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The optimal solution set of interval linear programming problems

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Abstract: Determining the set of all optimal solutions of a linear program with interval data is one of the main problems discussed in interval optimization. We review two methods based on duality in linear programming, which are used to approximate the optimal set. Additionally, another decomposition method based on complementary slackness is proposed. This method provides the exact description of the optimal set for problems with a fixed coefficient matrix. The second part of the thesis is focused on studying the topological and geometric properties of the optimal set. We examine sufficient conditions for closedness, boundedness, connectedness and convexity. We also prove that testing boundedness is co-NP-hard for inequality-constrained problems with free variables. Stronger results are derived for some special classes of interval linear programs, such as problems with a fixed coefficient matrix. Furthermore, we study the effect of transformations commonly used in linear programming on interval problems, which allows for a direct generalization of some results to different types of interval linear programs.

Keywords: interval linear programming, optimal set, topological properties

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Contents

Abbreviations

LP	linear program
ILP	interval linear program

Notation

Sets

$\mathbb{R}$	set of real numbers
$\mathbb{R}^n$	set of real $n \times 1$ vectors
$\mathbb{R}^{m \times n}$	set of real $m \times n$ matrices
$\mathbb{IR}$	set of intervals
$\mathbb{IR}^n$	set of interval $n \times 1$ vectors
$\mathbb{IR}^{m \times n}$	set of interval $m \times n$ matrices
2^X	power set of X
$\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$	set of optimal values of an interval linear program
$\mathcal{M}(\boldsymbol{A}, \boldsymbol{b})$	(weak) feasible set of an interval linear program

Reals

A	a real matrix
$b = (b_1, \dots, b_n)^T$	a real vector
$0 = (0, \dots, 0)^T$	vector of zeros
$e = (1, \dots, 1)^T$	vector of ones
I_n	$n \times n$ identity matrix
$f(A, b, c)$	optimal value of a linear program

Intervals

$\boldsymbol{A}$	an interval matrix
$\underline{A}, \overline{A}$	lower and upper bound of an interval matrix $\boldsymbol{A}$
A^c	central matrix of an interval matrix $\boldsymbol{A}$
A^Δ	radial matrix of an interval matrix $\boldsymbol{A}$
$\boldsymbol{b}$	an interval vector
$\overline{b}, \underline{b}$	lower and upper bound of an interval vector $\boldsymbol{b}$
b^c	central vector of an interval vector $\boldsymbol{b}$
b^Δ	radial vector of an interval vector $\boldsymbol{b}$

Introduction

Motivation

In the year 1980, mathematician and computer scientist László Lovász wrote [28]: "If one would take statistics about which mathematical problem is using up most of the computer time in the world, then (not counting database handling problems like sorting and searching) the answer would probably be linear programming. The problem is very simple and natural indeed: given a set of inequalities (constraints)

$$a_i x \leq b_i, \qquad (i = 1, \ldots, m)$$

where each a_i is a real n-vector, each b_i is a real number and $x = (x_1, \ldots, x_n)^T$ is unknown, find the maximum of

$$cx = c_1 x_1 + \cdots + c_n x_n$$

subject to the above constraints." Nowadays, more than 30 years later, linear programming continues to be a growing mathematical area of immense practical importance.

Unfortunately, real-world problems are often accompanied by various inaccuracies and measurement errors in the input data, which can impair the results obtained by solving a linear program. The need for extending classical linear programming to include tools for modelling uncertainty can be dated back to 1955, as documented by the paper *Linear programming under uncertainty* published by Dantzig [11].

There are several different approaches to handling uncertainty and inexactness in mathematical modelling, such as stochastic programming, fuzzy set theory, parametric programming or interval analysis, each bearing their own advantages and disadvantages. In this thesis, we will adopt the approach of interval linear programming, which can be viewed as a special case of multiparametric programming with interval domains, to obtain rigorous results covering all possible scenarios that can occur in a given realistic linear programming model. Instead of working with estimated or rounded quantities, we consider data in the form of intervals enclosing the exact real values. Interval linear programming has been applied in solving various practical problems, e.g. portfolio selection [18, 26], resources and environmental systems management [9], solid waste management [22] or chemical engineering problems [39].

In linear programming, there are two main questions connected with a given problem: calculating the optimal value of the objective function and finding one or more optimal solutions, provided the given problem is feasible and bounded. The first question has also been well-studied in the context of interval linear programming, yielding formulas for computing the lower and upper bound on the optimal value. The question of finding optimal solutions for a general interval linear program is, however, more complicated, and there are still many open problems related to it. In case of a classical linear program, the set of optimal solutions is a convex polyhedron, namely a face of the feasible region. However, the introduction of interval coefficients may cause the optimal set to become non-convex or even disconnected. This leads to the fact that determining the set of optimal solutions is one of the most challenging problems in interval linear programming.

Main goals

The goal of this thesis is to investigate the set of all possible optimal solutions to a linear program with the input data being perturbed in given intervals. We are interested in the geometric and topological properties of the optimal solution set, such as closedness, boundedness, connectedness or convexity. Additionally, we also study the effects of the transformations used in linear programming (rewriting inequalities into equations and vice versa, introducing non-negative variables) on the optimal set of an interval program.

Since it is in general difficult to determine the exact optimal solution set of an interval linear program, finding a tight approximation or enclosure of the optimal set is also desirable. We will review some of the known methods for computing an approximation and examine other possible approaches to this problem.

Related works

A thorough overview of interval linear programming, as well as some other techniques of dealing with uncertainty in linear programming, can be found in the book *Linear Optimization Problems with Inexact Data* by Fiedler et al. [12]. Regarding interval optimization, the book discusses mainly feasibility of interval linear systems and the problem of computing the optimal value range of interval linear programs.

A broader summary of the current state of the art is presented in the article *Interval Linear Programming: A Survey* by Hladík [19]. The survey is focused on three main topics: feasibility, boundedness and optimality. It also addresses the question of characterizing the optimal solution set in a general interval linear program, and in some special cases (basis stability, linear programs with interval objective or right-hand side).

From an algorithmic point of view, we are interested in computing an approximation of the optimal solution set of an interval linear program. The first algorithms designed to compute guaranteed bounds for optimal vertices and the optimal value used an interval extension of the simplex method [4, 23, 30]. Some methods use decomposition of the given interval program into submodels, such as the best and the worst case method [2, 10] or the enhanced-interval linear programming model [47]. In this thesis, we will review algorithms based on the description of the optimal set using duality [20]. Another approach is to exploit basis stability [21, 25], which significantly simplifies the problem of finding the optimal solutions.

Structure of the thesis

One of the basic tools used in interval linear programming is interval arithmetic and algebra. In Chapter 1, we extend the operations and objects known from real numbers to the case of intervals and introduce the necessary terminology of interval computations. We also briefly discuss the complications to be aware of when using interval arithmetic.

Chapter 2 summarizes the notions used in classical linear programming and introduces their counterparts and extensions, which are used in interval linear programming. Other approaches to dealing with uncertainty are also reviewed.

An overview of the theory of interval linear programming is presented in Chapter 3. We begin by characterizing the feasible set of an interval program and state the Oettli–Prager theorem and the Gerlach theorem, which provide a description of the feasible set for systems of interval linear equations and inequalities, respectively. We also investigate the interval variants of transformations commonly used in linear programming. Further, we review the notion of duality and use the principle of strong duality to derive a general description of the optimal solution set in interval linear programming. The last section of the chapter focuses on basis stability, which allows for a simplification of the description.

In Chapter 4, two existing methods for computing an outer approximation of the optimal solution set are reviewed: the orthant decomposition method and an iterative contractor. We also introduce another decomposition method, which is based on complementary slackness in linear programming.

The main topic of the thesis, topological, metric and geometric properties of the optimal solution set, are studied in Chapter 5 and Chapter 6. The properties discussed are: closedness, boundedness, connectedness, convexity and polyhedrality. We review the previously known properties and derive some new results. Apart from the general case, we study the class of interval linear programs with a fixed coefficient matrix, which allows us to strengthen some of the statements. We also show that in this special case, it is possible to transform equation constraints into inequalities and impose non-negativity.

1 Interval computations

1.1 Real intervals

This section introduces the necessary terminology and notation that will be used throughout the thesis. Since we will work with uncertainty represented by interval data, let us begin by defining a real interval and extending familiar arithmetic and set-theoretic operations to the set of intervals.

Definition 1.1. Given $\underline{x}, \overline{x} \in \mathbb{R}$ such that $\underline{x} \leq \overline{x}$, we define a *closed real interval* $\boldsymbol{x} = [\underline{x}, \overline{x}]$ as the set

$$\{x \in \mathbb{R} : \underline{x} \leq x \leq \overline{x}\}.$$

The values $\underline{x}, \overline{x}$ are called the *lower bound* and the *upper bound* of the interval $\boldsymbol{x}$, respectively.

Notation. Hereinafter, real intervals will be denoted by bold lowercase letters. The set of all closed real intervals will be denoted by $\mathbb{IR}$[1].

Other types of intervals can be obtained by replacing the weak inequality in the definition by a strict inequality (open or half-open intervals) or by changing the underlying set (e.g. complex intervals). However, our work will focus mainly on closed intervals over the set of real numbers. Unless specified otherwise, the term "interval" will refer to the sets characterized by Definition 1.1.

Definition 1.2. An interval $[\underline{x}, \overline{x}]$ with $\underline{x} = \overline{x}$ is said to be *degenerate*.

Since a degenerate interval $[x, x]$ only contains a single number, it is often identified with the number x itself, therefore it holds that $x = [x, x]$. We will also employ this convention throughout the thesis.

Definition 1.3. Let $\boldsymbol{x} = [\underline{x}, \overline{x}]$ be a real interval, then:

(a) the *midpoint* of $\boldsymbol{x}$ is defined as $x_c = \frac{1}{2}(\overline{x} + \underline{x})$,

(b) the *radius* of $\boldsymbol{x}$ is defined as $x_\Delta = \frac{1}{2}(\overline{x} - \underline{x})$,

(c) the *width* of $\boldsymbol{x}$ is defined as $w(\boldsymbol{x}) = \overline{x} - \underline{x}$,

(d) the *magnitude* of $\boldsymbol{x}$ is defined as $\operatorname{mag}(\boldsymbol{x}) = \max\{|\underline{x}|, |\overline{x}|\}$.

Using these terms, we can equivalently describe an interval $\boldsymbol{x}$ by its *midpoint-radius representation* (as opposed to the *endpoint representation* introduced in Definition 1.1) in the form

$$[x_c - x_\Delta, x_c + x_\Delta] = \{x \in \mathbb{R} : |x - x_c| \leq x_\Delta\}.$$

When using intervals to represent uncertainty, this is a natural approach to denote an approximate value x_c with an error of at most $\pm x_\Delta$.

[1]Some authors choose to include the empty set or unbounded intervals in $\mathbb{IR}$, however, we treat these cases separately.

1.1.1 Interval arithmetic

The arithmetic of real numbers can be naturally generalized to intervals. For a binary arithmetic operation $\circ \in \{+, -, \cdot, /\}$ defined on $\mathbb{R}$, we can introduce the corresponding interval operation as follows:

$$\boldsymbol{x} \circ \boldsymbol{y} = \{x \circ y : x \in \boldsymbol{x}, y \in \boldsymbol{y}\}.$$

This definition leads to explicit formulas for calculating the lower and upper bounds of the sum, difference, product and quotient of two intervals:

$$\begin{aligned}
\boldsymbol{x} + \boldsymbol{y} &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}],\\
\boldsymbol{x} - \boldsymbol{y} &= [\underline{x} - \overline{y}, \overline{x} - \underline{y}],\\
\boldsymbol{x} \;\cdot\; \boldsymbol{y} &= \left[\min\left\{\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy}\right\}, \max\left\{\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy}\right\}\right],\\
\boldsymbol{x} \;/\; \boldsymbol{y} &= \boldsymbol{x} \cdot 1/\boldsymbol{y}, \text{where } 1/\boldsymbol{y} = [1/\overline{y}, 1/\underline{y}] \text{ and } 0 \notin \boldsymbol{y}.
\end{aligned}$$

The formula for multiplication of two intervals can be further simplified by testing the signs of the lower and upper bounds of given intervals. In most cases it is possible to determine which of the four products will be the endpoints of the resulting interval.

Interval division, as defined above, assumes that the interval in the denominator does not contain 0, i.e., it is either strictly positive or strictly negative. If we need to allow division by an interval containing 0, we can employ the *extended interval arithmetic* (see [32, pp. 109–115]). In this case, division by the degenerate interval $[0, 0]$ returns the empty set, and unbounded intervals are used to describe the result of division when $0 \in \boldsymbol{y}$.

We can observe that some properties of real arithmetic are also transferred to intervals, as stated by Lemma 1.1 and Lemma 1.2. However, other properties, such as the existence of inverse elements or the distributivity of multiplication over addition are lost in the generalization.

Lemma 1.1. *Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{IR}$, then the following properties of interval addition and multiplication hold:*

$$\begin{aligned}
&\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}, && \boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}, && \textit{(commutativity)}\\
&(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}), && (\boldsymbol{x} \cdot \boldsymbol{y}) \cdot \boldsymbol{z} = \boldsymbol{x} \cdot (\boldsymbol{y} \cdot \boldsymbol{z}), && \textit{(associativity)}\\
&\boldsymbol{x} \cdot (\boldsymbol{y} + \boldsymbol{z}) \subseteq \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \cdot \boldsymbol{z}. && && \textit{(subdistributivity)}
\end{aligned}$$

Lemma 1.2. *For every $\boldsymbol{x} \in \mathbb{IR}$ it holds that*

$$\begin{aligned}
&[0, 0] + \boldsymbol{x} = \boldsymbol{x}, && \textit{(additive identity)}\\
&[1, 1] \cdot \boldsymbol{x} = \boldsymbol{x}. && \textit{(multiplicative identity)}
\end{aligned}$$

When interval arithmetic is implemented on a machine with limited precision, *outward rounding* is used in order to guarantee the enclosure of the result obtained in real interval arithmetic. During the computation, lower bound of the resulting interval is rounded down to its machine-representable predecessor and the upper bound is rounded up to its machine-representable successor. For more information about machine interval arithmetic see [1, pp. 39–49].

We can also introduce an interval generalization of other real-valued functions. We will define such a generalization by a natural requirement that the real function coincides with its interval counterpart for degenerate intervals (real numbers).

Definition 1.4. Let $f : \mathbb{R}^n \to \mathbb{R}$, the function $[f] : \mathbb{IR}^n \to \mathbb{IR}$ is an *interval extension* of f, if it satisfies the property

$$[f](x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in \mathbb{R}$.

1.1.2 Dependency problem

One of the main drawbacks of interval arithmetic is the so-called *dependency problem.* When calculating the image of a real function over an interval domain, the use of interval arithmetic can lead to overestimation of the result. This is due to the fact that interval evaluation of an expression does not preserve the dependency among multiple occurrences of a variable. As an illustration, consider the following example.

