# Mathematics of Infectious Diseases

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***Happy St. Patrick's Day!***

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- 2012-present: Assistant Professor of Mathematics, UCF
  *Mathematical biology, ecology and epidemiology*

# Core Research

- New models, new approaches, new results, and new applications to complex biological systems
- Developed new graph-theoretic approaches to investigate dynamics of coupled systems on networks
- Established sharp threshold results for many heterogeneous infectious disease models
- Formulated and analyzed new mathematical models for waterborne diseases such as cholera
- Defined a new concept target reproduction number to mathematically measure intervention and control strategies in order to eradicate infectious diseases in heterogeneous host populations

# Paper Passing Game

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  - hold the paper, or
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- How many persons have been infected by the "paper" disease?

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- An immune response against a pathogen, which can include a high fever, inflammation, and other damaging symptoms, can be more devastating than the direct damage caused by the pathogen
- Infectious diseases require a mode of transmission (direct or indirect transmission, waterborne, airborne, vector-borne, food-borne, etc.) to be transmitted to other individuals (infectious)

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- spread among heterogeneous host groups (heterogeneous model)

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  - idea of a "threshold"
- MacDonald [1957]
  - included adult and larval mosquitoes
  - showed that control of adult mosquitoes is more effective than control of larvae

Gonorrhea

A sexually transmitted bacterial disease

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- Hethcote and Yorke [1984]
  - found that a core group of very highly active individuals maintains the disease at an epidemic level
  - contact tracing is more efficient for control than routine screening

Measles

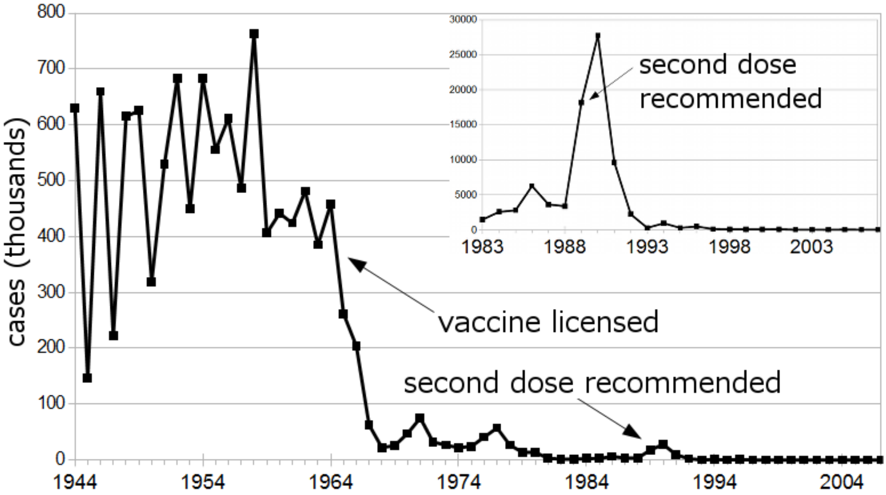
A viral childhood disease

Measles

A viral childhood disease

- Anderson and May [1992]
  *Infectious Diseases of Humans: Dynamics and Control*
  - guided design of vaccination programs
    (two dose strategy in UK)
  - two dose vaccination strategy started in US in 1989

# Measles cases in the United States, 1944-2007



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- Mathematical modeling of epidemics can lead to and motivate new results in mathematics e.g.,
  - ruling out periodic orbits in higher dimensional ODE systems

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In 1902 Sir Ronald Ross was awarded the Noble prize for medicine for proving that malaria is transmitted by mosquitoes

# The Kermack-McKendrick Epidemic Model [1927]

Constant population divided into 3 compartments:

- $S(t) =$ number of individuals *susceptible* to disease
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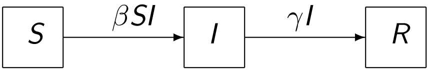


$$\begin{cases} S' = -\beta SI \\ I' = \beta SI - \gamma I \\ R' = \gamma I \end{cases}$$

- $\beta =$ effective contact coefficient
- $1/\gamma =$ average time of infection (depends on disease)
  Length of the infective period is exponentially distributed

# Dynamics of Kermack-McKendrick Model

- Asymptotic analysis for $I' = \beta SI - \gamma I$
  - If $\beta S(0) < \gamma$ then $I(t)$ decreases monotonically to 0
  - If $\beta S(0) > \gamma$ then $I(t) \to I_{max} \to 0$ as $t \to \infty$ leaving a positive number of susceptibles $S(\infty) > 0$

    $$I_{max} = I(0) + S(0) - \frac{\gamma}{\beta}\Big(1 + \ln\frac{\beta S(0)}{\gamma}\Big)$$

