene | 95 | 2.7 | 152 | 4.3 |

** Mean values from the data of Appendix A, Table A10*




The Modulus – Embodied energy chart: the one for stiffness at minimum embodied energy.

**E12.3** Identical casings for a power tool could be die-cast in aluminum or molded in ABS or polyester GFRP. Use the embodied-energy per unit volume bar-chart of Figure 15.8 to decide which choice minimizes the material embodied energy, assuming the same volume of material is used for each casing.

***Answer.*** The embodied energy /m³, $H_p \rho$, (where $H_p$ is the energy per kg and $\rho$ the density) is plotted Figure 15.8. It shows that ABS has the lower embodied energy per unit volume. For a casing using the same volume of material, ABS requires only one fifth of the energy of aluminum.

| Material | Embodied energy/m³, $H_p \rho$, MJ/m³ |
|---|---|
| ABS | $1 \times 10^5$ |
| GFRP | [illegible] x $10^5$ |
| Aluminum | [illegible] x $10^5$ |

**E12.4 Disposable knives and forks** (Figure E37). Disposable knives and forks are ordered by an environmentally-conscious pizza-house. The shape of each (and thus the length, width and profile) are fixed, but the thickness is free: it is chosen to give enough bending-stiffness to cut and impale the pizza without excessive flexure. The pizzeria-proprietor wishes to enhance the greenness of his image by minimizing the energy-content of his throw-away tableware, which could be molded from polystyrene (PS) or stampe[illegible]

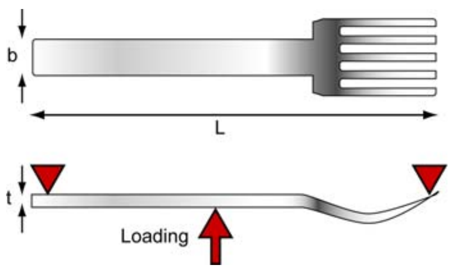



Figure E37

Establish an appropriate material index for selecting materials for energy-economic forks. Model the eating implement as a beam of fixed length $L$ and width $w$, but with a thickness $t$ that is free, loaded in bending, as in the figure. The objective-function is the volume of material in the fork times its energy content, $H_p \rho$, per unit volume ($H_p$ is the embodied energy per kg, and $\rho$ the density). The limit on flexure imposes a stiffness constraint (Appendix B, Section B3). Use this information to develop the index.

Flexure, in cutlery, is an inconvenience. Failure – whether by plastic deformation or by fracture – is more serious: it causes loss-of-function; it might even cause hunger. Repeat the analysis, deriving an index when a required strength is the constraint.

***Answer.*** This is a straightforward application of the method illustrated in the text by the derivation of equations 5.15 and 5.16; the only difference is that energy content, $mH_p$, rather than mass, $m$, is minimized. The free variable is the thickness $t$ of the shaft of the fork; all other dimensions are fixed. There are two alternative constraints, first, that the fork should not flex too much, second, that it should not fail.

| | |
|---|---|
| **Function** | • *Environmentally friendly disposable forks* |
| **Constraints** | • *Length L specified*<br>• *Width b specified*<br>• *Stiffness S specified, or*<br>• *Failure load F is specified* |
| **Objective** | • *Minimize the material energy-content* |
| **Free variables** | • *Shaft thickness t*<br>• *Choice of material* |

The resulting indices are

$$M_1 = \frac{E^{1/2}}{H_p \rho} \quad \text{and} \quad M_2 = \frac{\sigma_y^{2/3}}{H_p \rho}$$

The selection can be implemented using Figures 15.9 and 15.10 of the text. If the CES EduPack software is available, make a chart with the stiffness index as one axis and the strength index as the other. The materials that best meet *both* criteria lie at the top right