Example. Given a real-valued function $g(x) = x \cdot x$ and an interval $\boldsymbol{x} = [-5, 5]$, we would like to find the image of $\boldsymbol{x}$ under the function g. Substituting for x in the function rule we obtain the result $[-5, 5] \cdot [-5, 5] = [-25, 25]$.

However, if we use a simplified expression in the form $g(x) = x^2$ with a single occurrence of the variable x and evaluate $[-5, 5]^2 = \{x^2 : x \in [-5, 5]\}$, we obtain the exact image of $\boldsymbol{x}$, which is the interval $[0, 25]$. The overestimation in the first case appeared because we described the set $\{x \cdot y : x \in [-5, 5], y \in [-5, 5]\}$, thus losing the dependency between the two occurrences of x.

Due to the dependency problem, it is important to bear in mind that equivalent transformations used in real arithmetic may not result in an equivalent expression when working with intervals. The loss of dependency also causes the non-existence of additive inverses in interval arithmetic: the equality $\boldsymbol{x} - \boldsymbol{x} = 0$ only holds for degenerate intervals. In general, we have

$$\boldsymbol{x} - \boldsymbol{x} = [\underline{x} - \overline{x}, \overline{x} - \underline{x}] = [-1, 1] \cdot w(\boldsymbol{x}).$$

1.1.3 Set operations

Since intervals are defined as sets of real numbers, we can also apply set operations to them. Unfortunately, the resulting set is in general not an interval. This is true even for basic operations such as intersection or union.

Definition 1.5. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{IR}$, we define the *intersection* and *union* of $\boldsymbol{x}$, $\boldsymbol{y}$ as

$$\begin{aligned} \boldsymbol{x} \cap \boldsymbol{y} &= \{z \in \mathbb{R} : z \in \boldsymbol{x} \wedge z \in \boldsymbol{y}\}, \\ \boldsymbol{x} \cup \boldsymbol{y} &= \{z \in \mathbb{R} : z \in \boldsymbol{x} \vee z \in \boldsymbol{y}\}. \end{aligned}$$

However, if the intersection of $\boldsymbol{x}$ and $\boldsymbol{y}$ is non-empty, it can be equivalently described as the interval $\left[\max\{\underline{x}, \underline{y}\}, \min\{\overline{x}, \overline{y}\}\right]$. The union of any two disjoint non-empty intervals is not an interval itself, it is therefore often more convenient to work with an interval enclosing the union.

Definition 1.6. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{IR}$, we define the *interval hull* of $\boldsymbol{x}$, $\boldsymbol{y}$ as

$$\boldsymbol{x} \,\underline{\cup}\, \boldsymbol{y} = \left[\min\{\underline{x}, \underline{y}\}, \max\{\overline{x}, \overline{y}\}\right].$$

1.2 Interval matrices

Intervals can also be used in matrix theory, thus providing a mathematical tool for representing arrays of inexact data, e.g. in linear algebra or linear programming. We will begin this section by reviewing the necessary notation from the classical matrix theory and then extend it to the interval case.

Notation. The symbol $\mathbb{R}^{m\times n}$ will be used to denote the set of all real m-by-n matrices. For a matrix $A \in \mathbb{R}^{m\times n}$, we will denote by a_{ij} the coefficient of A in the i-th row and j-th column. For a vector $x = (x_1, \dots, x_n)$ the notation $\operatorname{diag}(x)$ will stand for the diagonal matrix

$$\begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_n \end{pmatrix}.$$

The inequality relations $\leq, <$ on the set of real (interval) matrices, as well as the absolute value $|A|$, are understood element-wise.

Definition 1.7. Given two matrices $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$ such that $\underline{A} \leq \overline{A}$, we define a *real interval matrix* $\boldsymbol{A} = [\underline{A}, \overline{A}]$ as the set $\{A \in \mathbb{R}^{m\times n} : \underline{A} \leq A \leq \overline{A}\}$. The matrices $\underline{A}, \overline{A}$ are called the *lower bound* and the *upper bound* of the interval matrix $\boldsymbol{A}$, respectively.

There exists an alternative definition of interval matrices: an interval matrix can be viewed as a matrix, whose coefficients are intervals. A real matrix then belongs to the interval matrix if each of its coefficients belongs to the corresponding interval coefficient. Using the bounds introduced in Definition 1.7 we can characterize the interval matrix $\boldsymbol{A}$ by its coefficients $[\underline{a}_{ij}, \overline{a}_{ij}]$, where $\underline{a}_{ij}, \overline{a}_{ij}$ are the coefficients of the lower and upper bound of $\boldsymbol{A}$, respectively.

Notation. Real interval matrices will be denoted by bold uppercase letters. The set of all real interval m-by-n matrices will be denoted by the symbol $\mathbb{IR}^{m\times n}$. The coefficient of the interval matrix $\boldsymbol{A}$ in the i-th row and j-th column will be denoted by $\boldsymbol{a}_{ij}$.

In analogy to the case of one-dimensional intervals, we can also describe an interval matrix using its central and radial matrix:

Definition 1.8. Let $\boldsymbol{A} = [\underline{A}, \overline{A}] \in \mathbb{IR}^{m\times n}$, then:

(a) the *central matrix* of $\boldsymbol{A}$ is defined as $A_c = \frac{1}{2}(\overline{A} + \underline{A})$,

(b) the *radial matrix* of $\boldsymbol{A}$ is defined as $A_\Delta = \frac{1}{2}(\overline{A} - \underline{A})$.

Furthermore, we can also naturally extend arithmetic operations on matrices to the interval case, using the definitions introduced in Section 1.1.1. A special case of an interval matrix is an *interval vector*, which is a matrix with the dimensions $n \times 1$ for some $n \in \mathbb{N}$. An interval vector can also be thought of as the Cartesian product of n real intervals. Geometrically, we can represent interval vectors as 2-dimensional rectangles, 3-dimensional rectangular cuboids or so-called orthotopes in higher dimensions. Thanks to this representation, interval vectors are also known as *interval boxes*.

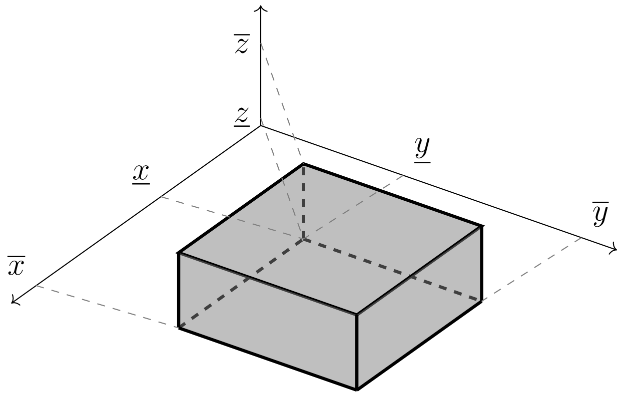

Figure 1.1: An interval vector (box) in $\mathbb{R}^3$.

Notation. Since interval vectors can be viewed as a generalization of real intervals into higher dimensions, we will denote them by bold lowercase letters. Should any confusion arise, the distinction will be made by stating the dimension of the element. The symbol $\mathbb{IR}^n$ will be used instead of $\mathbb{IR}^{n\times 1}$ to denote the set of all n-dimensional interval vectors.

Interval vectors play a crucial role in interval analysis: they allow us to extend the notion of an interval hull introduced in Definition 1.6 to arbitrary bounded sets, even in higher dimension:

Definition 1.9. Let a bounded set $\mathcal{S} \subseteq \mathbb{R}^n$ be given. An interval vector $\boldsymbol{x} \in \mathbb{IR}^n$ satisfying $\mathcal{S} \subseteq \boldsymbol{x}$ is said to be an *interval enclosure* of the set $\mathcal{S}$. The *interval hull* of $\mathcal{S}$ is defined as the interval vector

$$\square\mathcal{S} = \bigcap \{\boldsymbol{x} \in \mathbb{IR}^n : \mathcal{S} \subseteq \boldsymbol{x}\}\,.$$

2 Linear programming under uncertainty

2.1 Linear programming

Let us now review the basics of classical linear programming, which will be further generalized to include uncertainty described by interval coefficients.

Definition 2.1. Given $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$, a *linear program* is an optimization problem in the form

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned} \tag{LP}$$

A linear program (abbr. LP) is therefore the problem of minimizing a linear function $c^T x$ (called the *objective function*) over a set described by linear constraints. It is easy to see that a maximization problem with a linear objective and linear constraints can also be restated as a linear program in the form given in Definition 2.1. We will further define some terms related to the solution set of a linear program.

Definition 2.2. For $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, the *feasible set* of the corresponding linear program is the set

$$\mathcal{M}(A, b) = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}.$$

A linear program is said to be *feasible* if $\mathcal{M}(A, b) \neq \emptyset$, otherwise it is *infeasible*. A point $x \in \mathcal{M}(A, b)$ is called a *feasible solution*.

Notation. Note that Definition 2.1 can be equivalently restated with the feasible set in various different forms. In this thesis, we will refer to one of the following three basic types of (interval) linear programs:

(I) $\mathcal{M}(A, b) = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$,

(II) $\mathcal{M}(A, b) = \{x \in \mathbb{R}^n : Ax \leq b\}$,

(III) $\mathcal{M}(A, b) = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$.

In linear programming, we can define the feasible region of an LP using any of these sets. However, such a transformation is not always possible in the case of interval linear programs due to the dependency problem discussed in Section 1.1.2. A distinction between these forms will thus be necessary. Transformations between linear programs (and their interval counterparts) of type (I), (II) and (III) are further addressed in Section 3.2.

Definition 2.3. A *halfspace* in $\mathbb{R}^n$ is a set in the form

$$\{x \in \mathbb{R}^n : a^T x \leq b\}$$

for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. An intersection of finitely many halfspaces is called a *(convex) polyhedron*. Further, we define a *face* of a polyhedron P as the set

$$\{x \in P : a^T x = b\},$$

where $a \in \mathbb{R}^n, b \in \mathbb{R}$ satisfy $a^T y \leq b$ for all $y \in P$.

Using the description of $\mathcal{M}(A, b)$ in form (II), we directly see that the feasible set of a linear program is a polyhedron.

In linear programming, we are mainly interested in feasible solutions, which yield the best (in our case minimal) value of the objective function. Definition 2.4 gives a formal description of such solutions.

Definition 2.4. A point $x^* \in \mathcal{M}(A, b)$ is called *optimal*, if for each $y \in \mathcal{M}(A, b)$ the inequality $c^T x^* \leq c^T y$ holds.

Notation. We will denote by $\mathcal{S}(A, b, c)$ the set of all optimal solutions to (LP). The optimal objective value will be denoted by

$$f(A, b, c) = \inf_{x \in \mathcal{M}(A,b)} c^T x.$$

It is easy to see that an optimal solution does not always exist, for example if the given linear program is infeasible or if the objective function is unbounded on the feasible set. For each linear program, one of the three cases occurs:

- $f(A, b, c) = \infty$, then $\mathcal{M}(A, b) = \emptyset$ and the problem is infeasible,
- $f(A, b, c) = -\infty$, then the problem is said to be unbounded,
- $f(A, b, c)$ is finite, then there exists an optimal solution x^* with the objective value $f(A, b, c) = c^T x^*$.

Note that if $c = 0$, then the objective value for each feasible solution is also 0, and the linear program is reduced to a linear system.

The set of optimal solutions to a linear program is always a convex polyhedron, which is convenient when we formulate its description. Moreover, the optimal set possesses the following property (see also Figure 2.1):

Theorem 2.1 ([15, p. 15])**.** *The set of all optimal solutions $\mathcal{S}(A, b, c)$ to a linear program forms a face of the polyhedron $\mathcal{M}(A, b)$.*

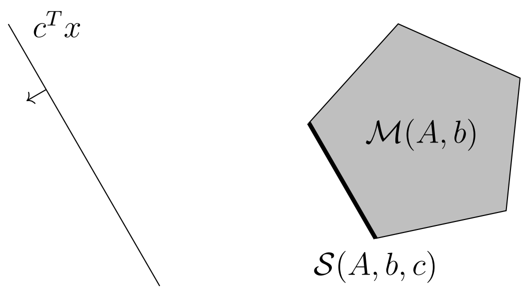

Figure 2.1: The set of all optimal solutions to a linear program.

A crucial part in the theory of linear programming is based on the notion of duality. Duality develops a relationship between optimal solutions of a given linear program and the optimal solutions of another program, and allows for an advantageous description of the optimal set. We postpone the overview of definitions and results related to the theory of duality in linear programming to Section 3.3, where we also present applications of duality in interval linear programming.

2.2 Linear programming with interval data

We now proceed to generalize Definition 2.1 to include problems with inexact coefficients. For this purpose, we will utilize interval arithmetic and algebra introduced in Chapter 1. Other approaches for working with inexact data in linear programs will be briefly reviewed in Section 2.3.

Definition 2.5. Given $\boldsymbol{A} \in \mathbb{IR}^{m \times n}, \boldsymbol{b} \in \mathbb{IR}^m, \boldsymbol{c} \in \mathbb{IR}^n$, an *interval linear program* is a family of linear programs

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & c^T x \\ \text{subject to} & x \in \mathcal{M}(A, b). \end{array}$$

with coefficients satisfying $A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}$ and the feasible set $\mathcal{M}(A, b)$ in one of the forms (I), (II) or (III). A particular linear program in such family is called a *scenario*.

If a scenario is uniquely determined by a subset of the coefficients, e.g. when some of the coefficients are fixed real values, we will also use the word "scenario" to refer to such subset.

Notation. We will also write an interval linear program (abbreviated by ILP[1]) defined by the triplet $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ as

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & \boldsymbol{c}^T x \\ \text{subject to} & x \in \mathcal{M}(\boldsymbol{A}, \boldsymbol{b}). \end{array}$$

Based on the form of the feasible set $\mathcal{M}(\boldsymbol{A}, \boldsymbol{b})$, we will refer to an ILP as a problem of type (I), (II) or (III).

Definition 2.6. An interval linear program is said to be *strongly feasible*, if it is feasible for each scenario. It is said to be *weakly feasible*, if there exists at least one feasible scenario.

For a characterization of strong and weak feasibility[2] in interval linear systems see [43]. Similarly, we can generalize other definitions of properties used in linear programming to ILPs, such as boundedness of the objective or existence of optimal solutions. An overview of results in this area can be found in [19]. Furthermore, the following chapters will be devoted to describing the properties of the optimal solution set in an ILP.

[1]Depending on the context, the abbreviation "ILP" is also commonly used to denote integer linear programs.

[2]Note that in [43], the term "feasibility" is reserved for non-negative solutions and the term "solvability" is used instead.

Notation. We will denote by $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ the set of optimal solutions to an ILP over all scenarios, i.e.,

$$\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) = \bigcup_{A \in \boldsymbol{A},\, b \in \boldsymbol{b},\, c \in \boldsymbol{c}} \mathcal{S}(A, b, c).$$

Definition 2.7. A solution $x \in \mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ is called *weakly optimal*.

