Understanding and Using Dynamic Stiffness – A Tutorial

[Editor's Note: The material in this tutorial has been adapted for our ORBIT readers by Steve Sabin, Corporate Marketing Manager. It is based on standard curriculum presented in Bently Nevada's Machinery Diagnostics Seminar. For more information about Dynamic Stiffness, contact our Training Department or write to steve.sabin@bently.com]

Aworking knowledge of Dynamic Stiffness for the machinery engineer is as useful and indispensable as Ohm's Law is for the electrical engineer. Unfortunately, the topic of Dynamic Stiffness generally conjures up images of complicated equations with limited practical value. Quite to the contrary, Dynamic Stiffness is neither hard to understand, nor of only theoretical interest. It is of significant practical value and teaches an important, fundamental way of approaching and diagnosing machinery problems.

This tutorial has been developed to help you understand what Dynamic Stiffness is, why it is important, and how to use it. While the mathematics are included, practical examples and analogies are used wherever possible. Anyone familiar with the basic concepts of machinery balancing and the calculations for determining an Influence Vector will be able to understand and apply the concepts in this tutorial. For this reason, we draw heavily upon machinery balancing examples to convey the concepts of Dynamic Stiffness. In fact, we'll show that the Influence Vector and Dynamic Stiffness are very closely related to one another and that the calculations to determine Dynamic Stiffness are really no more difficult than those involved in basic machinery balancing.

In short, this tutorial focuses on a simple, practical explanation of Dynamic Stiffness that anyone concerned with machinery behavior will find useful and applicable.

[Editor's Note: This tutorial requires familiarity with vector calculations and concepts, conversions between rectangular and polar coordinates, and the basics of complex number operations, i.e., those with both direct and quadrature (sometimes called "real" and "imaginary") components. If you are not familiar with these topics, you are encouraged to explore them via any college algebra or pre-calculus textbook.]

Dynamic Stiffness – Why Is It Important?

Sometimes, people concerned with "vibration analysis" of machinery lose sight of a very simple truth: vibration is merely a response to other conditions in a machine, it is not (and *should* not be) the fundamental concern for the machinery engineer. Instead vibration should be thought of as nothing more than a ratio of the forces acting on the machine to its stiffnesses. Mathematically, this is summarized as:

$$\text{Observed Vibration (Response)} = \frac{\text{Force}}{\text{Dynamic Stiffness (Restraint)}} \quad [1]$$

Unfortunately, as mentioned above, people all too often become concerned exclusively with the left side of the above equation (the vibration response) without giving appropriate thought to the right side of the equation (why did the vibration change?). As Equation 1 shows, vibration can only change as the result of two things: a change in force or a change in stiffness (or both). In other words, vibration is merely a result of other root causes occurring in a machine. One of those root causes can be changing stiffnesses, and it is for this reason that a solid grasp of Dynamic Stiffness is essential for the machinery specialist.

A change in unbalance is an example of a *force* changing in a machine. In this case, force increases (the once-per-turn unbalance excitation), stiffness stays the same, and vibration increases as a result.

In contrast, a cracked shaft, cracked foundation, or loose hold-down bolt is an example of *stiffness* changing in a machine. In these cases, the excitation force stays the same, but stiffness is reduced. However, the same result occurs: vibration increases.

When vibration is viewed as a ratio of forces to stiffnesses, the perspective changes and the focus becomes "what has changed in the machine, the forces acting on it or its stiffnesses?" rather than simply "more vibration is bad, less vibration is

good." Indeed, Equation 1 underscores why it is so important to understand *all* changes in vibration – they signify a change in Dynamic Stiffness or a change in force, or both. A sudden reduction in vibration could signify an increased stiffness, such as pipe strain or a cocked bearing or a frozen coupling. All of these are undesirable. To the individual looking only at a reduction in vibration as a favorable change, however, the underlying cause of the vibration change and its significance would be missed.

Synchronous and Nonsynchronous

The excitation forces acting on a machine can be separated into two broad categories: *synchronous* and *nonsynchronous*. Synchronous means that the input force is at the same frequency as the machine's rotational speed. The most common synchronous excitation force in rotating machinery is the residual imbalance in the rotor. As this imbalance rotates, it generates a once-per-turn centrifugal force. This is often why a machine's vibration is predominantly at 1X (once-per-turn) – because the machine's excitation force is predominantly at 1X as well.

Nonsynchronous, as you would expect, refers to any excitation force that is independent of the rotational speed of the machine. This tutorial will only concern itself with synchronous excitation forces, namely, unbalance. It will also confine itself to Synchronous Dynamic Stiffness, that is, the stiffness characteristics for synchronous excitation. Synchronous Dynamic Stiffness is a special case of Nonsynchronous Dynamic Stiffness: namely, when the excitation frequency equals the rotative speed of the machine.

Sometimes, exciting a machine with an oscillating force is referred to as "perturbating" the machine. Synchronous perturbation is as simple as creating a known imbalance in the machine. It results in a once-per-turn rotating imbalance force. Nonsynchronous perturbation is considerably more difficult and requires special apparatus (see the article on page 10 for an example of a nonsynchronous perturbator). For this reason, nonsynchronous perturbation is rarely done in the field, although it is an excellent tool when it can be applied.

[Editor's Note: Bently Nevada does provide expertise, equipment, and services for conducting synchronous and nonsynchronous perturbation in both laboratory and field environments. Please contact our Global Services Organization for further information.]