**E12.5.** Show that the index for selecting materials for a strong panel with the dimensions shown in Figure E.38, loaded in bending, with minimum embodied energy content is that with the largest value of

$$M = \frac{\sigma_y^{1/2}}{H_p \rho}$$

where $H_p$ is the embodied energy of the material, $\rho$ its density and $\sigma_y$ its yield strength. To do so, rework the panel derivation in Chapter 5 (equation 5.9) replacing the stiffness constraint with a constraint on failure load $F$ requiring that it exceed a chosen value $F^*$ where

$$F = C_2 \frac{I \sigma_y}{h L} > F^*$$

where $C_2$ is a constant and $I$ is the second moment of area of the panel

$$I = \frac{b h^3}{12} \,.$$

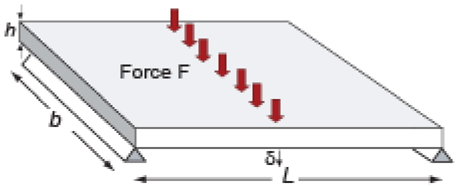



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Figure E38

***Answer.*** The objective function for the energy of the panel, $H$, is the volume $Lbh$ times the embodied energy of the material per unit volume, $\rho H_p$ :

$$H = b h L \rho H_p$$

Its failure load $F$ must be at least $F^*$, where

$$F = C_2 \frac{I \sigma_y}{h L} \geq F^*$$

Here $C_2$ is a constant that depends only on the distribution of the loads and $I$ is the second moment of area, which, for a rectangular section, is

$$I = \frac{b h^3}{12}$$

We can reduce the energy $H$ by reducing $h$, but only so far that the stiffness constraint is still met. Combining the last two equations and solving for $h$ gives

$$h \geq \left( \frac{12 F^* L}{C_2 b \sigma_y} \right)^{1./2}$$

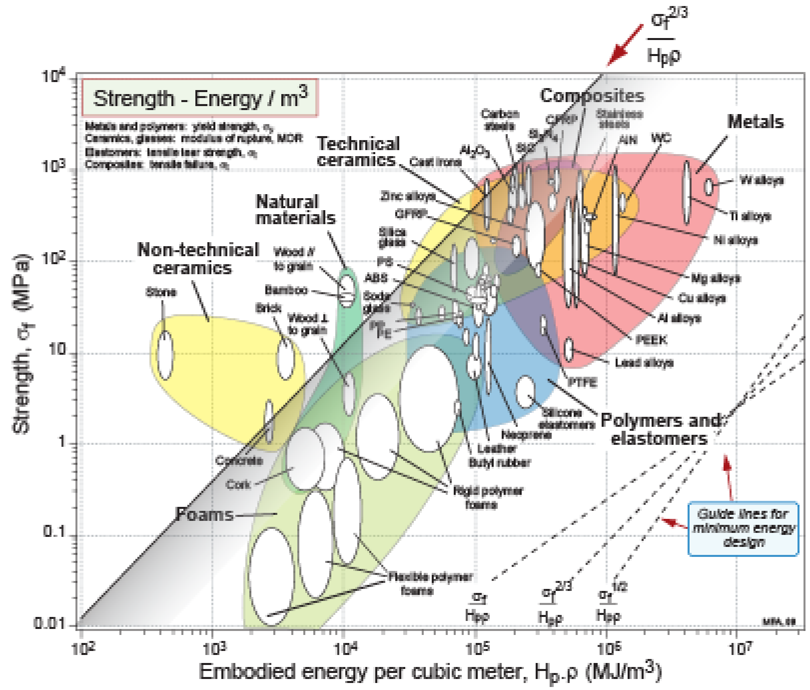
Using this to eliminate $h$ in the objective function gives

$$H \geq \left( \frac{12 F^* L^3 b}{C_2} \right)^{1/2} \frac{\rho H_p}{\sigma_y^{1/2}}$$

The quantities $F^*$, $L$, $b$ and $C_2$ are all specified; the only freedom of choice left is that of the material. The best materials for a strong panel with the lowest embodied energy are those with the smallest values of

$$M_p = \frac{\rho H_p}{\sigma_y^{1/2}}$$

**E12.6.** Use the indices for the crash barriers (equations 15.9 and 15.10) with the charts for strength and density (Figure 4.4) and strength and embodied energy (Figure 15.10) to select materials for each of the barriers. Position your selection line to include one metal for each. Reject ceramics and glass on the grounds of brittleness. List what you find for each barrier.

