CHAPTER

2 Mathematical Models of Systems

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PREVIEW

Mathematical models of physical systems are key elements in the design and analysis of control systems. The dynamic behavior is generally described by ordinary differential equations. We will consider a wide range of systems, including mechanical, hydraulic, and electrical. Since most physical systems are nonlinear, we will discuss linearization approximations, which allow us to use Laplace transform methods. We will then proceed to obtain the input–output relationship for components and subsystems in the form of transfer functions. The transfer function blocks can be organized into block diagrams or signal-flow graphs to graphically depict the interconnections. Block diagrams (and signal-flow graphs) are very convenient and natural tools for designing and analyzing complicated control systems. We conclude the chapter by developing transfer function models for the various components of the Sequential Design Example: Disk Drive Read System.

DESIRED OUTCOMES

Upon completion of Chapter 2, students should:

- ❑ Recognize that differential equations can describe the dynamic behavior of physical systems.
- ❑ Be able to utilize linearization approximations through the use of Taylor series expansions.
- ❑ Understand the application of Laplace transforms and their role in obtaining transfer functions.
- ❑ Be aware of block diagrams (and signal-flow graphs) and their role in analyzing control systems.
- ❑ Understand the important role of modeling in the control system design process.

2.1 INTRODUCTION

To understand and control complex systems, one must obtain quantitative **mathematical models** of these systems. It is necessary therefore to analyze the relationships between the system variables and to obtain a mathematical model. Because the systems under consideration are dynamic in nature, the descriptive equations are usually **differential equations**. Furthermore, if these equations can be **linearized**, then the **Laplace transform** can be used to simplify the method of solution. In practice, the complexity of systems and our ignorance of all the relevant factors necessitate the introduction of **assumptions** concerning the system operation. Therefore we will often find it useful to consider the physical system, express any necessary assumptions, and linearize the system. Then, by using the physical laws describing the linear equivalent system, we can obtain a set of linear differential equations. Finally, using mathematical tools, such as the Laplace transform, we obtain a solution describing the operation of the system. In summary, the approach to dynamic system modeling can be listed as follows:

1. Define the system and its components.
2. Formulate the mathematical model and fundamental necessary assumptions based on basic principles.
3. Obtain the differential equations representing the mathematical model.
4. Solve the equations for the desired output variables.
5. Examine the solutions and the assumptions.
6. If necessary, reanalyze or redesign the system.

2.2 DIFFERENTIAL EQUATIONS OF PHYSICAL SYSTEMS

The differential equations describing the dynamic performance of a physical system are obtained by utilizing the physical laws of the process [1–3]. This approach applies equally well to mechanical [1], electrical [3], fluid, and thermodynamic systems [4]. Consider the torsional spring–mass system in Figure 2.1 with applied torque $T_a(t)$. Assume the torsional spring element is massless. Suppose we want to measure the torque $T_s(t)$ transmitted to the mass m. Since the spring is massless, the sum of the torques acting on the spring itself must be zero, or

$$T_a(t) - T_s(t) = 0,$$

which implies that $T_s(t) = T_a(t)$. We see immediately that the external torque $T_a(t)$ applied at the end of the spring is transmitted *through* the torsional spring. Because of this, we refer to the torque as a **through-variable**. In a similar manner, the angular rate difference associated with the torsional spring element is

$$\omega(t) = \omega_s(t) - \omega_a(t).$$

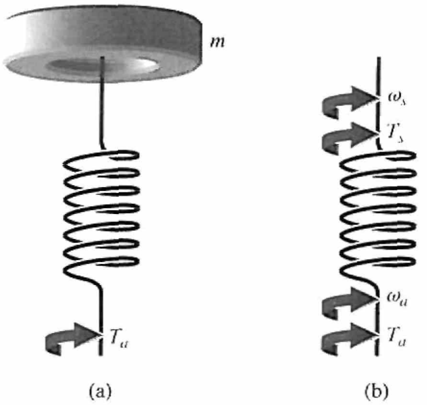


FIGURE 2.1 (a) Torsional spring–mass system. (b) Spring element.

Thus, the angular rate difference is measured across the torsional spring element and is referred to as an **across-variable**. These same types of arguments can be made for most common physical variables (such as force, current, volume, flow rate, etc.). A more complete discussion on through- and across-variables can be found in [26, 27]. A summary of the through- and across-variables of dynamic systems is given in Table 2.1 [5]. Information concerning the International System (SI) of units associated with the various variables discussed in this section can be found at the MCS website.† For example, variables that measure temperature are degrees Kelvin in SI units, and variables that measure length are meters. Important conversions between SI and English units are also given at the MCS website. A summary of the describing equations for lumped,

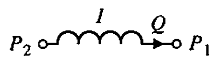
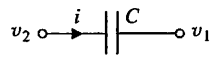
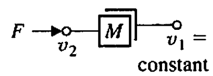
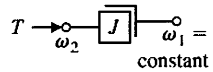
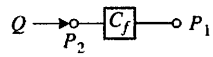
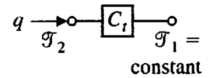
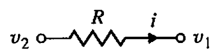
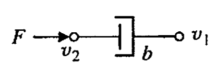
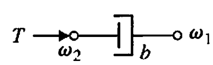
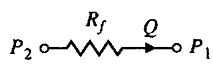
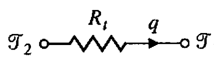
Table 2.1 Summary of Through- and Across-Variables for Physical Systems

System	Variable Through Element	Integrated Through-Variable	Variable Across Element	Integrated Across-Variable
Electrical	Current, i	Charge, q	Voltage difference, v_{21}	Flux linkage, λ_{21}
Mechanical translational	Force, F	Translational momentum, P	Velocity difference, v_{21}	Displacement difference, y_{21}
Mechanical rotational	Torque, T	Angular momentum, h	Angular velocity difference, ω_{21}	Angular displacement difference, θ_{21}
Fluid	Fluid volumetric rate of flow, Q	Volume, V	Pressure difference, P_{21}	Pressure momentum, γ_{21}
Thermal	Heat flow rate, q	Heat energy, H	Temperature difference, $\mathcal{T}_{21}$	

† The companion website is found at www.pearsonhighered.com/dorf.

linear, dynamic elements is given in Table 2.2 [5]. The equations in Table 2.2 are idealized descriptions and only approximate the actual conditions (for example, when a linear, lumped approximation is used for a distributed element).

Table 2.2 Summary of Governing Differential Equations for Ideal Elements

Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Inductive storage	Electrical inductance	$v_{21} = L\frac{di}{dt}$	$E = \frac{1}{2}Li^2$	
	Translational spring	$v_{21} = \frac{1}{k}\frac{dF}{dt}$	$E = \frac{1}{2}\frac{F^2}{k}$	
	Rotational spring	$\omega_{21} = \frac{1}{k}\frac{dT}{dt}$	$E = \frac{1}{2}\frac{T^2}{k}$	
	Fluid inertia	$P_{21} = I\frac{dQ}{dt}$	$E = \frac{1}{2}IQ^2$	
Capacitive storage	Electrical capacitance	$i = C\frac{dv_{21}}{dt}$	$E = \frac{1}{2}Cv_{21}^2$	
	Translational mass	$F = M\frac{dv_2}{dt}$	$E = \frac{1}{2}Mv_2^2$	
	Rotational mass	$T = J\frac{d\omega_2}{dt}$	$E = \frac{1}{2}J\omega_2^2$	
	Fluid capacitance	$Q = C_f\frac{dP_{21}}{dt}$	$E = \frac{1}{2}C_fP_{21}^2$	
	Thermal capacitance	$q = C_t\frac{d\mathcal{T}_2}{dt}$	$E = C_t\mathcal{T}_2$	
Energy dissipators	Electrical resistance	$i = \frac{1}{R}v_{21}$	$\mathcal{P} = \frac{1}{R}v_{21}^2$	
	Translational damper	$F = bv_{21}$	$\mathcal{P} = bv_{21}^2$	
	Rotational damper	$T = b\omega_{21}$	$\mathcal{P} = b\omega_{21}^2$	
	Fluid resistance	$Q = \frac{1}{R_f}P_{21}$	$\mathcal{P} = \frac{1}{R_f}P_{21}^2$	
	Thermal resistance	$q = \frac{1}{R_t}\mathcal{T}_{21}$	$\mathcal{P} = \frac{1}{R_t}\mathcal{T}_{21}$	

Nomenclature

- *Through-variable:* F = force, T = torque, i = current, Q = fluid volumetric flow rate, q = heat flow rate.
- *Across-variable:* v = translational velocity, ω = angular velocity, v = voltage, P = pressure, $\mathcal{T}$ = temperature.
- *Inductive storage:* L = inductance, $1/k$ = reciprocal translational or rotational stiffness, I = fluid inertance.
- *Capacitive storage:* C = capacitance, M = mass, J = moment of inertia, C_f = fluid capacitance, C_t = thermal capacitance.
- *Energy dissipators:* R = resistance, b = viscous friction, R_f = fluid resistance, R_t = thermal resistance.

The symbol v is used for both voltage in electrical circuits and velocity in translational mechanical systems and is distinguished within the context of each differential equation. For mechanical systems, one uses Newton's laws; for electrical systems, Kirchhoff's voltage laws. For example, the simple spring-mass-damper mechanical system shown in Figure 2.2(a) is described by Newton's second law of motion. (This system could represent, for example, an automobile shock absorber.) The free-body diagram of the mass M is shown in Figure 2.2(b). In this spring-mass-damper example, we model the wall friction as a **viscous damper**, that is, the friction force is linearly proportional to the velocity of the mass. In reality the friction force may behave in a more complicated fashion. For example, the wall friction may behave as a **Coulomb damper**. Coulomb friction, also known as dry friction, is a nonlinear function of the mass velocity and possesses a discontinuity around zero velocity. For a well-lubricated, sliding surface, the viscous friction is appropriate and will be used here and in subsequent spring-mass-damper examples. Summing the forces acting on M and utilizing Newton's second law yields

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t), \tag{2.1}$$

where k is the spring constant of the ideal spring and b is the friction constant. Equation (2.1) is a second-order linear constant-coefficient differential equation.

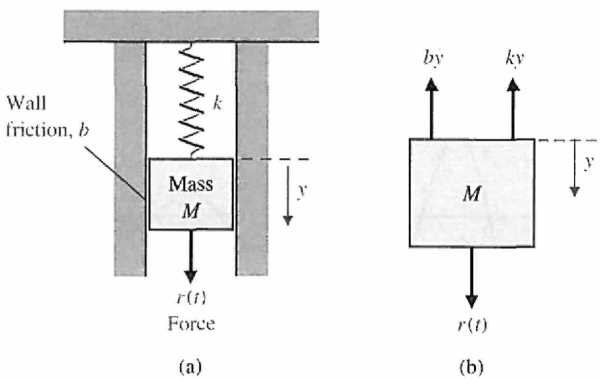


FIGURE 2.2 (a) Spring-mass-damper system. (b) Free-body diagram.

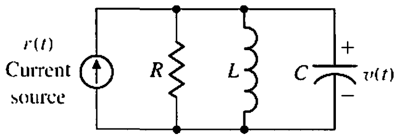


FIGURE 2.3
RLC circuit.

Alternatively, one may describe the electrical *RLC* circuit of Figure 2.3 by utilizing Kirchhoff's current law. Then we obtain the following integrodifferential equation:

$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)\,dt = r(t). \tag{2.2}$$

The solution of the differential equation describing the process may be obtained by classical methods such as the use of integrating factors and the method of undetermined coefficients [1]. For example, when the mass is initially displaced a distance $y(0) = y_0$ and released, the dynamic response of the system can be represented by an equation of the form

$$y(t) = K_1 e^{-\alpha_1 t}\sin(\beta_1 t + \theta_1). \tag{2.3}$$

A similar solution is obtained for the voltage of the *RLC* circuit when the circuit is subjected to a constant current $r(t) = I$. Then the voltage is

$$v(t) = K_2 e^{-\alpha_2 t}\cos(\beta_2 t + \theta_2). \tag{2.4}$$

A voltage curve typical of an *RLC* circuit is shown in Figure 2.4.

To reveal further the close similarity between the differential equations for the mechanical and electrical systems, we shall rewrite Equation (2.1) in terms of velocity:

$$v(t) = \frac{dy(t)}{dt}.$$

Then we have

$$M\frac{dv(t)}{dt} + bv(t) + k\int_0^t v(t)\,dt = r(t). \tag{2.5}$$

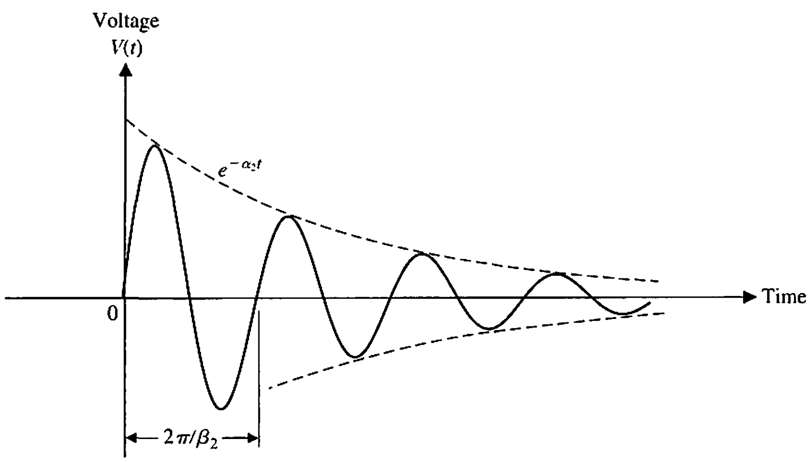


FIGURE 2.4
Typical voltage response for an *RLC* circuit.

One immediately notes the equivalence of Equations (2.5) and (2.2) where velocity $v(t)$ and voltage $v(t)$ are equivalent variables, usually called **analogous variables**, and the systems are analogous systems. Therefore the solution for velocity is similar to Equation (2.4), and the response for an underdamped system is shown in Figure 2.4. The concept of analogous systems is a very useful and powerful technique for system modeling. The voltage–velocity analogy, often called the force–current analogy, is a natural one because it relates the analogous through- and across-variables of the electrical and mechanical systems. Another analogy that relates the velocity and current variables is often used and is called the force–voltage analogy [21, 23].

Analogous systems with similar solutions exist for electrical, mechanical, thermal, and fluid systems. The existence of analogous systems and solutions provides the analyst with the ability to extend the solution of one system to all analogous systems with the same describing differential equations. Therefore what one learns about the analysis and design of electrical systems is immediately extended to an understanding of fluid, thermal, and mechanical systems.

2.3 LINEAR APPROXIMATIONS OF PHYSICAL SYSTEMS

A great majority of physical systems are linear within some range of the variables. In general, systems ultimately become nonlinear as the variables are increased without limit. For example, the spring-mass-damper system of Figure 2.2 is linear and described by Equation (2.1) as long as the mass is subjected to small deflections $y(t)$. However, if $y(t)$ were continually increased, eventually the spring would be overextended and break. Therefore the question of linearity and the range of applicability must be considered for each system.

A system is defined as linear in terms of the system excitation and response. In the case of the electrical network, the excitation is the input current $r(t)$ and the response is the voltage $v(t)$. In general, a **necessary condition** for a linear system can be determined in terms of an excitation $x(t)$ and a response $y(t)$. When the system at rest is subjected to an excitation $x_1(t)$, it provides a response $y_1(t)$. Furthermore, when the system is subjected to an excitation $x_2(t)$, it provides a corresponding response $y_2(t)$. For a linear system, it is necessary that the excitation $x_1(t) + x_2(t)$ result in a response $y_1(t) + y_2(t)$. This is usually called the **principle of superposition.**

Furthermore, the magnitude scale factor must be preserved in a **linear system**. Again, consider a system with an input $x(t)$ that results in an output $y(t)$. Then the response of a linear system to a constant multiple β of an input x must be equal to the response to the input multiplied by the same constant so that the output is equal to βy. This is called the property of **homogeneity**.

A linear system satisfies the properties of superposition and homogeneity.

A system characterized by the relation $y = x^2$ is not linear, because the superposition property is not satisfied. A system represented by the relation $y = mx + b$ is not linear, because it does not satisfy the homogeneity property. However, this second system may be considered linear about an operating point x_0, y_0 for small changes Δx and Δy. When $x = x_0 + \Delta x$ and $y = y_0 + \Delta y$, we have

$$y = mx + b$$

or

$$y_0 + \Delta y = mx_0 + m\,\Delta x + b.$$

Therefore, $\Delta y = m\,\Delta x$, which satisfies the necessary conditions.

The linearity of many mechanical and electrical elements can be assumed over a reasonably large range of the variables [7]. This is not usually the case for thermal and fluid elements, which are more frequently nonlinear in character. Fortunately, however, one can often linearize nonlinear elements assuming small-signal conditions. This is the normal approach used to obtain a linear equivalent circuit for electronic circuits and transistors. Consider a general element with an excitation (through-) variable $x(t)$ and a response (across-) variable $y(t)$. Several examples of dynamic system variables are given in Table 2.1. The relationship of the two variables is written as

$$y(t) = g(x(t)), \tag{2.6}$$

where $g(x(t))$ indicates $y(t)$ is a function of $x(t)$. The normal operating point is designated by x_0. Because the curve (function) is continuous over the range of interest, a **Taylor series** expansion about the operating point may be utilized [7]. Then we have

$$y = g(x) = g(x_0) + \left.\frac{dg}{dx}\right|_{x=x_0} \frac{(x - x_0)}{1!} + \left.\frac{d^2g}{dx^2}\right|_{x=x_0} \frac{(x - x_0)^2}{2!} + \cdots. \tag{2.7}$$

The slope at the operating point,

$$\left.\frac{dg}{dx}\right|_{x=x_0},$$

is a good approximation to the curve over a small range of $(x - x_0)$, the deviation from the operating point. Then, as a reasonable approximation, Equation (2.7) becomes

$$y = g(x_0) + \left.\frac{dg}{dx}\right|_{x=x_0} (x - x_0) = y_0 + m(x - x_0), \tag{2.8}$$

where m is the slope at the operating point. Finally, Equation (2.8) can be rewritten as the linear equation

$$(y - y_0) = m(x - x_0)$$

or

$$\Delta y = m\,\Delta x. \tag{2.9}$$

Consider the case of a mass, M, sitting on a nonlinear spring, as shown in Figure 2.5(a). The normal operating point is the equilibrium position that occurs when the spring force balances the gravitational force Mg, where g is the gravitational constant. Thus, we obtain $f_0 = Mg$, as shown. For the nonlinear spring with $f = y^2$, the equilibrium position is $y_0 = (Mg)^{1/2}$. The linear model for small deviation is

$$\Delta f = m\,\Delta y,$$

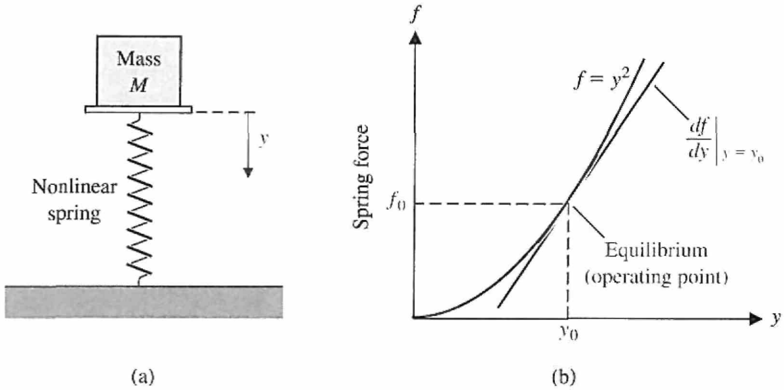


FIGURE 2.5 (a) A mass sitting on a nonlinear spring. (b) The spring force versus *y*.

where

$$m = \frac{df}{dy}\bigg|_{y_0},$$

as shown in Figure 2.5(b). Thus, $m = 2y_0$. A **linear approximation** is as accurate as the assumption of small signals is applicable to the specific problem.

If the dependent variable y depends upon several excitation variables, $x_1, x_2, \ldots, x_n$, then the functional relationship is written as

$$y = g(x_1, x_2, \ldots, x_n). \tag{2.10}$$

The Taylor series expansion about the operating point $x_{1_0}, x_{2_0}, \ldots, x_{n_0}$ is useful for a linear approximation to the nonlinear function. When the higher-order terms are neglected, the linear approximation is written as

$$y = g(x_{1_0}, x_{2_0}, \ldots, x_{n_0}) + \frac{\partial g}{\partial x_1}\bigg|_{x=x_0}(x_1 - x_{1_0}) + \frac{\partial g}{\partial x_2}\bigg|_{x=x_0}(x_2 - x_{2_0}) \tag{2.11}$$
$$+ \cdots + \frac{\partial g}{\partial x_n}\bigg|_{x=x_0}(x_n - x_{n_0}),$$

where x_0 is the operating point. Example 2.1 will clearly illustrate the utility of this method.

EXAMPLE 2.1 **Pendulum oscillator model**

Consider the pendulum oscillator shown in Figure 2.6(a). The torque on the mass is

$$T = MgL \sin\theta, \tag{2.12}$$

where g is the gravity constant. The equilibrium condition for the mass is $\theta_0 = 0°$. The nonlinear relation between T and θ is shown graphically in Figure 2.6(b). The first derivative evaluated at equilibrium provides the linear approximation, which is

$$T - T_0 \cong MgL\frac{\partial \sin\theta}{\partial\theta}\bigg|_{\theta=\theta_0}(\theta - \theta_0),$$

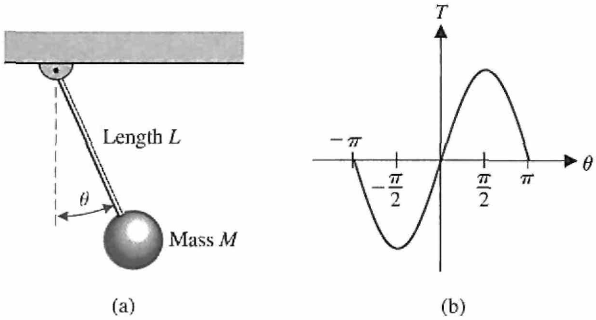


FIGURE 2.6 Pendulum oscillator.

where $T_0 = 0$. Then, we have

$$T = MgL(\cos 0°)(\theta - 0°)$$
$$= MgL\theta. \tag{2.13}$$

This approximation is reasonably accurate for $-\pi/4 \leq \theta \leq \pi/4$. For example, the response of the linear model for the swing through $\pm 30°$ is within 5% of the actual nonlinear pendulum response. ■

2.4 THE LAPLACE TRANSFORM

The ability to obtain linear approximations of physical systems allows the analyst to consider the use of the **Laplace transformation**. The Laplace transform method substitutes relatively easily solved algebraic equations for the more difficult differential equations [1, 3]. The time-response solution is obtained by the following operations:

1. Obtain the linearized differential equations.
2. Obtain the Laplace transformation of the differential equations.
3. Solve the resulting algebraic equation for the transform of the variable of interest.