Why would we want to excite a machine at various frequencies? Because its Dynamic Stiffness is a function of the excitation frequency, and it is useful to see how Dynamic Stiffness changes as a result of perturbation frequency. The relationship between Dynamic Stiffness and frequency is easily observed in the field during machine startups or coast-downs. A machine will have characteristic resonances where it responds to a given input force more strongly (with more vibration amplitude) than at other frequencies. This is because its Synchronous Dynamic Stiffness is at a minimum at these excitation frequencies (rotational speeds).

Keeping It Simple – Balancing

Looking again at Equation 1 and considering a typical balancing exercise, three things immediately become apparent:

1. In a balancing project we introduce a known synchronous force (the centrifugal force generated by a calibration weight).

2. We then measure the change in synchronous response (the 1X vibration amplitude and phase) due to that force.

3. If we know the synchronous force and the synchronous response, we can calculate Synchronous Dynamic Stiffness.

"When vibration is viewed as a ratio of forces to stiffnesses, the perspective changes and the focus becomes 'what has changed in the machine, the forces acting on it or its stiffnesses?' rather than simply 'more vibration is bad, less vibration is good.'"

The only real difference in a typical balancing project is that we use a relationship between force and response called an Influence Vector. The Influence Vector provides a relation-ship between weight placement and response, rather than force and response, but it is intimately related to Dynamic Stiffness. As we mentioned at the beginning of this tutorial, we'll later show the exact relationship between an Influence Vector and Dynamic Stiffness.

Vectors – How Much and Which Direction?

Believe it or not, we've already covered the most fundamental concept in our discussion of Dynamic Stiffness: Dynamic Stiffness is nothing more than the forces acting on

a rotor divided by its vibration response. It's that simple. Now, all that is left is the mathematics. While we indicated earlier that this tutorial does not teach the basics of vector mathematics or complex number operations, some basic review is appropriate.

Forces and responses (vibration) are *vector* quantities. That is, they have both magnitude and direction. Scalar quantities have only magnitude, not direction. In this tutorial, we'll denote vector quantities with an arrow (such as $\vec{F}$) and scalar quantities in regular text (such as F).

If we apply an oscillating force to a machine, it has both a magnitude and a direction. A force that acts straight down at a certain point on a machine is quite different from a force that pushes across the machine at the same location, rather than down on it. Thus, force is a vector quantity.

We also find that the response is a vector. Not only is the response a vector, but it has both a different magnitude *and* direction than the input force. This confirms the observation that when you push down on a rotor, it moves not only downward (in the direction of the force) but also *to the side* (in a direction at right angles to the force).

Since the rotor's response differs in both magnitude and direction from that of the applied force, this tells us that the relationship between force and response (the machine's Dynamic Stiffness) is also a vector quantity. If it were not a vector quantity, only the magnitude would change, not the direction. Thus, we have established by way of observation, rather than mathematics, that forces, responses, and the Dynamic Stiffness relating the two are all vector quantities.

Vectors are often treated mathematically as complex numbers. Figure 1 shows how a force $\vec{F}$ can be separated into two orthogonal (i.e., at right angles to one another) components: $\vec{F}_x$ (the force acting in the x-direction) and $\vec{F}_y$ (the force acting in the y-direction).

Direct and Quadrature – At Right Angles

Like the force in Figure 1, we also speak about Dynamic Stiffness in terms of its orthogonal components, since Dynamic Stiffness is a vector. The component of Dynamic Stiffness that acts in the *same* direction as the applied force is called *Direct* Dynamic Stiffness (DDS) and is denoted as $\vec{K}_D$. The component of Dynamic Stiffness
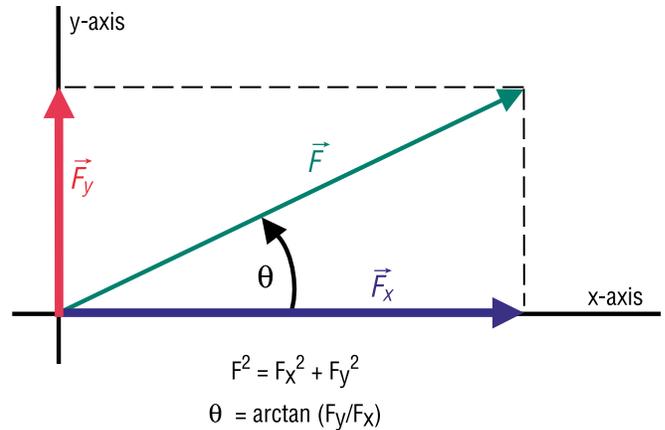

$$F^2 = F_x{}^2 + F_y{}^2$$
$$\theta = \arctan (F_y/F_x)$$

Figure 1. A force $\vec{F}$ resolved into its orthogonal components, $\vec{F}_x$ and $\vec{F}_y$.

orthogonal to this is known as the *Quadrature* Dynamic Stiffness (QDS) and is denoted as $\vec{K}_Q$. When both Direct and Quadrature components are included, we generally refer to Dynamic Stiffness as *Complex* Dynamic Stiffness (DS), $\vec{K}_{DS}$, or sometimes just Dynamic Stiffness. Figure 2 illustrates this relationship.

Incidentally, recall from our earlier discussions that we concern ourselves only with synchronous excitation forces in this tutorial. Therefore, even though we don't designate it as such, when we refer to Dynamic Stiffness throughout, we are specifically referring to *Synchronous* Dynamic Stiffness.

Returning to Figure 2 and the two orthogonal components of Dynamic Stiffness, DDS determines how far the rotor moves in the direction of the applied force and QDS determines how far the rotor moves to the side (orthogonal to the applied force).