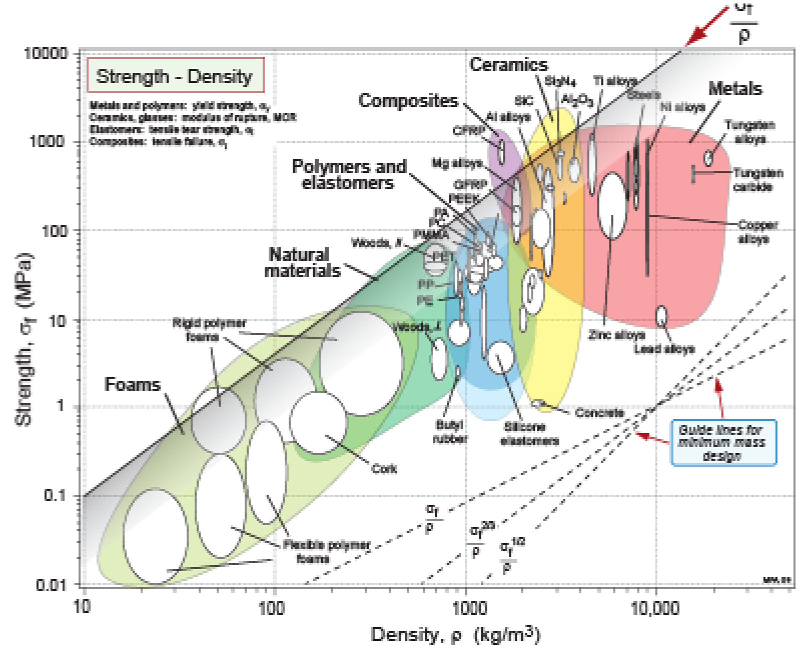



***Answer.*** The figure above shows the two charts with the indices

$$M_1 = \frac{H_p\rho}{\sigma_y^{2/3}} \qquad \text{and} \qquad M_2 = \frac{\rho}{\sigma_y^{2/3}}$$

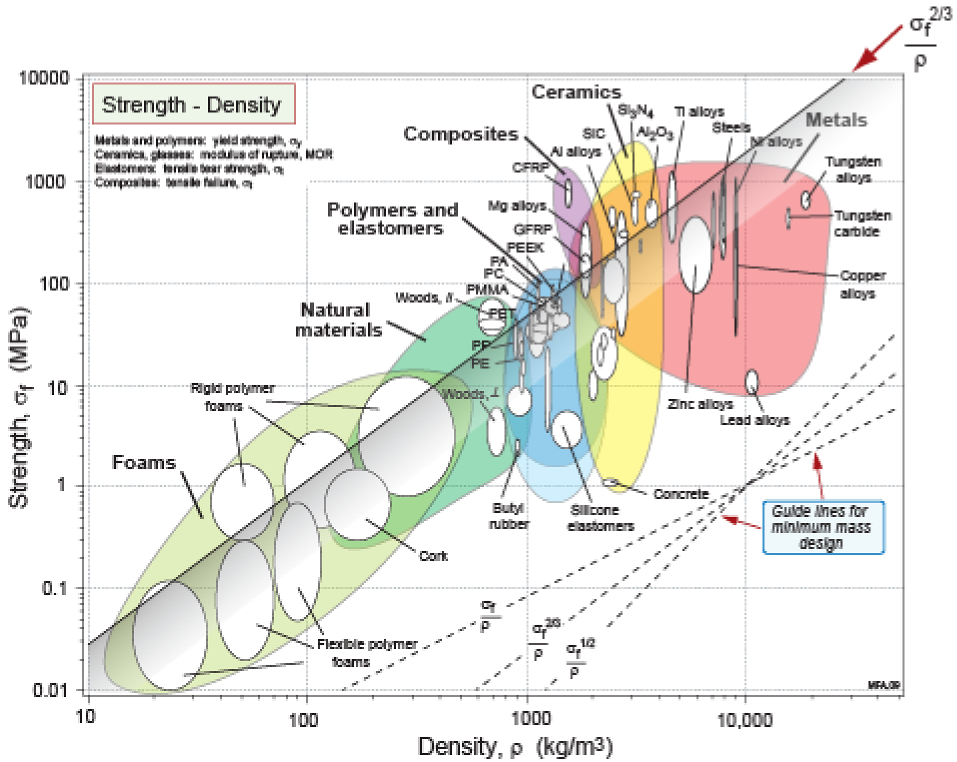
marked. Each is position to leave one class of metal exposed. The upper selection is that for the mobile barrier with minimizing mass as the objective. The best choice is CFRP (carbon fiber reinforced polymers); among metals, magnesium alloys offer the lightest solution . The lower selection is that for the static barrier with minimizing embodied energy as the objective. The selection is wood; among , metals cast irons offer the solution with lowest embodied energy.