The Laplace transform exists for linear differential equations for which the transformation integral converges. Therefore, for $f(t)$ to be transformable, it is sufficient that

$$\int_{0^-}^{\infty} |f(t)|e^{-\sigma_1 t}\, dt < \infty,$$

for some real, positive σ_1 [1]. The 0^- indicates that the integral should include any discontinuity, such as a delta function at $t = 0$. If the magnitude of $f(t)$ is $|f(t)| < Me^{\alpha t}$ for all positive t, the integral will converge for $\sigma_1 > \alpha$. The region of convergence is therefore given by $\infty > \sigma_1 > \alpha$, and σ_1 is known as the abscissa of absolute convergence. Signals that are physically realizable always have a Laplace transform. The Laplace transformation for a function of time, $f(t)$, is

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\, dt = \mathcal{L}\{f(t)\}. \tag{2.14}$$

The **inverse Laplace transform** is written as

$$f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{+st}\, ds. \tag{2.15}$$

The transformation integrals have been employed to derive tables of Laplace transforms that are used for the great majority of problems. A table of important Laplace transform pairs is given in Table 2.3, and a more complete list of Laplace transform pairs can be found at the MCS website.

Table 2.3 Important Laplace Transform Pairs

$f(t)$	$F(s)$
Step function, $u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$f^{(k)}(t) = \frac{d^k f(t)}{dt^k}$	$s^k F(s) - s^{k-1} f(0^-) - s^{k-2} f'(0^-) - \ldots - f^{(k-1)}(0^-)$
$\int_{-\infty}^{t} f(t)\, dt$	$\frac{F(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} f(t)\, dt$
Impulse function $\delta(t)$	1
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\frac{1}{\omega}[(\alpha - a)^2 + \omega^2]^{1/2} e^{-at} \sin(\omega t + \phi)$, $\phi = \tan^{-1}\frac{\omega}{\alpha - a}$	$\frac{s+\alpha}{(s+a)^2+\omega^2}$
$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$, $\zeta < 1$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
$\frac{1}{a^2+\omega^2} + \frac{1}{\omega\sqrt{a^2+\omega^2}} e^{-at} \sin(\omega t - \phi)$, $\phi = \tan^{-1}\frac{\omega}{-a}$	$\frac{1}{s[(s+a)^2+\omega^2]}$
$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \phi\right)$, $\phi = \cos^{-1}\zeta$, $\zeta < 1$	$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
$\frac{\alpha}{a^2+\omega^2} + \frac{1}{\omega}\left[\frac{(\alpha - a)^2 + \omega^2}{a^2+\omega^2}\right]^{1/2} e^{-at} \sin(\omega t + \phi)$. $\phi = \tan^{-1}\frac{\omega}{\alpha - a} - \tan^{-1}\frac{\omega}{-a}$	$\frac{s+\alpha}{s[(s+a)^2+\omega^2]}$

Alternatively, the Laplace variable s can be considered to be the differential operator so that

$$s \equiv \frac{d}{dt}. \tag{2.16}$$

Then we also have the integral operator

$$\frac{1}{s} \equiv \int_{0^-}^{t} dt. \tag{2.17}$$

The inverse Laplace transformation is usually obtained by using the Heaviside partial fraction expansion. This approach is particularly useful for systems analysis and design because the effect of each characteristic root or eigenvalue can be clearly observed.

To illustrate the usefulness of the Laplace transformation and the steps involved in the system analysis, reconsider the spring-mass-damper system described by Equation (2.1), which is

$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = r(t). \tag{2.18}$$

We wish to obtain the response, y, as a function of time. The Laplace transform of Equation (2.18) is

$$M\left(s^2Y(s) - sy(0^-) - \frac{dy}{dt}(0^-)\right) + b(sY(s) - y(0^-)) + kY(s) = R(s). \tag{2.19}$$

When

$$r(t) = 0, \quad \text{and} \quad y(0^-) = y_0, \quad \text{and} \quad \left.\frac{dy}{dt}\right|_{t=0^-} = 0,$$

we have

$$Ms^2Y(s) - Msy_0 + bsY(s) - by_0 + kY(s) = 0. \tag{2.20}$$

Solving for $Y(s)$, we obtain

$$Y(s) = \frac{(Ms + b)y_0}{Ms^2 + bs + k} = \frac{p(s)}{q(s)}. \tag{2.21}$$

The denominator polynomial $q(s)$, when set equal to zero, is called the **characteristic equation** because the roots of this equation determine the character of the time response. The roots of this characteristic equation are also called the **poles** of the system. The roots of the numerator polynomial $p(s)$ are called the **zeros** of the system; for example, $s = -b/M$ is a zero of Equation (2.21). Poles and zeros are critical frequencies. At the poles, the function $Y(s)$ becomes infinite, whereas at the zeros, the function becomes zero. The complex frequency **s-plane** plot of the poles and zeros graphically portrays the character of the natural transient response of the system.

For a specific case, consider the system when $k/M = 2$ and $b/M = 3$. Then Equation (2.21) becomes

$$Y(s) = \frac{(s + 3)y_0}{(s + 1)(s + 2)}. \tag{2.22}$$

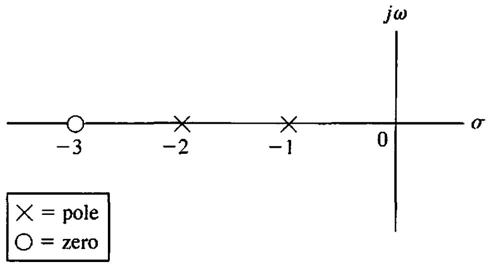


FIGURE 2.7 An s-plane pole and zero plot.

The poles and zeros of $Y(s)$ are shown on the s-plane in Figure 2.7.

Expanding Equation (2.22) in a partial fraction expansion, we obtain

$$Y(s) = \frac{k_1}{s+1} + \frac{k_2}{s+2}, \tag{2.23}$$

where k_1 and k_2 are the coefficients of the expansion. The coefficients k_i are called **residues** and are evaluated by multiplying through by the denominator factor of Equation (2.22) corresponding to k_i and setting s equal to the root. Evaluating k_1 when $y_0 = 1$, we have

$$k_1 = \left.\frac{(s - s_1)p(s)}{q(s)}\right|_{s=s_1} \tag{2.24}$$

$$= \left.\frac{(s+1)(s+3)}{(s+1)(s+2)}\right|_{s_1=-1} = 2$$

and $k_2 = -1$. Alternatively, the residues of $Y(s)$ at the respective poles may be evaluated graphically on the s-plane plot, since Equation (2.24) may be written as

$$k_1 = \left.\frac{s+3}{s+2}\right|_{s=s_1=-1} \tag{2.25}$$

$$= \left.\frac{s_1+3}{s_1+2}\right|_{s_1=-1} = 2.$$

The graphical representation of Equation (2.25) is shown in Figure 2.8. The graphical method of evaluating the residues is particularly valuable when the order of the characteristic equation is high and several poles are complex conjugate pairs.

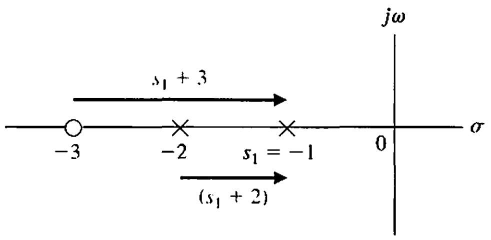


FIGURE 2.8 Graphical evaluation of the residues.

The inverse Laplace transform of Equation (2.22) is then

$$y(t) = \mathscr{L}^{-1}\left\{\frac{2}{s+1}\right\} + \mathscr{L}^{-1}\left\{\frac{-1}{s+2}\right\}. \tag{2.26}$$

Using Table 2.3, we find that

$$y(t) = 2e^{-t} - 1e^{-2t}. \tag{2.27}$$

Finally, it is usually desired to determine the **steady-state** or **final value** of the response of $y(t)$. For example, the final or steady-state rest position of the spring-mass-damper system may be calculated. The **final value theorem** states that

$$\boxed{\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s),} \tag{2.28}$$

where a simple pole of $Y(s)$ at the origin is permitted, but poles on the imaginary axis and in the right half-plane and repeated poles at the origin are excluded. Therefore, for the specific case of the spring-mass-damper, we find that

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = 0. \tag{2.29}$$

Hence the final position for the mass is the normal equilibrium position $y = 0$.

Reconsider the spring-mass-damper system. The equation for $Y(s)$ may be written as

$$Y(s) = \frac{(s + b/M)y_0}{s^2 + (b/M)s + k/M} = \frac{(s + 2\zeta\omega_n)y_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{2.30}$$

where ζ is the dimensionless **damping ratio**, and ω_n is the **natural frequency** of the system. The roots of the characteristic equation are

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}, \tag{2.31}$$

where, in this case, $\omega_n = \sqrt{k/M}$ and $\zeta = b/(2\sqrt{kM})$. When $\zeta > 1$, the roots are real and the system is **overdamped**; when $\zeta < 1$, the roots are complex and the system is **underdamped**. When $\zeta = 1$, the roots are repeated and real, and the condition is called **critical damping**.

When $\zeta < 1$, the response is underdamped, and

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}. \tag{2.32}$$

The s-plane plot of the poles and zeros of $Y(s)$ is shown in Figure 2.9, where $\theta = \cos^{-1}\zeta$. As ζ varies with ω_n constant, the complex conjugate roots follow a circular

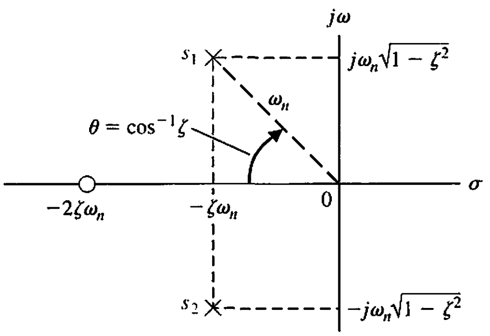


FIGURE 2.9 An s-plane plot of the poles and zeros of $Y(s)$.

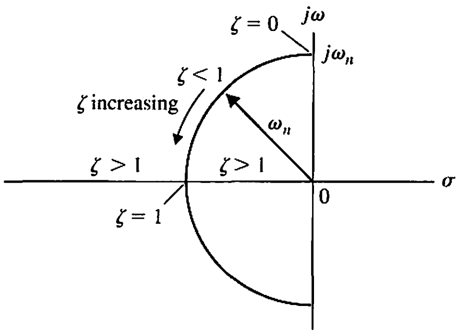


FIGURE 2.10 The locus of roots as ζ varies with ω_n constant.

locus, as shown in Figure 2.10. The transient response is increasingly oscillatory as the roots approach the imaginary axis when ζ approaches zero.

The inverse Laplace transform can be evaluated using the graphical residue evaluation. The partial fraction expansion of Equation (2.30) is

$$Y(s) = \frac{k_1}{s - s_1} + \frac{k_2}{s - s_2}. \tag{2.33}$$

Since s_2 is the complex conjugate of s_1, the residue k_2 is the complex conjugate of k_1 so that we obtain

$$Y(s) = \frac{k_1}{s - s_1} + \frac{k_1^*}{s - s_1^*},$$

where the asterisk indicates the conjugate relation. The residue k_1 is evaluated from Figure 2.11 as

$$k_1 = \frac{y_0(s_1 + 2\zeta\omega_n)}{s_1 - s_1^*} = \frac{y_0 M_1 e^{j\theta}}{M_2 e^{j\pi/2}}, \tag{2.34}$$

where M_1 is the magnitude of $s_1 + 2\zeta\omega_n$, and M_2 is the magnitude of $s_1 - s_1^*$. (A review of complex numbers can be found on the MCS website.) In this case, we obtain

$$k_1 = \frac{y_0(\omega_n e^{j\theta})}{2\omega_n\sqrt{1 - \zeta^2}e^{j\pi/2}} = \frac{y_0}{2\sqrt{1 - \zeta^2}e^{j(\pi/2 - \theta)}}, \tag{2.35}$$

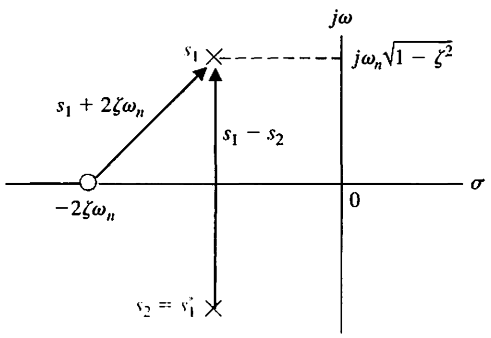


FIGURE 2.11 Evaluation of the residue k_1.

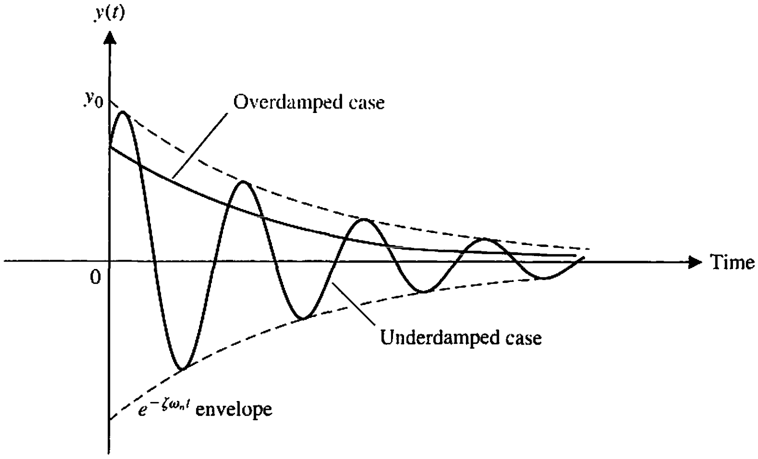


FIGURE 2.12 Response of the spring-mass-damper system.

where $\theta = \cos^{-1}\zeta$. Therefore,

$$k_2 = \frac{y_0}{2\sqrt{1-\zeta^2}} e^{j(\pi/2-\theta)}. \tag{2.36}$$

Finally, letting $\beta = \sqrt{1-\zeta^2}$, we find that

$$\begin{aligned} y(t) &= k_1 e^{s_1 t} + k_2 e^{s_2 t} \\ &= \frac{y_0}{2\sqrt{1-\zeta^2}} \left(e^{j(\theta-\pi/2)} e^{-\zeta\omega_n t} e^{j\omega_n \beta t} + e^{j(\pi/2-\theta)} e^{-\zeta\omega_n t} e^{-j\omega_n \beta t}\right) \\ &= \frac{y_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2}\,t + \theta). \end{aligned} \tag{2.37}$$

The solution, Equation (2.37), can also be obtained using item 11 of Table 2.3. The transient responses of the overdamped ($\zeta > 1$) and underdamped ($\zeta < 1$) cases are shown in Figure 2.12. The transient response that occurs when $\zeta < 1$ exhibits an oscillation in which the amplitude decreases with time, and it is called a **damped oscillation**.

The relationship between the s-plane location of the poles and zeros and the form of the transient response can be interpreted from the s-plane pole–zero plots. For example, as seen in Equation (2.37), adjusting the value of $\zeta\omega_n$ varies the $e^{-\zeta\omega_n t}$ envelope, hence the response $y(t)$ shown in Figure 2.12. The larger the value of $\zeta\omega_n$, the faster the damping of the response, $y(t)$. In Figure 2.9, we see that the location of the complex pole s_1 is given by $s_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$. So, making $\zeta\omega_n$ larger moves the pole further to the left in the s-plane. Thus, the connection between the location of the pole in the s-plane and the step response is apparent—moving the pole s_1 farther in the left half-plane leads to a faster damping of the transient step response. Of course, most control systems will have more than one complex pair of poles, so the transient response will be the result of the contributions of all the poles. In fact, the magnitude of the response of each pole, represented by the residue, can be visualized by examining the graphical residues on the s-plane. We will discuss the connection between the pole and zero locations and the transient and steady-state response more in subsequent chapters. We will find that the Laplace transformation and the s-plane approach are very useful techniques for system analysis and design where emphasis is placed on the transient and steady-state performance. In fact, because the study of control systems is concerned primarily with the transient and steady-state performance of dynamic systems, we have real cause to appreciate the value of the Laplace transform techniques.

2.5 THE TRANSFER FUNCTION OF LINEAR SYSTEMS

The **transfer function** of a linear system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero. The transfer function of a system (or element) represents the relationship describing the dynamics of the system under consideration.

A transfer function may be defined only for a linear, stationary (constant parameter) system. A nonstationary system, often called a time-varying system, has one or more time-varying parameters, and the Laplace transformation may not be utilized. Furthermore, a transfer function is an input–output description of the behavior of a system. Thus, the transfer function description does not include any information concerning the internal structure of the system and its behavior.

The transfer function of the spring-mass-damper system is obtained from the original Equation (2.19), rewritten with zero initial conditions as follows:

$$Ms^2Y(s) + bsY(s) + kY(s) = R(s). \tag{2.38}$$

Then the transfer function is

$$\frac{\text{Output}}{\text{Input}} = G(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + bs + k}. \tag{2.39}$$

The transfer function of the RC network shown in Figure 2.13 is obtained by writing the Kirchhoff voltage equation, yielding

$$V_1(s) = \left(R + \frac{1}{Cs}\right)I(s), \tag{2.40}$$

expressed in terms of transform variables. We shall frequently refer to variables and their transforms interchangeably. The transform variable will be distinguishable by the use of an uppercase letter or the argument (s).

The output voltage is

$$V_2(s) = I(s)\left(\frac{1}{Cs}\right). \tag{2.41}$$

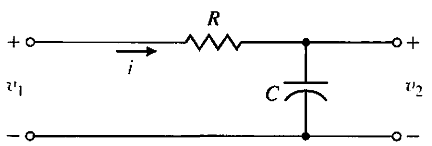


FIGURE 2.13
An *RC* network.

Therefore, solving Equation (2.40) for $I(s)$ and substituting in Equation (2.41), we have

$$V_2(s) = \frac{(1/Cs)V_1(s)}{R + 1/Cs}.$$

Then the transfer function is obtained as the ratio $V_2(s)/V_1(s)$, which is

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{RCs + 1} = \frac{1}{\tau s + 1} = \frac{1/\tau}{s + 1/\tau}, \tag{2.42}$$

where $\tau = RC$, the **time constant** of the network. The single pole of $G(s)$ is $s = -1/\tau$. Equation (2.42) could be immediately obtained if one observes that the circuit is a voltage divider, where

$$\frac{V_2(s)}{V_1(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}, \tag{2.43}$$

and $Z_1(s) = R, Z_2 = 1/Cs$.

A multiloop electrical circuit or an analogous multiple-mass mechanical system results in a set of simultaneous equations in the Laplace variable. It is usually more convenient to solve the simultaneous equations by using matrices and determinants [1, 3, 15]. An introduction to matrices and determinants can be found on the MCS website.

Let us consider the long-term behavior of a system and determine the response to certain inputs that remain after the transients fade away. Consider the dynamic system represented by the differential equation

$$\frac{d^n y}{dt^n} + q_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + q_0 y = p_{n-1}\frac{d^{n-1}r}{dt^{n-1}} + p_{n-2}\frac{d^{n-2}r}{dt^{n-2}} + \cdots + p_0 r, \tag{2.44}$$

where $y(t)$ is the response, and $r(t)$ is the input or forcing function. If the initial conditions are all zero, then the transfer function is the coefficient of $R(s)$ in

$$Y(s) = G(s)R(s) = \frac{p(s)}{q(s)}R(s) = \frac{p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_0}{s^n + q_{n-1}s^{n-1} + \cdots + q_0}R(s). \tag{2.45}$$

The output response consists of a natural response (determined by the initial conditions) plus a forced response determined by the input. We now have

$$Y(s) = \frac{m(s)}{q(s)} + \frac{p(s)}{q(s)}R(s),$$

where $q(s) = 0$ is the characteristic equation. If the input has the rational form

$$R(s) = \frac{n(s)}{d(s)},$$

then

$$Y(s) = \frac{m(s)}{q(s)} + \frac{p(s)}{q(s)}\frac{n(s)}{d(s)} = Y_1(s) + Y_2(s) + Y_3(s), \tag{2.46}$$

where $Y_1(s)$ is the partial fraction expansion of the natural response, $Y_2(s)$ is the partial fraction expansion of the terms involving factors of $q(s)$, and $Y_3(s)$ is the partial fraction expansion of terms involving factors of $d(s)$.

Taking the inverse Laplace transform yields

$$y(t) = y_1(t) + y_2(t) + y_3(t).$$

The transient response consists of $y_1(t) + y_2(t)$, and the steady-state response is $y_3(t)$.

EXAMPLE 2.2 **Solution of a differential equation**

Consider a system represented by the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2r(t),$$

where the initial conditions are $y(0) = 1, \frac{dy}{dt}(0) = 0$, and $r(t) = 1, t \geq 0$.

The Laplace transform yields

$$[s^2Y(s) - sy(0)] + 4[sY(s) - y(0)] + 3Y(s) = 2R(s).$$

Since $R(s) = 1/s$ and $y(0) = 1$, we obtain

$$Y(s) = \frac{s + 4}{s^2 + 4s + 3} + \frac{2}{s(s^2 + 4s + 3)},$$

where $q(s) = s^2 + 4s + 3 = (s + 1)(s + 3) = 0$ is the characteristic equation, and $d(s) = s$. Then the partial fraction expansion yields

$$Y(s) = \left[\frac{3/2}{s + 1} + \frac{-1/2}{s + 3}\right] + \left[\frac{-1}{s + 1} + \frac{1/3}{s + 3}\right] + \frac{2/3}{s} = Y_1(s) + Y_2(s) + Y_3(s).$$

Hence, the response is

$$y(t) = \left[\frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}\right] + \left[-1e^{-t} + \frac{1}{3}e^{-3t}\right] + \frac{2}{3},$$

and the steady-state response is

$$\lim_{t\to\infty} y(t) = \frac{2}{3}. \blacksquare$$

EXAMPLE 2.3 **Transfer function of an op-amp circuit**

The operational amplifier (op-amp) belongs to an important class of analog integrated circuits commonly used as building blocks in the implementation of control systems and in many other important applications. Op-amps are active elements (that is, they have external power sources) with a high gain when operating in their linear regions. A model of an ideal op-amp is shown in Figure 2.14.

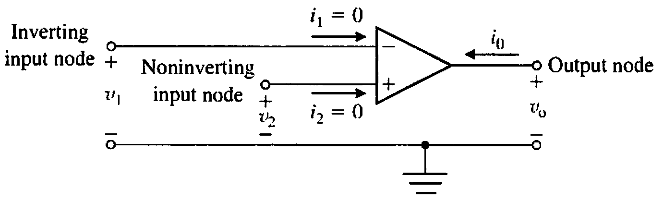


FIGURE 2.14 The ideal op-amp.