Unless stated otherwise, we use the word "optimal" to refer to weakly optimal solutions. Unlike the set of all optimal solutions of a linear program, the set $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ of an ILP is, in general, not a polyhedron. This is illustrated by the following problem:

Example. Consider the interval linear program

$$\begin{array}{ll} \text{minimize} & [0,1]x_1 + x_2 \\ \text{subject to} & x_1 + x_2 \geq 2, \\ & \qquad\;\; x_1 \geq 0, \\ & \qquad\;\; x_2 \geq 1. \end{array} \tag{1}$$

The optimal solution set for the scenario determined by the objective function $0x_1 + x_2$ is the ray $(1+t, 1)$ with $t \geq 0$. For the scenario $1x_1 + x_2$, we have the optimal set

$$\mathcal{M}(A, b) \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 2\}.$$

For any other scenario $\alpha x_1 + x_2$ with $0 < \alpha < 1$, there is a unique optimal solution in the vertex $(1, 1)$. Obviously, this set is non-convex (see Figure 2.2) and therefore not a polyhedron.

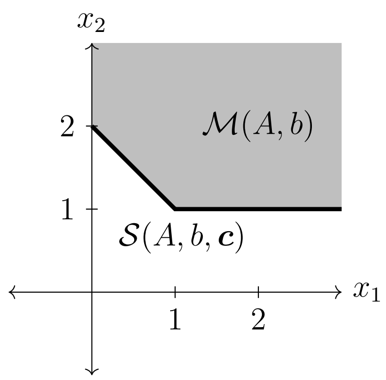

Figure 2.2: The feasible set (gray) and the set of optimal solutions (thick black) of ILP (1).

Notation. We will denote by $\underline{f}, \overline{f}$ the lower and the upper bound on the optimal objective value of an interval linear program, respectively. For an ILP defined by the triplet $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ we have

$$\underline{f}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) = \inf \{f(A, b, c) : A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}\},$$
$$\overline{f}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) = \sup \{f(A, b, c) : A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}\}.$$

Even though the set of optimal values does not always form an interval, we are often interested in the best and the worst possible value of the objective function, which is optimal for some scenario. The problem of computing the range of the optimal value $[\underline{f}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}), \overline{f}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})]$ is well-studied, and is one of the main topics adressed in interval linear programming, see [19, 42].

2.3 Other approaches to handling uncertainty

For the sake of completeness, we also briefly present other approaches to handling inexact or uncertain data in (linear) optimization.

One of the fields dealing with optimization problems affected by uncertainty is *robust optimization* [5]. The main idea of robust optimization is to consider solutions, which are feasible for any scenario of the given problem and thus are immune to perturbations in the data. Instead of solving a family of linear programs with interval coefficients, we can formulate a single optimization problem (called the robust counterpart) and compute a stable solution, which is also optimal in some sense.

Another popular approach to optimization under uncertainty is represented by *stochastic programming* [7]. The stochastic approach is useful for problems, in which the probabilistic distribution of the uncertain data is either available from the statistical data or can be estimated. While in interval linear programming we assume that the values are distributed uniformly, in some cases more complex distributions can help better capture the nature of uncertainty in the problem.

Fuzzy optimization [37] is an approach, which uses the theory of fuzzy sets to model ambiguity and vagueness present in the problem formulation. By employing membership functions, a generalization of characteristic functions for classical sets, we are able to describe the degree to which an element belongs to a fuzzy set. Such a membership function can then be used to characterize the constraints and also the objective function of an optimization problem.

In the following chapters, we will focus solely on uncertainty modelled by interval linear programs. Unlike robust optimization, our interest is the union of all optimal solution sets over all possible scenarios of a given problem. There are no further assumptions on the probability distribution of the given data. The inexactness is represented crisply by the lower and upper bounds on the interval coefficients.

3 Interval linear programming

3.1 Feasible set

We have already encountered the question of weak and strong feasibility of an interval linear program in Section 2.2. In this thesis, we will further address only weak feasibility and related weakly feasible solutions. Later, we will employ the results presented in this section to derive a description of the set of all optimal solutions.

Definition 3.1. Given $\boldsymbol{A} \in \mathbb{IR}^{m \times n}, \boldsymbol{b} \in \mathbb{IR}^m$, the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ is a family of linear systems $\{Ax = b : A \in \boldsymbol{A}, b \in \boldsymbol{b}\}$. A vector $x \in \mathbb{R}^n$ is a *weak solution* to the interval linear system $\boldsymbol{A}x = \boldsymbol{b}$ if there exist $A \in \boldsymbol{A}, b \in \boldsymbol{b}$ such that $Ax = b$.

We can formulate a similar definition for the feasible set of a system of linear interval inequalities $\boldsymbol{A}x \leq \boldsymbol{b}$. The following two theorems provide a characterization of weak solutions in interval linear systems. Theorem 3.1, which was proved in 1964 by Oettli and Prager, is one of the most important tools concerning the description of weak solutions. Theorem 3.2 is due to Gerlach (1981) and gives an analogous description of the solution set for systems of interval linear inequalities.

Theorem 3.1 (Oettli and Prager [35])**.** *Let $\boldsymbol{A} \in \mathbb{IR}^{m \times n}, \boldsymbol{b} \in \mathbb{IR}^m$ be given. A vector $x \in \mathbb{R}^n$ is a weak solution to the system $\boldsymbol{A}x = \boldsymbol{b}$ if and only if it satisfies*

$$|A_c x - b_c| \leq A_\Delta |x| + b_\Delta.$$

Theorem 3.2 (Gerlach [14])**.** *Let $\boldsymbol{A} \in \mathbb{IR}^{m \times n}, \boldsymbol{b} \in \mathbb{IR}^m$ be given. A vector $x \in \mathbb{R}^n$ is a weak solution to the system $\boldsymbol{A}x \leq \boldsymbol{b}$ if and only if it satisfies*

$$A_c x \leq A_\Delta |x| + \overline{b}.$$

These theorems also show the basic geometric properties of the feasible set of an interval system: it can be non-convex, but it becomes a convex polyhedron if we restrict the signs of the variables.

Example (Inspired by [8])**.** Consider the interval linear system

$$\begin{aligned} [-1, 1]x_1 + x_2 &= 0, \\ [-1, 1]x_2 &= 1, \\ x_1 &= [-3, 3]. \end{aligned} \tag{2}$$

By applying the Oettli–Prager theorem, we obtain a description of the feasible set by inequalities

$$\left| \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right| \leq \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix},$$

or equivalently $|x_2| \leq |x_1|$, $1 \leq |x_2|$ and $|x_1| \leq 3$. The feasible set, which is clearly non-convex, is illustrated in Figure 3.1.

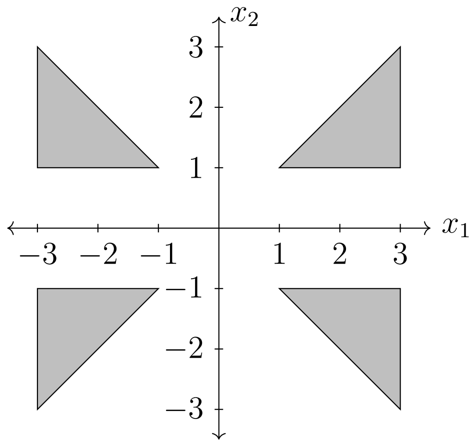

Figure 3.1: The feasible set of interval system (2).

The characterizations of weak feasible sets given by Theorem 3.1 and Theorem 3.2 imply that a system of interval linear equations or inequalities can be solved by decomposition into an exponential number of linear problems, as illustrated by Theorem 3.3. In general, there is no known polynomial-time algorithm to solve these problems, since both of them are NP-hard.

Theorem 3.3 ([43, p. 57])**.** *A vector $x \in \mathbb{R}^n$ is a weak solution to the interval system $\boldsymbol{A}x \leq \boldsymbol{b}$ if and only if it solves the linear system*

$$(A_c - A_\Delta \operatorname{diag}(p))x \leq \overline{b}$$

for some $p \in \{\pm 1\}^n$.

However, the situation is quite different for systems with non-negative variables, for which polynomial-time methods are available.

Theorem 3.4 ([43, p. 49])**.** *A vector $x \in \mathbb{R}^n$ is a weak solution to the interval system $\boldsymbol{A}x = \boldsymbol{b}, x \geq 0$ if and only if it solves the linear system*

$$\underline{A}x \leq \overline{b}, -\overline{A}x \leq -\underline{b}, x \geq 0.$$

Theorem 3.5 ([43, p. 58])**.** *A vector $x \in \mathbb{R}^n$ is a weak solution to the interval system $\boldsymbol{A}x \leq \boldsymbol{b}, x \geq 0$ if and only if it solves the linear system*

$$\underline{A}x \leq \overline{b}, x \geq 0.$$

3.2 Transformations

As already mentioned, an ILP may not have an equivalent representation in all of the forms (I), (II) and (III), due to the dependency problem described in Section 1.1.2. We will now discuss the basic transformations used in linear programming in the context of ILP.

Clearly, we can still rewrite a maximization problem into a minimization problem by taking the negative of the objective function. Similarly, the multiplication of a constraint by -1 can be used to reverse the inequality relation.

We postpone the results on the transformations of interval linear programs with a fixed coefficient matrix to Section 6.1.3. Note that in this special case, some of the transformations, which are not possible in the general case, become valid, in the sense that they preserve the optimal set.

3.2.1 Equations to inequalities

Let us consider an interval linear equation $\boldsymbol{a}^T x = \boldsymbol{b}$. Following the technique used in linear programming, we would replace the equation by two inequalities: $\boldsymbol{a}^T x \leq \boldsymbol{b}$ and $\boldsymbol{a}^T x \geq \boldsymbol{b}$. As we can see, this transformation has introduced two occurrences of the interval coefficients $\boldsymbol{a}, \boldsymbol{b}$, which are in the latter formulation independent.

Example. Consider the following interval linear program:

$$\begin{aligned} \text{maximize} \quad & x_1 \\ \text{subject to} \quad & [0,1]x_1 - x_2 = 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned} \tag{3}$$

The set of all optimal solutions of problem (3) comprises the values $x_1 \in [1, \infty)$ and $x_2 = 1$. Splitting the equation into two inequality constraints, we obtain the program

$$\begin{aligned} \text{maximize} \quad & x_1 \\ \text{subject to} \quad & [0,1]x_1 - x_2 \leq 0, \\ & [0,1]x_1 - x_2 \geq 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned} \tag{4}$$

Let us now look at the scenario with inequalities $1x_1 - x_2 \leq 0$ and $0x_1 - x_2 \geq 0$. It is obvious that the only feasible solution (and therefore also the optimal solution) is the vector $(0, 0)$. This shows that problem (4) is, indeed, a relaxation of the original problem.

However, if we are only interested in the set of all feasible solutions to an interval system, this type of transformation may still be usable. By applying Theorem 3.1 to the equation $\boldsymbol{a}^T x = \boldsymbol{b}$, we obtain the following description of the weak solution set:

$$\begin{aligned} a_c x - b_c &\leq a_\Delta |x| + b_\Delta, \\ -a_c x + b_c &\leq a_\Delta |x| + b_\Delta. \end{aligned}$$

If we compare this description to the one obtained by applying Theorem 3.2 to the two inequalities $\boldsymbol{a}^T x \leq \boldsymbol{b}, \boldsymbol{a}^T x \geq \boldsymbol{b}$:

$$\begin{aligned} a_c^T x &\leq a_\Delta^T |x| + \overline{b}, \\ -a_c^T x &\leq a_\Delta^T |x| - \underline{b}, \end{aligned}$$

we can see that both descriptions are identical. Therefore, it is possible to use this type of transformation when studying the weak feasible set, although it changes the structure of the interval system. This type of transformation was also addressed in [27].

3.2.2 Inequalities to equations

The transformation of an inequality constraint into an equation by adding slack variables does not break any dependencies, and can therefore be used in the same manner as in linear programming. A system of interval linear inequalities in the form $\boldsymbol{A}x \leq \boldsymbol{b}$ can be equivalently restated (with respect to the set of weak feasible solutions) as the interval system with non-negativity constraints

$$\begin{aligned} \boldsymbol{A}x + Iy &= \boldsymbol{b}, \\ y &\geq 0. \end{aligned}$$

3.2.3 Non-negative variables

In linear programming, an often used trick is to replace an unrestricted variable $x \in \mathbb{R}$ by the difference of two non-negative variables x^+, x^-, thus introducing the substitution $x = x^+ - x^-$. However, this substitution is not always possible in interval linear programming, since we would choose the values from the interval coefficients of x^+ and x^- independently. The following example provides a better insight into the nature of dependency in this context and shows that this type of transformation can change the feasible set of an interval system.

Example. Consider the interval system

$$[1,2]x \leq 0, \tag{5}$$

and its transformation

$$\begin{aligned} [1,2]x^+ - [1,2]x^- &\leq 0, \\ x^+, x^- &\geq 0. \end{aligned} \tag{6}$$

The point $x^+ = 2, x^- = 1$ solves system (6) for the scenario $(1,2)$, since it holds that $1 \cdot 2 - 2 \cdot 1 = 0$. Yet, it is easy to see that $x = x^+ - x^- = 1$ does not belong to the solution set of the original system (5).

3.3 Duality

Recall that the optimal solution set $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ of an ILP is the union of all optimal solutions over all possible scenarios. We will now review an important notion used in linear programming: the principle of duality. Together with Theorem 3.1 and Theorem 3.2, giving the characterization of the solution sets in interval linear systems, we will use duality to derive a characterization and an approximation of the set $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$. This approach was also presented in [19]. Let us begin by defining the dual counterpart of a linear program.

Definition 3.2. Given a linear programming problem in the form (LP) (called *primal*), we define the *dual* problem as the linear program

$$\begin{aligned} \underset{y \in \mathbb{R}^m}{\text{maximize}} \quad & b^T y \\ \text{subject to} \quad & A^T y \leq c. \end{aligned} \tag{LP$_d$}$$

Similarly, we can define a dual interval linear program to an ILP. For example, a dual to an ILP of type (I) is the problem

$$\begin{aligned} \underset{y \in \mathbb{R}^m}{\text{maximize}} \quad & \boldsymbol{b}^T y \\ \text{subject to} \quad & \boldsymbol{A}^T y \leq \boldsymbol{c}. \end{aligned}$$

The following theorems summarize the relationship between the primal and the dual problem and show the importance of duality in linear programming. The proofs of Theorem 3.6 and Theorem 3.7 can be found in [42].

Theorem 3.6 (Weak duality). *Let $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ be feasible solutions of the primal and dual problem, respectively. Then the inequality $c^T x \geq b^T y$ holds.*

Theorem 3.7 (Strong duality). *Consider a linear program* (LP) *along with its dual* (LP_d) *and let $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ be feasible solutions to the respective problems. Then (x, y) is a pair of primal and dual optimal solutions if and only if*

$$c^T x = b^T y,$$

and also, if and only if

$$x^T(c - A^T y) = 0.$$

Using the principle of duality, we can show that for each linear program, one of the following situations occurs:

- (LP) is infeasible and (LP_d) is unbounded,
- (LP) is unbounded and (LP_d) is infeasible,
- both problems are infeasible,
- both problems are feasible and have optimal solutions with the same optimal objective value.