The exercise brings out how strongly the selection to minimize environmental burden depends on the application, as did the selection method used in the text.




| Selected materials: mobile barrier | Selected materials: static barrier |
|---|---|
| Carbon-fiber reinforced polymers (CFRP) | Wood |
| Magnesium alloys | Cast irons |
| (Aluminum alloys) | (Carbon steels) |
| (Wood) | |

**E12.7.** The makers of a small electric car wish to make bumpers out of a molded thermoplastic. Which index is the one to guide this selection if the aim is to maximize the range for a given battery storage capacity? Plot it on the Strength / Density chart of Figure 4.4, and make a selection.




***Answer.*** The use-energy of any vehicle, be it gasoline or electric powered, increases with the mass of the vehicle. The bumper contributes to this mass. Maximizing range for a given battery capacity means minimizing mass. The function of the bumper is to sustain bending loads. The required index (equation 15.10 of the text) is

$$M = \frac{\sigma_y^{2/3}}{\rho}$$

This is plotted on the strength – density chart below, picking up the slope of 1.5 from the guide line at the lower right. The selection line is position such that a few polymers remain above it – they are the best choice. They are:

- Polycarbonate, PC
- Polyamide (nylon) PA and
- Polyetheretherketone PEEK

In practice bumpers are made of blends of Polycarbonate with other polymers such as Polypropylene or Polyamide, or of fiber reinforced thermosets such as Polyester. PEEK is too expensive for applications such as this.

**E12.8. Energy-efficient floor joists**. Floor joists are beams loaded in bending. They can be made of wood, of steel, or of steel-reinforced concrete, with the shape factors listed below. For a given bending stiffness and strength, which of these carries the lowest production-energy burden? The relevant data, drawn from the tables of Appendix A, are listed.

(a) Start with stiffness. Locate from equation 9.20 of the text the material index for stiffness-limited, shaped beams of minimum mass. Adapt this to make the index for stiffness-limited, shaped beams of embodied energy by multiplying density $\rho$ by embodied energy /kg, $H_p$. Use the modified index to rank the three beams.

(b) Repeat the procedure, this time for strength, creating the appropriate index for strength-limited shaped beams at minimum energy content by adapting equation 9.28.

What do you conclude about the relative energy-penalty of design with wood and with steel?

| Material | Density $\rho$ kg/m$^3$ | Modulus $E$ GPa | Strength $\sigma_f$ MPa | Energy $H_p$ MJ/kg | $\phi_B^e$ | $\phi_B^f$ |
|---|---|---|---|---|---|---|
| Soft wood | 700 | 10 | 40 | 7.5 | 2 | 1.4 |
| Reinforced concrete | 2900 | 35 | 10 | 5 | 2 | 1.4 |
| Steel | 7900 | 210 | 200 | 30 | 15 | 4 |

**Answer.** (a) The index for an energy-efficient beam with a specified stiffness, modified to include shape, is

$$M_1 = \frac{(\phi_B^e\, E)^{1/2}}{H_p \rho}$$

where $E$ is the modulus, $\rho$ the density, $H_p$ the energy content per unit weight, and $\phi_B^e$ is the shape factor for stiffness controlled design. The first Table shows values of $M_1$ for the wood, the concrete and the shaped steel beam. Wood wins.

| *Material* | $E$ (GPa) | $H_p \rho$ (GJ/m$^3$) | $\phi_B^e$ | $M_1$ (GPa$^{1/2}$/GJ/m$^3$) |
|---|---|---|---|---|
| Wood | 10 | 5.25 | 2 | 0.86 |
| Concrete | 35 | 14.5 | 2 | 0.57 |
| Steel | 210 | 238 | 15 | 0.23 |

(b) The index for an energy efficient beam of prescribed strength is, f

$$M_2 = \frac{\left(\phi_B^f\, \sigma_f\right)^{2/3}}{H_p \rho}$$

Where $\phi_B^f$ is the shape factor for failure in bending. The next table shows that, as before, wood is the most efficient material.

| *Material* | $\sigma_f$ (MPa) | $H_p \rho$ (GJ/m$^3$) | $\phi_B^e$ | $M_2$ (MPa$^{2/3}$/GJ/m$^3$) |
|---|---|---|---|---|
| Wood | 40 | 5.25 | 1.4 | 2.8 |
| Concrete | 10 | 14.5 | 1.4 | 0.4 |
| Steel | 200 | 238 | 4 | 0.36 |