The operating conditions for the ideal op-amp are (1) $i_1 = 0$ and $i_2 = 0$, thus implying that the input impedance is infinite, and (2) $v_2 - v_1 = 0$ (or $v_1 = v_2$). The input–output relationship for an ideal op-amp is

$$v_0 = K(v_2 - v_1) = -K(v_1 - v_2),$$

where the gain K approaches infinity. In our analysis, we will assume that the linear op-amps are operating with high gain and under idealized conditions.

Consider the inverting amplifier shown in Figure 2.15. Under ideal conditions, we have $i_1 = 0$, so that writing the node equation at v_1 yields

$$\frac{v_1 - v_{in}}{R_1} + \frac{v_1 - v_0}{R_2} = 0.$$

Since $v_2 = v_1$ (under ideal conditions) and $v_2 = 0$ (see Figure 2.15 and compare it with Figure 2.14), it follows that $v_1 = 0$. Therefore,

$$-\frac{v_{in}}{R_1} - \frac{v_0}{R_2} = 0,$$

and rearranging terms, we obtain

$$\frac{v_0}{v_{in}} = -\frac{R_2}{R_1}.$$

We see that when $R_2 = R_1$, the ideal op-amp circuit inverts the sign of the input, that is, $v_0 = -v_{in}$ when $R_2 = R_1$. ■

EXAMPLE 2.4 **Transfer function of a system**

Consider the mechanical system shown in Figure 2.16 and its electrical circuit analog shown in Figure 2.17. The electrical circuit analog is a force–current analog as outlined in Table 2.1. The velocities $v_1(t)$ and $v_2(t)$ of the mechanical system are directly

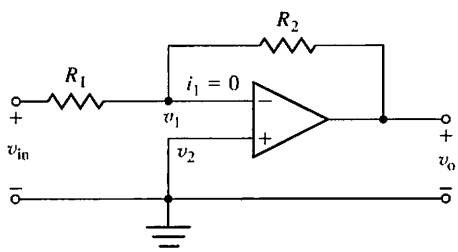


FIGURE 2.15 An inverting amplifier operating with ideal conditions.

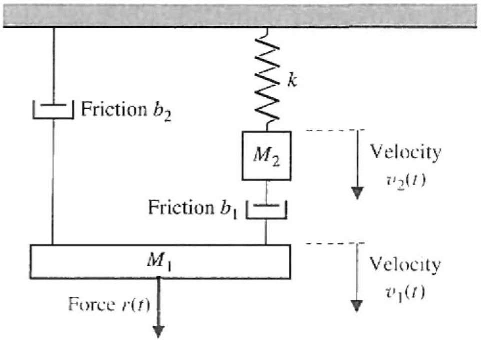


FIGURE 2.16 Two-mass mechanical system.

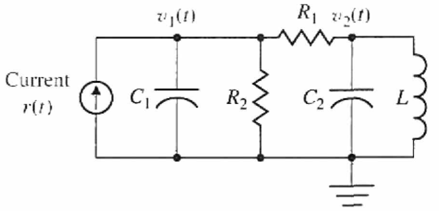


FIGURE 2.17 Two-node electric circuit analog $C_1 = M_1$, $C_2 = M_2$, $L = 1/k$, $R_1 = 1/b_1$, $R_2 = 1/b_2$.

analogous to the node voltages $v_1(t)$ and $v_2(t)$ of the electrical circuit. The simultaneous equations, assuming that the initial conditions are zero, are

$$M_1 s V_1(s) + (b_1 + b_2)V_1(s) - b_1 V_2(s) = R(s), \tag{2.47}$$

and

$$M_2 s V_2(s) + b_1(V_2(s) - V_1(s)) + k\frac{V_2(s)}{s} = 0. \tag{2.48}$$

These equations are obtained using the force equations for the mechanical system of Figure 2.16. Rearranging Equations (2.47) and (2.48), we obtain

$$(M_1 s + (b_1 + b_2))V_1(s) + (-b_1)V_2(s) = R(s),$$

$$(-b_1)V_1(s) + \left(M_2 s + b_1 + \frac{k}{s}\right)V_2(s) = 0,$$

or, in matrix form,

$$\begin{bmatrix} M_1 s + b_1 + b_2 & -b_1 \\ -b_1 & M_2 s + b_1 + \dfrac{k}{s} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix}. \tag{2.49}$$

Assuming that the velocity of M_1 is the output variable, we solve for $V_1(s)$ by matrix inversion or Cramer's rule to obtain [1, 3]

$$V_1(s) = \frac{(M_2 s + b_1 + k/s)R(s)}{(M_1 s + b_1 + b_2)(M_2 s + b_1 + k/s) - b_1^2}. \tag{2.50}$$

Then the transfer function of the mechanical (or electrical) system is

$$G(s) = \frac{V_1(s)}{R(s)} = \frac{(M_2 s + b_1 + k/s)}{(M_1 s + b_1 + b_2)(M_2 s + b_1 + k/s) - b_1^2}$$

$$= \frac{(M_2 s^2 + b_1 s + k)}{(M_1 s + b_1 + b_2)(M_2 s^2 + b_1 s + k) - b_1^2 s}. \tag{2.51}$$

If the transfer function in terms of the position $x_1(t)$ is desired, then we have

$$\frac{X_1(s)}{R(s)} = \frac{V_1(s)}{sR(s)} = \frac{G(s)}{s}. \tag{2.52}$$ ■

As an example, let us obtain the transfer function of an important electrical control component, the **DC motor** [8]. A DC motor is used to move loads and is called an **actuator**.

An actuator is a device that provides the motive power to the process.

EXAMPLE 2.5 **Transfer function of the DC motor**

The DC motor is a power actuator device that delivers energy to a load, as shown in Figure 2.18(a); a sketch of a DC motor is shown in Figure 2.18(b). The DC motor converts direct current (DC) electrical energy into rotational mechanical energy. A major fraction of the torque generated in the rotor (armature) of the motor is available to drive an external load. Because of features such as high torque, speed controllability over a wide range, portability, well-behaved speed–torque characteristics, and adaptability to various types of control methods, DC motors are widely used in numerous control applications, including robotic manipulators, tape transport mechanisms, disk drives, machine tools, and servovalve actuators.

The transfer function of the DC motor will be developed for a linear approximation to an actual motor, and second-order effects, such as hysteresis and the voltage drop across the brushes, will be neglected. The input voltage may be applied to the field or armature terminals. The air-gap flux ϕ of the motor is proportional to the field current, provided the field is unsaturated, so that

$$\phi = K_f i_f. \tag{2.53}$$

The torque developed by the motor is assumed to be related linearly to ϕ and the armature current as follows:

$$T_m = K_1 \phi i_a(t) = K_1 K_f i_f(t) i_a(t). \tag{2.54}$$

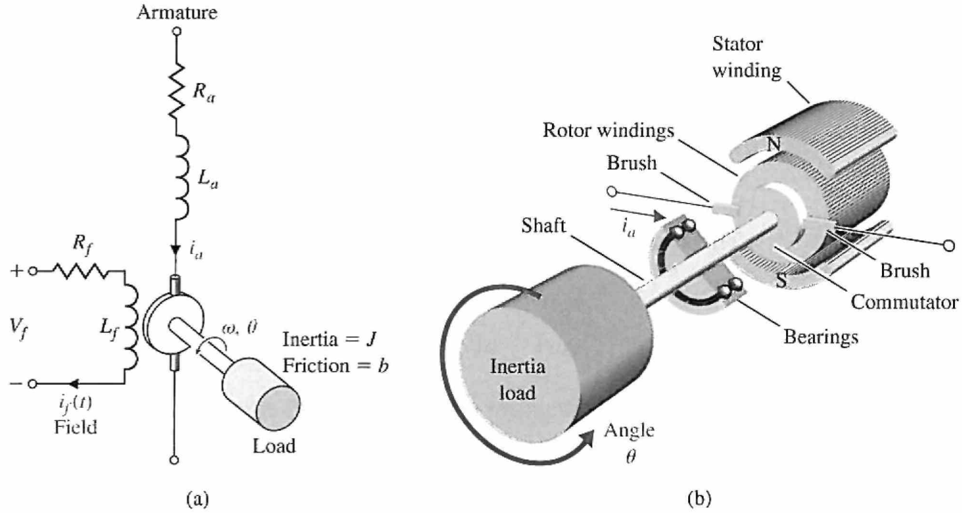


FIGURE 2.18 A DC motor (a) electrical diagram and (b) sketch.

It is clear from Equation (2.54) that, to have a linear system, one current must be maintained constant while the other current becomes the input current. First, we shall consider the field current controlled motor, which provides a substantial power amplification. Then we have, in Laplace transform notation,

$$T_m(s) = (K_1 K_f I_a) I_f(s) = K_m I_f(s), \tag{2.55}$$

where $i_a = I_a$ is a constant armature current, and K_m is defined as the motor constant. The field current is related to the field voltage as

$$V_f(s) = (R_f + L_f s) I_f(s). \tag{2.56}$$

The motor torque $T_m(s)$ is equal to the torque delivered to the load. This relation may be expressed as

$$T_m(s) = T_L(s) + T_d(s), \tag{2.57}$$

where $T_L(s)$ is the load torque and $T_d(s)$ is the disturbance torque, which is often negligible. However, the disturbance torque often must be considered in systems subjected to external forces such as antenna wind-gust forces. The load torque for rotating inertia, as shown in Figure 2.18, is written as

$$T_L(s) = Js^2\theta(s) + bs\theta(s). \tag{2.58}$$

Rearranging Equations (2.55)–(2.57), we have

$$T_L(s) = T_m(s) - T_d(s), \tag{2.59}$$

$$T_m(s) = K_m I_f(s), \tag{2.60}$$

$$I_f(s) = \frac{V_f(s)}{R_f + L_f s}. \tag{2.61}$$

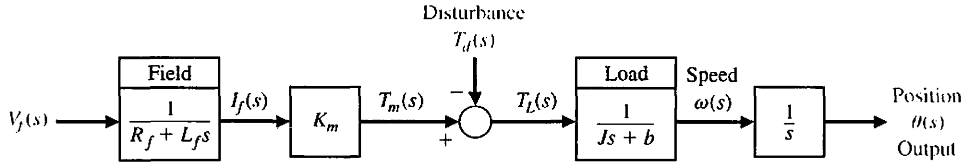


FIGURE 2.19 Block diagram model of field-controlled DC motor.

Therefore, the transfer function of the motor–load combination, with $T_d(s) = 0$, is

$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js + b)(L_f s + R_f)} = \frac{K_m/(JL_f)}{s(s + b/J)(s + R_f/L_f)}. \tag{2.62}$$

The block diagram model of the field-controlled DC motor is shown in Figure 2.19. Alternatively, the transfer function may be written in terms of the time constants of the motor as

$$\frac{\theta(s)}{V_f(s)} = G(s) = \frac{K_m/(bR_f)}{s(\tau_f s + 1)(\tau_L s + 1)}, \tag{2.63}$$

where $\tau_f = L_f/R_f$ and $\tau_L = J/b$. Typically, one finds that $\tau_L > \tau_f$ and often the field time constant may be neglected.

The armature-controlled DC motor uses the armature current i_a as the control variable. The stator field can be established by a field coil and current or a permanent magnet. When a constant field current is established in a field coil, the motor torque is

$$T_m(s) = (K_1 K_f I_f) I_a(s) = K_m I_a(s). \tag{2.64}$$

When a permanent magnet is used, we have

$$T_m(s) = K_m I_a(s),$$

where K_m is a function of the permeability of the magnetic material.

The armature current is related to the input voltage applied to the armature by

$$V_a(s) = (R_a + L_a s) I_a(s) + V_b(s), \tag{2.65}$$

where $V_b(s)$ is the back electromotive-force voltage proportional to the motor speed. Therefore, we have

$$V_b(s) = K_b \omega(s), \tag{2.66}$$

where $\omega(s) = s\theta(s)$ is the transform of the angular speed and the armature current is

$$I_a(s) = \frac{V_a(s) - K_b \omega(s)}{R_a + L_a s}. \tag{2.67}$$

Equations (2.58) and (2.59) represent the load torque, so that

$$T_L(s) = Js^2\theta(s) + bs\theta(s) = T_m(s) - T_d(s). \tag{2.68}$$

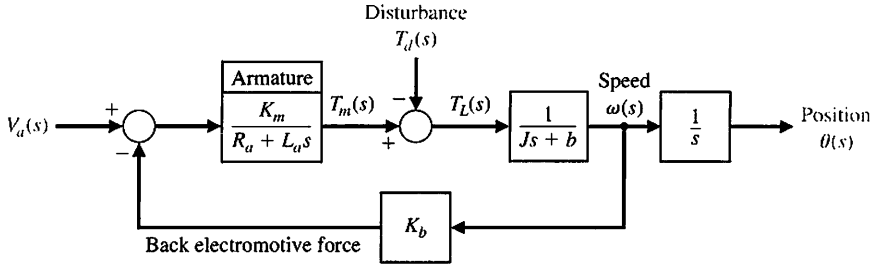


FIGURE 2.20 Armature-controlled DC motor.

The relations for the armature-controlled DC motor are shown schematically in Figure 2.20. Using Equations (2.64), (2.67), and (2.68) or the block diagram, and letting $T_d(s) = 0$, we solve to obtain the transfer function

$$G(s) = \frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[(R_a + L_a s)(Js + b) + K_b K_m]}$$
$$= \frac{K_m}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}. \quad (2.69)$$

However, for many DC motors, the time constant of the armature, $\tau_a = L_a/R_a$, is negligible; therefore,

$$G(s) = \frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[R_a(Js + b) + K_b K_m]} = \frac{K_m/(R_a b + K_b K_m)}{s(\tau_1 s + 1)}, \quad (2.70)$$

where the equivalent time constant $\tau_1 = R_a J/(R_a b + K_b K_m)$.

Note that K_m is equal to K_b. This equality may be shown by considering the steady-state motor operation and the power balance when the rotor resistance is neglected. The power input to the rotor is $(K_b\omega)i_a$, and the power delivered to the shaft is $T\omega$. In the steady-state condition, the power input is equal to the power delivered to the shaft so that $(K_b\omega)i_a = T\omega$; since $T = K_m i_a$ (Equation 2.64), we find that $K_b = K_m$.

Electric motors are used for moving loads when a rapid response is not required and for relatively low power requirements. Typical constants for a fractional horsepower motor are provided in Table 2.4. Actuators that operate as a result of hydraulic pressure are used for large loads. Figure 2.21 shows the usual ranges of use for electromechanical drives as contrasted to electrohydraulic drives. Typical applications are also shown on the figure. ■

Table 2.4 Typical Constants for a Fractional Horsepower DC Motor

Motor constant K_m	50×10^{-3} N·m/A
Rotor inertia J_m	1×10^{-3} N·m·s^2/rad
Field time constant τ_f	1 ms
Rotor time constant τ	100 ms
Maximum output power	$^1/_4$ hp, 187 W

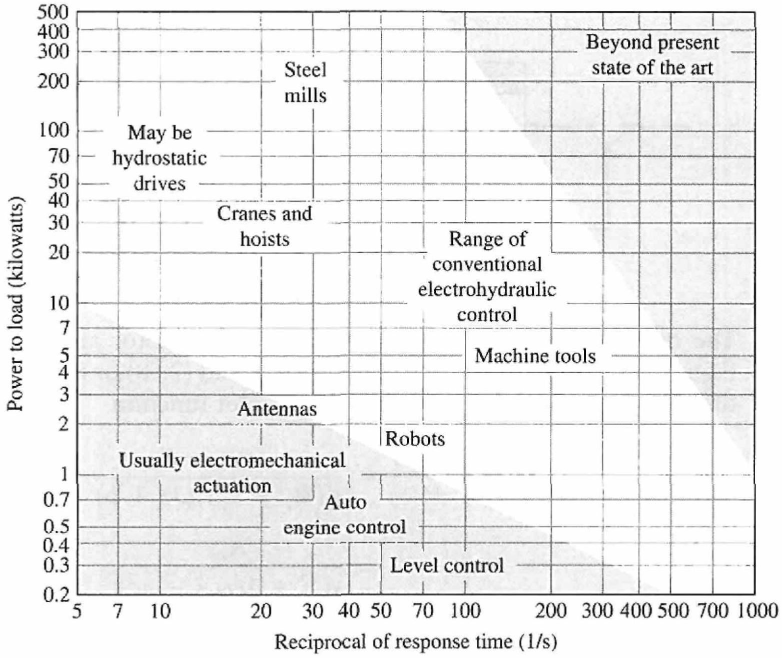


FIGURE 2.21 Range of control response time and power to load for electromechanical and electrohydraulic devices.

EXAMPLE 2.6 **Transfer function of a hydraulic actuator**

A useful actuator for the linear positioning of a mass is the hydraulic actuator shown in Table 2.5, item 9 [9, 10]. The hydraulic actuator is capable of providing a large power amplification. It will be assumed that the hydraulic fluid is available from a constant pressure source and that the compressibility of the fluid is negligible. A downward input displacement x moves the control valve; thus, fluid passes into the upper part of the cylinder, and the piston is forced downward. A small, low-power displacement of $x(t)$ causes a larger, high-power displacement, $y(t)$. The volumetric fluid flow rate Q is related to the input displacement $x(t)$ and the differential pressure across the piston as $Q = g(x, P)$. Using the Taylor series linearization as in Equation (2.11), we have

$$Q = \left(\frac{\partial g}{\partial x}\right)_{x_0, P_0} x + \left(\frac{\partial g}{\partial P}\right)_{x_0, P_0} P = k_x x - k_P P, \tag{2.71}$$

where $g = g(x, P)$ and (x_0, P_0) is the operating point. The force developed by the actuator piston is equal to the area of the piston, A, multiplied by the pressure, P. This force is applied to the mass, so we have

$$AP = M\frac{d^2y}{dt^2} + b\frac{dy}{dt}. \tag{2.72}$$

Thus, substituting Equation (2.71) into Equation (2.72), we obtain

$$\frac{A}{k_P}(k_x x - Q) = M\frac{d^2y}{dt^2} + b\frac{dy}{dt}. \quad (2.73)$$

Furthermore, the volumetric fluid flow is related to the piston movement as

$$Q = A\frac{dy}{dt}. \quad (2.74)$$

Then, substituting Equation (2.74) into Equation (2.73) and rearranging, we have

$$\frac{Ak_x}{k_P}x = M\frac{d^2y}{dt^2} + \left(b + \frac{A^2}{k_P}\right)\frac{dy}{dt}. \quad (2.75)$$

Therefore, using the Laplace transformation, we have the transfer function

$$\frac{Y(s)}{X(s)} = \frac{K}{s(Ms + B)}, \quad (2.76)$$

where

$$K = \frac{Ak_x}{k_P} \quad \text{and} \quad B = b + \frac{A^2}{k_P}.$$

Note that the transfer function of the hydraulic actuator is similar to that of the electric motor. For an actuator operating at high pressure levels and requiring a rapid response of the load, we must account for the effect of the compressibility of the fluid [4, 5].

Symbols, units, and conversion factors associated with many of the variables in Table 2.5 are located at the MCS website. The symbols and units for each variable can be found in tables with corresponding conversions between SI and English units. ■

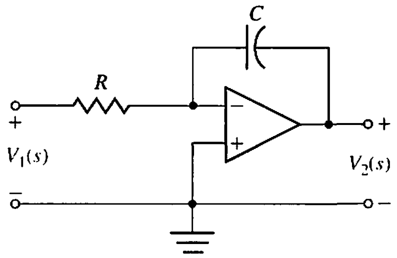
The transfer function concept and approach is very important because it provides the analyst and designer with a useful mathematical model of the system elements. We shall find the transfer function to be a continually valuable aid in the attempt to model dynamic systems. The approach is particularly useful because the s-plane poles and zeros of the transfer function represent the transient response of the system. The transfer functions of several dynamic elements are given in Table 2.5.

In many situations in engineering, the transmission of rotary motion from one shaft to another is a fundamental requirement. For example, the output power of an automobile engine is transferred to the driving wheels by means of the gearbox and differential. The gearbox allows the driver to select different gear ratios depending on the traffic situation, whereas the differential has a fixed ratio. The speed of the engine in this case is not constant, since it is under the control of the driver. Another example is a set of gears that transfer the power at the shaft of an electric motor to the shaft of a rotating antenna. Examples of mechanical converters are gears, chain drives, and belt drives. A commonly used electric converter is the electric transformer. An example of a device that converts rotational motion to linear motion is the rack-and-pinion gear shown in Table 2.5, item 17.