Let us now use the weak and strong duality properties to derive a description of the optimal solution set of a linear program by means of a system of linear equations and inequalities.

Corollary 3.8. *Let $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ be given. Then x is an optimal solution to the primal problem* (LP) *and y is an optimal solution to the dual problem* (LP_d) *if and only if they solve the system*

$$\begin{aligned} & c^T x = b^T y, \\ & Ax = b,\ x \geq 0, \\ & A^T y \leq c. \end{aligned} \tag{7}$$

We will now use the theory of duality in linear programming to derive a characterization of the optimal solution set of an interval linear program, without any further assumptions about the structure or properties of the problem. As we have seen, duality can be used to transform the problem of finding the set of optimal solutions to a linear program into a linear feasibility problem. Using

Corollary 3.8, we employ a similar technique for finding the optimal solution set of an ILP.

Let us now consider an ILP of type (I), i.e.,

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & \boldsymbol{c}^T x \\ \text{subject to} \quad & \boldsymbol{A}x = \boldsymbol{b}, \\ & x \geq 0. \end{aligned}$$

The optimal solutions to every linear program in this family can be characterized as x-solutions to system (7). By formulating an interval counterpart of this system, we introduce an overestimation of the optimal solution set due to the dependency problem. Therefore, the optimal set $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ is a subset of the set of x-solutions to the interval system

$$\boldsymbol{c}^T x = \boldsymbol{b}^T y,\ \boldsymbol{A}x = \boldsymbol{b},\ x \geq 0,\ \boldsymbol{A}^T y \leq \boldsymbol{c}.$$

Using Theorem 3.1 and Theorem 3.2, it is possible to rewrite this system as

$$\begin{aligned} &|c_c^T x - b_c^T y| \leq c_\Delta^T |x| + b_\Delta^T |y|, \\ &|A_c x - b_c| \leq A_\Delta |x| + b_\Delta, x \geq 0, \\ &A_c^T y - A_\Delta^T |y| \leq \overline{c}. \end{aligned}$$

We can further simplify the description by employing the fact that x is non-negative, therefore we have $|x| = x$. The inequality $|A_c x - b_c| \leq A_\Delta x + b_\Delta$ can then be restated as

$$-A_\Delta x - b_\Delta \leq A_c x - b_c \leq A_\Delta x + b_\Delta,$$

which is by definition of the central and radial matrix equivalent to $-\overline{A}x \leq -\underline{b}$ and $\underline{A}x \leq \overline{b}$. Similarly, we can split the objective value constraint into two inequalities, which yields the following characterization:

$$\begin{aligned} &c_c^T x - b_c^T y \leq c_\Delta^T x + b_\Delta^T |y|, \\ &c_c^T x - b_c^T y \geq -c_\Delta^T x - b_\Delta^T |y|, \\ &\underline{A}x \leq \overline{b}, -\overline{A}x \leq -\underline{b}, x \geq 0, \\ &A_c^T y - A_\Delta^T |y| \leq \overline{c}. \end{aligned} \tag{8}$$

We can also formulate a similar characterization for an ILP of type (II), based on the following primal–dual system:

$$\boldsymbol{c}^T x = \boldsymbol{b}^T y,\ \boldsymbol{A}x \leq \boldsymbol{b},\ \boldsymbol{A}^T y = \boldsymbol{c},\ y \leq 0.$$

The resulting system is

$$\begin{aligned} &c_c^T x - b_c^T y \leq c_\Delta^T |x| - b_\Delta^T y, \\ &c_c^T x - b_c^T y \geq -c_\Delta^T |x| + b_\Delta^T y, \\ &A_c x - A_\Delta |x| \leq \overline{b}, \\ &\overline{A}^T y \leq \overline{c}, -\underline{A}^T y \leq -\underline{c}, y \leq 0. \end{aligned}$$

For an ILP of type (III), we have the interval system

$$\boldsymbol{c}^T x = \boldsymbol{b}^T y,\ \boldsymbol{A}x \le \boldsymbol{b},\ x \ge 0,\ \boldsymbol{A}^T y \le \boldsymbol{c},\ y \le 0$$

and the simplified reformulation

$$\begin{aligned} &\underline{c}^T x - \overline{b}^T y \le 0, -\overline{c}^T x + \underline{b}^T y \le 0,\\ &\underline{A}x \le \overline{b}, x \ge 0,\\ &\overline{A}^T y \le \overline{c}, y \le 0. \end{aligned} \tag{9}$$

Notice that system (9) consists of linear constraints only. This is thanks to the sign restrictions $x \ge 0$ and $y \le 0$, which allowed us to express the absolute values. If we were able to impose such a restriction on the free variables in the previous cases, we would also obtain a purely linear system. This idea leads to a decomposition method for dealing with the non-linearity caused by the absolute values, which is further discussed in Section 4.1.

3.4 Basis stability

The set of all optimal solutions of an interval linear program may be difficult to describe exactly, in general. However, under some additional assumptions, we may be able to obtain a simple explicit description. In this section, we consider a class of ILPs having a stable basis, which is optimal for every scenario.

In the following definitions, we consider an (interval) linear program of type (I). First, we review the definition of a basis used in conventional linear programming and the conditions for its optimality.

Definition 3.3. A matrix $A \in \mathbb{R}^{n\times n}$ is called *non-singular*, if there exist a matrix $B \in \mathbb{R}^{n\times n}$ such that $AB = BA = I_n$. Otherwise, the matrix A is *singular*. An interval matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ is called *regular*, if each $A \in \boldsymbol{A}$ is non-singular. If there exists a singular matrix $A \in \boldsymbol{A}$, then $\boldsymbol{A}$ is said to be *singular*.

Notation. Given a matrix $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$ and a set $B \subseteq \{1,\dots,n\}$, we denote by the symbol $\boldsymbol{A}_B$ the restriction of $\boldsymbol{A}$ to the columns indexed by elements of the set B.

Definition 3.4. Let a linear program be given by the triplet (A, b, c). The index set $B \subseteq \{1,\dots,n\}$ is said to be a *basis*, if the matrix A_B is non-singular. Furthermore, B is called *feasible*, if $A_B^{-1}b \ge 0$ holds and it is called *optimal*, if it is feasible and for $N = \{1,\dots,n\}\backslash B$ we have

$$c_N^T - c_B^T A_B^{-1} A_N \ge 0.$$

A basic solution (x_B, x_N) with $x_B = A_B^{-1}b$ and $x_N = 0$ is called *non-degenerate*, if $x_B > 0$ holds.

The feasibility condition implies that the basic solution defined as $x_B = A_B^{-1}b$, $x_N = 0$ is feasible and, moreover, the optimality condition ensures that such a solution is also optimal. Theorem 3.9, which is also commonly referred to as "the fundamental theorem of linear programming", emphasizes the importance of basic solutions in linear programming. It also provides the idea behind the simplex method, which is one of the main algorithms used for solving linear programs.

Theorem 3.9 ([29, p. 21]). *Let a linear program be given in the form* (LP)*, where $A \in \mathbb{R}^{m \times n}$ has full row rank m. Then, the following properties hold:*

a) if there is a feasible solution, then there also exists a basic feasible solution,

b) if there is an optimal solution, then there also exists a basic optimal solution.

Let us now extend the definition of a basis to the interval case, by introducing the related concept of basis stability.

Definition 3.5. Let a basis $B \subseteq \{1, \ldots, n\}$ be given. An ILP is said to be *B-stable*, if B is an optimal basis for each scenario of the ILP. Furthermore, it is called *non-degenerate B-stable*, if each scenario has a non-degenerate optimal basic solution x with the basis B, and it is called *unique B-stable* if it is B-stable and the optimal solution in each scenario is unique.

Basis stability is an important notion in interval linear programming, since it allows for a simplification of many fundamental problems. When assuming a stronger type of stability, so-called unique B-stability, we obtain an exact description of the optimal solution set by a system of linear inequalities. If we consider a unique B-stable ILP with a basis B, then the set of optimal solutions is the set of feasible solutions to the interval system

$$\begin{aligned} &\boldsymbol{A}_B x = \boldsymbol{b}, \\ &x_B \geq 0, x_N = 0. \end{aligned}$$

Theorem 3.10 ([19]). *Let an ILP of type (I) be given by the triplet $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$. If there exists a basis B, such that the ILP is unique B-stable, then the optimal solution set $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ can be described by the linear system*

$$\begin{aligned} &\underline{A}_B x_B \leq \overline{b},\ \overline{A}_B x_B \geq \underline{b}, \\ &x_B \geq 0,\ x_N = 0. \end{aligned} \tag{10}$$

If the ILP is B-stable, then each solution in the set described by (10) *is optimal for some scenario, and conversely, each scenario has at least one optimal solution contained in this set.*

In order to exploit the characterization of the optimal solution set under basis stability given by Theorem 3.10, we need to be able to test the existence of a stable basis for a given problem. Unfortunately, the problem of checking basis stability is co-NP-hard, and it can take exponential time to perform such a test. However, there exist methods which work well on large subclasses of interval linear programs, see for example [21].

Let an index set $B \subseteq \{1, \ldots, n\}$ be given. By Definition 3.4 and Definition 3.5, the problem of testing B-stability comprises three subproblems:

- testing non-singularity of the restricted matrix A_B,
- verifying feasibility of the basis, i.e. $A_B^{-1} b \geq 0$,
- verifying optimality of the basis, i.e. $c_N^T - c_B^T A_B^{-1} A_N \geq 0$,

for each $A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}$. While testing regularity of an interval matrix $\boldsymbol{A}_B$ is also a co-NP-hard problem [36], it is well-studied, and many characterizations of regularity are known (see [44] for an exhaustive list of characterizations). Furthermore, there are also some sufficient conditions, which may be efficiently tested [38]. For conditions on feasibility and optimality of the basis, see [21].

Conditions for testing stronger types of basis stability were also studied. Theorem 3.11 states a characterization of (unique) non-degenerate B-stability proposed by Rohn [40] by checking the property for a finite subset of scenarios of the given problem. Additionally, some sufficient conditions were also derived by Koníčková [25].

Theorem 3.11 (Rohn [40])**.** *Given an ILP of type (I) and a basis $B \subseteq \{1, \ldots, n\}$, the ILP is (unique) non-degenerate B-stable if the property holds for every scenario in the form*

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & (c_c + \operatorname{diag}(q) c_\Delta)^T x \\ \text{subject to} \quad & (A_c - \operatorname{diag}(p) A_\Delta \operatorname{diag}(q)) x = b_c + \operatorname{diag}(p) b_\Delta, \\ & x \geq 0, \end{aligned}$$

where $p \in \{\pm 1\}^m$ and $q \in \{\pm 1\}^n$ with $q_j = 1$ for each $j \notin B$.

4 Approximating the optimal set

4.1 Orthant decomposition

Since computing the exact set of optimal solutions of an ILP may be difficult, we can also search for a tight approximation in the form of an interval hull or enclosure. This can be achieved by adapting standard linear programming algorithms to involve interval arithmetic [23], but there also exist other techniques developed for the purposes of ILP [2, 47].

In this chapter, we present some approximation methods based on the primal–dual description introduced in Section 3.3. For simplicity, let us assume that the optimal solution set is bounded. The first method employs the idea of decomposing the non-linear description of $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ into several linear programs, in order to obtain the interval hull of the primal–dual system.

Definition 4.1. Let $s \in \{\pm 1\}^n$, an *orthant* defined by s is the set

$$\{x \in \mathbb{R}^n : \operatorname{diag}(s)x \geq 0\}.$$

The vector s is called the *signature* of the orthant.

Let us consider an ILP of type (I). We proceed by restricting the characterization obtained in system (8) to a single orthant, yielding a system of linear inequalities. For an orthant with the signature s, we substitute $\operatorname{diag}(s)y$ for $|y|$ and add the constraint $\operatorname{diag}(s)y \geq 0$. We obtain the following system:

$$\begin{aligned}
&c_c^T x - b_c^T y \leq c_\Delta^T x + b_\Delta^T \operatorname{diag}(s)y,\\
&c_c^T x - b_c^T y \geq -c_\Delta^T x - b_\Delta^T \operatorname{diag}(s)y,\\
&\underline{A}x \leq \overline{b}, -\overline{A}x \leq -\underline{b}, x \geq 0,\\
&A_c^T y - A_\Delta^T \operatorname{diag}(s)y \leq \overline{c},\\
&\operatorname{diag}(s)y \geq 0.
\end{aligned} \tag{11}$$

To systematically generate all possible signatures, we can use Algorithm 1, in which any two successively generated vectors differ by exactly one entry. A proof of correctness of the algorithm can be found in [43].

Algorithm 1 Generating $\{\pm 1\}^n$

$z \leftarrow (0, \dots, 0) \in \mathbb{R}^n$, $y \leftarrow (1, \dots, 1) \in \mathbb{R}^n$, $Y \leftarrow \{y\}$
while $z \neq (1, \dots, 1)$ **do**
  $k \leftarrow \min\{i : z_i = 0\}$
  for all $i \in \{1, \dots, k-1\}$ **do**
    $z_i \leftarrow 0$
  $z_k \leftarrow 1$, $y_k \leftarrow -y_k$
  $Y \leftarrow Y \cup \{y\}$
return Y

As presented so far, the method provides an approximation of the optimal set by means of a union of convex polyhedra. In order to find an interval enclosure of the optimal set, we can compute the interval hull of the union of the feasible sets from (11). This can be achieved by solving $2n$ linear programs with objective functions minimize x_i and maximize x_i for each $i \in \{1, \dots, n\}$ and finding the overall minimal and maximal values for each variable. For an ILP of type (I) this amounts to computing $2n$ linear programs for each of the 2^m orthants, for an ILP of type (II) we have $2n \cdot 2^n$ linear programs.

4.2 Iterative contractor

The orthant decomposition is based on an exponential characterization of the optimal set. However, sometimes it may be desirable to sacrifice preciseness of the approximation in order to reduce the computation time needed. In this section, we present a polynomial-time iterative algorithm for computing an interval enclosure of $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ by Hladík [20], based on a linearization of the primal–dual description. We will focus on computing the optimal set for an ILP of type (I), however, the method can easily be adapted for a problem of type (II) as well.

Recall the description of the optimal solution set for an ILP of type (I) by system (8):

$$\begin{aligned}
&|c_c^T x - b_c^T y| \le c_\Delta^T x + b_\Delta^T |y|, \\
&\underline{A}x \le \overline{b}, -\overline{A}x \le -\underline{b}, x \ge 0, \\
&A_c^T y - A_\Delta^T |y| \le \overline{c}.
\end{aligned}$$

To replace the term $|y|$ in the constraints by a linear approximation, we will use a theorem by Beaumont providing an upper bound on the absolute value (for an illustration of the theorem, see Figure 4.1).

Theorem 4.1 (Beaumont [3]). *Let $\boldsymbol{y} = [\underline{y}, \overline{y}] \in \mathbb{IR}$ with $\underline{y} < \overline{y}$. Then for every $y \in \boldsymbol{y}$ it holds that*

$$|y| \le \alpha y + \beta, \tag{12}$$

where

$$\alpha = \frac{|\overline{y}| - |\underline{y}|}{\overline{y} - \underline{y}}, \quad \beta = \frac{\overline{y}|\underline{y}| - \underline{y}|\overline{y}|}{\overline{y} - \underline{y}}.$$

Moreover, if $\underline{y} \ge 0$ or $\overline{y} \le 0$, then (12) holds as equation.