If you are familiar with complex numbers, you have probably already realized from the foregoing discussions that Complex Dynamic Stiffness is a complex quantity with its direct (*real*) component and quadrature (*imaginary*) component.
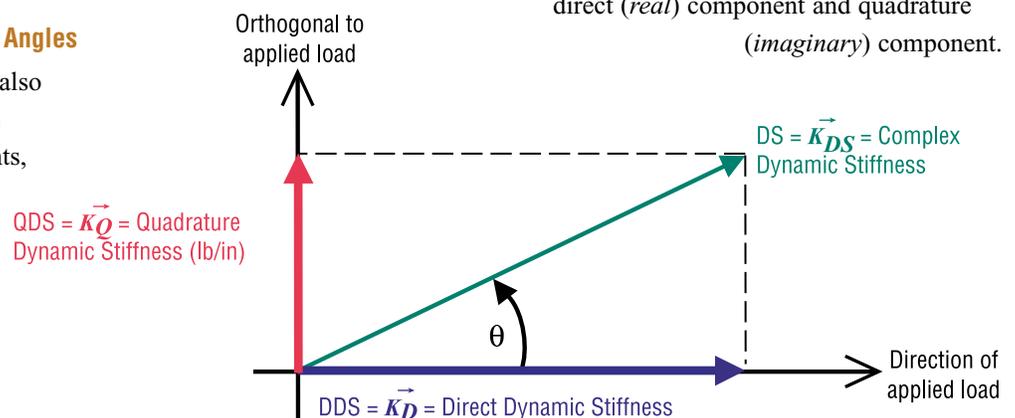

Figure 2. The Relationship between Complex, Direct, and Quadrature Dynamic Stiffness.

Bently Nevada avoids using the terminology "real" and "imaginary" since it implies one component is somehow more "real" than the other. Both are measurable, physical quantities, readily observable. The terms "direct" and "quadrature" are used instead. Mathematically, Complex Dynamic Stiffness can be written as shown in Equation 2:

$$\text{Complex Dynamic Stiffness } = \text{ DDS } + \text{ j QDS} \qquad [2]$$

$$\text{where } j = \sqrt{-1}$$

What Is Dynamic Stiffness?

If the force in Figure 3 were a steady preload, the spring would slowly compress and the system would come to rest at a new position. This static response is controlled only by the static spring stiffness, K.

A dynamic force is a force that changes in magnitude or direction with time. A dynamic input force will cause dynamic output motion. Because of this dynamic motion, both the Quadrature Stiffness due to damping and the mass stiffness effects come into play.

The force, $\vec{F}$, and the response, $\vec{R}$, are *vectors*, and they have both magnitude and direction.

The mechanical system is a "black box." We can obtain information about the system by perturbing, or shaking it. Shortly, we will see how this can be done to a rotor system using a balance shot.
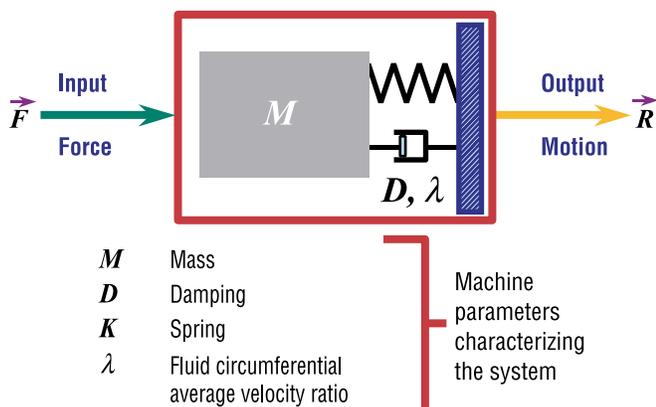

M	Mass	
D	Damping	Machine
K	Spring	parameters
λ	Fluid circumferential	characterizing
	average velocity ratio	the system

Figure 3. Dynamic Stiffness is the static spring stiffness of the mechanical system complemented by the *dynamic* effects of mass and damping.

Synchronous Dynamic Stiffness

As we have said, the Synchronous Dynamic Stiffness, $\vec{K}_{DS}$, is the ratio of the applied force to the rotor system response:

$$\vec{K}_{DS} = \frac{\vec{F}}{\vec{R}} \qquad [3]$$

Another way of remembering this is to look at Hooke's Law for springs:

$$F = Kx \qquad [4]$$

By Hooke's Law, the force created is proportional to the *static* spring deflection (or response), x, where K is the spring stiffness. The *Dynamic* Stiffness, $\vec{K}_{DS}$, is the rotor dynamic equivalent of the simple, static, linear spring in the Hooke's Law equation. Rotor systems have both static and dynamic springs, but you can derive the basic Force/Response relationship from the Hooke's Law relation.

Synchronous Rotor Response Is Controlled by Synchronous Dynamic Stiffness

The Synchronous Dynamic Stiffness is in the denominator of the rotor response equation and controls the rotor response to the unbalance force.

$$\text{Dynamic Stiffness:} \quad \vec{K}_{DS} = \underbrace{K - M\Omega^2}_{\text{Direct term}} + \underbrace{j\, D(1-\lambda)\Omega}_{\text{Quadrature term}} \qquad [6]$$

In Equation 5, the phase lag of the unbalance mass is given by the Greek letter delta, δ. The other terms in the numerator are the unbalance mass, m, unbalance radius, r_u, and rotative speed, Ω.