Table 2.5 Transfer Functions of Dynamic Elements and Networks

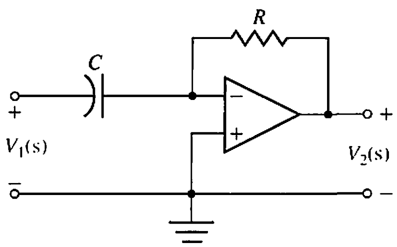
Element or System | **G(s)**

1. Integrating circuit, filter



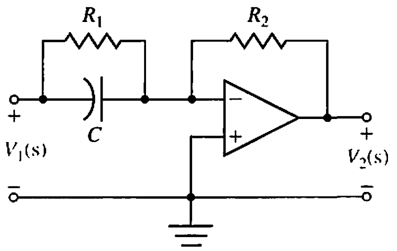
$$\frac{V_2(s)}{V_1(s)} = -\frac{1}{RCs}$$

2. Differentiating circuit



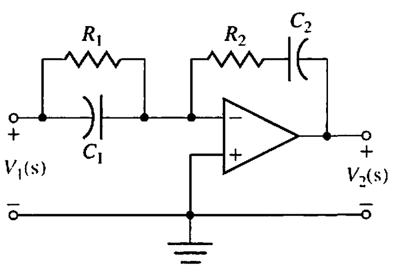
$$\frac{V_2(s)}{V_1(s)} = -RCs$$

3. Differentiating circuit



$$\frac{V_2(s)}{V_1(s)} = -\frac{R_2(R_1Cs + 1)}{R_1}$$

4. Integrating filter



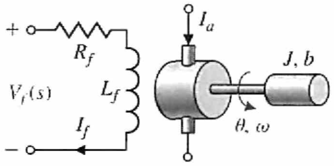
$$\frac{V_2(s)}{V_1(s)} = -\frac{(R_1C_1s + 1)(R_2C_2s + 1)}{R_1C_2s}$$

(continued)

Table 2.5 *Continued*

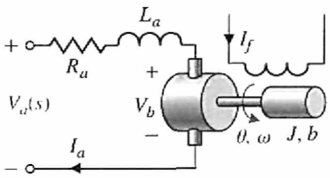
Element or System	G(s)

5. DC motor, field-controlled, rotational actuator



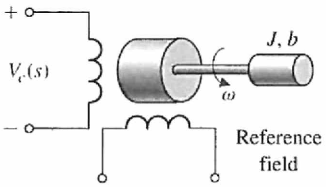
$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js + b)(L_f s + R_f)}$$

6. DC motor, armature-controlled, rotational actuator



$$\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[(R_a + L_a s)(Js + b) + K_b K_m]}$$

7. AC motor, two-phase control field, rotational actuator

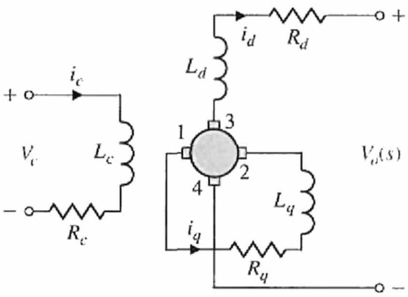


$$\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau s + 1)}$$

$$\tau = J/(b - m)$$

m = slope of linearized torque-speed curve (normally negative)

8. Rotary Amplifier (Amplidyne)



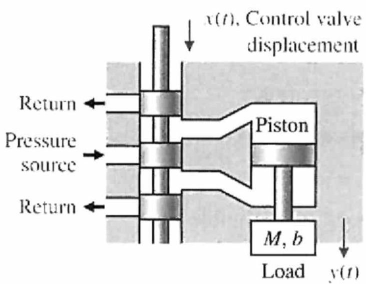
$$\frac{V_o(s)}{V_c(s)} = \frac{K/(R_c R_q)}{(s\tau_c + 1)(s\tau_q + 1)}$$

$$\tau_c = L_c/R_c, \qquad \tau_q = L_q/R_q$$

for the unloaded case, $i_d \approx 0$, $\tau_c \approx \tau_q$, $0.05\text{ s} < \tau_c < 0.5\text{ s}$

$$V_q,\ V_{34} = V_d$$

9. Hydraulic actuator



$$\frac{Y(s)}{X(s)} = \frac{K}{s(Ms + B)}$$

$$K = \frac{Ak_x}{k_p}, \qquad B = \left(b + \frac{A^2}{k_p}\right)$$

$$k_x = \left.\frac{\partial g}{\partial x}\right|_{x_0}, \qquad k_p = \left.\frac{\partial g}{\partial P}\right|_{P_0},$$

$$g = g(x, P) = \text{flow}$$

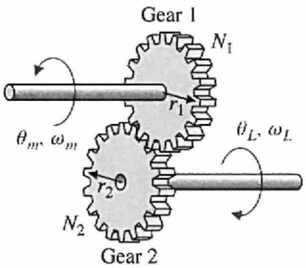
$$A = \text{area of piston}$$

(continued)

Table 2.5 *Continued*

Element or System	G(s)

10. Gear train, rotational transformer

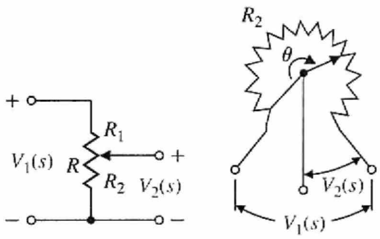


$$\text{Gear ratio} = n = \frac{N_1}{N_2}$$

$$N_2\theta_L = N_1\theta_m, \qquad \theta_L = n\theta_m$$

$$\omega_L = n\omega_m$$

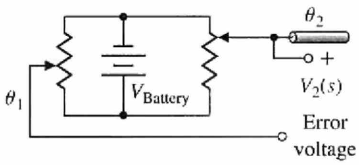
11. Potentiometer, voltage control



$$\frac{V_2(s)}{V_1(s)} = \frac{R_2}{R} = \frac{R_2}{R_1 + R_2}$$

$$\frac{R_2}{R} = \frac{\theta}{\theta_{max}}$$

12. Potentiometer, error detector bridge

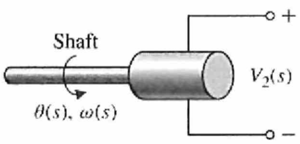


$$V_2(s) = k_s(\theta_1(s) - \theta_2(s))$$

$$V_2(s) = k_s\theta_{error}(s)$$

$$k_s = \frac{V_{Battery}}{\theta_{max}}$$

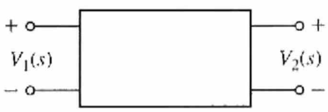
13. Tachometer, velocity sensor



$$V_2(s) = K_t\omega(s) = K_t s\theta(s)$$

$$K_t = \text{constant}$$

14. DC amplifier



$$\frac{V_2(s)}{V_1(s)} = \frac{k_a}{s\tau + 1}$$

R_o = output resistance

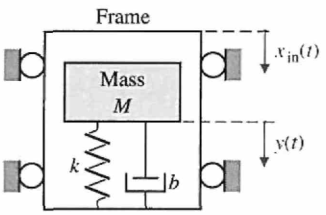
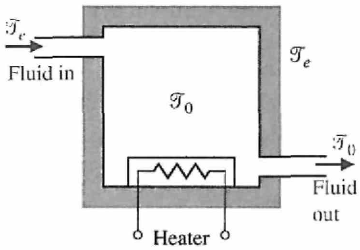
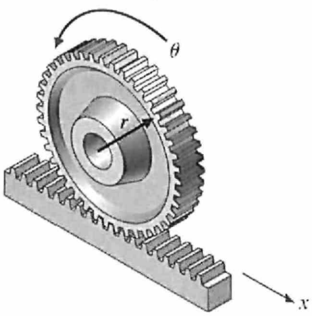
C_o = output capacitance

$\tau = R_oC_o, \tau \ll 1s$

and is often negligible for controller amplifier

(continued)

Table 2.5 *Continued*

Element or System	*G(s)*
15. Accelerometer, acceleration sensor 	$x_o(t) = y(t) - x_{in}(t)$, $\frac{X_o(s)}{X_{in}(s)} = \frac{-s^2}{s^2 + (b/M)s + k/M}$ For low-frequency oscillations, where $\omega < \omega_n$, $\frac{X_o(j\omega)}{X_{in}(j\omega)} \simeq \frac{\omega^2}{k/M}$
16. Thermal heating system 	$\frac{\mathcal{T}(s)}{q(s)} = \frac{1}{C_t s + (QS + 1/R_t)}$, where $\mathcal{T} = \mathcal{T}_o - \mathcal{T}_e$ = temperature difference due to thermal process; C_t = thermal capacitance; Q = fluid flow rate = constant; S = specific heat of water; R_t = thermal resistance of insulation; $q(s)$ = transform of rate of heat flow of heating element
17. Rack and pinion 	$x = r\theta$ converts radial motion to linear motion

2.6 BLOCK DIAGRAM MODELS

The dynamic systems that comprise automatic control systems are represented mathematically by a set of simultaneous differential equations. As we have noted in the previous sections, the Laplace transformation reduces the problem to the solution of a set of linear algebraic equations. Since control systems are concerned with the control of specific variables, the controlled variables must relate to the controlling variables. This relationship is typically represented by the transfer function of the subsystem relating

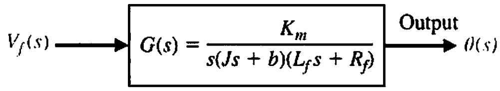


FIGURE 2.22 Block diagram of a DC motor.

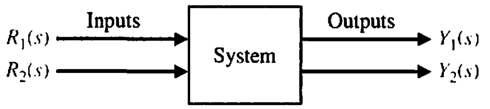


FIGURE 2.23 General block representation of two-input, two-output system.

the input and output variables. Therefore, one can correctly assume that the transfer function is an important relation for control engineering.

The importance of this cause-and-effect relationship is evidenced by the facility to represent the relationship of system variables by diagrammatic means. The **block diagram** representation of the system relationships is prevalent in control system engineering. Block diagrams consist of unidirectional, operational blocks that represent the transfer function of the variables of interest. A block diagram of a field-controlled DC motor and load is shown in Figure 2.22. The relationship between the displacement $\theta(s)$ and the input voltage $V_f(s)$ is clearly portrayed by the block diagram.

To represent a system with several variables under control, an interconnection of blocks is utilized. For example, the system shown in Figure 2.23 has two input variables and two output variables [6]. Using transfer function relations, we can write the simultaneous equations for the output variables as

$$Y_1(s) = G_{11}(s)R_1(s) + G_{12}(s)R_2(s), \tag{2.77}$$

and

$$Y_2(s) = G_{21}(s)R_1(s) + G_{22}(s)R_2(s), \tag{2.78}$$

where $G_{ij}(s)$ is the transfer function relating the ith output variable to the jth input variable. The block diagram representing this set of equations is shown in Figure 2.24. In general, for J inputs and I outputs, we write the simultaneous equation in matrix form as

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_I(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & \cdots & G_{1J}(s) \\ G_{21}(s) & \cdots & G_{2J}(s) \\ \vdots & & \vdots \\ G_{I1}(s) & \cdots & G_{IJ}(s) \end{bmatrix} \begin{bmatrix} R_1(s) \\ R_2(s) \\ \vdots \\ R_J(s) \end{bmatrix} \tag{2.79}$$

or simply

$$\mathbf{Y} = \mathbf{GR}. \tag{2.80}$$

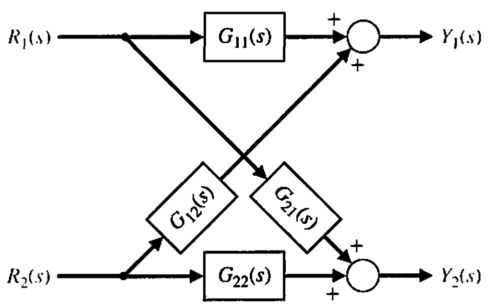


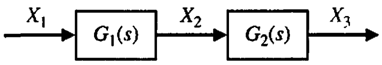
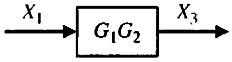
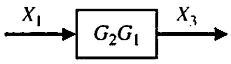
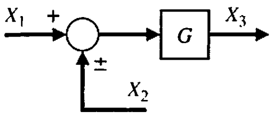
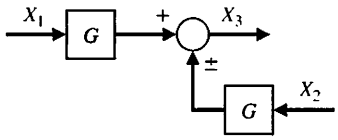
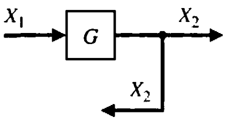
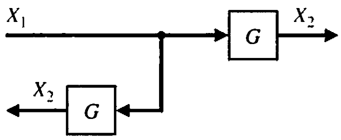
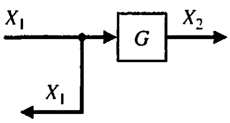
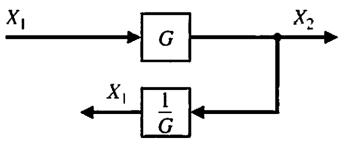
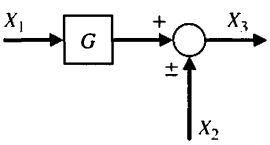
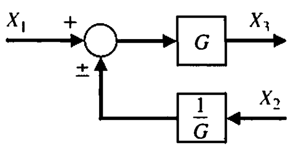
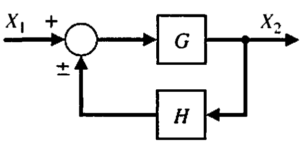
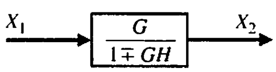
FIGURE 2.24 Block diagram of interconnected system.

Here the **Y** and **R** matrices are column matrices containing the I output and the J input variables, respectively, and **G** is an I by J transfer function matrix. The matrix representation of the interrelationship of many variables is particularly valuable for complex multivariable control systems. An introduction to matrix algebra is provided on the MCS website for those unfamiliar with matrix algebra or who would find a review helpful [21].

The block diagram representation of a given system often can be reduced to a simplified block diagram with fewer blocks than the original diagram. Since the transfer functions represent linear systems, the multiplication is commutative. Thus, in Table 2.6, item 1, we have

$$X_3(s) = G_2(s)X_2(s) = G_1(s)G_2(s)X_1(s).$$

Table 2.6 Block Diagram Transformations

Transformation	Original Diagram	Equivalent Diagram
1. Combining blocks in cascade		 or 
2. Moving a summing point behind a block		
3. Moving a pickoff point ahead of a block		
4. Moving a pickoff point behind a block		
5. Moving a summing point ahead of a block		
6. Eliminating a feedback loop		

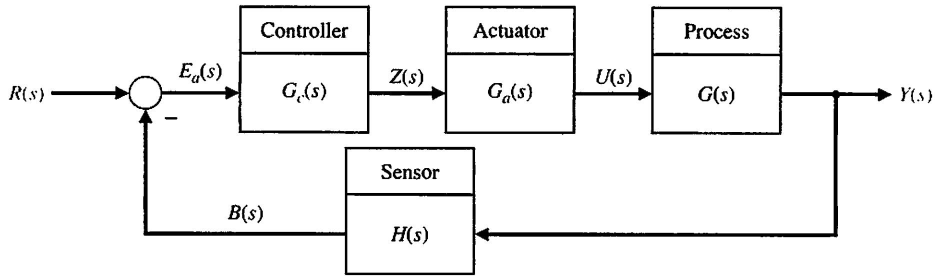


FIGURE 2.25 Negative feedback control system.

When two blocks are connected in cascade, as in Table 2.6, item 1, we assume that

$$X_3(s) = G_2(s)G_1(s)X_1(s)$$

holds true. This assumes that when the first block is connected to the second block, the effect of loading of the first block is negligible. Loading and interaction between interconnected components or systems may occur. If the loading of interconnected devices does occur, the engineer must account for this change in the transfer function and use the corrected transfer function in subsequent calculations.

Block diagram transformations and reduction techniques are derived by considering the algebra of the diagram variables. For example, consider the block diagram shown in Figure 2.25. This negative feedback control system is described by the equation for the actuating signal, which is

$$E_a(s) = R(s) - B(s) = R(s) - H(s)Y(s). \tag{2.81}$$

Because the output is related to the actuating signal by $G(s)$, we have

$$Y(s) = G(s)U(s) = G(s)G_a(s)Z(s) = G(s)G_a(s)G_c(s)E_a(s); \tag{2.82}$$

thus,

$$Y(s) = G(s)G_a(s)G_c(s)[R(s) - H(s)Y(s)]. \tag{2.83}$$

Combining the $Y(s)$ terms, we obtain

$$Y(s)[1 + G(s)G_a(s)G_c(s)H(s)] = G(s)G_a(s)G_c(s)R(s). \tag{2.84}$$

Therefore, the transfer function relating the output $Y(s)$ to the input $R(s)$ is

$$\boxed{\frac{Y(s)}{R(s)} = \frac{G(s)G_a(s)G_c(s)}{1 + G(s)G_a(s)G_c(s)H(s)}}. \tag{2.85}$$

This **closed-loop transfer function** is particularly important because it represents many of the existing practical control systems.

The reduction of the block diagram shown in Figure 2.25 to a single block representation is one example of several useful techniques. These diagram transformations are given in Table 2.6. All the transformations in Table 2.6 can be derived by simple algebraic manipulation of the equations representing the blocks. System analysis by the method of block diagram reduction affords a better understanding of the contribution of each component element than possible by the manipulation of

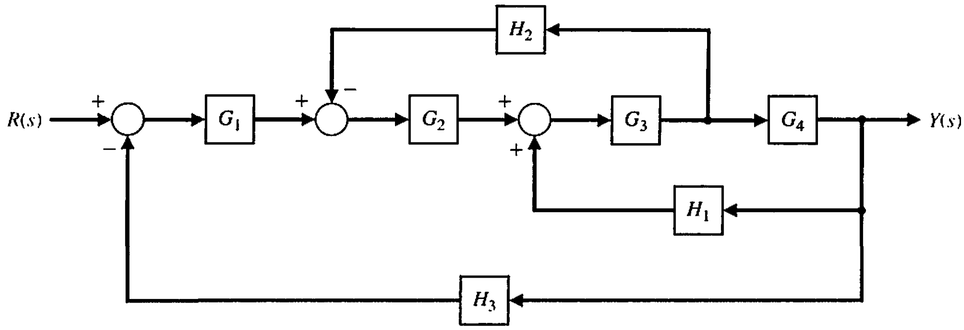


FIGURE 2.26 Multiple-loop feedback control system.

equations. The utility of the block diagram transformations will be illustrated by an example using block diagram reduction.

EXAMPLE 2.7 **Block diagram reduction**

The block diagram of a multiple-loop feedback control system is shown in Figure 2.26. It is interesting to note that the feedback signal $H_1(s)Y(s)$ is a positive feedback signal, and the loop $G_3(s)G_4(s)H_1(s)$ is a **positive feedback loop**. The block diagram reduction procedure is based on the use of Table 2.6, transformation 6, which eliminates feedback loops. Therefore the other transformations are used to transform the diagram to a form ready for eliminating feedback loops. First, to eliminate the loop $G_3G_4H_1$, we move H_2 behind block G_4 by using transformation 4, and obtain Figure 2.27(a). Eliminating the loop $G_3G_4H_1$ by using transformation 6, we obtain Figure 2.27(b). Then, eliminating the inner loop containing H_2/G_4, we obtain Figure 2.27(c). Finally, by reducing the loop containing H_3, we obtain the closed-loop system transfer function as shown in Figure 2.27(d). It is worthwhile to examine the form of the numerator and denominator of this closed-loop transfer function. We note that the numerator is composed of the cascade transfer function of the feedforward elements connecting the input $R(s)$ and the output $Y(s)$. The denominator is composed of 1 minus the sum of each loop transfer function. The loop $G_3G_4H_1$ has a plus sign in the sum to be subtracted because it is a positive feedback loop, whereas the loops $G_1G_2G_3G_4H_3$ and $G_2G_3H_2$ are negative feedback loops. To illustrate this point, the denominator can be rewritten as

$$q(s) = 1 - (+G_3G_4H_1 - G_2G_3H_2 - G_1G_2G_3G_4H_3). \tag{2.86}$$

This form of the numerator and denominator is quite close to the general form for multiple-loop feedback systems, as we shall find in the following section. ∎

The block diagram representation of feedback control systems is a valuable and widely used approach. The block diagram provides the analyst with a graphical representation of the interrelationships of controlled and input variables. Furthermore, the designer can readily visualize the possibilities for adding blocks to the existing system block diagram to alter and improve the system performance. The transition from the block diagram method to a method utilizing a line path representation instead of a block representation is readily accomplished and is presented in the following section.

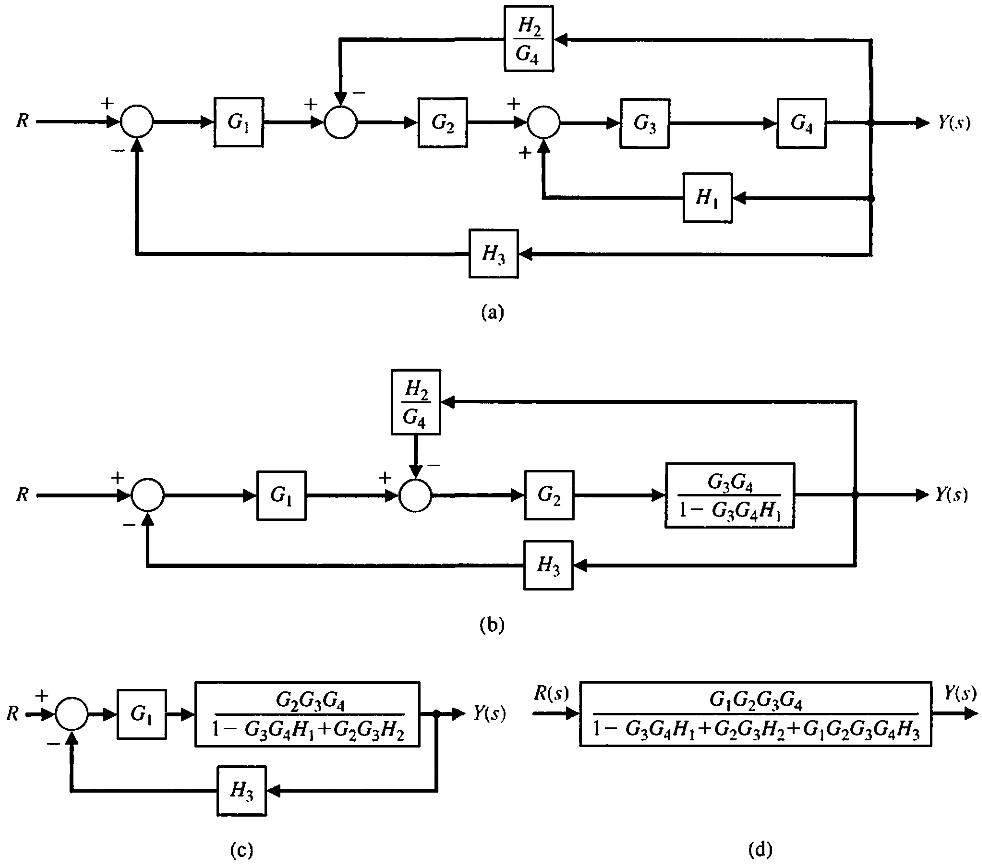


FIGURE 2.27 Block diagram reduction of the system of Figure 2.26.

2.7 SIGNAL-FLOW GRAPH MODELS

Block diagrams are adequate for the representation of the interrelationships of controlled and input variables. However, for a system with reasonably complex interrelationships, the block diagram reduction procedure is cumbersome and often quite difficult to complete. An alternative method for determining the relationship between system variables has been developed by Mason and is based on a representation of the system by line segments [4, 25]. The advantage of the line path method, called the signal-flow graph method, is the availability of a flow graph gain formula, which provides the relation between system variables without requiring any reduction procedure or manipulation of the flow graph.

The transition from a block diagram representation to a directed line segment representation is easy to accomplish by reconsidering the systems of the previous section. A **signal-flow graph** is a diagram consisting of nodes that are connected by several directed branches and is a graphical representation of a set of linear relations. Signal-flow graphs are particularly useful for feedback control systems because feedback theory is primarily concerned with the flow and processing of signals in systems. The basic element of a signal-flow graph is a unidirectional path segment called a **branch**, which relates the dependency of an input and an output variable in

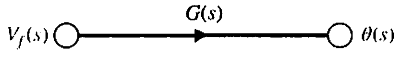


FIGURE 2.28 Signal-flow graph of the DC motor.