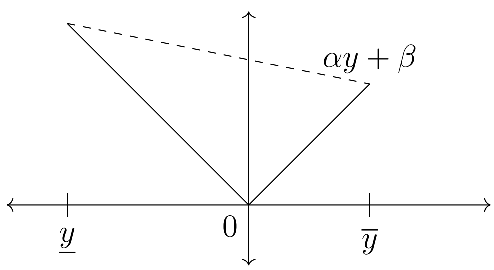

Figure 4.1: Illustration of Beaumont's theorem.

Since we have $A_\Delta^T|y| \ge 0$ and $b_\Delta^T|y| \ge 0$, replacing the absolute value by an upper bound yields a relaxation of the constraints. For a vector $y \in \boldsymbol{y}$ with

$\boldsymbol{y} \in \mathbb{IR}^m$, we define the coefficients α_i, β_i for $i \in \{1, \dots, m\}$ as follows:

$$\begin{aligned} \alpha_i &= \begin{cases} \dfrac{|\overline{y}_i| - |\underline{y}_i|}{\overline{y}_i - \underline{y}_i} & \text{if } \underline{y}_i < \overline{y}_i, \\ \operatorname{sgn}(\overline{y}_i) & \text{if } \underline{y}_i = \overline{y}_i, \end{cases} \\ \beta_i &= \begin{cases} \dfrac{\overline{y}_i|\underline{y}_i| - \underline{y}_i|\overline{y}_i|}{\overline{y}_i - \underline{y}_i} & \text{if } \underline{y}_i < \overline{y}_i, \\ 0 & \text{if } \underline{y}_i = \overline{y}_i. \end{cases} \end{aligned} \tag{13}$$

Let $\alpha = (\alpha_1, \dots, \alpha_m)^T$ and $\beta = (\beta_1, \dots, \beta_m)^T$ be coefficient vectors defined by (13). Using Theorem 4.1, we can rewrite the absolute value of the dual variable in the constraints as

$$|c_c^T x - b_c^T y| \le c_\Delta^T x + b_\Delta^T \operatorname{diag}(\alpha) y + b_\Delta^T \beta, \tag{14}$$

$$(A_c^T - A_\Delta^T \operatorname{diag}(\alpha)) y \le \overline{c} + A_\Delta^T \beta. \tag{15}$$

Furthermore, we can split the absolute value in (14) into two linear inequalities. It is possible to further simplify the system, by using the definition of central and radial vectors, which yields the following:

$$\begin{aligned} &\underline{c}^T x + (-b_c^T - b_\Delta^T \operatorname{diag}(\alpha)) y \le b_\Delta^T \beta, \\ &-\overline{c}^T x + (b_c^T - b_\Delta^T \operatorname{diag}(\alpha)) y \le b_\Delta^T \beta. \end{aligned}$$

We have thus obtained a linearized version of the original system (8). Summarizing the results, an enclosure of the optimal solution set can be computed as the interval hull of the linear system

$$\begin{aligned} &\underline{c}^T x + (-b_c^T - b_\Delta^T \operatorname{diag}(\alpha)) y \le b_\Delta^T \beta, \\ &-\overline{c}^T x + (b_c^T - b_\Delta^T \operatorname{diag}(\alpha)) y \le b_\Delta^T \beta, \\ &\underline{A} x \le \overline{b}, -\overline{A} x \le -\underline{b}, x \ge 0, \\ &(A_c^T - A_\Delta^T \operatorname{diag}(\alpha)) y \le \overline{c} + A_\Delta^T \beta. \end{aligned} \tag{16}$$

For the initialization of an iterative algorithm, we need to provide an interval enclosure $(\boldsymbol{x}^0, \boldsymbol{y}^0)$ of system (8). Since x is a non-negative variable, we can use for the initial enclosure an interval box in the form

$$\begin{aligned} \boldsymbol{x}^0 &= ([0, K], \dots, [0, K])^T, \\ \boldsymbol{y}^0 &= ([-K, K], \dots, [-K, K])^T, \end{aligned}$$

for a sufficiently large $K \gg 0$. The choice of an appropriate constant K is further discussed in [20].

Using the initial enclosure, we can calculate the coefficient vectors α, β and compute the interval hull of system (16). If this step yields a significant contraction of the enclosure, we can iterate the procedure and calculate new coefficients to obtain an even tighter enclosure (see Algorithm 2).

Algorithm 2 Optimal solution set contractor

$(\boldsymbol{x}^0, \boldsymbol{y}^0) \leftarrow$ an initial interval enclosure of (8)
$i \leftarrow 0$
repeat
    $\alpha, \beta \leftarrow$ coefficients defined in (13) for $\boldsymbol{y}^i$
    $i \leftarrow i + 1$
    $(\boldsymbol{x}^i, \boldsymbol{y}^i) \leftarrow$ interval hull of (16)
until improvement is non-significant
return $\boldsymbol{x}^i$

4.3 Decomposition by complementarity

In this section, we discuss the idea of approximating the optimal solution set using a decomposition method based on the complementarity constraint in the primal–dual description. This approach also provides an exact description of the optimal set for a special class of interval linear programs with a fixed coefficient matrix.

Again, we consider an ILP of type (I). By Theorem 3.7, we can formulate the following parametric description of the optimal solution set:

$$\begin{aligned}
&Ax = b,\ x \geq 0,\\
&A^T y \leq c,\\
&x^T(c - A^T y) = 0,\\
&A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c},
\end{aligned}$$

which can also be viewed as a non-linear system, with the variables A, b and c constrained by the respective lower and upper bounds (this idea is also further developed in the proof of Theorem 6.1). Using non-negativity of the variable x and the inequality $c - A^T y \geq 0$, we can see that the constraint $x^T(c - A^T y) = 0$ is satisfied if and only if $x_i = 0$ or $(c - A^T y)_i = 0$ holds for each $i \in \{1, \ldots, n\}$. This implies that for a fixed subset $I \subseteq \{1, \ldots, n\}$ with $x_i = 0$ for $i \in I$, we only need to consider the primal and dual feasibility conditions with the remaining equation constraints from the complementarity condition to obtain the corresponding subset of optimal solutions. In other words, we need to solve the 2^n problems in the form

$$\begin{aligned}
&Ax = b, &&\\
&x_i = 0, && \text{for } i \in I,\\
&x_j \geq 0, && \text{for } j \notin I,\\
&(A^T y)_i \leq c_i, && \text{for } i \in I,\\
&(A^T y)_j = c_j, && \text{for } j \notin I,\\
&A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}.
\end{aligned} \tag{17}$$

Consider now a special case of ILP, in which the entries of the matrix A are only degenerate intervals. In this case, we can fix the value of the variable A, thus reducing system (17) to a linear problem. Therefore, we can obtain the exact optimal set of a linear program with interval objective and right-hand side by solving 2^n linear subproblems. Similarly, we can also compute the exact interval hull of the optimal solution set.

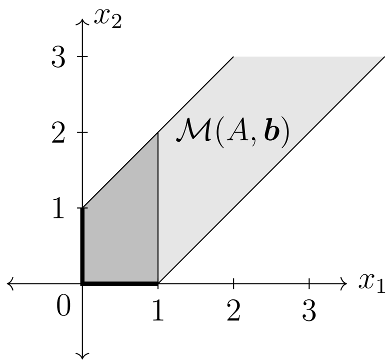

Figure 4.2: The optimal set of ILP (18) (thick black) and its approximation obtained by orthant decomposition (dark gray).

Example. Consider the ILP

$$\begin{aligned} \text{minimize} \quad & x_1 \\ \text{subject to} \quad & x_1 - x_2 = [-1, 1], \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned} \tag{18}$$

When using orthant decomposition presented in Section 4.1, we approximate the optimal solution set of (18) by the union of feasible sets of linear systems in the form

$$\begin{aligned} & x_1 \leq sy, \; x_1 \geq -sy, \\ & x_1 - x_2 \leq 1, \; -x_1 + x_2 \leq 1, \; x_1 \geq 0, \; x_2 \geq 0, \\ & y \leq 1, \; -y \leq 0, \\ & sy \geq 0, \end{aligned}$$

with $s \in \{-1, 1\}$. For the choice $s = -1$, we have $y = 0$ and the feasible set of x-solutions is formed by all pairs (x_1, x_2) with $x_1 = 0$ and $x_2 \in [0, 1]$. In the case of $s = 1$, we obtain the set described by $x_1 \in [0, 1]$, $x_2 \geq 0$ and $x_1 - x_2 \in [-1, 1]$. Due to the dependency problem, this set also contains solutions, which are not optimal for the original ILP (see Figure 4.2). Even if we only consider the interval enclosure of the optimal set generated by orthant decomposition to the exact interval hull, it is still an overestimation of the exact interval hull.

Let us now consider the general case. For an ILP with an interval coefficient matrix $\boldsymbol{A}$, the system (17) remains non-linear. In order to simplify the problem, we can formulate an interval relaxation of the system by breaking the dependencies between multiple occurrences of the variable A. Moreover, since we have fixed the value of some of the primal variables to 0, the corresponding columns of the system $Ax = b$ are also equal to 0 independent of the coefficient value. Therefore, the only relaxed dependencies are present for indices $j \notin I$.

Furthermore, let us examine the constraints

$$\begin{aligned} (\boldsymbol{A}^T y)_i &\leq \boldsymbol{c}_i, && \text{for } i \in I, \\ (\boldsymbol{A}^T y)_j &= \boldsymbol{c}_j, && \text{for } j \notin I. \end{aligned} \tag{19}$$

Since we are only interested in the optimal solution set, which is formed by the projection of the feasible set onto the primal variables, we only need to test weak feasibility of system (19). If there are no feasible solutions, then the interval relaxation is also strongly infeasible. Otherwise, we can fix a feasible scenario and solve the remaining primal constraints.

Unfortunately, testing weak feasibility is also difficult due to the fact that the variables in system (19) are unrestricted. However, we can again employ the linear approximation introduced in Theorem 4.1, thus creating another relaxation. This leads to a general method for approximating the optimal solution set of an ILP, which is exponential in the number of variables. Nevertheless, it may provide a tighter enclosure than the orthant decomposition method.

5 Properties of the optimal set

5.1 Closedness

In this chapter, we study some topological, metric and geometric properties of the optimal solution set $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$. The theoretical results are accompanied by examples of interval programs, which illustrate the sources of complexity in the problem of finding a description of the optimal set. Some stronger results, which are only proved for special classes of interval linear programs, are presented in Chapter 6. Hereinafter, we consider the space $\mathbb{R}^n$ equipped with the Euclidean distance

$$d(x, s) = \sqrt{\sum_{i=1}^{n} (x_i - s_i)^2}.$$

The first studied property is closedness. There are several equivalent ways of characterizing a closed set. One of the commonly used definitions characterizes closed sets by introducing the complementary notion of an open set.

Definition 5.1. A set $M \subseteq \mathbb{R}^n$ is said to be *open*, if for every $x \in M$ there exists a real number $\varepsilon > 0$, such that every $y \in \mathbb{R}^n$ satisfying $d(x, y) < \varepsilon$ belongs to M. A set $M \subseteq \mathbb{R}^n$ is said to be *closed*, if the complement of M is an open set.

Note that the properties of being open and closed are not mutually exclusive, i.e., a set may be both open and closed (such as the empty set). Moreover, a set may be neither open nor closed. Let us first motivate this study by an example of a problem showing that if we allow unbounded intervals, obtaining an optimal set that is not closed is, indeed, possible.

Example. Consider the optimization problem

$$\begin{array}{ll} \text{maximize} & x \\ \text{subject to} & [1, \infty) x \leq 1. \end{array}$$

If we fix a scenario $\alpha x \leq 1$ with $\alpha \in [1, \infty)$, the optimal solution of the corresponding linear program will be $x = \frac{1}{\alpha}$. Therefore, the optimal solution set of the problem is the half-open interval $(0, 1]$, which is not a closed set.

Before we proceed to formulating a theorem about closedness of $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ under additional assumptions, we state some properties of continuous functions, which will be needed for the proof. The proofs of the following lemmas, as well as a detailed introduction to topology, can be found in [34].

Lemma 5.1. *Let a function $f : \mathbb{R}^n \to \mathbb{R}^m$ be defined by the equation*

$$f(x) = (f_i(x))_{i=1}^m,$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$. Then the function f is continuous if and only if each component function f_i is continuous.

Lemma 5.2. *Let X, Y be topological spaces and let $f : X \to Y$ be given. The function f is continuous if and only if for every closed set $M \subseteq Y$, the preimage $f^{-1}(M)$ is closed in X.*

Lemma 5.3. *Let X, Y be topological spaces and assume Y is compact. Consider the projection $\pi_X : X \times Y \to X$. If $M \subseteq X \times Y$ is a closed set, then $\pi_X(M)$ is a closed subset of X.*

For the forthcoming statement of Theorem 5.4, assume we have an ILP of type (I):

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & \boldsymbol{c}^T x \\ \text{subject to} \quad & \boldsymbol{A} x = \boldsymbol{b}, \\ & x \geq 0, \end{aligned} \tag{ILP$_p$}$$

and its interval dual:

$$\begin{aligned} \underset{y \in \mathbb{R}^m}{\text{maximize}} \quad & \boldsymbol{b}^T y \\ \text{subject to} \quad & \boldsymbol{A}^T y \leq \boldsymbol{c}. \end{aligned} \tag{ILP$_d$}$$

The proof of the theorem can easily be adapted for other types of ILPs as well. However, it remains an open question, whether it is possible to weaken or even drop the assumption on the boundedness of the dual optimal set and state the theorem in general.

Theorem 5.4. *Assume the set of optimal solutions of* (ILP$_d$) *is bounded. Then the set of optimal solutions of* (ILP$_p$) *is closed.*

Proof. Using strong duality presented in Theorem 3.7, we can characterize the set of all optimal solutions of (ILP$_p$) by the following parametric linear system:

$$\begin{aligned} & c^T x = b^T y, \\ & Ax = b, \ x \geq 0, \\ & A^T y \leq c, \\ & A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}. \end{aligned}$$

We can also view this system as a non-linear system by replacing the parameters with new variables, without affecting the x-projection of the set of feasible solutions. Applying this approach, we obtain the system

$$\begin{aligned} & c^T x = b^T y, \\ & Ax = b, \ x \geq 0, \\ & A^T y \leq c, \\ & \underline{A} \leq A \leq \overline{A}, \ \underline{b} \leq b \leq \overline{b}, \ \underline{c} \leq c \leq \overline{c}. \end{aligned} \tag{20}$$

We will now prove that each of the constraints in (20) describes a closed set. Since the feasible set $\mathcal{M}$ of system (20) is a finite intersection of the sets defined by the individual constraints, this argument will also yield its closedness. Let us consider the function $g(A, x, b) := Ax - b$. All components of g are polynomials of the form

$$\sum_{k=1}^{n} A_{ik} x_k - b_i$$

for $i \in \{1, \ldots, m\}$ and therefore continuous functions. By Lemma 5.1, the function g is also continuous. The set of all triplets (A, x, b) satisfying $Ax = b$, or equivalently $Ax - b = 0$, can also be viewed as the preimage of $\{0\}$ under g.