The Dynamic Stiffness parameters are the rotor mass, M, radial direct stiffness, K, damping, D, and Fluid Circumferential Average Velocity Ratio, λ (lambda). λ is

the ratio of the average fluid angular velocity divided by the rotor angular velocity. Or, alternatively, the average fluid angular velocity is equal to $\lambda\Omega$.

The "j" of the quadrature part is mathematically the $\sqrt{-1}$, but merely means that the action occurs +90°. This is due to a combination of two factors. First, the force on the rotor due to damping is proportional to velocity, which leads displacement by +90°. Second, the oil wedge stiffness can be separated into two parts. The radial part (included in the spring stiffness, K, in the Direct Dynamic Stiffness) acts along the same line as the displacement; however, the tangential part, $+jD\lambda\Omega$, acts at +90° relative to the displacement. The combination of the stiffness due to the damping force and the tangential stiffness produces the Quadrature Dynamic Stiffness that acts at +90° relative to the displacement vector.

Synchronous Dynamic Stiffness Versus Speed

While Synchronous Dynamic Stiffness can be calculated for a single speed, in fact both the Direct and Quadrature parts are functions of the rotative speed, Ω.

The Direct Dynamic Stiffness (Figure 4) is a parabola with y-intercept equal to the radial spring stiffness, K. The zero crossing is important and occurs when

$$\Omega = \sqrt{\frac{K}{M}} \qquad [7]$$

At speeds above resonance, the Direct Dynamic Stiffness is negative.
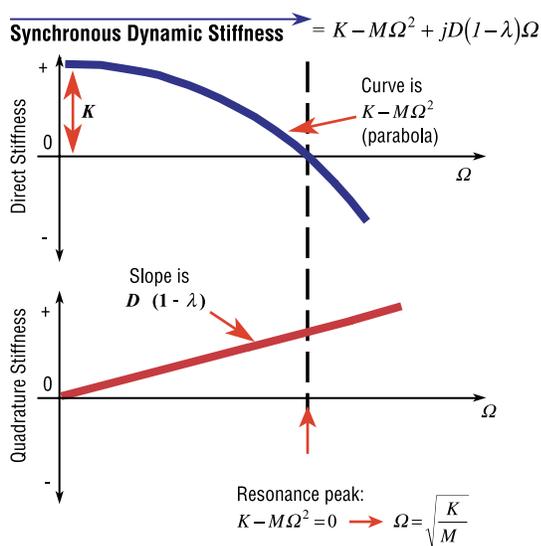

Figure 4. Synchronous Dynamic Stiffness versus speed.

The Quadrature Dynamic Stiffness (Figure 4) is a straight line which starts at the origin at zero speed and has a slope equal to $D(1-\lambda)$.

All of the rotor parameters can be obtained from measurements of the Synchronous Direct and Quadrature Dynamic Stiffness graphs except for λ, which must be estimated.

Why Identify Dynamic Stiffness?

There are numerous reasons why knowing the Dynamic Stiffness for your machine is valuable:

- Dynamic Stiffness is the relationship between machine parameters and measured vibration response.
- Parameter identification provides information for analytical modeling.
- Trending of Dynamic Stiffness can provide valuable information on changes in machine parameters.
- Dynamic Stiffness can be used to estimate the dynamic forces acting in a machine.

Knowledge of your rotor parameters together with an understanding of what they *mean* in terms of rotor behavior allows you to minimize the guesswork when diagnosing machine problems and proposing solutions.
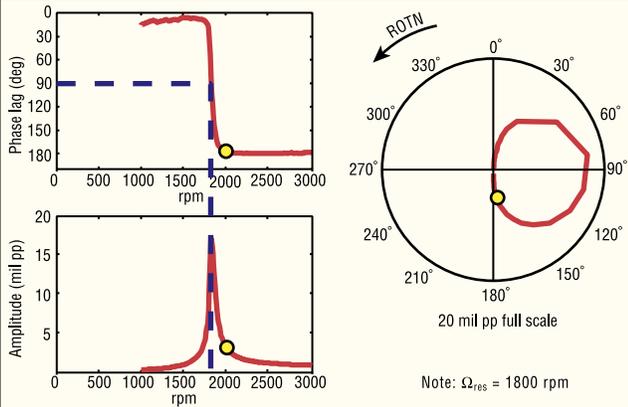
How Do We Measure Synchronous Dynamic Stiffness?

There are five basic steps involved in determining Dynamic Stiffness:

Step 1 – Obtain reference data (startup or shutdown).

Step 2 – Attach a known unbalance mass to the rotor in a known location and get new data.

Step 3 – Calculate response due to calibration weight alone.

Step 4 – Calculate applied force from this calibration weight.

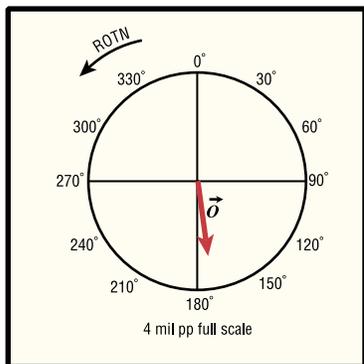*Step 5* – Use data to calculate the Synchronous Dynamic Stiffness.

The information necessary to calculate Synchronous Dynamic Stiffness is obtained every time you balance your machine. You start with an existing unbalanced machine condition, produce a known force by installing a known unbalance weight, and start up the machine again. The change in vibration will be due to the weight that has been added to the machine. The vibration data (1X-filtered amplitude and phase) for the two runs can be used to calculate the Synchronous Dynamic Stiffness.