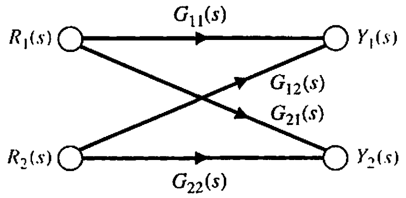


FIGURE 2.29 Signal-flow graph of interconnected system.

a manner equivalent to a block of a block diagram. Therefore, the branch relating the output $\theta(s)$ of a DC motor to the field voltage $V_f(s)$ is similar to the block diagram of Figure 2.22 and is shown in Figure 2.28. The input and output points or junctions are called **nodes**. Similarly, the signal-flow graph representing Equations (2.77) and (2.78), as well as Figure 2.24, is shown in Figure 2.29. The relation between each variable is written next to the directional arrow. All branches leaving a node will pass the nodal signal to the output node of each branch (unidirectionally). The summation of all signals entering a node is equal to the node variable. A **path** is a branch or a continuous sequence of branches that can be traversed from one signal (node) to another signal (node). A **loop** is a closed path that originates and terminates on the same node, with no node being met twice along the path. Two loops are said to be **nontouching** if they do not have a common node. Two touching loops share one or more common nodes. Therefore, considering Figure 2.29 again, we obtain

$$Y_1(s) = G_{11}(s)R_1(s) + G_{12}(s)R_2(s), \tag{2.87}$$

and

$$Y_2(s) = G_{21}(s)R_1(s) + G_{22}(s)R_2(s). \tag{2.88}$$

The flow graph is simply a pictorial method of writing a system of algebraic equations that indicates the interdependencies of the variables. As another example, consider the following set of simultaneous algebraic equations:

$$a_{11}x_1 + a_{12}x_2 + r_1 = x_1 \tag{2.89}$$

$$a_{21}x_1 + a_{22}x_2 + r_2 = x_2. \tag{2.90}$$

The two input variables are r_1 and r_2, and the output variables are x_1 and x_2. A signal-flow graph representing Equations (2.89) and (2.90) is shown in Figure 2.30. Equations (2.89) and (2.90) may be rewritten as

$$x_1(1 - a_{11}) + x_2(-a_{12}) = r_1, \tag{2.91}$$

and

$$x_1(-a_{21}) + x_2(1 - a_{22}) = r_2. \tag{2.92}$$

The simultaneous solution of Equations (2.91) and (2.92) using Cramer's rule results in the solutions

$$x_1 = \frac{(1 - a_{22})r_1 + a_{12}r_2}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} = \frac{1 - a_{22}}{\Delta}r_1 + \frac{a_{12}}{\Delta}r_2, \tag{2.93}$$

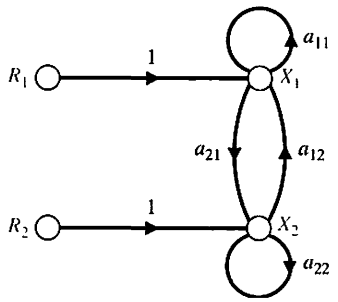


FIGURE 2.30 Signal-flow graph of two algebraic equations.

and

$$x_2 = \frac{(1 - a_{11})r_2 + a_{21}r_1}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} = \frac{1 - a_{11}}{\Delta}r_2 + \frac{a_{21}}{\Delta}r_1. \tag{2.94}$$

The denominator of the solution is the determinant Δ of the set of equations and is rewritten as

$$\Delta = (1 - a_{11})(1 - a_{22}) - a_{12}a_{21} = 1 - a_{11} - a_{22} + a_{11}a_{22} - a_{12}a_{21}. \tag{2.95}$$

In this case, the denominator is equal to 1 minus each self-loop a_{11}, a_{22}, and $a_{12}a_{21}$, plus the product of the two nontouching loops a_{11} and a_{22}. The loops a_{22} and $a_{21}a_{12}$ are touching, as are a_{11} and $a_{21}a_{12}$.

The numerator for x_1 with the input r_1 is 1 times $1 - a_{22}$, which is the value of Δ excluding terms that touch the path 1 from r_1 to x_1. Therefore the numerator from r_2 to x_1 is simply a_{12} because the path through a_{12} touches all the loops. The numerator for x_2 is symmetrical to that of x_1.

In general, the linear dependence T_{ij} between the independent variable x_i (often called the input variable) and a dependent variable x_j is given by Mason's signal-flow gain formula [11, 12],

$$T_{ij} = \frac{\sum_k P_{ijk}\,\Delta_{ijk}}{\Delta}, \tag{2.96}$$

P_{ijk} = gain of kth path from variable x_i to variable x_j,

Δ = determinant of the graph,

Δ_{ijk} = cofactor of the path P_{ijk},

and the summation is taken over all possible k paths from x_i to x_j. The path gain or transmittance P_{ijk} is defined as the product of the gains of the branches of the path, traversed in the direction of the arrows with no node encountered more than once. The cofactor Δ_{ijk} is the determinant with the loops touching the kth path removed. The determinant Δ is

$$\Delta = 1 - \sum_{n=1}^{N} L_n + \sum_{\substack{n,\,m \\ \text{nontouching}}} L_n L_m - \sum_{\substack{n,\,m,\,p \\ \text{nontouching}}} L_n L_m L_p + \cdots, \tag{2.97}$$

where L_q equals the value of the qth loop transmittance. Therefore the rule for evaluating Δ in terms of loops $L_1, L_2, L_3, \ldots, L_N$ is

$\Delta = 1 -$ (sum of all different loop gains)
  $+$ (sum of the gain products of all combinations of two nontouching loops)
  $-$ (sum of the gain products of all combinations of three nontouching loops)
  $+ \cdots$.

The gain formula is often used to relate the output variable $Y(s)$ to the input variable $R(s)$ and is given in somewhat simplified form as

$$\boxed{T = \frac{\Sigma_k P_k \Delta_k}{\Delta},} \tag{2.98}$$

where $T(s) = Y(s)/R(s)$.

Several examples will illustrate the utility and ease of this method. Although the gain Equation (2.96) appears to be formidable, one must remember that it represents a summation process, not a complicated solution process.

EXAMPLE 2.8 **Transfer function of an interacting system**

A two-path signal-flow graph is shown in Figure 2.31(a) and the corresponding block diagram is shown in Figure 2.31(b). An example of a control system with multiple signal paths is a multilegged robot. The paths connecting the input $R(s)$ and output $Y(s)$ are

$$P_1 = G_1G_2G_3G_4 \text{ (path 1)} \quad \text{and} \quad P_2 = G_5G_6G_7G_8 \text{ (path 2)}.$$

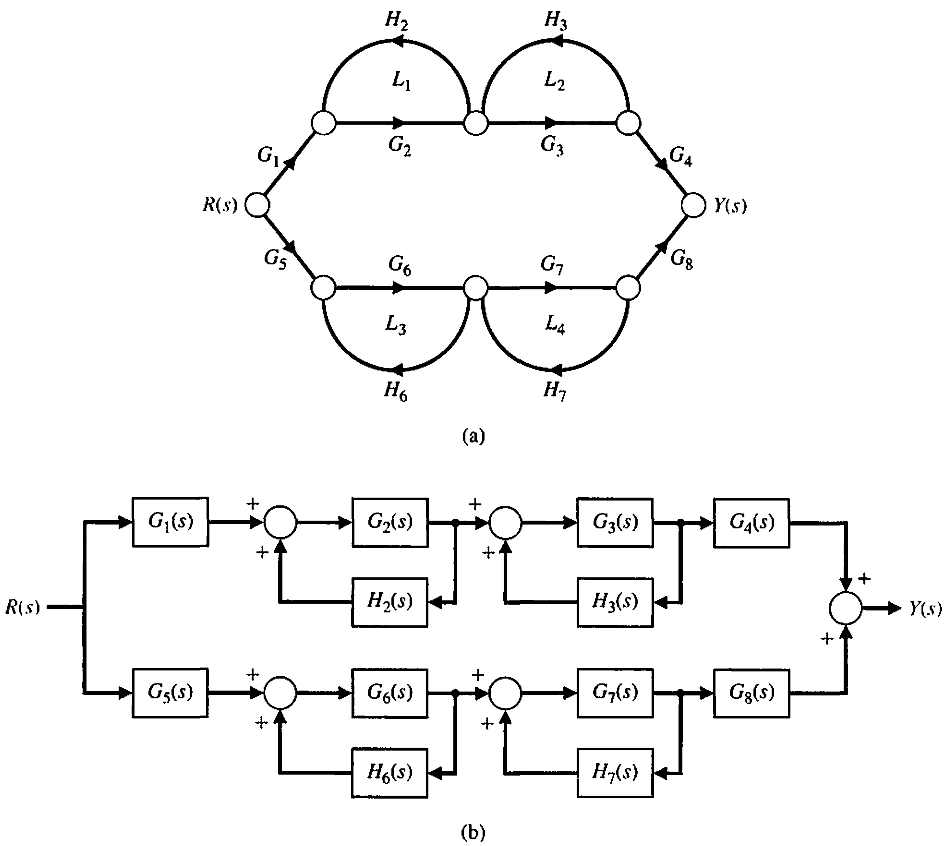


FIGURE 2.31 Two-path interacting system. (a) Signal-flow graph. (b) Block diagram.

There are four self-loops:

$$L_1 = G_2H_2, \quad L_2 = H_3G_3, \quad L_3 = G_6H_6, \quad \text{and} \quad L_4 = G_7H_7.$$

Loops L_1 and L_2 do not touch L_3 and L_4. Therefore, the determinant is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4). \tag{2.99}$$

The cofactor of the determinant along path 1 is evaluated by removing the loops that touch path 1 from Δ. Hence, we have

$$L_1 = L_2 = 0 \quad \text{and} \quad \Delta_1 = 1 - (L_3 + L_4).$$

Similarly, the cofactor for path 2 is

$$\Delta_2 = 1 - (L_1 + L_2).$$

Therefore, the transfer function of the system is

$$\frac{Y(s)}{R(s)} = T(s) = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} \tag{2.100}$$

$$= \frac{G_1G_2G_3G_4(1 - L_3 - L_4) + G_5G_6G_7G_8(1 - L_1 - L_2)}{1 - L_1 - L_2 - L_3 - L_4 + L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4}.$$

A similar analysis can be accomplished using block diagram reduction techniques. The block diagram shown in Figure 2.31(b) has four inner feedback loops within the overall block diagram. The block diagram reduction is simplified by first reducing the four inner feedback loops and then placing the resulting systems in series. Along the top path, the transfer function is

$$Y_1(s) = G_1(s)\left[\frac{G_2(s)}{1 - G_2(s)H_2(s)}\right]\left[\frac{G_3(s)}{1 - G_3(s)H_3(s)}\right]G_4(s)R(s)$$

$$= \left[\frac{G_1(s)G_2(s)G_3(s)G_4(s)}{(1 - G_2(s)H_2(s))(1 - G_3(s)H_3(s))}\right]R(s).$$

Similarly across the bottom path, the transfer function is

$$Y_2(s) = G_5(s)\left[\frac{G_6(s)}{1 - G_6(s)H_6(s)}\right]\left[\frac{G_7(s)}{1 - G_7(s)H_7(s)}\right]G_8(s)R(s)$$

$$= \left[\frac{G_5(s)G_6(s)G_7(s)G_8(s)}{(1 - G_6(s)H_6(s))(1 - G_7(s)H_7(s))}\right]R(s).$$

The total transfer function is then given by

$$Y(s) = Y_1(s) + Y_2(s) = \left[\frac{G_1(s)G_2(s)G_3(s)G_4(s)}{(1 - G_2(s)H_2(s))(1 - G_3(s)H_3(s))}\right.$$

$$\left. + \frac{G_5(s)G_6(s)G_7(s)G_8(s)}{(1 - G_6(s)H_6(s))(1 - G_7(s)H_7(s))}\right]R(s). \blacksquare$$

EXAMPLE 2.9 **Armature-controlled motor**

The block diagram of the armature-controlled DC motor is shown in Figure 2.20. This diagram was obtained from Equations (2.64)–(2.68). The signal-flow diagram can be obtained either from Equations (2.64)–(2.68) or from the block diagram and is shown in Figure 2.32. Using Mason's signal-flow gain formula, let us obtain the transfer function for $\theta(s)/V_a(s)$ with $T_d(s) = 0$. The forward path is $P_1(s)$, which touches the one loop, $L_1(s)$, where

$$P_1(s) = \frac{1}{s}G_1(s)G_2(s) \quad \text{and} \quad L_1(s) = -K_bG_1(s)G_2(s).$$

Therefore, the transfer function is

$$T(s) = \frac{P_1(s)}{1 - L_1(s)} = \frac{(1/s)G_1(s)G_2(s)}{1 + K_bG_1(s)G_2(s)} = \frac{K_m}{s[(R_a + L_as)(Js + b) + K_bK_m]},$$

which is exactly the same as that derived earlier (Equation 2.69). ∎

The signal-flow graph gain formula provides a reasonably straightforward approach for the evaluation of complicated systems. To compare the method with block diagram reduction, which is really not much more difficult, let us reconsider the complex system of Example 2.7.

EXAMPLE 2.10 **Transfer function of a multiple-loop system**

A multiple-loop feedback system is shown in Figure 2.26 in block diagram form. There is no need to redraw the diagram in signal-flow graph form, and so we shall proceed as usual by using Mason's signal-flow gain formula, Equation (2.98). There is one forward path $P_1 = G_1G_2G_3G_4$. The feedback loops are

$$L_1 = -G_2G_3H_2, \quad L_2 = G_3G_4H_1, \quad \text{and} \quad L_3 = -G_1G_2G_3G_4H_3. \tag{2.101}$$

All the loops have common nodes and therefore are all touching. Furthermore, the path P_1 touches all the loops, so $\Delta_1 = 1$. Thus, the closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{P_1\Delta_1}{1 - L_1 - L_2 - L_3} = \frac{G_1G_2G_3G_4}{1 + G_2G_3H_2 - G_3G_4H_1 + G_1G_2G_3G_4H_3}. \tag{2.102}$$ ∎

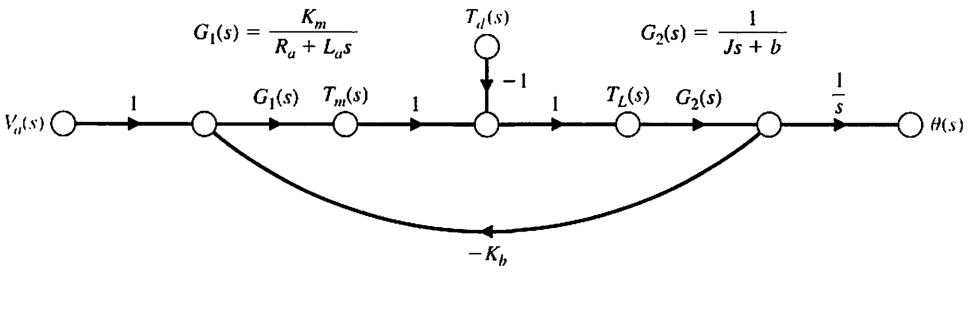


FIGURE 2.32 The signal-flow graph of the armature-controlled DC motor.

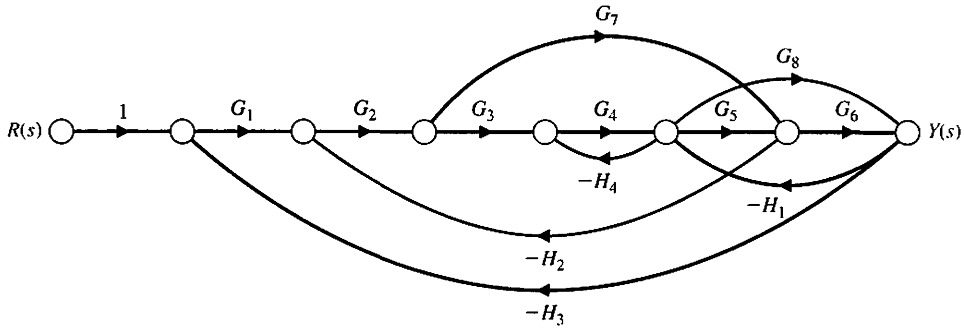


FIGURE 2.33 Multiple-loop system.

EXAMPLE 2.11 **Transfer function of a complex system**

Finally, we shall consider a reasonably complex system that would be difficult to reduce by block diagram techniques. A system with several feedback loops and feedforward paths is shown in Figure 2.33. The forward paths are

$$P_1 = G_1G_2G_3G_4G_5G_6, \quad P_2 = G_1G_2G_7G_6, \quad \text{and} \quad P_3 = G_1G_2G_3G_4G_8.$$

The feedback loops are

$$L_1 = -G_2G_3G_4G_5H_2, \quad L_2 = -G_5G_6H_1, \quad L_3 = -G_8H_1, \quad L_4 = -G_7H_2G_2,$$
$$L_5 = -G_4H_4, \quad L_6 = -G_1G_2G_3G_4G_5G_6H_3, \quad L_7 = -G_1G_2G_7G_6H_3, \quad \text{and}$$
$$L_8 = -G_1G_2G_3G_4G_8H_3.$$

Loop L_5 does not touch loop L_4 or loop L_7, and loop L_3 does not touch loop L_4; but all other loops touch. Therefore, the determinant is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8) + (L_5L_7 + L_5L_4 + L_3L_4). \tag{2.103}$$

The cofactors are

$$\Delta_1 = \Delta_3 = 1 \quad \text{and} \quad \Delta_2 = 1 - L_5 = 1 + G_4H_4.$$

Finally, the transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{P_1 + P_2\Delta_2 + P_3}{\Delta}. \tag{2.104}$$ ■

Signal-flow graphs and Mason's signal-flow gain formula may be used profitably for the analysis of feedback control systems, electronic amplifier circuits, statistical systems, and mechanical systems, among many other examples.

2.8 DESIGN EXAMPLES

In this section, we present six illustrative design examples. The first example describes modeling of a photovoltaic generator in a manner amenable to feedback control to achieve maximum power delivery as the sunlight varies over time. Using feedback control to improve the efficiency of producing electricity using solar energy in areas of abundant sunlight is a valuable contribution to green engineering (discussed in Chapter 1). In the second example, we present a detailed look at modeling of the fluid level in a reservoir. The modeling is presented in a very detailed manner to emphasize the effort required to obtain a linear model in the form of a transfer function. The design process depicted in Figure 1.17 is highlighted in this example. The remaining four examples include an electric traction motor model development, a look at a mechanical accelerometer aboard a rocket sled, an overview of a laboratory robot and the associated hardware specifications, and the design of a low-pass filter.

EXAMPLE 2.12 **Photovoltaic generators**

Photovoltaic cells were developed at Bell Laboratories in 1954. Solar cells are one example of photovoltaic cells and convert solar light to electricity. Other types of photovoltaic cells can detect radiation and measure light intensity. The use of solar cells to produce energy supports the principles of green engineering by minimizing pollution. Solar panels minimize the depletion of natural resources and are effective in areas where sunlight is abundant. Photovoltaic generators are systems that provide electricity using an assortment of photovoltaic modules comprised of interconnected solar cells. Photovoltaic generators can be used to recharge batteries, they can be directly connected to an electrical grid, or they can drive electric motors without a battery [34–42].

The power output of a solar cell varies with available solar light, temperature, and external loads. To increase the overall efficiency of the photovoltaic generator, feedback control strategies can be employed to seek to maximize the power output. This is known as maximum power point tracking (MPPT) [34–36]. There are certain values of current and voltage associated with the solar cells corresponding to the maximum power output. The MPPT uses closed-loop feedback control to seek the optimal point to allow the power converter circuit to extract the maximum power from the photovoltaic generator system. We will discuss the control design in later chapters, but here we focus on the modeling of the system.

The solar cell can be modeled as an equivalent circuit shown in Figure 2.34 composed of a current generator, I_{PH}, a light sensitive diode, a resistance series, R_s, and a shunt resistance, R_P [34, 36–38].

The output voltage, V_{PV}, is given by

$$V_{PV} = \frac{N}{\lambda}\ln\left(\frac{I_{PH} - I_{PV} + MI_0}{MI_0}\right) - \frac{N}{M}R_S I_{PV}, \tag{2.105}$$

where the photovoltaic generator is comprised of M parallel strings with N series cells per string, I_0 is the reverse saturation current of the diode, I_{PH} represents the insolation level, and λ is a known constant that depends on the cell material [34–36].

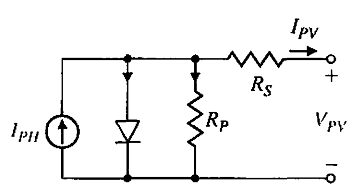


FIGURE 2.34 Equivalent circuit of the photovoltaic generator.

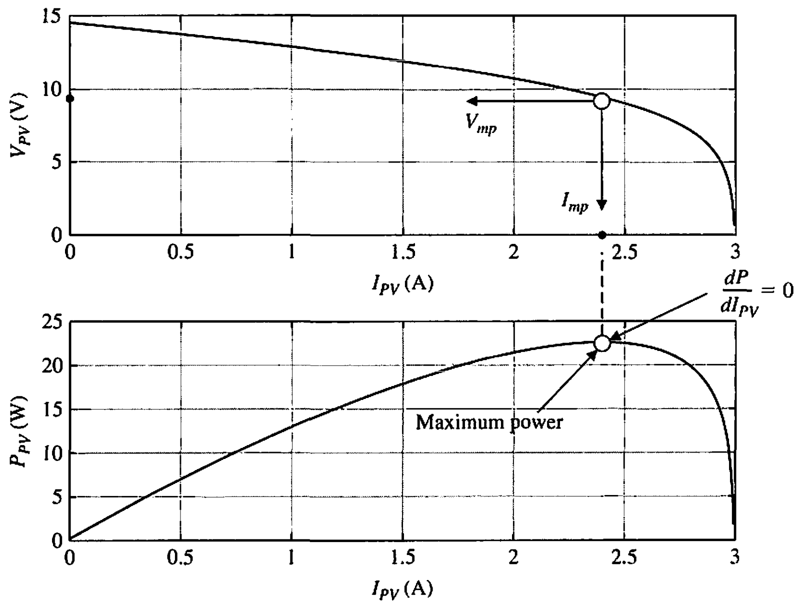


FIGURE 2.35 Voltage versus current and power versus current for an example photovoltaic generator at a specific insolation level.

The insolation level is a measure of the amount of incident solar radiation on the solar cells.

Suppose that we have a single silicon solar panel ($M = 1$) with 10 series cells ($N = 10$) and the parameters given by $1/\lambda = 0.05$ V, $R_s = 0.025\ \Omega$, $I_{PH} = 3$ A, and $I_0 = 0.001$ A. The voltage versus current relationship in Equation (2.105) and the power versus voltage are shown in Figure 2.35 for one particular insolation level where $I_{PH} = 3$ A. In Figure 2.35, we see that when $dP/dI_{PV} = 0$ we are at the maximum power level with an associated $V_{PV} = V_{mp}$ and $I_{PV} = I_{mp}$, the values of voltage and current at the maximum power, respectively. As the sunlight varies, the insolation level, I_{PH}, varies resulting in different power curves.

The goal of the power point tracking is to seek the voltage and current condition that maximizes the power output as conditions vary. This is accomplished by varying the reference voltage as a function of the insolation level. The reference voltage is the voltage at the maximum power point as shown in Figure 2.36. The feedback control system should track the reference voltage in a rapid and accurate fashion.