Using Lemma 5.2 and the fact that the set $\{0\}$ is closed, we can also infer the closedness of the set

$$\{(A, x, b) \in \mathbb{R}^{m \times n + n + m} : Ax = b\},$$

and of the set of all quintuples (A, b, c, x, y) satisfying $Ax = b$. By a similar argument, we can also prove closedness of the sets defined by the constraints $A^T y \leq c$ and $c^T x = b^T y$. It is easy to see that the remaining inequality constraints in (20) also define closed sets, so the feasible set $\mathcal{M}$ is closed.

The set of all dual optimal solutions is bounded by assumption, and thus we can restrict the space of dual variables y to a compact subset Y of $\mathbb{R}^m$ (e.g. an interval envelope). Consider the set $\mathcal{M}$ as a subset of the space

$$Z := \mathbf{A} \times \mathbf{b} \times \mathbf{c} \times \mathbb{R}^n \times Y$$

and let $\pi_x : Z \to \mathbb{R}^n$ denote the projection into the space of primal variables x. Since $\mathbf{A}, \mathbf{b}, \mathbf{c}$ and Y are compact, their product is also compact and π_x is a closed map by Lemma 5.3. Therefore, the set $\pi_x(\mathcal{M}) = \mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ is also closed. $\square$

Clearly, Theorem 5.4 holds under the assumption that the dual feasible set is bounded. Other sufficient conditions for boundedness of the optimal set will be presented in Section 5.2.

5.2 Boundedness

This section is devoted to another important set property called boundedness. Boundedness is a property related to the "size" of a set, bounded sets are therefore usually defined in the context of metric spaces. Let us formalize this notion for sets in $\mathbb{R}^n$ equipped with the Euclidean metric.

Definition 5.2. A set $M \subseteq \mathbb{R}^n$ is said to be *bounded*, if there exists a point $s \in \mathbb{R}^n$ and a real number $r > 0$, such that every $x \in M$ satisfies $d(x, s) < r$.

Since the optimal solution set of a classical linear program may be unbounded, the set $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ may be unbounded as well. However, it is also possible for an ILP to have an unbounded optimal solution set, even if the optimal set of each scenario is bounded.

Example. Consider the following ILP:

$$\begin{array}{ll} \text{maximize} & x \\ \text{subject to} & [0, 1]x \leq 1. \end{array} \tag{21}$$

For the scenario $0x \leq 1$, the feasible solution set is $\mathbb{R}$, the corresponding linear program is unbounded and thus it has no optimal solutions. For any other scenario, we have a constraint in the form $x \leq \frac{1}{\alpha}$ for some $\alpha \in (0, 1]$ and the corresponding linear program has a unique optimal solution $x = \frac{1}{\alpha}$. Even though each scenario has a bounded (possibly empty) set of optimal solutions, the united optimal solution set is the unbounded interval $[1, \infty)$.

Note that ILP (21) in the example above is not strongly optimal, i.e. not all scenarios have non-empty optimal set. The unboundedness of $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ in the example is caused by unboundedness of the objective function in a particular scenario. The following characterization of a bounded optimal solution set using duality was formulated by Mostafaee, Hladík and Černý [33], based on the results on continuity of some set-valued functions in linear programming proved by Wets [46]. Again, we consider an interval program in the form (ILP_p) and its dual program (ILP_d).

Theorem 5.5 ([33, 46]). *Assume that for every $A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}$ the following holds:*

$$\{x \in \mathbb{R}^n : Ax = 0, x \geq 0, c^T x \leq 0\} = \{0\}, \tag{22}$$

$$\{y \in \mathbb{R}^m : A^T y \leq 0, b^T y \geq 0\} = \{0\}. \tag{23}$$

Then $f(A, b, c)$ is continuous on $\boldsymbol{A} \times \boldsymbol{b} \times \boldsymbol{c}$ and the optimal solution set $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ is bounded.

It is easy to see that the characterization of boundedness of optimal solution sets given by Theorem 5.5 is not complete. Therefore, an ILP with a bounded optimal set does not necessarily satisfy conditions (22) and (23), as illustrated by the following example.

Example. Consider the ILP

$$\begin{array}{ll} \text{minimize} & -x \\ \text{subject to} & [0,1]x = 0, \\ & \quad\quad x \geq 0. \end{array}$$

The scenario with constraint $0x = 0$ is unbounded, therefore it does not have any optimal solutions. For any other scenario, the constraint reads $\alpha x = 0$ with $\alpha \neq 0$, and the optimal solution is $x = 0$. Clearly, the optimal solution set $\{0\}$ is bounded. However, condition (22) is violated for the scenario $A = 0$:

$$\{x \in \mathbb{R}^n : 0x = 0, x \geq 0, -1x \leq 0\} = [0, \infty).$$

For an ILP of type (III), we have obtained an interval enclosure of the optimal solution set by using duality, which comprises only linear constraints. Based on this approximation, we can state a sufficient condition of boundedness of the optimal solution set. For other types of ILPs, we can formulate a similar condition by orthant decomposition. However, such test would require solving an exponential number of linear systems. In this case, it is possible to weaken and simplify the condition by using the linearization technique introduced in Section 4.2.

Theorem 5.6. *Let an ILP of type (III) be given by the triplet $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$. Assume that the set of feasible x-solutions to the linear system*

$$\begin{aligned} &\underline{c}^T x - \overline{b}^T y \leq 0, -\overline{c}^T x + \underline{b}^T y \leq 0, \\ &\underline{A} x \leq \overline{b}, x \geq 0, \\ &\overline{A}^T y \leq \overline{c}, y \leq 0, \end{aligned}$$

is bounded. Then the optimal solution set $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ of the corresponding ILP is also bounded.

Let us now study the complexity of testing boundedness of the optimal solution set to an interval linear program. Theorem 5.7 establishes a relationship between boundedness of the feasible set of an interval system and non-singularity of the coefficient matrix. Recall that an interval matrix $\boldsymbol{A}$ is non-singular if every real matrix $A \in \boldsymbol{A}$ is non-singular, otherwise it is singular.

Theorem 5.7 ([41])**.** *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ contain at least one non-singular matrix and denote by $\mathcal{M}(\boldsymbol{A}, \boldsymbol{b})$ the feasible set of the interval system $\boldsymbol{A}x = \boldsymbol{b}$. Then the following assertions are equivalent:*

a) $\boldsymbol{A}$ is non-singular,

b) $\mathcal{M}(\boldsymbol{A}, \boldsymbol{b})$ is bounded for some $\boldsymbol{b} \in \mathbb{IR}^n$,

c) $\mathcal{M}(\boldsymbol{A}, \boldsymbol{b})$ is bounded for each $\boldsymbol{b} \in \mathbb{IR}^n$.

Before we proceed, let us review a complexity result, which is often used to establish NP-hardness of decision problems related to interval linear systems.

Definition 5.3. A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is said to be *positive definite* if the inequality $x^T Ax > 0$ holds for every $x \in \mathbb{R}^n \backslash \{0\}$.

Lemma 5.8 ([43, p. 39])**.** *Let $e = (1, \dots, 1)^T$. The problem of checking whether the system of inequalities*

$$\begin{aligned} -e \leq Ax &\leq e, \\ e^T|x| &\geq 1, \end{aligned} \tag{24}$$

has a feasible solution is NP-hard on the set of non-negative positive definite rational matrices.

We continue with a complexity result for inequality-constrained interval linear programs of type (II). The proof exploits the fact that a feasibility problem can be formulated as an optimization problem with a constant objective function, thus, the result also holds for testing boundedness of the feasible set of interval systems.

Theorem 5.9. *The problem of checking boundedness of the optimal set $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ for an ILP of type (II) is co-NP-hard.*

Proof. From the proof of [43, Theorem 2.33], we have that system (24) has a feasible solution for a given matrix $A \in \mathbb{R}^{n\times n}$ if and only if the interval matrix $\boldsymbol{A} = [A - ee^T, A + ee^T]$ is singular. This result implies that testing regularity of interval matrices is a co-NP-hard problem.

Further, let A be a non-negative positive definite rational matrix. By the properties of positive definite matrices, A is non-singular. Therefore, the interval matrix $\boldsymbol{A} = [A - ee^T, A + ee^T]$ contains a non-singular matrix, namely the central matrix $A^c = A$. This allows us to use the characterization of non-singularity of an interval matrix stated in Theorem 5.7.

Let us choose $b = 0$. Then, $\boldsymbol{A}$ is regular if and only if the feasible set of the system $\boldsymbol{A}x = 0$ is bounded. Using the results of Section 3.2, we can split the equation constraint into $\boldsymbol{A}x \leq 0, \boldsymbol{A}x \geq 0$, while preserving the same feasible set.

Therefore, the interval matrix $\boldsymbol{A}$ is singular if and only if the optimal solution set of the interval linear program

$$\begin{array}{ll} \text{minimize} & 0^T x \\ \text{subject to} & \boldsymbol{A}x \leq 0, \\ & -\boldsymbol{A}x \leq 0, \end{array} \tag{25}$$

is unbounded, implying that testing boundedness of the optimal set for an ILP of type (II) is co-NP-hard. □

5.3 Connectedness and convexity

In this section, we study the conditions under which an interval linear program has a convex, or at least connected optimal solution set. To characterize the property of connectedness, we can use one of the following definitions:

Definition 5.4. A set $M \subseteq \mathbb{R}^n$ is said to be *connected*, if for each pair of sets $X, Y \subseteq \mathbb{R}^n$ with $M = X \cup Y$ and $X \cap Y = \emptyset$, which are open in the subset topology induced on M, it holds that $X = \emptyset$ or $Y = \emptyset$.

Definition 5.5. A set $M \subseteq \mathbb{R}^n$ is said to be *path-connected*, if for every $x, y \in M$ there exists a continuous function $f : [0, 1] \to M$ with $f(0) = x$ and $f(1) = y$.

Even though there are many examples, for which Definition 5.4 and Definition 5.5 coincide, only one implication holds in general.

Lemma 5.10 ([34, p. 153])**.** *Every path-connected set is connected.*

Convexity is a strengthening of the (path-)connectedness property. Recall that a set is convex, if for every pair of points from the set, the line segment joining the points is also contained within the set.

Definition 5.6. A set $M \subseteq \mathbb{R}^n$ is said to be *convex*, if for every $x, y \in M$ and every $\lambda \in [0, 1]$ it holds that $\lambda x + (1 - \lambda)y \in M$.

Since the feasible set of an interval linear program may be disconnected, this is also true for the optimal solution set. However, even if the feasible set is connected, it is still possible for the optimal set to be disconnected.

Example. Consider the ILP

$$\begin{array}{ll} \text{maximize} & x_2 \\ \text{subject to} & [-1, 1]x_1 + x_2 \leq 0 \\ & \qquad\qquad x_2 \leq 1. \end{array} \tag{26}$$

For the scenario involving the constraint $0x_1 + x_2 \leq 0$, the set of optimal solutions is formed by the line $x_2 = 0$. Further, consider a scenario with the constraint $\alpha x_1 + x_2 \leq 0$ for $\alpha \neq 0$. If we take the union of all optimal sets for $\alpha > 0$, we obtain the ray $(-1 - t, 1)$ with $t \geq 0$. For $\alpha < 0$, we have the united optimal set $(1 + t, 1)$ with $t \geq 0$. The overall optimal solution set of the interval program, which is formed by the union of the two rays and the line, is (path-)disconnected (see Figure 5.1).

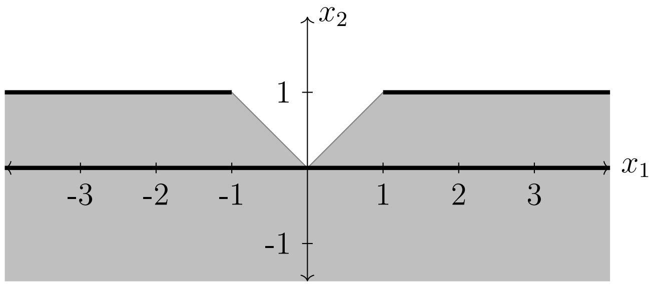

Figure 5.1: The union of all feasible sets (gray) and the set of all optimal solutions (thick black) of ILP (26).

We have already encountered a sufficient condition for convexity of the optimal solution set for an ILP of type (I) in Section 3.4, where basis stability was introduced. Theorem 3.10 states that if there exists a basis B, for which the given interval program is unique B-stable, then the optimal set can be described by a linear system. Therefore, it forms a convex polyhedron. Let us now show that if the problem is B-stable, but not necessarily unique B-stable, then the optimal solution set is path-connected.

Theorem 5.11. *Let an ILP of type (I) be given by the triplet $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$. If there exists a basis $B \subseteq \{1, \dots, n\}$, for which the problem is B-stable, then the optimal solution set $\mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ is path-connected.*

Proof. Let B be a basis, which is optimal for each scenario of the ILP and let $\mathcal{S}(B)$ denote the set of all optimal basic solutions with the basis B. Furthermore, let $x_1, x_2 \in \mathcal{S}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ be arbitrary solutions optimal for some scenarios (A_1, b_1, c_1) and (A_2, b_2, c_2), respectively.

Since the problem is B-stable, there exist basic solutions $x_1^B, x_2^B \in \mathcal{S}(B)$, which are optimal for the scenarios (A_1, b_1, c_1) and (A_2, b_2, c_2). From the theory of linear programming, we know that the optimal solution set of a fixed scenario is convex, and therefore also path-connected. Thus, there exists a continuous mapping (path) $p_1 : [0, 1] \to \mathbb{R}^n$ with $p_1(0) = x_1$ and $p_1(1) = x_1^B$ and also a path p_2 connecting x_2^B to x_2. By Theorem 3.10, the set $\mathcal{S}(B)$ is convex, which implies that there also exists a path p_B connecting x_1^B to x_2^B. Using transitivity of the path-connectedness relation, we obtain a path p_3 from $p_3(0) = x_1$ to $p_3(1) = x_2$. $\square$

6 Special cases

6.1 Interval objective and right-hand side

In many practical applications, uncertainty only affects the objective function coefficients or the right-hand side vector of a linear programming model. These include, for example, various transportation problems [24, 45] or minimum-cost flow problems [16]. We will now examine this class of interval linear programs and strenghten some of the results we have obtained for general problems. Following the topics of the previous chapter, we focus mainly on the properties of the optimal solution set. For more details about computing optimal solutions and testing basis stability in these special cases, see [19, pp. 107 – 110] and references therein. Similar types of problems have also been studied in the context of (multi-)parametric programming [13].