Next, we'll illustrate these five steps using actual data.
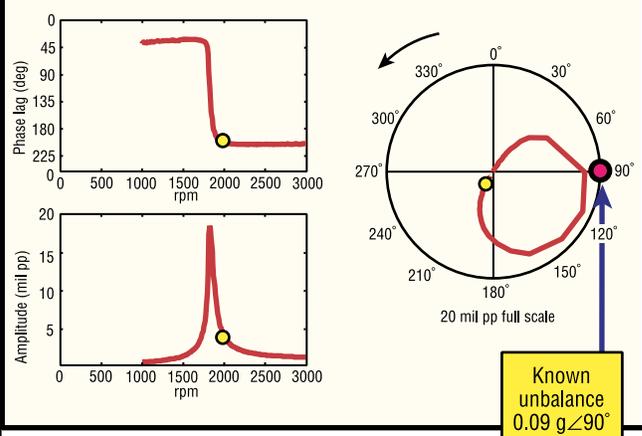
Note: Ω_{res} = 1800 rpm

a.) Obtain data from startup or shutdown or single speed.

b.) Select a point. In this case, the point is:
2000 rpm: 3.19 mil pp $\angle 177°$
This is the original response vector, $\vec{O}$.

c.) Convert from polar form to rectangular.
2000 rpm: 3.19 mil pp $\angle 177°$ = -3.186 + j0.167 mil pp
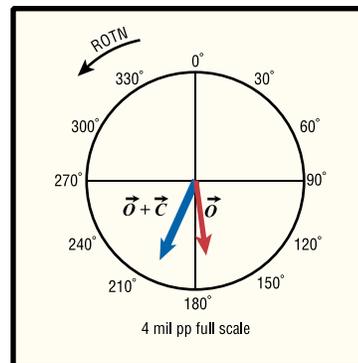
d.) Denote $\vec{O}$ on a polar plot.

4 mil pp full scale

Original response vector
$\vec{O}$ = 3.19 mil pp $\angle 177°$

Known unbalance 0.09 g$\angle 90°$

a.) Add known calibration weight at known location.
0.09 grams $\angle 90°$.

b.) Obtain new data at same speed.
2000 rpm: 3.74 mil pp $\angle 206°$
This new response is equal to the original response vector ($\vec{O}$) *plus* the response due to the calibration weight ($\vec{C}$).

c.) Convert from polar form to rectangular.
2000 rpm: 3.74 mil pp $\angle 206°$= -3.361 - j1.640 mil pp

d.) Denote $\vec{O} + \vec{C}$ (the new response) on your polar plot.

4 mil pp full scale

New response vector
$\vec{O} + \vec{C}$ = 3.74 mil pp $\angle 206°$

Step 3: Calculate Response Due to Calibration Weight Alone

a.) $\vec{O} + \vec{C}$ is a response to calibration weight ($\vec{C}$) plus original response ($\vec{O}$). Therefore, subtract $\vec{O}$ from $\vec{O} + \vec{C}$ to obtain $\vec{C}$.
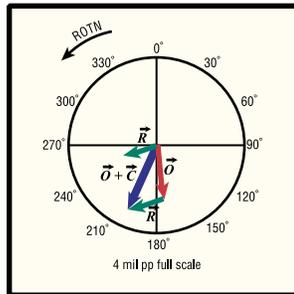
	Polar	Rectangular	
$\overrightarrow{O + C}$	(3.74 ∠206°)	(- 3.361 – j1.640)	mil pp
$-\vec{O}$	– (3.19 ∠177°)	– (- 3.186 + j0.167)	mil pp
$\vec{R} = \vec{C}$	1.81 ∠264°	– 0.175 – j1.807	mil pp

This is the *change* in the rotor system response due to the weight that was added.

b.) $\vec{C}$ is the rotor's response to the calibration weight alone. We denote it as $\vec{R}$ for consistency with Equation 3 (page 47).

c.) Denote $\vec{R}$ on your polar plot (translate $\vec{R}$ to the origin to easily see amplitude and phase).

Response vector due to calibration weight only
$\vec{R}$ = 1.81 mil pp ∠264°

4 mil pp full scale

Step 4: Calculate Applied Force

$$\vec{F} = mr_u\Omega^2 e^{j\delta}$$

Phase Angle of Unbalance

Mass of Unbalance · Radius of Unbalance · Rotative Speed

$$\vec{F} = \left(0.09\times10^{-3}\,\text{kg}\right)\left(30\times10^{-3}\,\text{m}\right)\left[\left(2000\,\frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi\,\text{rad}}{\text{rev}}\right)\left(\frac{1\text{min}}{60\text{sec}}\right)\right]^2 \angle 90°$$

Metric result: $\vec{F} = 118.4\times10^{-3}\,\text{N} \angle 90°$

$\left(\dfrac{1\text{lb}}{4.448\,\text{N}}\right)$ Convert to English

$$\vec{F} = 0.0266\,\text{lb} \angle 90°$$

Step 5: Calculate Synchronous Dynamic Stiffness

a.) Because Dynamic Stiffness is a zero-to-peak type of parameter (lb/in or N/m), first convert the vibration amplitude of the response vector ($\vec{R}$) from peak-to-peak to zero-to-peak (i.e., divide by 2). While we are at it, we convert from mils to inches as well:

$$\vec{R} = \left(\frac{1.81\,\text{mil pp}\,\angle 264°}{2}\right)\left(\frac{1\,\text{in}}{1000\,\text{mil}}\right) = 0.905\times10^{-3}\,\text{in}\,\angle 264°$$

b.) Calculate the Dynamic Stiffness:

The response vector, $\vec{R}$, can be expressed as $\vec{R} = Ae^{j\alpha}$ where A is the amplitude of the response and α is the phase lag of the response.