Figure 2.37 illustrates a simplified block diagram of the controlled system. The main components are a power circuit (e.g., a phase control IC and a thyristor bridge), photovoltaic generator, and current transducer. The plant including the power circuit, photovoltaic generator, and current transducer is modeled as a second-order transfer function given by

$$G(s) = \frac{K}{s(s + p)}, \tag{2.106}$$

where K and p depend on the photovoltaic generator and associated electronics [35]. The controller, $G_c(s)$, in Figure 2.37 is designed such that as the insolation levels varies (that is, as I_{PH} varies), the voltage output will approach the reference input voltage, V_{ref}, which has been set to the voltage associated with the maximum

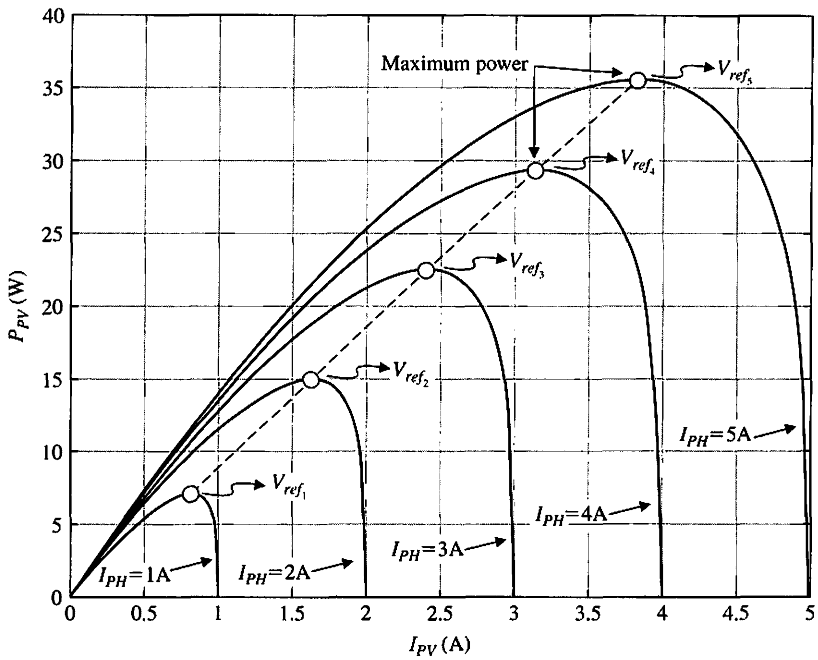


FIGURE 2.36 Maximum power point for varying values of I_{PH} specifies V_{ref}.

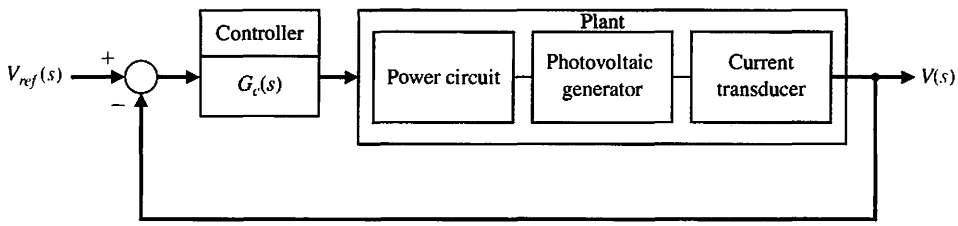


FIGURE 2.37 Block diagram of feedback control system for maximum power transfer with parameters K and p.

power point resulting in maximum power transfer. If, for example, the controller is the proportional plus integral controller

$$G_c(s) = K_P + \frac{K_I}{s},$$

the closed-loop transfer function is

$$T(s) = \frac{K(K_P s + K_I)}{s^3 + ps^2 + KK_P s + KK_I}. \tag{2.107}$$

We can select the controller gains in Equation (2.107) to place the poles of $T(s)$ in the desired locations (see Chapters 4 and 5) to meet the desired performance specifications.

EXAMPLE 2.13 **Fluid flow modeling**

A fluid flow system is shown in Figure 2.38. The reservoir (or tank) contains water that evacuates through an output port. Water is fed to the reservoir through a pipe controlled by an input valve. The variables of interest are the fluid velocity V (m/s), fluid height in the reservoir H (m), and pressure p (N/m^2). The pressure is defined as the force per unit area exerted by the fluid on a surface immersed (and at rest with respect to) the fluid. Fluid pressure acts normal to the surface. For further reading on fluid flow modeling, see [28–30].

The elements of the control system design process emphasized in this example are shown in Figure 2.39. The strategy is to establish the system configuration and then obtain the appropriate mathematical models describing the fluid flow reservoir from an input–output perspective.

The general equations of motion and energy describing fluid flow are quite complicated. The governing equations are coupled nonlinear partial differential equations. We must make some selective assumptions that reduce the complexity of the mathematical model. Although the control engineer is not required to be a fluid dynamicist, and a deep understanding of fluid dynamics is not necessarily acquired during the control system design process, it makes good engineering sense to gain at least a rudimentary understanding of the important simplifying assumptions. For a more complete discussion of fluid motion, see [31–33].

To obtain a realistic, yet tractable, mathematical model for the fluid flow reservoir, we first make several key assumptions. We assume that the water in the tank is incompressible and that the flow is inviscid, irrotational and steady. An incompressible fluid has a constant density ρ (kg/m^3). In fact, all fluids are compressible to some extent. The compressibility factor, k, is a measure of the compressibility of a fluid. A smaller value of k indicates less compressibility. Air (which is a compressible fluid) has a compressibility factor of $k_{\text{air}} = 0.98$ m^2/N, while water has a compressibility factor of $k_{H_2O} = 4.9 \times 10^{-10}$ m^2/N $= 50 \times 10^{-6}$ atm^{-1}. In other words, a given volume of water decreases by 50 one-millionths of the original volume for each atmosphere (atm) increase in pressure. Thus the assumption that the water is incompressible is valid for our application.

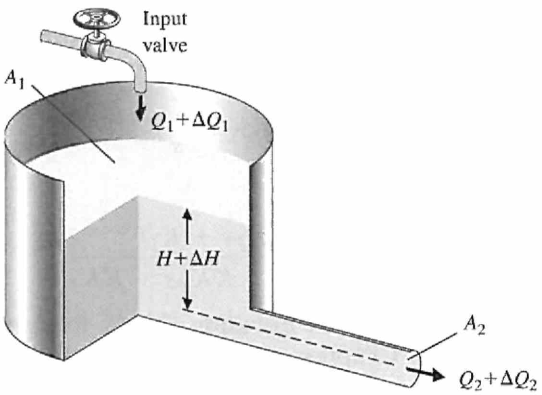


FIGURE 2.38 The fluid flow reservoir configuration.

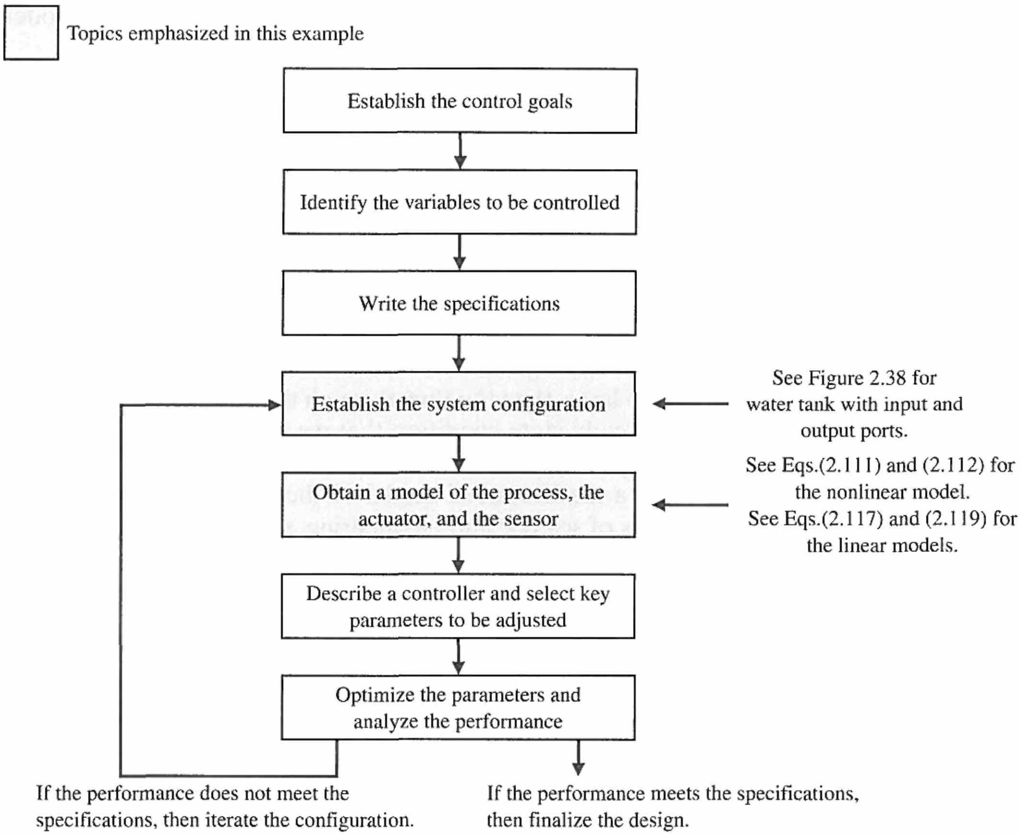


FIGURE 2.39 Elements of the control system design process emphasized in the fluid flow reservoir example.

Consider a fluid in motion. Suppose that initially the flow velocities are different for adjacent layers of fluid. Then an exchange of molecules between the two layers tends to equalize the velocities in the layers. This is internal friction, and the exchange of momentum is known as viscosity. Solids are more viscous than fluids, and fluids are more viscous than gases. A measure of viscosity is the coefficient of viscosity μ (N s/m^2). A larger coefficient of viscosity implies higher viscosity. The coefficient of viscosity (under standard conditions, 20°C) for air is

$$\mu_{\text{air}} = 0.178 \times 10^{-4} \text{ N s/m}^2,$$

and for water we have

$$\mu_{H_2O} = 1.054 \times 10^{-3} \text{ N s/m}^2.$$

So water is about 60 times more viscous than air. Viscosity depends primarily on temperature, not pressure. For comparison, water at 0°C is about 2 times more viscous than water at 20°C. With fluids of low viscosity, such as air and water, the effects of friction are important only in the boundary layer, a thin layer adjacent to the wall of the reservoir and output pipe. We can neglect viscosity in our model development. We say our fluid is inviscid.

If each fluid element at each point in the flow has no net angular velocity about that point, the flow is termed irrotational. Imagine a small paddle wheel immersed in the fluid (say in the output port). If the paddle wheel translates without rotating, the flow is irrotational. We will assume the water in the tank is irrotational. For an inviscid fluid, an initially irrotational flow remains irrotational.

The water flow in the tank and output port can be either steady or unsteady. The flow is steady if the velocity at each point is constant in time. This does not necessarily imply that the velocity is the same at every point but rather that at any given point the velocity does not change with time. Steady-state conditions can be achieved at low fluid speeds. We will assume steady flow conditions. If the output port area is too large, then the flow through the reservoir may not be slow enough to establish the steady-state condition that we are assuming exists and our model will not accurately predict the fluid flow motion.

To obtain a mathematical model of the flow within the reservoir, we employ basic principles of science and engineering, such as the principle of conservation of mass. The mass of water in the tank at any given time is

$$m = \rho A_1 H, \tag{2.108}$$

where A_1 is the area of the tank, ρ is the water density, and H is the height of the water in the reservoir. The constants for the reservoir system are given in Table 2.7.

In the following formulas, a subscript 1 denotes quantities at the input, and a subscript 2 refers to quantities at the output. Taking the time derivative of m in Equation (2.108) yields

$$\dot{m} = \rho A_1 \dot{H},$$

where we have used the fact that our fluid is incompressible (that is, $\dot{\rho} = 0$) and that the area of the tank, A_1, does not change with time. The change in mass in the reservoir is equal to the mass that enters the tank minus the mass that leaves the tank, or

$$\dot{m} = \rho A_1 \dot{H} = Q_1 - \rho A_2 v_2, \tag{2.109}$$

where Q_1 is the steady-state input mass flow rate, v_2 is the exit velocity, and A_2 is the output port area. The exit velocity, v_2, is a function of the water height. From Bernoulli's equation [39] we have

$$\frac{1}{2}\rho v_1^2 + P_1 + \rho g H = \frac{1}{2}\rho v_2^2 + P_2,$$

where v_1 is the water velocity at the mouth of the reservoir, and P_1 and P_2 are the atmospheric pressures at the input and output, respectively. But P_1 and P_2 are equal to

Table 2.7 Water Tank Physical Constants

ρ (kg/m^3)	g (m/s^2)	A_1 (m^2)	A_2 (m^2)	H^* (m)	Q^* (kg/s)
1000	9.8	$\pi/4$	$\pi/400$	1	34.77

atmospheric pressure, and A_2 is sufficiently small ($A_2 = A_1/100$), so the water flows out slowly and the velocity v_1 is negligible. Thus Bernoulli's equation reduces to

$$v_2 = \sqrt{2gH}. \tag{2.110}$$

Substituting Equation (2.110) into Equation (2.109) and solving for $\dot{H}$ yields

$$\dot{H} = -\left[\frac{A_2}{A_1}\sqrt{2g}\right]\sqrt{H} + \frac{1}{\rho A_1}Q_1. \tag{2.111}$$

Using Equation (2.110), we obtain the exit mass flow rate

$$Q_2 = \rho A_2 v_2 = (\rho\sqrt{2g}A_2)\sqrt{H}. \tag{2.112}$$

To keep the equations manageable, define

$$k_1 := -\frac{A_2\sqrt{2g}}{A_1},$$
$$k_2 := \frac{1}{\rho A_1},$$
$$k_3 := \rho\sqrt{2g}A_2.$$

Then, it follows that

$$\dot{H} = k_1\sqrt{H} + k_2 Q_1,$$
$$Q_2 = k_3\sqrt{H}. \tag{2.113}$$

Equation (2.113) represents our model of the water tank system, where the input is Q_1 and the output is Q_2. Equation (2.113) is a nonlinear, first-order, ordinary differential equation model. The nonlinearity comes from the $H^{1/2}$ term. The model in Equation (2.113) has the functional form

$$\dot{H} = f(H, Q_1),$$
$$Q_2 = h(H, Q_1),$$

where

$$f(H, Q_1) = k_1\sqrt{H} + k_2 Q_1 \quad \text{and} \quad h(H, Q_1) = k_3\sqrt{H}.$$

A set of linearized equations describing the height of the water in the reservoir is obtained using Taylor series expansions about an equilibrium flow condition. When the tank system is in equilibrium, we have $\dot{H} = 0$. We can define Q^* and H^* as the equilibrium input mass flow rate and water level, respectively. The relationship between Q^* and H^* is given by

$$Q^* = -\frac{k_1}{k_2}\sqrt{H^*} = \rho\sqrt{2g}A_2\sqrt{H^*}. \tag{2.114}$$

This condition occurs when just enough water enters the tank in A_1 to make up for the amount leaving through A_2. We can write the water level and input mass flow rate as

$$H = H^* + \Delta H, \tag{2.115}$$
$$Q_1 = Q^* + \Delta Q_1,$$

where ΔH and ΔQ_1 are small deviations from the equilibrium (steady-state) values. The Taylor series expansion about the equilibrium conditions is given by

$$\dot{H} = f(H, Q_1) = f(H^*, Q^*) + \left.\frac{\partial f}{\partial H}\right|_{\substack{H=H^* \\ Q_1=Q^*}} (H - H^*) \tag{2.116}$$

$$+ \left.\frac{\partial f}{\partial Q_1}\right|_{\substack{H=H^* \\ Q_1=Q^*}} (Q_1 - Q^*) + \cdots,$$

where

$$\left.\frac{\partial f}{\partial H}\right|_{\substack{H=H^* \\ Q_1=Q^*}} = \left.\frac{\partial(k_1\sqrt{H} + k_2Q_1)}{\partial H}\right|_{\substack{H=H^* \\ Q_1=Q^*}} = \frac{1}{2}\frac{k_1}{\sqrt{H^*}},$$

and

$$\left.\frac{\partial f}{\partial Q_1}\right|_{\substack{H=H^* \\ Q_1=Q^*}} = \left.\frac{\partial(k_1\sqrt{H} + k_2Q_1)}{\partial Q_1}\right|_{\substack{H=H^* \\ Q_1=Q^*}} = k_2.$$

Using Equation (2.114), we have

$$\sqrt{H^*} = \frac{Q^*}{\rho\sqrt{2g}A_2},$$

so that

$$\left.\frac{\partial f}{\partial H}\right|_{\substack{H=H^* \\ Q_1=Q^*}} = -\frac{A_2^2}{A_1}\frac{g\rho}{Q^*}.$$

It follows from Equation (2.115) that

$$\dot{H} = \Delta\dot{H},$$

since H^* is constant. Also, the term $f(H^*, Q^*)$ is identically zero, by definition of the equilibrium condition. Neglecting the higher order terms in the Taylor series expansion yields

$$\Delta\dot{H} = -\frac{A_2^2}{A_1}\frac{g\rho}{Q^*}\Delta H + \frac{1}{\rho A_1}\Delta Q_1. \tag{2.117}$$

Equation (2.117) is a linear model describing the deviation in water level ΔH from the steady-state due to a deviation from the nominal input mass flow rate ΔQ_1.

Similarly, for the output variable Q_2 we have

$$Q_2 = Q_2^* + \Delta Q_2 = h(H, Q_1) \tag{2.118}$$

$$\approx h(H^*, Q^*) + \left.\frac{\partial h}{\partial H}\right|_{\substack{H=H^* \\ Q_1=Q^*}} \Delta H + \left.\frac{\partial h}{\partial Q_1}\right|_{\substack{H=H^* \\ Q_1=Q^*}} \Delta Q_1,$$

where ΔQ_2 is a small deviation in the output mass flow rate and

$$\left.\frac{\partial h}{\partial H}\right|_{\substack{H=H^* \\ Q_1=Q^*}} = \frac{g\rho^2 A_2^2}{Q^*},$$

and

$$\left.\frac{\partial h}{\partial Q_1}\right|_{\substack{H=H^* \\ Q_1=Q^*}} = 0.$$

Therefore, the linearized equation for the output variable Q_2 is

$$\Delta Q_2 = \frac{g\rho^2 A_2^2}{Q^*}\Delta H. \tag{2.119}$$

For control system design and analysis, it is convenient to obtain the input–output relationship in the form of a transfer function. The tool to accomplish this is the Laplace transform, discussed in Section 2.4. Taking the time-derivative of Equation (2.119) and substituting into Equation (2.117) yields the input–output relationship

$$\Delta\dot{Q}_2 + \frac{A_2^2}{A_1}\frac{g\rho}{Q^*}\Delta Q_2 = \frac{A_2^2 g\rho}{A_1 Q^*}\Delta Q_1.$$

If we define

$$\Omega := \frac{A_2^2}{A_1}\frac{g\rho}{Q^*}, \tag{2.120}$$

then we have

$$\Delta\dot{Q}_2 + \Omega\Delta Q_2 = \Omega\Delta Q_1. \tag{2.121}$$

Taking the Laplace transform (with zero initial conditions) yields the transfer function

$$\Delta Q_2(s)/\Delta Q_1(s) = \frac{\Omega}{s + \Omega}. \tag{2.122}$$

Equation (2.122) describes the relationship between the change in the output mass flow rate $\Delta Q_2(s)$ due to a change in the input mass flow rate $\Delta Q_1(s)$. We can also obtain a transfer function relationship between the change in the input mass flow rate and the change in the water level in the tank, $\Delta H(s)$. Taking the Laplace transform (with zero initial conditions) of Eq. (2.117) yields

$$\Delta H(s)/\Delta Q_1(s) = \frac{k_2}{s + \Omega}. \tag{2.123}$$

Given the linear time-invariant model of the water tank system in Equation (2.121), we can obtain solutions for step and sinusoidal inputs. Remember that our input $\Delta Q_1(s)$ is actually a change in the input mass flow rate from the steady-state value Q^*.

Consider the step input

$$\Delta Q_1(s) = q_o/s,$$

where q_o is the magnitude of the step input, and the initial condition is $\Delta Q_2(0) = 0$. Then we can use the transfer function form given in Eq. (2.122) to obtain

$$\Delta Q_2(s) = \frac{q_o\Omega}{s(s + \Omega)}.$$

The partial fraction expansion yields

$$\Delta Q_2(s) = \frac{-q_o}{s + \Omega} + \frac{q_o}{s}.$$

Taking the inverse Laplace transform yields

$$\Delta Q_2(t) = -q_o e^{-\Omega t} + q_o.$$

Note that $\Omega > 0$ (see Equation (2.120)), so the term $e^{-\Omega t}$ approaches zero as t approaches ∞. Therefore, the steady-state output due to the step input of magnitude q_o is

$$\Delta Q_{2_{ss}} = q_o.$$

We see that in the steady state, the deviation of the output mass flow rate from the equilibrium value is equal to the deviation of the input mass flow rate from the equilibrium value. By examining the variable Ω in Equation (2.120), we find that the larger the output port opening A_2, the faster the system reaches steady state. In other words, as Ω gets larger, the exponential term $e^{-\Omega t}$ vanishes more quickly, and steady state is reached faster.

Similarly for the water level we have

$$\Delta H(s) = \frac{-q_o k_2}{\Omega}\left(\frac{1}{s + \Omega} - \frac{1}{s}\right).$$

Taking the inverse Laplace transform yields

$$\Delta H(t) = \frac{-q_o k_2}{\Omega}(e^{-\Omega t} - 1).$$

The steady-state change in water level due to the step input of magnitude q_o is

$$\Delta H_{ss} = \frac{q_o k_2}{\Omega}.$$

Consider the sinusoidal input

$$\Delta Q_1(t) = q_o \sin \omega t,$$

which has Laplace transform

$$\Delta Q_1(s) = \frac{q_o \omega}{s^2 + \omega^2}.$$

Suppose the system has zero initial conditions, that is, $\Delta Q_2(0) = 0$. Then from Equation (2.122) we have

$$\Delta Q_2(s) = \frac{q_o \omega \Omega}{(s + \Omega)(s^2 + \omega^2)}.$$

Expanding in a partial fraction expansion and taking the inverse Laplace transform yields

$$\Delta Q_2(t) = q_o \Omega \omega \left(\frac{e^{-\Omega t}}{\Omega^2 + \omega^2} + \frac{\sin(\omega t - \phi)}{\omega(\Omega^2 + \omega^2)^{1/2}}\right),$$

where $\phi = \tan^{-1}(\omega/\Omega)$. So, as $t \to \infty$, we have

$$\Delta Q_2(t) \quad \to \quad \frac{q_o\Omega}{\sqrt{\Omega^2 + \omega^2}}\sin(\omega t - \phi).$$

The maximum change in output flow rate is

$$|\Delta Q_2(t)|_{\max} = \frac{q_o\Omega}{\sqrt{\Omega^2 + \omega^2}}. \tag{2.124}$$

The above analytic analysis of the linear system model to step and sinusoidal inputs is a valuable way to gain insight into the system response to test signals. Analytic analysis is limited, however, in the sense that a more complete representation can be obtained with carefully constructed numerical investigations using computer simulations of both the linear and nonlinear mathematical models. A computer simulation uses a model and the actual conditions of the system being modeled, as well as actual input commands to which the system will be subjected.