Let us begin by studying the optimal solution set for a class of interval linear programs with intervals occurring only in the objective function and the right-hand-side vector. In other words, we consider an interval linear program with a fixed real coefficient matrix:

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & \boldsymbol{c}^T x \\ \text{subject to} \quad & Ax = \boldsymbol{b}, \\ & x \geq 0. \end{aligned} \tag{27}$$

6.1.1 Polyhedrality and closedness

The optimal solution set of (27) may still be more complex than the one of a classical linear program, but we can prove some additional properties to those which hold for a general interval linear program. Although the optimal set can be non-convex, we now have the following result:

Theorem 6.1. *The set of optimal solutions of interval linear program* (27) *is a union of at most* 2^n *convex polyhedra.*

Proof. According to Theorem 3.7, we can describe the optimal solution set of interval program (27) by the parametric linear system

$$\begin{aligned} & Ax = b,\ x \geq 0, \\ & A^T y \leq c, \\ & x^T(c - A^T y) = 0, \\ & b \in \mathbf{b}, c \in \mathbf{c}. \end{aligned}$$

Further, we replace the parameters b and c by variables with given lower and upper bounds. Note that this transformation does not change the set of feasible solution vectors x and y. We obtain the following non-linear system:

$$\begin{aligned} & Ax = b,\ x \geq 0, \\ & A^T y \leq c, \\ & x^T(c - A^T y) = 0, \\ & \underline{b} \leq b \leq \overline{b},\ \underline{c} \leq c \leq \overline{c}. \end{aligned}$$

To deal with the non-linearity in the constraints, we introduce an auxiliary variable $z = c - A^T y$, which leads to the characterization

$$\begin{aligned} &Ax = b,\ \ x \geq 0, \\ &z = c - A^T y,\ \ z \geq 0, \\ &x^T z = 0, \\ &\underline{b} \leq b \leq \overline{b},\ \ \underline{c} \leq c \leq \overline{c}. \end{aligned} \tag{28}$$

Since A is a fixed real matrix, the only non-linear constraint left is $x^T z = 0$. Taking into account the non-negativity conditions on x and z, the constraint can be equivalently restated as

$$\forall i \in \{1, \dots, n\} : x_i = 0 \vee z_i = 0.$$

This restatement leads to 2^n linear programs obtained by replacing $x^T z = 0$ with a collection of constraints $x_i = 0$ or $z_i = 0$ for each index i. Therefore, the feasible set of (28) is a union of 2^n convex polyhedra. The projection $\pi_x : \mathbb{R}^{2m+3n} \to \mathbb{R}^n$, which maps solutions of (28) onto the x-variable, preserves convexity and polyhedrality. Thus, the set of optimal solutions of (27) is also a union of 2^n convex polyhedra. □

Corollary 6.2. *The set of optimal solutions of interval linear program*

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & \boldsymbol{c}^T x \\ \textit{subject to} \quad & Ax \leq \boldsymbol{b}, \end{aligned} \tag{29}$$

is a union of at most 2^m convex polyhedra.

From Theorem 6.1, we can see that the optimal set of (27) has a finite number of vertices, since it is a finite union of convex polyhedra. Thus, it forms a (generally non-convex) polyhedral set. Moreover, it also implies closedness of the optimal set.

Corollary 6.3. *The optimal solution set of* (27) *is closed.*

Proof. By Theorem 6.1, the optimal solution set is a finite union of closed sets, therefore, it is also closed. □

6.1.2 Connectedness and boundedness

Let us now examine connectedness of the optimal set of ILP (27). To do this, we will use some properties of the optimal set function. First, we can consider the notation $\mathcal{S}_A(b, c) = \mathcal{S}(A, b, c)$ as a function of the parameters b and c, since the coefficient matrix A is fixed. The value of $\mathcal{S}_A(b, c)$ is then the optimal solution set of the corresponding linear program. The following definition provides a formalization of such set-valued functions.

Definition 6.1. A function f from a set $X \subseteq \mathbb{R}^m$ into the power set of $Y \subseteq \mathbb{R}^n$ is called a *multifunction*. Moreover, $f(x) \neq \emptyset$ for every $x \in X$, then f is said to be a *correspondence*.[1]

[1] The terms “multifunction”, “set-valued function” or “correspondence” are sometimes used interchangeably.

To prove connectedness, we will employ continuity of $\mathcal{S}_A(b,c)$. Definition 6.2 extends the continuity property known from single-valued functions to multifunctions by introducing the notion of lower and upper hemicontinuity.

Definition 6.2. A correspondence $f : X \to 2^Y$ is called *upper hemicontinuous* at $x \in X$, if for every open neighborhood U of $f(x)$ there exists an open neighborhood V of x such that $f(z) \subseteq U$ holds for all $z \in V$.

The correspondence f is *lower hemicontinuous* at x, if for every open set M in Y such that $M \cap f(x) \neq \emptyset$, there exists an open neighborhood V of x such that $M \cap f(z) \neq \emptyset$ holds for all $z \in V$.

The correspondence f is called *continuous* at x, if it is lower and upper hemicontinuous at x.

We say that f is lower/upper hemicontinuous or continuous (on X), if the corresponding property holds for every $x \in X$. For classical single-valued functions, the notion of continuity is commonly used to prove that some topological properties of the preimage are preserved under a given mapping. Lemma 6.4 provides a generalization of two results relating continuity to topological properties of sets, in the context of multifunctions.

Lemma 6.4 ([17])**.**

(a) The image of a compact set under an upper hemicontinuous compact-valued correspondence is compact.

(b) The image of a connected set under an upper hemicontinuous (or a lower hemicontinuous) connected-valued correspondence is connected.

Since we are interested in the optimal solution set, not all possible scenarios in a given ILP need to be considered. For some values in the given intervals, the corresponding linear program can be unbounded or infeasible, and these do not contribute any solutions to the optimal set. Using the theory of duality in linear programming, we can describe the relevant scenarios by such parameters $b \in \boldsymbol{b}$ and $c \in \boldsymbol{c}$, for which both the primal and the dual linear program are feasible. Let us define the following sets:

$$\mathcal{B} = \{b \in \mathbb{R}^m : Ax = b, x \geq 0 \text{ for some } x \in \mathbb{R}^n\},$$
$$\mathcal{C} = \{c \in \mathbb{R}^n : A^T y \leq c \text{ for some } y \in \mathbb{R}^m\}.$$

Given two parameter vectors $b \in \boldsymbol{b}$ and $c \in \boldsymbol{c}$, the corresponding scenario of (27) has an optimal solution if and only if $(b,c) \in \mathcal{B} \times \mathcal{C}$. Therefore, the optimal solution set can be obtained as an image of the set $\mathcal{B} \times \mathcal{C}$ restricted to the given interval box $\boldsymbol{b} \times \boldsymbol{c}$ under the multifunction $\mathcal{S}_A(b,c)$. To derive some results concerning the optimal set, we will also need some properties of the preimage $(\mathcal{B} \cap \boldsymbol{b}) \times (\mathcal{C} \cap \boldsymbol{c})$, namely its closedness and connectedness.

Lemma 6.5. *The set $(\mathcal{B} \cap \boldsymbol{b}) \times (\mathcal{C} \cap \boldsymbol{c})$ is a closed convex set.*

Proof. From the theory of parametric programming (see e.g. [13, p. 179]), we know that the sets $\mathcal{B}$ and $\mathcal{C}$ are closed convex sets. Furthermore, the interval vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ are closed and convex, too. As finite intersections and Cartesian products preserve these properties, the resulting set $(\mathcal{B} \cap \boldsymbol{b}) \times (\mathcal{C} \cap \boldsymbol{c})$ is also closed and convex. $\square$

Continuity properties of the optimal value function and optimal solution sets of linear programs, in which the right-hand side vector or the objective function coefficients are subjected to data perturbations, were studied by Böhm [6] and Meyer [31]. Some of these results will serve as a basis for deriving the topological properties of the optimal set of an ILP.

Theorem 6.6 (Meyer [31])**.** *The correspondence $\mathcal{S}_A$ from $\mathcal{B} \times \mathcal{C}$ to the power set of $\mathbb{R}^n$ is upper hemicontinuous on $\mathcal{B} \times \mathcal{C}$.*

Now, we can proceed with the proof of connectedness of the optimal solution set in the special case with a fixed coefficient matrix. This property does not hold for interval linear programs in general, perturbations in the coefficients of matrix A are therefore the cause of possible disconnected optimal sets.

Corollary 6.7. *The optimal solution set of* (27) *is connected.*

Proof. We will use the observation, that the optimal solution set $\mathcal{S}(A, \boldsymbol{b}, \boldsymbol{c})$ can be equivalently described as

$$\bigcup \{\mathcal{S}_A(b, c) : b \in \mathcal{B} \cap \boldsymbol{b}, c \in \mathcal{C} \cap \boldsymbol{c}\}.$$

According to Theorem 6.6, the correspondence $\mathcal{S}_A(b, c)$ is upper hemicontinuous on $\mathcal{B} \times \mathcal{C}$. For fixed vectors b, c, the value $\mathcal{S}_A(b, c)$ is the optimal set of a linear program, which is convex, and therefore also connected. The optimal set correspondence thus satisfies the assumptions of Lemma 6.4, part (b).

Lemma 6.5 implies that the set $(\mathcal{B} \cap \boldsymbol{b}) \times (\mathcal{C} \cap \boldsymbol{c})$ is connected. Hence, the image of the set $(\mathcal{B} \cap \boldsymbol{b}) \times (\mathcal{C} \cap \boldsymbol{c})$, i.e. the optimal solution set, is also connected. □

It is easy to see that the optimal set may still be unbounded, even for this special case. However, we can formulate a strengthening of Theorem 5.5, which characterizes boundedness of $\mathcal{S}(A, \boldsymbol{b}, \boldsymbol{c})$ using boundedness of the optimal sets of individual scenarios. As it was shown in Section 5.2, the property stated in Corollary 6.8 does not hold for general ILPs either.

Corollary 6.8. *The optimal solution set $\mathcal{S}(A, \boldsymbol{b}, \boldsymbol{c})$ of* (27) *is bounded if and only if for each $b \in \boldsymbol{b}$ and $c \in \boldsymbol{c}$ the optimal solution set $\mathcal{S}(A, b, c)$ of the corresponding linear program is bounded.*

Proof. Obviously, if $\mathcal{S}(A, \boldsymbol{b}, \boldsymbol{c})$ is bounded, then the set $\mathcal{S}(A, b, c)$ is also bounded for each $b \in \boldsymbol{b}, c \in \boldsymbol{c}$.

Conversely, assume that each $\mathcal{S}(A, b, c)$ is bounded. Since $\mathcal{S}(A, b, c)$ is the optimal set of a linear program, it is also closed, and thus compact. Therefore, the correspondence $\mathcal{S}_A$ is compact-valued.

By Lemma 6.5, the set $(\mathcal{B} \cap \boldsymbol{b}) \times (\mathcal{C} \cap \boldsymbol{c})$ is closed. Moreover, it is also bounded by the interval boxes $\boldsymbol{b}$ and $\boldsymbol{c}$. Hence, it is compact, and we can apply Lemma 6.4, part (a), to show that the image of $(\mathcal{B} \cap \boldsymbol{b}) \times (\mathcal{C} \cap \boldsymbol{c})$ under the correspondence $\mathcal{S}_A$ is compact. Specifically, we have shown that the optimal set $\mathcal{S}(A, \boldsymbol{b}, \boldsymbol{c})$ is bounded. □

6.1.3 Transformations

We can also strengthen some results concerning the transformations of problems of different types, which were discussed in Section 3.2. Thus far, we have assumed that the ILP is given in formulation (27). This is an equality-constrained problem (with non-negative variables) of type (I), i.e.

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & \boldsymbol{c}^T x \\ \text{subject to} \quad & Ax = \boldsymbol{b}, \\ & x \geq 0. \end{aligned}$$

We follow by showing that ILP (27) can be transformed to an ILP of type (III) in the form

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & \boldsymbol{c}^T x \\ \text{subject to} \quad & Ax \leq \boldsymbol{b}, \\ & -Ax \leq -\boldsymbol{b}, \\ & x \geq 0, \end{aligned} \tag{30}$$

with the same set of optimal solutions (see Theorem 6.9). Trivially, this also allows for a transformation from an ILP of type (I) to type (II), since we can include the non-negativity condition as a constraint. Recall that such transformation was not possible for an ILP with an interval coefficient matrix. Note that, due to the dependency problem, the transformation may also generate new infeasible linear programs. Thus, the two ILPs are not entirely equivalent.

Theorem 6.9. *Let $\mathcal{S}(A, \boldsymbol{b}, \boldsymbol{c})$ and $\mathcal{S}(A', \boldsymbol{b}', \boldsymbol{c})$ be the optimal solution sets of the programs (27) and (30), respectively. Then, the equality $\mathcal{S}(A, \boldsymbol{b}, \boldsymbol{c}) = \mathcal{S}(A', \boldsymbol{b}', \boldsymbol{c})$ holds.*

Proof. Clearly, any linear program contained in (27) is also contained in (30), therefore $\mathcal{S}(A, \boldsymbol{b}, \boldsymbol{c}) \subseteq \mathcal{S}(A', \boldsymbol{b}', \boldsymbol{c})$ holds. Conversely, let $x' \in \mathcal{S}(A', \boldsymbol{b}', \boldsymbol{c})$, then there exist $b_1, b_2 \in \boldsymbol{b}$ and $c \in \boldsymbol{c}$ such that x' is optimal for the scenario

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b_1, \\ & -Ax \leq -b_2, \\ & x \geq 0. \end{aligned} \tag{31}$$

Since $b_2 \leq Ax' \leq b_1$, there exists a vector $b_3 \in [b_2, b_1] \subseteq \boldsymbol{b}$ with $Ax' = b_3$. We claim that $x' \in \mathcal{S}(A, \boldsymbol{b}, \boldsymbol{c})$, because it is optimal for the scenario

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b_3, \\ & x \geq 0. \end{aligned} \tag{32}$$

Suppose for the sake of contradiction that there exists x^* with $Ax^* = b_3, x^* \geq 0$ and $c^T x^* < c^T x'$. By the choice of b_3, the vector x^* is feasible for scenario (31). Since the objective function is the same for both programs, this yields a contradiction with the assumption that x' is optimal for scenario (31). □

Furthermore, we show that the transformation of free variables to non-negative variables by substitution is also applicable for an ILP with a fixed coefficient matrix. The substitution does not violate any dependencies in the constraints, but it splits the objective coefficients into two independent interval vectors, which may seem to create a dependency problem.

Let an ILP be given in the form of type (II) as

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & \boldsymbol{c}^T x \\ \text{subject to} \quad & Ax \leq \boldsymbol{b}, \end{aligned} \tag{33}$$

and consider its transformation into an ILP of type (III) in the form

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & \boldsymbol{c}^T x^+ - \boldsymbol{c}^T x^- \\ \text{subject to} \quad & Ax^+ - Ax^- \leq \boldsymbol{b}, \\ & x^+, x^- \geq 0. \end{aligned} \tag{34}$$

We will prove that the two ILPs have equivalent optimal sets in the sense that for each optimal solution x of ILP (33) there exist non-negative vectors x^+, x^- with $x = x^+ - x^-$, which form an optimal solution of (34) and vice versa.