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- Basic reproduction number (threshold):
  $\mathcal{R}_0 = \beta S(0)\frac{1}{\gamma}$ is the average number infected when one infective enters a susceptible population
  - $\mathcal{R}_0 < 1$: disease dies out
  - $\mathcal{R}_0 > 1$: there is a disease epidemic

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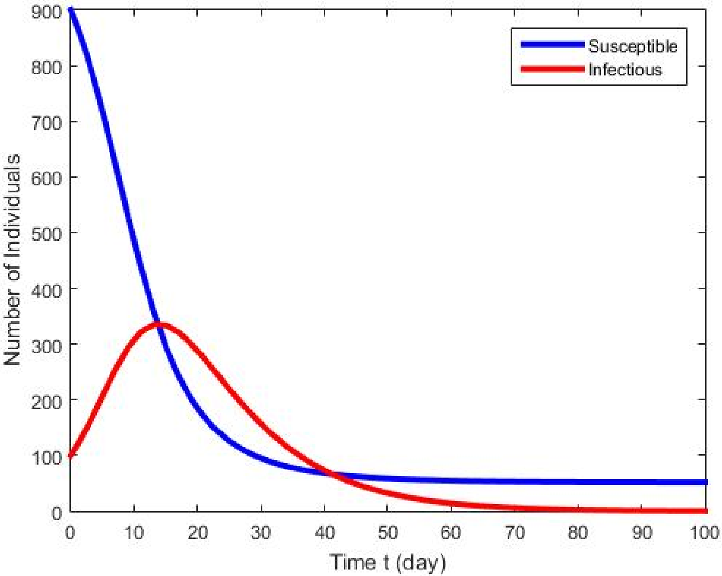
- The final size relation

$$1 - R(\infty) = \exp(-\mathcal{R}_0 R(\infty))$$

# Simulations with $\mathcal{R}_0 = 2.7$



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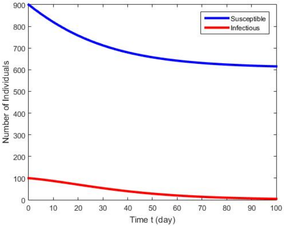


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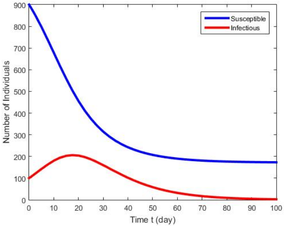


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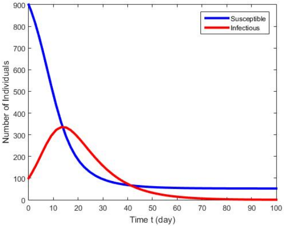

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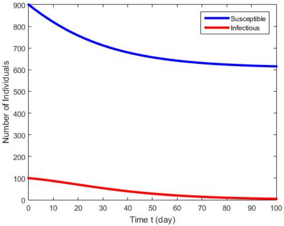


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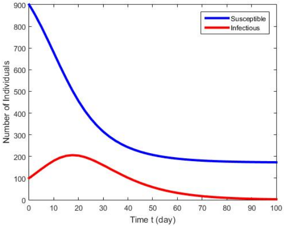


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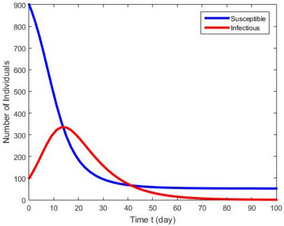
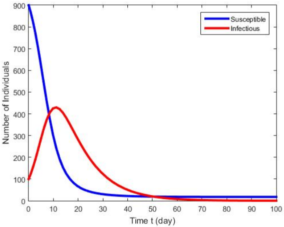
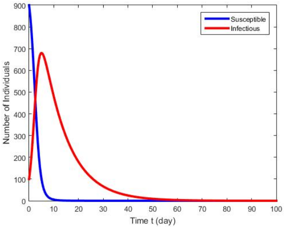


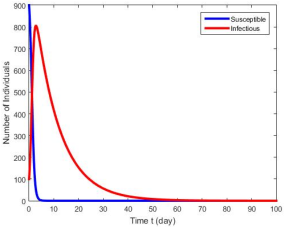

$\mathcal{R}_0 = 3.6$




$\mathcal{R}_0 = 9$




$\mathcal{R}_0 = 18$



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  - For ebola $\mathcal{R}_0 \approx 2$
    need to vaccinate over only 50% (hypothetically)

Thanks!