Various levels of simulation fidelity (that is, accuracy) are available to the control engineer. In the early stages of the design process, highly interactive design software packages are effective. At this stage, computer speed is not as important as the time it takes to obtain an initial valid solution and to iterate and fine tune that solution. Good graphics output capability is crucial. The analysis simulations are generally low fidelity in the sense that many of the simplifications (such as linearization) made in the design process are retained in the simulation.

As the design matures usually it is necessary to conduct numerical experiments in a more realistic simulation environment. At this point in the design process, the computer processing speed becomes more important, since long simulation times necessarily reduce the number of computer experiments that can be obtained and correspondingly raise costs. Usually these high-fidelity simulations are programmed in FORTRAN, C, C++, Matlab, LabVIEW or similar languages.

Assuming that a model and the simulation are reliably accurate, computer simulation has the following advantages [13]:

1. System performance can be observed under all conceivable conditions.
2. Results of field-system performance can be extrapolated with a simulation model for prediction purposes.
3. Decisions concerning future systems presently in a conceptual stage can be examined.
4. Trials of systems under test can be accomplished in a much-reduced period of time.
5. Simulation results can be obtained at lower cost than real experimentation.
6. Study of hypothetical situations can be achieved even when the hypothetical situation would be unrealizable at present.
7. Computer modeling and simulation is often the only feasible or safe technique to analyze and evaluate a system.

The nonlinear model describing the water level flow rate is as follows (using the constants given in Table 2.7):

$$\dot{H} = -0.0443\sqrt{H} + 1.2732 \times 10^{-3} Q_1, \tag{2.125}$$

$$Q_2 = 34.77\sqrt{H}.$$

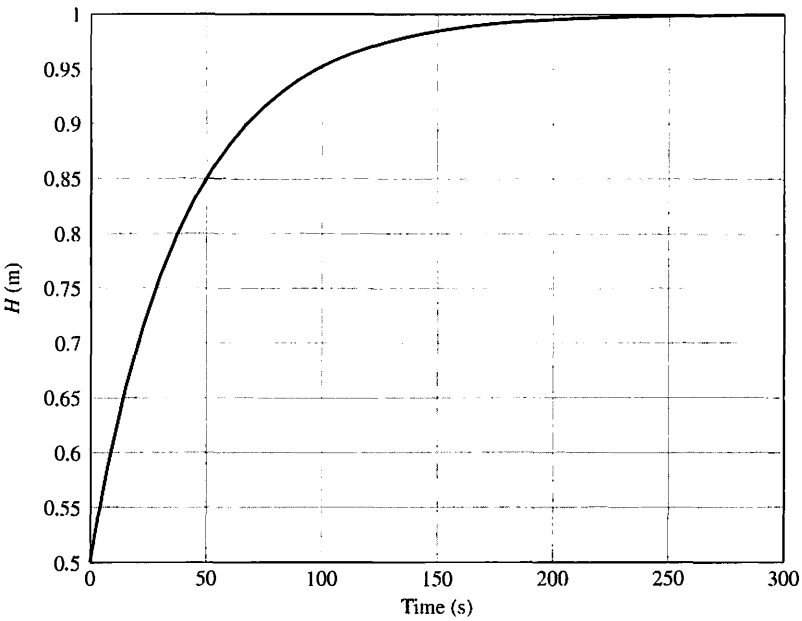


FIGURE 2.40 The tank water level time history obtained by integrating the nonlinear equations of motion in Equation (2.125) with $H(0) = 0.5$ m and $Q_1(t) = Q^* = 34.77$ kg/s.

With $H(0) = 0.5$ m and $Q_1(t) = 34.77$ kg/s, we can numerically integrate the nonlinear model given by Equation (2.125) to obtain the time history of $H(t)$ and $Q_2(t)$. The response of the system is shown in Figure 2.40. As expected from Equation (2.114), the system steady-state water level is $H^* = 1$ m when $Q^* = 34.77$ kg/m^3.

It takes about 250 seconds to reach steady-state. Suppose that the system is at steady state and we want to evaluate the response to a step change in the input mass flow rate. Consider

$$\Delta Q_1(t) = 1 \text{ kg/s}.$$

Then we can use the transfer function model to obtain the unit step response. The step response is shown in Figure 2.41 for both the linear and nonlinear models. Using the linear model, we find that the steady-state change in water level is $\Delta H = 5.75$ cm. Using the nonlinear model, we find that the steady-state change in water level is $\Delta H = 5.84$ cm. So we see a small difference in the results obtained from the linear model and the more accurate nonlinear model.

As the final step, we consider the system response to a sinusoidal change in the input flow rate. Let

$$\Delta Q_1(s) = \frac{q_o \omega}{s^2 + \omega^2},$$

where $\omega = 0.05$ rad/s and $q_o = 1$. The total water input flow rate is

$$Q_1(t) = Q^* + \Delta Q_1(t),$$

where $Q^* = 34.77$ kg/s. The output flow rate is shown in Figure 2.42.

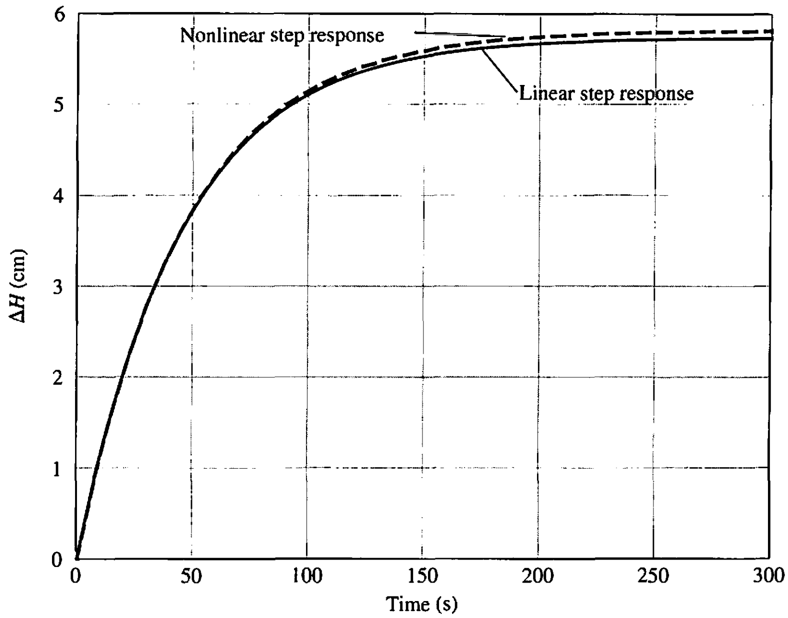


FIGURE 2.41 The response showing the linear versus nonlinear response to a step input.

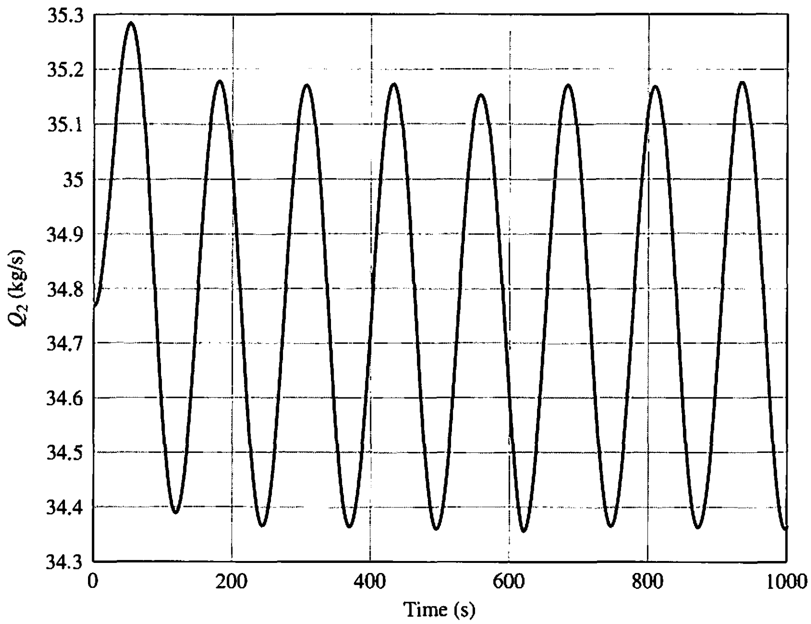


FIGURE 2.42 The output flow rate response to a sinusoidal variation in the input flow.

The response of the water level is shown in Figure 2.43. The water level is sinusoidal, with an average value of $H_{av} = H^* = 1$ m. As shown in Equation (2.124), the output flow rate is sinusoidal in the steady-state, with

$$|\Delta Q_2(t)|_{\max} = \frac{q_o \Omega}{\sqrt{\Omega^2 + \omega^2}} = 0.4 \text{ kg/s}.$$

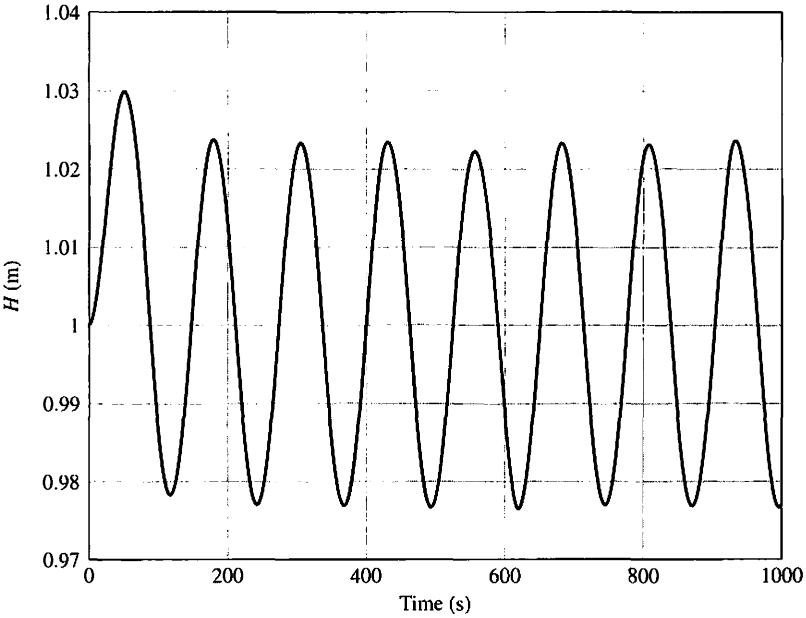


FIGURE 2.43 The water level response to a sinusoidal variation in the input flow.

Thus in the steady-state (see Figure 2.42) we expect that the output flow rate will oscillate at a frequency of $\omega = 0.05$ rad/s, with a maximum value of

$$Q_{2_{\max}} = Q^* + |\Delta Q_2(t)|_{\max} = 35.18 \text{ kg/s. } \blacksquare$$

EXAMPLE 2.14 **Electric traction motor control**

A majority of modern trains and local transit vehicles utilize electric traction motors. The electric motor drive for a railway vehicle is shown in block diagram form in Figure 2.44(a), incorporating the necessary control of the velocity of the vehicle. The goal of the design is to obtain a system model and the closed-loop transfer function of the system, $\omega(s)/\omega_d(s)$, select appropriate resistors R_1, R_2, R_3, and R_4, and then predict the system response.

The first step is to describe the transfer function of each block. We propose the use of a tachometer to generate a voltage proportional to velocity and to connect that voltage, v_t, to one input of a difference amplifier, as shown in Figure 2.44(b). The power amplifier is nonlinear and can be approximately represented by $v_2 = 2e^{3v_1} = g(v_1)$, an exponential function with a normal operating point, $v_{10} = 1.5$ V. Using the technique in Section 2.3, we then obtain a linear model:

$$\Delta v_2 = \left.\frac{dg(v_1)}{dv_1}\right|_{v_{10}} \Delta v_1 = 2[3\exp(3v_{10})]\,\Delta v_1 = 2(270)\,\Delta v_1 = 540\,\Delta v_1. \tag{2.126}$$

Then, discarding the delta notation and using the Laplace transform, we find that

$$V_2(s) = 540V_1(s).$$

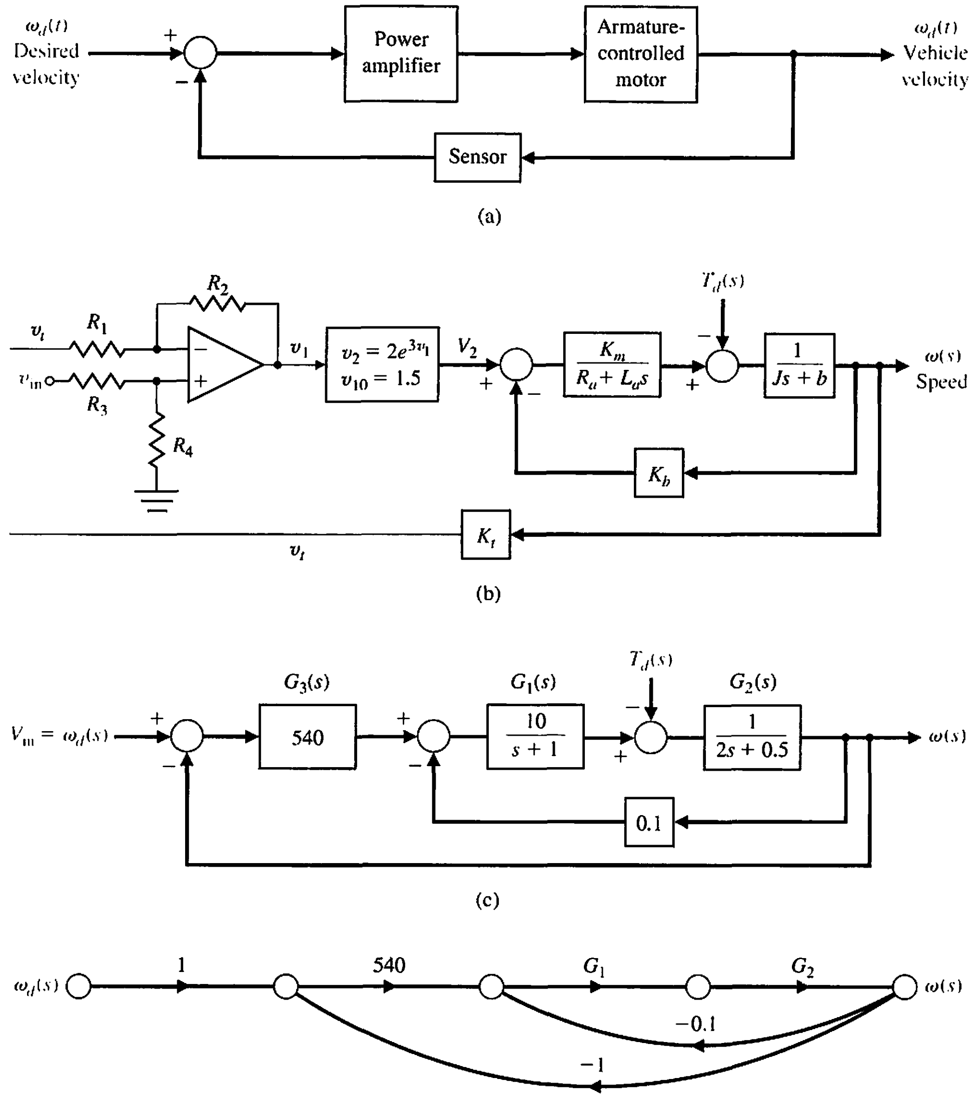


FIGURE 2.44 Speed control of an electric traction motor.

Also, for the differential amplifier, we have

$$v_1 = \frac{1 + R_2/R_1}{1 + R_3/R_4} v_{in} - \frac{R_2}{R_1} v_t. \tag{2.127}$$

We wish to obtain an input control that sets $\omega_d(t) = v_{in}$, where the units of ω_d are rad/s and the units of v_{in} are volts. Then, when $v_{in} = 10$ V, the steady-state speed is $\omega = 10$ rad/s. We note that $v_t = K_t\omega_d$ in steady state, and we expect, in balance, the steady-state output to be

$$v_1 = \frac{1 + R_2/R_1}{1 + R_3/R_4} v_{in} - \frac{R_2}{R_1} K_t v_{in}. \tag{2.128}$$

Table 2.8 Parameters of a Large DC Motor

$K_m = 10$	$J = 2$
$R_a = 1$	$b = 0.5$
$L_a = 1$	$K_b = 0.1$

When the system is in balance, $v_1 = 0$, and when $K_t = 0.1$, we have

$$\frac{1 + R_2/R_1}{1 + R_3/R_4} = \frac{R_2}{R_1}K_t = 1.$$

This relation can be achieved when

$$R_2/R_1 = 10 \quad \text{and} \quad R_3/R_4 = 10.$$

The parameters of the motor and load are given in Table 2.8. The overall system is shown in Figure 2.44(b). Reducing the block diagram in Figure 2.44(c) or the signal-flow graph in Figure 2.44(d) yields the transfer function

$$\begin{aligned}\frac{\omega(s)}{\omega_d(s)} &= \frac{540G_1(s)G_2(s)}{1 + 0.1G_1G_2 + 540G_1G_2} = \frac{540G_1G_2}{1 + 540.1G_1G_2} \\ &= \frac{5400}{(s + 1)(2s + 0.5) + 5401} = \frac{5400}{2s^2 + 2.5s + 5401.5} \\ &= \frac{2700}{s^2 + 1.25s + 2700.75}. \end{aligned} \tag{2.129}$$

Since the characteristic equation is second order, we note that $\omega_n = 52$ and $\zeta = 0.012$, and we expect the response of the system to be highly oscillatory (underdamped). ■

EXAMPLE 2.15 **Mechanical accelerometer**

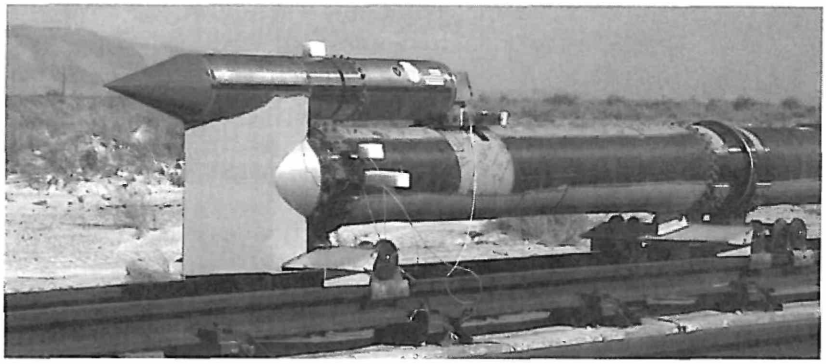
A mechanical accelerometer is used to measure the acceleration of a rocket test sled, as shown in Figure 2.45. The test sled maneuvers above a guide rail a small distance δ. The accelerometer provides a measurement of the acceleration $a(t)$ of the sled, since the position y of the mass M, with respect to the accelerometer case, is proportional to the acceleration of the case (and the sled). The goal is to design an accelerometer with an appropriate dynamic responsiveness. We wish to design an accelerometer with an acceptable time for the desired measurement characteristic, $y(t) = qa(t)$, to be attained (q is a constant).

The sum of the forces acting on the mass is

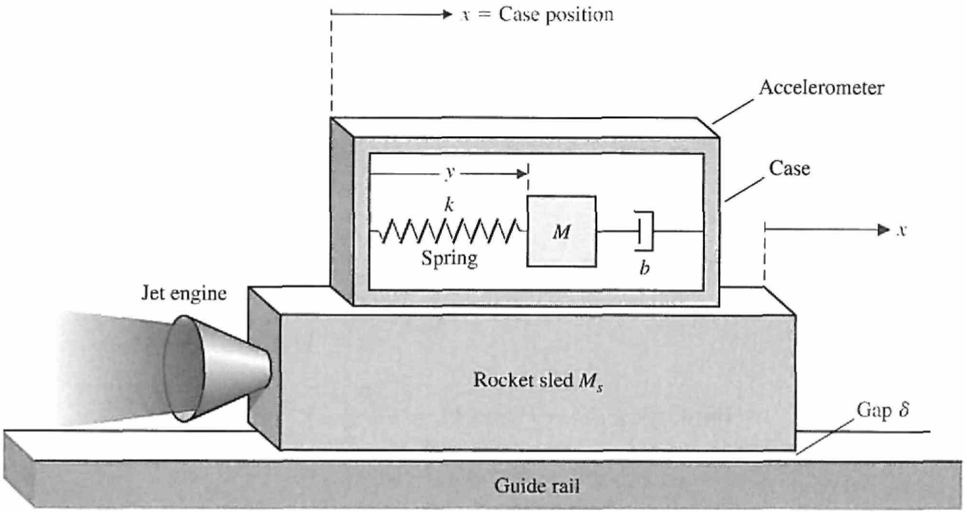
$$-b\frac{dy}{dt} - ky = M\frac{d^2}{dt^2}(y + x)$$

or

$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = -M\frac{d^2x}{dt^2}. \tag{2.130}$$



(a)



(b)

FIGURE 2.45 (a) This rocket-propelled sled set a world land speed record for a railed vehicle at 6,453 mph (U.S. Air Force photo by 2nd Lt. Heather Newcomb). (b) A schematic of an accelerometer mounted on the rocket sled.