Theorem 6.10. *Let $\mathcal{S}(A, \boldsymbol{b}, \boldsymbol{c})$ and $\mathcal{S}(A^\pm, \boldsymbol{b}, \boldsymbol{c}^\pm)$ denote the optimal solution sets of ILP (33) and (34), respectively. The following two properties hold:*

a) If $x \in \mathcal{S}(A, \boldsymbol{b}, \boldsymbol{c})$, then there exists $(x^+, x^-) \in \mathcal{S}(A^\pm, \boldsymbol{b}, \boldsymbol{c}^\pm)$ with $x = x^+ - x^-$.

b) Conversely, if $(x^+, x^-) \in \mathcal{S}(A^\pm, \boldsymbol{b}, \boldsymbol{c}^\pm)$, then $x^+ - x^- \in \mathcal{S}(A, \boldsymbol{b}, \boldsymbol{c})$.

Proof.

a) Let $x_0 \in \mathcal{S}(A, \boldsymbol{b}, \boldsymbol{c})$ be an optimal solution of (33) for a scenario determined by objective vector $c \in \boldsymbol{c}$ and right-hand side $b \in \boldsymbol{b}$. We define the vector $x_0^+ = \max(0, x_0)$ and $x_0^- = -\min(0, x_0)$, where the operations max and min are understood element-wise. Clearly, x_0^+ and x_0^- are non-negative and the equality $x_0 = x_0^+ - x_0^-$ holds. Furthermore, the feasible set of (34) for any scenario with right-hand-side vector b is equivalent to the feasible set of program (33) with the same right-hand side: Since we have

$$Ax^+ - Ax^- = A(x^+ - x^-),$$

if (x^+, x^-) is a feasible solution of (34) for a vector b, then $A(x^+ - x^-) \leq b$ holds and $x^+ - x^-$ is a feasible solution of (33). Conversely, let x be a feasible solution of (33) for some b. Then we can find non-negative vectors x^+, x^- with $x = x^+ - x^-$ and the inequality $Ax^+ - Ax^- \leq b$ also holds.

By a similar reasoning, if we consider the objective $(c, -c)$, we also have $c^T x^+ - c^T x^- = c^T (x^+ - x^-)$. This implies that the transformed problem is equivalent to the original scenario (A, b, c) and (x_0^+, x_0^-) is thus an optimal solution of (34).

b) On the other hand, let $(x_0^+, x_0^-) \in \mathcal{S}(A^\pm, \boldsymbol{b}, \boldsymbol{c}^\pm)$ be optimal for some $b \in \boldsymbol{b}$ and $c_1, c_2 \in \boldsymbol{c}$. Using duality in linear programming, there exists a dual feasible vector y_0, which satisfies the following system:

$$\begin{aligned} & c_1^T x_0^+ - c_2^T x_0^- = b^T y, \\ & Ax_0^+ - Ax_0^- \leq b, \; x_0^+ \geq 0, x_0^- \geq 0, \\ & A^T y \leq c_1, \; -A^T y \leq -c_2, \; y \leq 0. \end{aligned} \tag{35}$$

Let us define an objective vector $c_3 := A^T y_0$. Since $c_2 \leq A^T y_0 \leq c_1$ and $c_1, c_2 \in \boldsymbol{c}$, we also have $c_3 \in \boldsymbol{c}$. We will now show that the vector $x_0^+ - x_0^-$ is optimal for the scenario determined by right-hand side b and objective vector c_3, or equivalently, that $x_0^+ - x_0^-$ and y_0 satisfy the system

$$c_3^T x = b^T y,\ Ax \leq b,\ A^T y = c_3,\ y \leq 0.$$

Namely, it remains to show that the equality $c_3^T(x_0^+ - x_0^-) = b^T y_0$ holds. From system (35) we know that $b^T y_0 = c_1^T x_0^+ - c_2^T x_0^-$. The complementarity condition (see Theorem 3.7) implies that for each $i \in \{1, \ldots, n\}$ we have

$$\begin{aligned} (x_0^+)_i = 0 \vee (c_1 - A^T y_0)_i = 0,\ \text{and} \\ (x_0^-)_i = 0 \vee (A^T y_0 - c_2)_i = 0. \end{aligned}$$

Therefore, if $(x_0^+)_i > 0$ and $(x_0^-)_i > 0$ for some index i, then the corresponding entries of c_1 and c_2 are equal, $(c_1)_i = (c_2)_i$ and also $(c_1)_i = (c_3)_i$ by definition of c_3. It follows that $(c_3)_i(x_0^+ - x_0^-)_i = (c_1)_i(x_0^+)_i - (c_2)_i(x_0^-)_i$. Further assume that $(x_0^+)_i = 0$ and $(x_0^-)_i > 0$ (the symmetric case with $(x_0^+)_i > 0$ and $(x_0^-)_i = 0$ can be treated analogically). Then, complementarity implies $(c_3)_i = (c_2)_i$ and since $(x_0^+)_i = 0$, we obtain the equality

$$(c_3)_i(x_0^+ - x_0^-)_i = (c_2)_i(x_0^+ - x_0^-)_i = (c_1)_i(x_0^+)_i - (c_2)_i(x_0^-)_i.$$

For the last case, suppose we have $(x_0^+)_i = 0$ and $(x_0^-)_i = 0$. Trivially, the expressions $(c_3)_i(x_0^+ - x_0^-)_i$ and $(c_1)_i(x_0^+)_i - (c_2)_i(x_0^-)_i$ are both equal to 0. We have thus proved that $c_3^T(x_0^+ - x_0^-) = c_1^T x_0^+ - c_2^T x_0^-$, and therefore also $c_3^T(x_0^+ - x_0^-) = b^T y_0$. Using strong duality, the vector $x_0^+ - x_0^-$ is an optimal solution of (33) for the scenario with b and c_3. $\square$

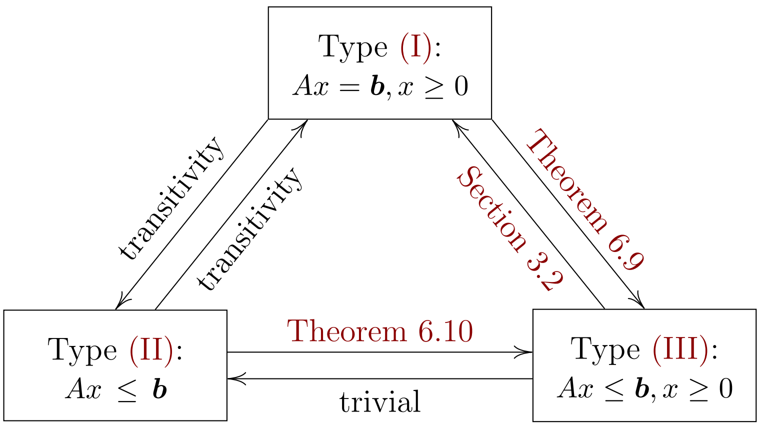

Figure 6.1: An overview of applicable transformations for an ILP with a fixed coefficient matrix.

Together with the cases addressed in Section 3.2, we have proved that all presented transformations preserve the optimal solution set for an ILP with a fixed coefficient matrix. This is a significant difference from the general case, in which the only possible transformation (apart from the trivial ones) was the introduction of slack variables in order to convert inequalities into equations. For an overview of the results relevant to each of the transformations, see Figure 6.1.

6.2 Interval objective function

We will now consider a class of programs, where interval coefficients occur only in the objective function. In this case, the feasible set is described by a linear system, and the description can therefore be transformed from equations to inequalities and conversely, e.g. into the form

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & \boldsymbol{c}^T x \\ \text{subject to} \quad & Ax \le b, \\ & x \ge 0. \end{aligned} \tag{36}$$

Note that if $0 \in \boldsymbol{c}$, then the optimal solution set of (36) is identical to its feasible set $\mathcal{M}(A, b)$, which forms a convex polyhedron. In general, the optimal set of the ILP possesses the following properties:

Theorem 6.11. *The optimal solution set $\mathcal{S}(A, b, \boldsymbol{c})$ of interval program (36) is formed by a union of convex polyhedra, which are faces of the feasible set $\mathcal{M}(A, b)$.*

Proof. Let $c \in \boldsymbol{c}$ be fixed. From the theory of linear programming, we know that the optimal solution set of the corresponding LP forms a face of the feasible polyhedron $\mathcal{M}(A, b)$. Because the feasible polyhedron is fixed, the optimal set $\mathcal{S}(A, b, \boldsymbol{c})$ is simply the union of such faces over all $c \in \boldsymbol{c}$. □

As problem (36) is a special case of ILP (27), we can obtain a characterization of boundedness of the optimal set from Corollary 6.8. However, in this setting, we can also formulate a significantly simplified proof:

Corollary 6.12. *The set $\mathcal{S}(A, b, \boldsymbol{c})$ is bounded if and only if for each $c \in \boldsymbol{c}$ the optimal set $\mathcal{S}(A, b, c)$ of the corresponding linear program is bounded.*

Proof. Assume there exists $c \in \boldsymbol{c}$, such that the set $\mathcal{S}(A, b, c)$ is unbounded. Then, since $\mathcal{S}(A, b, c) \subseteq \mathcal{S}(A, b, \boldsymbol{c})$, the latter is also unbounded.

On the other hand, let $\mathcal{S}(A, b, c)$ be bounded for every $c \in \boldsymbol{c}$. By Theorem 6.11, the optimal solution set $\mathcal{S}(A, b, \boldsymbol{c})$ of the ILP is then a finite union of bounded sets, therefore also a bounded set. □

6.3 Interval right-hand side

Let us now continue with the class of interval linear programs, where intervals only occur in the right-hand-side vector $\boldsymbol{b}$, i.e.

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = \boldsymbol{b}, \\ & x \ge 0. \end{aligned} \tag{37}$$

In this case, the (weak) feasible set is also a convex polyhedron, since we can rewrite the constraint $Ax = \boldsymbol{b}$ as two inequalities $\underline{b} \le Ax \le \overline{b}$, thus obtaining a linear system.

The first result presented in this section is a reformulation of Corollary 6.8 concerning boundedness of the optimal set. For this special case, the following equivalent characterization holds:

Theorem 6.13. *The optimal set $\mathcal{S}(A, \boldsymbol{b}, c)$ of ILP (37) is bounded if and only if there exists some $b \in \boldsymbol{b}$ such that the optimal set $\mathcal{S}(A, b, c)$ is non-empty and bounded or $\mathcal{S}(A, b, c) = \emptyset$ for each $b \in \boldsymbol{b}$.*

Proof. We prove that if $\mathcal{S}(A, b, c)$ is non-empty and bounded for some $b \in \boldsymbol{b}$, then the optimal set is bounded for every $b \in \boldsymbol{b}$. The statement then follows by applying Corollary 6.8.

Let $b_0 \in \boldsymbol{b}$ be such that $\mathcal{S}(A, b_0, c)$ is non-empty and bounded. For contradiction, assume that there exists $b_1 \in \boldsymbol{b}$, for which the optimal set $\mathcal{S}(A, b_1, c)$ is unbounded. Then there is a point $x_1 \in \mathcal{S}(A, b_1, c)$ and a direction $d \in \mathbb{R}^n$ with $x_1 + td \in \mathcal{S}(A, b_1, c)$ for each $t \geq 0$. Since

$$Ax_1 = b_1 = Ax_1 + tAd$$

holds, we obtain the equality $tAd = 0$. Similarly, using optimality of the solutions, we have $c^T x_1 = c^T x_1 + tc^T d$, and therefore $c^T d = 0$.

Choose $x_0 \in \mathcal{S}(A, b_0, c)$ arbitrarily and consider $x_0 + td$ with $t \geq 0$. We have $Ax_0 + tAd = b_0$ and $c^T x_0 + tc^T d = c^T x_0$ for every choice of $t \geq 0$. Therefore, the set $\mathcal{S}(A, b_0, c)$ is also unbounded, which is a contradiction. $\square$

We have seen that even a linear program with an interval objective function can still have a non-convex optimal set. Unfortunately, this is also true for linear programs with interval right-hand sides.

Example. Consider the following ILP:

$$\begin{aligned} \text{minimize} \quad & x_1 \\ \text{subject to} \quad & x_1 - x_2 = [-1, 1], \\ & x_1, x_2 \geq 0. \end{aligned} \tag{38}$$

Let a scenario of (38) be determined by the constraint $x_1 - x_2 = \alpha$ for $\alpha \in [-1, 1]$. If $\alpha \leq 0$, then the corresponding linear program has a unique optimal solution with $x_1 = 0$ and $x_2 = -\alpha$. For $\alpha > 0$, there is a unique optimal solution $(\alpha, 0)$. Therefore, the optimal solution set of the ILP is $(0, [0, 1]) \cup ([0, 1], 0)$, which is non-convex (see Figure 6.2).

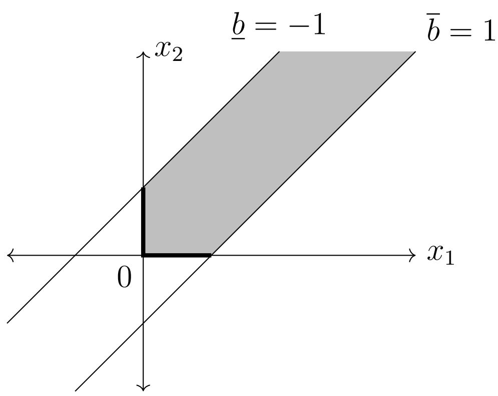

Figure 6.2: The feasible set (gray) and the set of optimal solutions (thick black) of ILP (38).

Conclusion

We have studied various properties of the weak optimal solution set of an interval linear program. Some new sufficient conditions on closedness, boundedness and connectedness of the optimal set were presented in Chapter 5, which also includes a review of the known results on this topic. For a special class of ILPs with a fixed coefficient matrix, we have proved that the optimal set is polyhedral and connected (see Theorem 6.1 and Corollary 6.7). Regarding the theoretical complexity of testing the properties, we have proved in Theorem 5.9 that the problem of checking boundedness of the optimal set is co-NP-hard for inequality-constrained interval programs. We have also studied transformations between different types of ILPs, in order to apply the obtained results to a wider set of problems. In Section 6.1.3, we have shown that all the standard transformations are valid for problems with a fixed coefficient matrix, in the sense that they preserve the set of optimal solutions.

Another topic addressed in the thesis was the problem of approximating or enclosing the optimal solution set. In Section 4.1 and Section 4.2, have reviewed two methods for generating an approximation using the theory of duality in linear programming, namely the orthant decomposition and the interval contractor by Hladík. Furthermore, the theoretical results have led to another decomposition method based on complementary slackness, which is presented in Section 4.3. This method can be used to find the exact optimal solution set for problems with a fixed coefficient matrix, but is suitable mainly for smaller problems due to the exponential time complexity.

The main open questions which arose during the work on this thesis are related to closedness and polyhedrality of the optimal set in the general case. Further research shall therefore be devoted to strengthening the results obtained in this area. From an algorithmic point of view, methods providing a tighter approximation, which preserves at least some of the properties of the exact optimal set, are desirable.

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