The force vector, $\vec{F}$, can be expressed as $Fe^{j\delta}$ where F is the amplitude of the force and δ is the phase lag of the force.

$$\vec{K}_{DS} = \frac{\vec{F}}{\vec{R}} = \frac{Fe^{j\delta}}{Ae^{j\alpha}} = \frac{F}{A}e^{j(\delta-\alpha)}$$

$$\vec{K}_{DS} = \frac{F}{A}\angle(\delta-\alpha)$$

Phase Lag of Response Vector

Note! Because the phase angles were measured as phase *lag*, use the negative of the calculated Dynamic Stiffness phase angle.

$$\vec{K}_{DS} = \frac{F}{A}\angle-(\delta-\alpha)$$

$$\vec{K}_{DS} = \frac{0.0266\,\text{lb}}{0.905\times10^{-3}\,\text{in}}\angle-(90°-264°)$$

$$\vec{K}_{DS} = 29\,\text{lb/in}\,\angle 174°$$

* In Bently Nevada instrumentation, phase angles are displayed as positive phase lag. However, the expression for Synchronous Dynamic Stiffness is derived using a standard mathematical coordinate system. In that coordinate system, phase angles are measured positive counterclockwise from the horizontal right axis. This is equivalent to negative phase lag for CCW (X to Y) rotor precession. So, when calculating Synchronous Dynamic Stiffness using positive phase lag data, we must take the negative of the resulting Dynamic Stiffness phase angle to get the correct result.

Synchronous Dynamic Stiffness Phase

A rotor operating well below the balance resonance will have a Synchronous Dynamic Stiffness phase between 0° and 90°. At the resonance, the Dynamic Stiffness vector phase will be 90°, and above the resonance the Dynamic Stiffness vector phase will be greater than 90°. Note that the phase angle of the Synchronous Dynamic Stiffness in this example is 174°. This shows that the machine is operating above the first balance.

Finding Direct and Quadrature Dynamic Stiffness

As noted previously, Dynamic Stiffness consists of two parts:

$$\vec{K}_{DS} = K_{Direct} + jK_{Quadrature} \qquad [8]$$

It can be separated into Direct and Quadrature parts by taking the cosine and sine, respectively, of the Synchronous Dynamic Stiffness phase and multiplying by the amplitude of the Synchronous Dynamic Stiffness.

Direct	Quadrature
$K_D = \left\|\vec{K}_{DS}\right\|\cos(\theta_K)$	$K_Q = \left\|\vec{K}_{DS}\right\|\sin(\theta_K)$
$K_D = (29\,\text{lb}/\text{in})\cos(174°)$	$K_Q = (29\,\text{lb}/\text{in})\sin(174°)$
$K_D = -28.8\,\text{lb}/\text{in}$	$K_Q = +3.0\,\text{lb}/\text{in}$

Calculating Synchronous Dynamic Stiffness Over the Entire Speed Range

Using the same data from our previous example, Figure 5, the Direct and Quadrature Dynamic Stiffness have been calculated for each speed sample and are plotted separately. The plots can then be used to find rotor parameters.

The rotor modal spring stiffness, K, can be found from the value of the Direct Dynamic Stiffness at zero rotor speed. The modal mass is then found using this value of K and the rotor speed (in radians/sec) at the balance resonance, where the Direct Dynamic Stiffness is equal to zero:

$$M = \frac{K}{\Omega^2_{res}} \qquad [9]$$

Note the very low slope of the Quadrature Stiffness. The slope of the Synchronous Quadrature Dynamic Stiffness plot
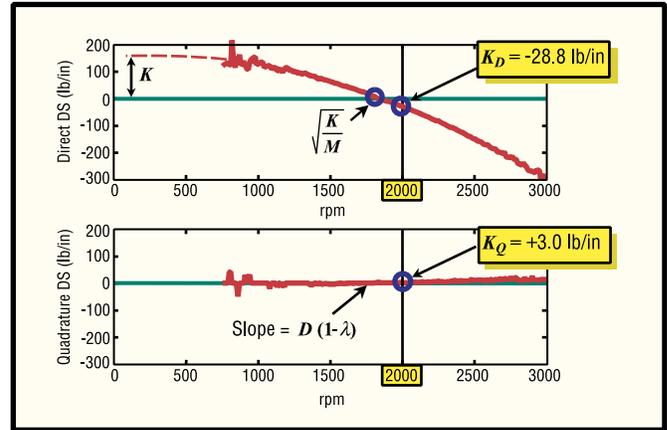
Figure 5. Calculating Synchronous Dynamic Stiffness over the entire speed range.

yields the value of $D(1-\lambda)$. To find D, the value of λ must be estimated[1]. Since the data in this example was obtained from a rotor kit with bronze bearings where there is very little fluid interaction with the rotor, we can set $\lambda = 0$. Because the slope is very low and lambda is zero, we can conclude that the damping is very low for this machine.

Obtaining Rotor Parameters from Two Points

Alternatively, we can estimate rotor parameters using only two points: the resonance and any other point away from the resonance.

Note that, because amplitude and phase can change very rapidly in the vicinity of a resonance, the data there may not be accurate if a narrow bandpass filter was used together with a fast ramp rate. Narrow bandpass filters require long settling times (about 7 seconds for a 12 rpm filter compared to about 0.7 seconds for a 120 rpm filter).

Also, nonlinearities are more likely to occur at resonances because of the higher vibration amplitudes. The effective spring stiffness at a high eccentricity resonance can be different from the spring stiffness at low eccentricity.

For these reasons, in general it is better to calculate rotor parameters from a *complete graph* of Synchronous Dynamic Stiffness over the full range of machine speeds. That way, noisy or bad data can be adjusted for, and anomalies at the resonance can easily be seen.

The following example will use two points from Step 1 of our previous example:

- the balance resonance at 1800 rpm.
- the previously studied data point at 2000 rpm.

Finding Rotor System Modal Stiffness

Two expressions are written for the Direct Dynamic Stiffness that correspond to the two speeds of interest. The Direct Dynamic Stiffness, $\vec{K}_D$, is equal to zero at the balance resonance, and it is equal to some nonzero value at another speed (2000 rpm in this example). This gives us two equations and two unknowns.

$$\text{Resonance} = 1800 \text{ rpm} \qquad 2000 \text{ rpm}$$

$$K_D = K - M\Omega^2_{res} = 0 \qquad K_D = K - M\Omega^2_{2000}$$

One way to solve this system is to eliminate the mass term, M. To do this, solve the two equations for M:

$$\frac{K}{\Omega^2_{res}} = M = \frac{1}{\Omega^2_{2000}}[K - K_D] \qquad [10]$$

Now, eliminating M, the resulting equation is solved for K, the rotor system *modal* stiffness. The solution is found to be:

$$K = \frac{K_D}{\left[1 - \left(\dfrac{\Omega^2_{2000}}{\Omega^2_{res}}\right)\right]} \qquad [11]$$

Finally, plug in the numbers.

$$K = \frac{-28.8 \text{ lb/in}}{\left[1 - \left(\dfrac{2000 \text{ rpm}}{1800 \text{ rpm}}\right)^2\right]} = 120 \text{ lb/in} \,(21{,}000 \text{ N/m})$$

What is modal stiffness? Each rotor system balance resonance is associated with a vibration *mode*. A freely vibrating rotor will usually have several natural frequencies of vibration. Each of these frequencies constitutes a mode with its own deflected *mode shape*. For example, first lateral bending mode is often bow-shaped, and the second bending mode is often s-shaped. Because the s-shaped mode has an effective nodal point near the midspan of the rotor, the rotor has much greater resistance to bending in the second mode; thus, the *modal stiffness* is higher.

Similarly, the two modes will have different distributions of moving mass; thus, the *modal mass* for the second mode is lower than for the first mode.

Because of the higher modal stiffness and the lower modal mass of the second mode, the natural frequency, $\sqrt{K/M}$, of the second mode is higher.

The K, M, D, and λ in our models represent *modal* parameters.

Finding Rotor System Modal Mass

Start with the basic relation for the natural frequency of the system. Then solve for M. We already know the modal spring stiffness, K, from our last calculation.

Substitute known K into (previous mass) Equation 9:

$$M = \frac{K}{\Omega^2_{res}} = \frac{120 \text{ lb/in}}{(188 \text{ rad/s})^2} = 0.0034 \text{ lb} \bullet \text{s}^2/\text{in} \,(0.59 \text{ kg})$$

Finding Rotor System Modal Damping

1) Use the Quadrature Dynamic Stiffness expression (refer to Equation 6):

$$K_Q = D(1 - \lambda)\Omega \qquad [12]$$

2) For this machine (rotor kit with bronze bearings), we have already noted that $\lambda = 0$. Also, Synchronous Quadrature Dynamic Stiffness at zero speed (i.e., $\Omega = 0$) is zero.

Damping is: $\quad D(1 - \lambda) = \dfrac{K_Q}{\Omega} = \dfrac{3.0 \text{ lb/in}}{209 \text{ rad/s}} \qquad [13]$

$$D = 0.014 \text{ lb} \bullet \text{s/in} \qquad (2.5 \text{ N} \bullet \text{s/m})$$

λ (lambda) must be estimated when *significant* fluid interaction with the rotor is present; that is, fluid that is trapped in a way which allows fluid swirling to take place. This can happen, for example, in fluid bearings, seals, or around pump impellers.

Synchronous Dynamic Stiffness and the Influence Vector

The Synchronous Dynamic Stiffness, $\vec{K}_{DS}$, and the Influence Vector, $\vec{H}$, are very closely related. Dynamic Stiffness is the ratio of the Force to the Response, while the Influence Vector is the ratio of the Response to the unbalance weight vector. In fact, one can be calculated from the other.

The relationship can be derived as shown below:

Dynamic Stiffness	Influence Vector
$\vec{K}_{DS} = \dfrac{mr_u\Omega^2 e^{j\delta}}{Ae^{j\alpha}} \quad \boxed{\text{Force}} \atop \boxed{\text{Response}}$	$\vec{H} = \dfrac{Ae^{j\alpha}}{me^{j\delta}} \quad \boxed{\text{Response}} \atop \boxed{\text{Weight}}$

Solve both equations for the response vector:

$$Ae^{j\alpha} = \frac{m r_u \Omega^2 e^{j\delta}}{\vec{K}_{DS}} = \vec{H} m e^{j\delta}$$

[14]

Here, m is the known calibration weight, r_u is the radius of the calibration weight that is mounted on the machine, Ω is the rotor speed, α is the response phase, and δ is the phase (angular mounting location) of the calibration weight. Now the equation can be solved to show the relationship of the Synchronous Dynamic Stiffness to the Influence Vector. Remember that both the Synchronous Dynamic Stiffness and the Influence Vector are vectors with amplitude and phase.

Relationship Between Synchronous Dynamic Stiffness and Influence Vector

Further simplifying Equation 14, we find:

$$\vec{K}_{DS} = \frac{r_u \Omega^2}{\vec{H}} = \frac{r_u \Omega^2}{H e^{j\theta_H}} = \frac{r_u \Omega^2}{H} e^{j(-\theta_H)}$$

[15]

$$\vec{K}_{DS} = \frac{r_u \Omega^2}{H} \angle \theta_H$$

[16]

$$\vec{H} = \frac{r_u \Omega^2}{K_{DS}} \angle \theta_K$$

[17]

Notice! Influence Vectors are usually calculated using positive phase lag data. The expressions for Dynamic Stiffness require phase lag to be negative. For this reason, the negative sign in Equation 15 has been eliminated in Equations 16 and 17.

The final Equations (16 and 17) show that $\vec{K}_{DS}$ and $\vec{H}$ are inversely related through the acceleration of the unbalance mass. In these equations, H is the *magnitude* of the Influence vector, $\vec{H}$, and K_{DS} is the *magnitude* of the Synchronous Dynamic Stiffness, $\vec{K}_{DS}$.

The phase angles θ_H and θ_K are the angles of the Influence Vector and Synchronous Dynamic Stiffness, respectively. $r_u \Omega^2$ is the centripetal acceleration of the calibration weight, and it is the connection between the two vectors.

Example: Calculating the Dynamic Stiffness from the Influence Vector

The Influence Vector has units of peak-to-peak that must be converted to zero-to-peak for the Dynamic Stiffness. There is a factor of ½ in the denominator to do this.

Given:

Influence vector	$\vec{H}$ =	4.68 mil pp/gram ∠46° lag
Radius of unbalance	r_u =	30 mm
Rotative speed	Ω =	2000 rpm

Find: $\vec{K}_{DS}$

Solution: Use Equation 16 as shown below

$$\vec{K}_{DS} = \frac{r_u \Omega^2}{H} \angle \theta_H$$

[16]

$$= \frac{(30 \times 10^{-3}\,\text{m}) \left[\left(2000\,\frac{\text{rev}}{\text{min}}\right) \left(2\pi\,\frac{\text{rad}}{\text{rev}}\right) \left(\frac{1\,\text{min}}{60\,\text{s}}\right) \right]^2}{\left(4.68\,\frac{\text{mil pp}}{\text{gram}}\right) \left(\frac{1}{2}\right) \left(\frac{1\,\text{in}}{1000\,\text{mil}}\right) \left(2.54 \times 10^{-2}\,\frac{\text{m}}{\text{in}}\right) \left(\frac{1000\,\text{grams}}{\text{kg}}\right)} \angle 46°$$

Metric result $\vec{K}_{DS} = 22{,}100\,\frac{\text{N}}{\text{m}} \angle 46°$ $\left[\left(\frac{1\,\text{lb}}{4.448\,\text{N}}\right) \left(2.54 \times 10^{-2}\,\frac{\text{m}}{\text{in}}\right) \right]$ Convert to English

$$\vec{K}_{DS} = 126\,\text{lb/in} \angle 46°$$

Estimating Forces Using Dynamic Stiffness

Recall that Dynamic Stiffness is the ratio of the force to the response from Equation 3:

$$\vec{K}_{DS} = \frac{\vec{F}}{\vec{R}}$$

[3]

This expression can be rearranged to give us an equation for estimating the force acting in a machine.

$$\vec{F} = \vec{K}_{DS} \vec{R}$$

[18]

Newly measured Response

Original Dynamic Stiffness

This is the procedure:

- The Dynamic Stiffness is calculated at some point in time, for example when the machine is balanced.
- The value of the Dynamic Stiffness is saved for future reference.
- At some *later* time, the machine vibration is measured using the same transducer that was used to calculate the Dynamic Stiffness.
- That filtered vibration vector is multiplied by the previously measured Dynamic Stiffness vector to obtain an estimate of the force acting in the machine. This method can be used with either shaft relative or casing transducers.

It is important to keep in mind that the original Dynamic Stiffness was calculated using a force that occurred in the machine *at a specific location* (for example, a balancing plane) on the rotor. At some other time, the force acting on the machine may originate in a *different* location on the rotor. Because of the different transmission path, the Dynamic Stiffness may not be exactly the same between this new force location and the measurement transducer. Thus, this method only provides an *estimate* of the dynamic force acting in the machine.

Summary

- The Synchronous Dynamic Stiffness is simply the ratio of the applied force to the rotor system response.
- Rotor parameters can be calculated from the Synchronous Dynamic Stiffness.
- Synchronous Dynamic Stiffness and Influence Vectors can be calculated from each other.
- Both Dynamic Stiffness and Influence Vectors can be calculated every time a balance shot is performed.

Changes in machine behavior are a *result* of changes in Dynamic Stiffness or changes in force. For example, suppose a balance resonance shifts downward over time. Because the resonant frequency is $\sqrt{K / M}$, either the modal stiffness, K, has decreased, or the modal mass, M, has increased. A significant change in mass of the rotor is unlikely, so it is probable that something has changed that has lowered the modal stiffness of the rotor system. It could be a loosening foundation or bearing support, a reduction in bearing stiffness, or a shaft crack.

As we have shown, the calculations for determining Dynamic Stiffness and various rotor parameters are no more involved than those for rotor balancing. Like balancing, command of the basic principles in Dynamic Stiffness is an important part of the machinery engineer's available tools. ↻