Since

$$M_s \frac{d^2x}{dt^2} = F(t),$$

is the engine force, we have

$$M\ddot{y} + b\dot{y} + ky = -\frac{M}{M_s}F(t),$$

or

$$\ddot{y} + \frac{b}{M}\dot{y} + \frac{k}{M}y = -\frac{F(t)}{M_s}. \tag{2.131}$$

We select the coefficients where $b/M = 3$, $k/M = 2$, $F(t)/M_s = Q(t)$, and we consider the initial conditions $y(0) = -1$ and $\dot{y}(0) = 2$. We then obtain the Laplace transform equation, when the force, and thus $Q(t)$, is a step function, as follows:

$$(s^2Y(s) - sy(0) - \dot{y}(0)) + 3(sY(s) - y(0)) + 2Y(s) = -Q(s). \quad (2.132)$$

Since $Q(s) = P/s$, where P is the magnitude of the step function, we obtain

$$(s^2Y(s) + s - 2) + 3(sY(s) + 1) + 2Y(s) = -\frac{P}{s},$$

or

$$(s^2 + 3s + 2)Y(s) = \frac{-(s^2 + s + P)}{s}. \quad (2.133)$$

Thus the output transform is

$$Y(s) = \frac{-(s^2 + s + P)}{s(s^2 + 3s + 2)} = \frac{-(s^2 + s + P)}{s(s + 1)(s + 2)}. \quad (2.134)$$

Expanding in partial fraction form yields

$$Y(s) = \frac{k_1}{s} + \frac{k_2}{s + 1} + \frac{k_3}{s + 2}. \quad (2.135)$$

We then have

$$k_1 = \left.\frac{-(s^2 + s + P)}{(s + 1)(s + 2)}\right|_{s=0} = -\frac{P}{2}. \quad (2.136)$$

Similarly, $k_2 = +P$ and $k_3 = \frac{-P - 2}{2}$. Thus,

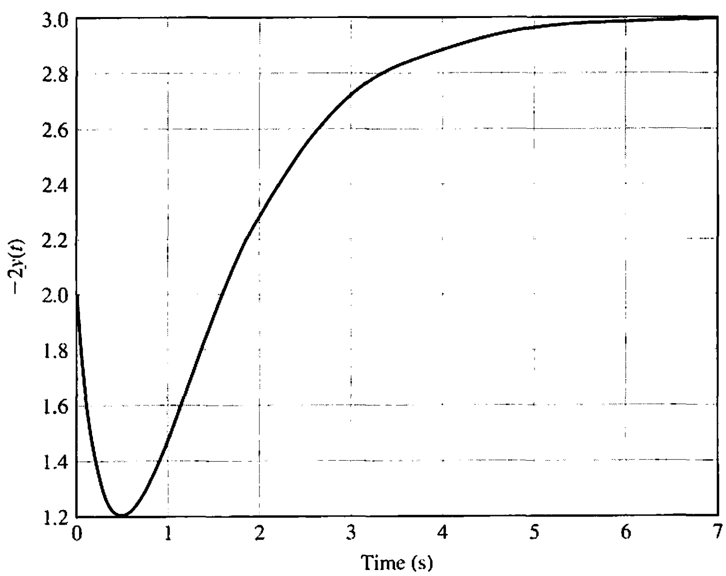


FIGURE 2.46 Accelerometer response.

$$Y(s) = \frac{-P}{2s} + \frac{P}{s+1} + \frac{-P-2}{2(s+2)}. \tag{2.137}$$

Therefore, the output measurement is

$$y(t) = \frac{1}{2}[-P + 2Pe^{-t} - (P+2)e^{-2t}], \quad t \geq 0.$$

A plot of $y(t)$ is shown in Figure 2.46 for $P = 3$. We can see that $y(t)$ is proportional to the magnitude of the force after 5 seconds. Thus in steady state, after 5 seconds, the response $y(t)$ is proportional to the acceleration, as desired. If this period is excessively long, we must increase the spring constant, k, and the friction, b, while reducing the mass, M. If we are able to select the components so that $b/M = 12$ and $k/M = 32$, the accelerometer will attain the proportional response in 1 second. (It is left to the reader to show this.) ■

EXAMPLE 2.16 **Design of a laboratory robot**

In this example, we endeavor to show the physical design of a laboratory device and demonstrate its complex design. We will also exhibit the many components commonly used in a control system.

A robot for laboratory use is shown in Figure 2.47. A laboratory robot's work volume must allow the robot to reach the entire bench area and access existing analytical instruments. There must also be sufficient area for a stockroom of supplies for unattended operation.

The laboratory robot can be involved in three types of tasks during an analytical experiment. The first is sample introduction, wherein the robot is trained to accept a number of different sample trays, racks, and containers and to introduce them into the system. The second set of tasks involves the robot transporting the samples between individual dedicated automated stations for chemical preparation and instrumental analysis. Samples must be scheduled and moved between these stations as necessary to complete the analysis. In the third set of tasks for the robot, flexible automation provides new capability to the analytical laboratory. The robot must be programmed to emulate the human operator or work with various devices. All of these types of operations are required for an effective laboratory robot.

The ORCA laboratory robot is an anthropomorphic arm, mounted on a rail, designed as the optimum configuration for the analytical laboratory [14]. The rail can be located at the front or back of a workbench, or placed in the middle of a table when access to both sides of the rail is required. Simple software commands permit moving the arm from one side of the rail to the other while maintaining the wrist position (to transfer open containers) or locking the wrist angle (to transfer objects in virtually any orientation). The rectilinear geometry, in contrast to the cylindrical geometry used by many robots, permits more accessories to be placed within the robot workspace and provides an excellent match to the laboratory bench. Movement of all joints is coordinated through software, which simplifies the use of the robot by representing the robot positions and movements in the more familiar Cartesian coordinate space.

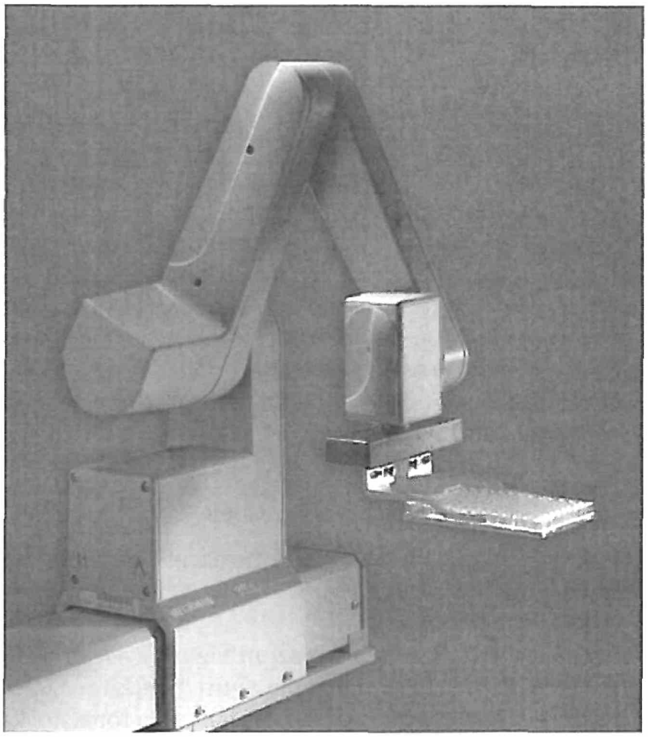


FIGURE 2.47 Laboratory robot used for sample preparation. The robot manipulates small objects, such as test tubes, and probes in and out of tight places at relatively high speeds [15]. (Photo *courtesy of* Beckman Coulter, Inc.)

Table 2.9 ORCA Robot Arm Hardware Specifications

Arm	Articulated, Rail-Mounted	Teach Pendant	Joy Stick with Emergency Stop
Degrees of freedom	6	Cycle time	4 s (move 1 inch up, 12 inch across, 1 inch down, and back)
Reach	±54 cm	Maximum speed	75 cm/s
Height	78 cm	Dwell time	50 ms typical (for moves within a motion)
Rail	1 and 2 m	Payload	0.5 kg continuous, 2.5 kg transient (with restrictions)
Weight	8.0 kg	Vertical deflection	<1.5 mm at continuous payload
Precision	±0.25 mm	Cross-sectional work envelope	1 m^2
Finger travel (gripper)	40 mm		
Gripper rotation	±77 revolutions		

The physical and performance specifications of the ORCA system are shown in Table 2.9. The design for the ORCA laboratory robot progressed to the selection of the component parts required to obtain the total system. The exploded view of the robot is shown in Figure 2.48. This device uses six DC motors, gears, belt drives, and a rail and carriage. The specifications are challenging and require the designer to model the system components and their interconnections accurately. ■

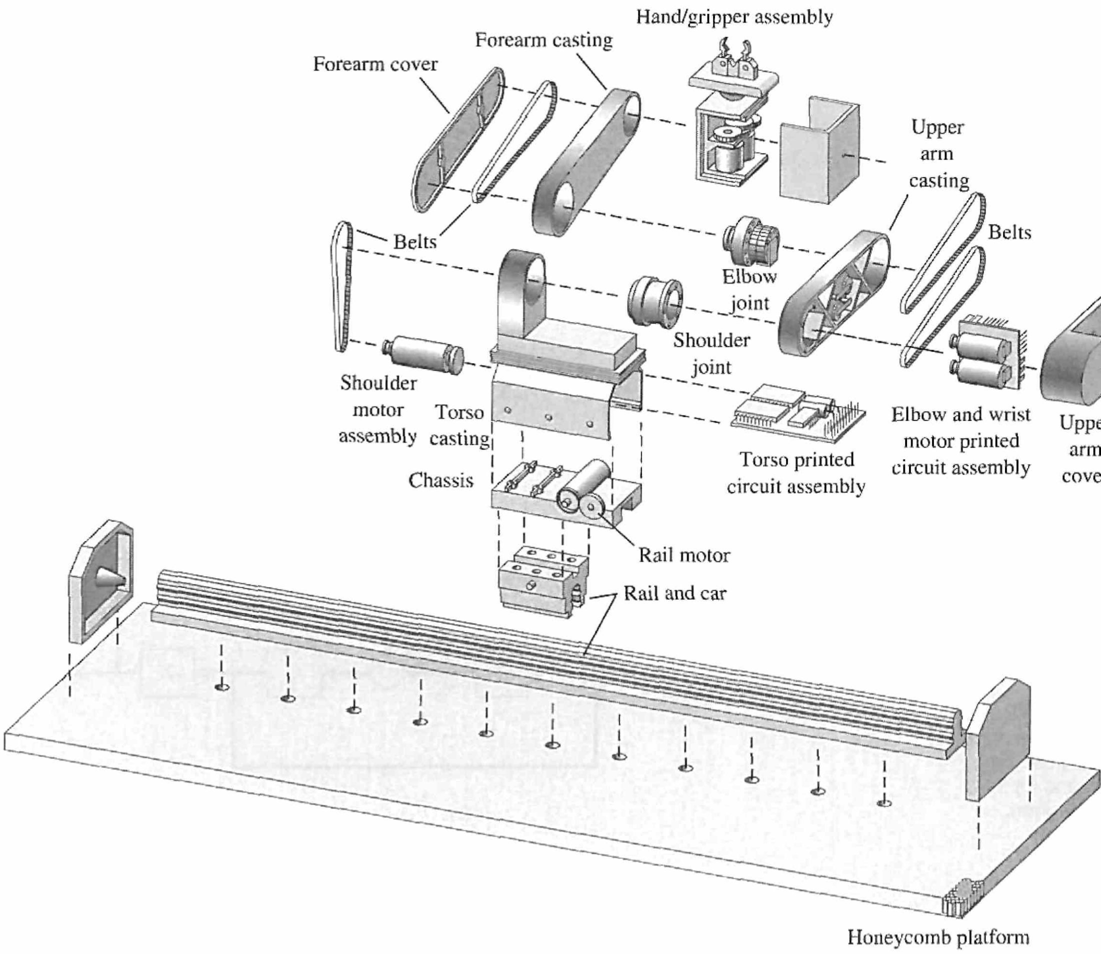


FIGURE 2.48 Exploded view of the ORCA robot showing the components [15]. (Courtesy of Beckman Coulter, Inc.)

EXAMPLE 2.17 **Design of a low-pass filter**

Our goal is to design a first-order low-pass filter that passes signals at a frequency below 106.1 Hz and attenuates signals with a frequency above 106.1 Hz. In addition, the DC gain should be $\frac{1}{2}$.

A ladder network with one energy storage element, as shown in Figure 2.49(a), will act as a first-order low-pass network. Note that the DC gain will be equal to $\frac{1}{2}$ (open-circuit the capacitor). The current and voltage equations are

$$I_1 = (V_1 - V_2)G,$$

$$I_2 = (V_2 - V_3)G,$$

$$V_2 = (I_1 - I_2)R,$$

$$V_3 = I_2 Z,$$

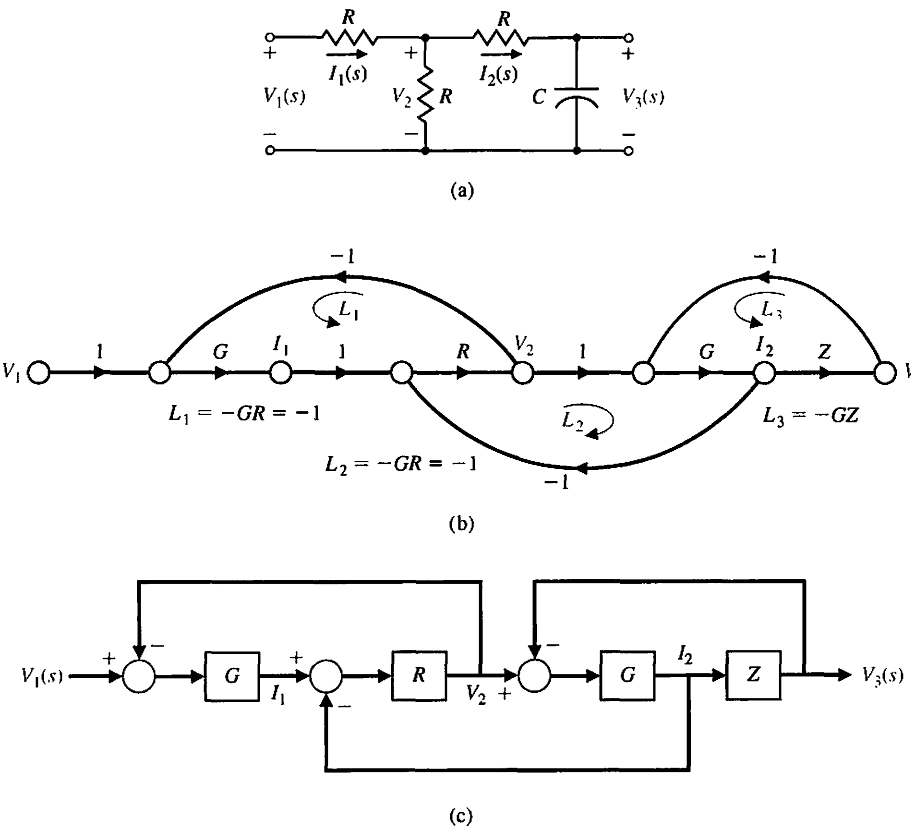


FIGURE 2.49 (a) Ladder network, (b) its signal-flow graph, and (c) its block diagram.

where $G = 1/R$, $Z(s) = 1/Cs$, and $I_1(s) = I_1$ (we omit the (s)). The signal-flow graph constructed for the four equations is shown in Figure 2.49(b), and the corresponding block diagram is shown in Figure 2.49(c). The three loops are $L_1 = -GR = -1$, $L_2 = -GR = -1$, and $L_3 = -GZ$. All loops touch the forward path. Loops L_1 and L_3 are nontouching. Therefore, the transfer function is

$$T(s) = \frac{V_3}{V_1} = \frac{P_1}{1 - (L_1 + L_2 + L_3) + L_1 L_3} = \frac{GZ}{3 + 2GZ}$$
$$= \frac{1}{3RCs + 2} = \frac{1/(3RC)}{s + 2/(3RC)}.$$

If one prefers to utilize block diagram reduction techniques, one can start at the output with

$$V_3(s) = ZI_2(s).$$

But the block diagram shows that

$$I_2(s) = G(V_2(s) - V_3(s)).$$

Therefore,

$$V_3(s) = ZGV_2(s) - ZGV_3(s)$$

so

$$V_2(s) = \frac{1 + ZG}{ZG} V_3(s).$$

We will use this relationship between $V_3(s)$ and $V_2(s)$ in the subsequent development. Continuing with the block diagram reduction, we have

$$V_3(s) = -ZGV_3(s) + ZGR(I_1(s) - I_2(s)),$$

but from the block diagram, we see that

$$I_1 = G(V_1(s) - V_2(s)), \qquad I_2 = \frac{V_3(s)}{Z}.$$

Therefore,

$$V_3(s) = -ZGV_3(s) + ZG^2R(V_1(s) - V_2(s)) - GRV_3(s).$$

Substituting for $V_2(s)$ yields

$$V_3(s) = \frac{(GR)(GZ)}{1 + 2GR + GZ + (GR)(GZ)} V_1(s).$$

But we know that $GR = 1$; hence, we obtain

$$V_3(s) = \frac{GZ}{3 + 2GZ} V_1(s).$$

Note that the DC gain is $1/2$, as expected. The pole is desired at $p = 2\pi(106.1) = 666.7 = 2000/3$. Therefore, we require $RC = 0.001$. Select $R = 1\ \text{k}\Omega$ and $C = 1\ \mu\text{F}$. Hence, we achieve the filter

$$T(s) = \frac{333.3}{(s + 666.7)}. \quad \blacksquare$$

2.9 THE SIMULATION OF SYSTEMS USING CONTROL DESIGN SOFTWARE

Application of the many classical and modern control system design and analysis tools is based on mathematical models. Most popular control design software packages can be used with systems given in the form of transfer function descriptions. In this book, we will focus on m-file scripts containing commands and functions to analyze and design control systems. Various commercial control system packages are available for student use. The m-files described here are compatible with the MATLAB[†] Control System Toolbox and the LabVIEW MathScript RT Module.[‡]

[†]See Appendix A for an introduction to MATLAB.

[‡]See Appendix B for an introduction to LabVIEW MathScipt RT Module.

We begin this section by analyzing a typical spring-mass-damper mathematical model of a mechanical system. Using an m-file script, we will develop an interactive analysis capability to analyze the effects of natural frequency and damping on the unforced response of the mass displacement. This analysis will use the fact that we have an analytic solution that describes the unforced time response of the mass displacement.

Later, we will discuss transfer functions and block diagrams. In particular, we are interested in manipulating polynomials, computing poles and zeros of transfer functions, computing closed-loop transfer functions, computing block diagram reductions, and computing the response of a system to a unit step input. The section concludes with the electric traction motor control design of Example 2.14.

The functions covered in this section are roots, poly, conv, polyval, tf, pzmap, pole, zero, series, parallel, feedback, minreal, and step.

Spring-Mass-Damper System. A spring-mass-damper mechanical system is shown in Figure 2.2. The motion of the mass, denoted by $y(t)$, is described by the differential equation

$$M\ddot{y}(t) + b\dot{y}(t) + ky(t) = r(t).$$

The unforced dynamic response $y(t)$ of the spring-mass-damper mechanical system is

$$\boxed{y(t) = \frac{y(0)}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \theta\right),}$$

where $\omega_n = \sqrt{k/M}$, $\zeta = b/(2\sqrt{kM})$, and $\theta = \cos^{-1}\zeta$. The initial displacement is $y(0)$. The transient system response is underdamped when $\zeta < 1$, overdamped when $\zeta > 1$, and critically damped when $\zeta = 1$. We can visualize the unforced time response of the mass displacement following an initial displacement of $y(0)$. Consider the underdamped case:

- $y(0) = 0.15 \text{ m}, \quad \omega_n = \sqrt{2}\dfrac{\text{rad}}{\text{sec}}, \quad \zeta = \dfrac{1}{2\sqrt{2}} \quad \left(\dfrac{k}{M} = 2, \dfrac{b}{M} = 1\right).$

The commands to generate the plot of the unforced response are shown in Figure 2.50. In the setup, the variables $y(0)$, ω_n, t, and ζ are input at the command level. Then the script unforced.m is executed to generate the desired plots. This creates an interactive analysis capability to analyze the effects of natural frequency and damping on the unforced response of the mass displacement. One can investigate the effects of the natural frequency and the damping on the time response by simply entering new values of ω_n and ζ at the command prompt and running the script unforced.m again. The time-response plot is shown in Figure 2.51. Notice that the script automatically labels the plot with the values of the damping coefficient and natural frequency. This avoids confusion when making many interactive simulations. Using scripts is an important aspect of developing an effective interactive design and analysis capability.

For the spring-mass-damper problem, the unforced solution to the differential equation was readily available. In general, when simulating closed-loop feedback

```
>>y0=0.15;
>>wn=sqrt(2);                      <-- ωn
>>zeta=1/(2*sqrt(2));              <-- ζ
>>t=[0:0.1:10];
>>unforced
```

unforced.m

```
%Compute Unforced Response to an Initial Condition
%
c=(y0/sqrt(1-zeta^2));             <-- y(0)/√(1 - ζ^2)
y=c*exp(-zeta*wn*t).*sin(wn*sqrt(1-zeta^2)*t+acos(zeta));
%
bu=c*exp(-zeta*wn*t);bl=-bu;       <-- e^(-ζωn t) envelope
%
plot(t,y,t,bu,'--',t,bl,'--'), grid
xlabel('Time (s)'), ylabel('y(t) (m)')
legend(['\omega_n=',num2str(wn),'   \zeta=',num2str(zeta)])
```

FIGURE 2.50 Script to analyze the spring-mass-damper.

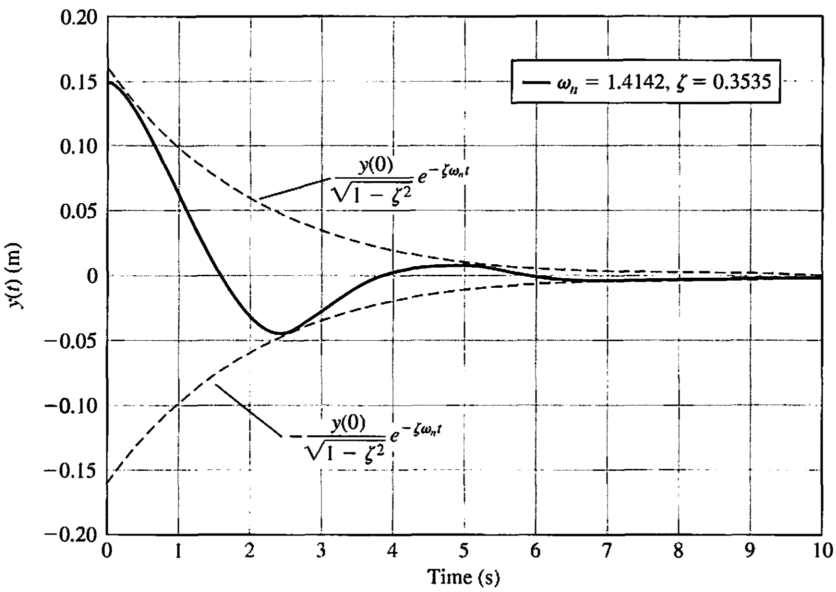


FIGURE 2.51 Spring-mass-damper unforced response.

control systems subject to a variety of inputs and initial conditions, it is difficult to obtain the solution analytically. In these cases, we can compute the solutions numerically and to display the solution graphically.

Most systems considered in this book can be described by transfer functions. Since the transfer function is a ratio of polynomials, we begin by investigating how to manipulate polynomials, remembering that working with transfer functions means that both a numerator polynomial and a denominator polynomial must be specified.

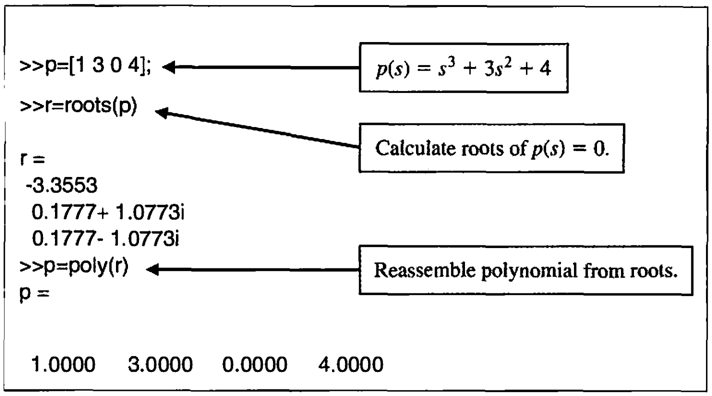


FIGURE 2.52 Entering the polynomial $p(s) = s^3 + 3s^2 + 4$ and calculating its roots.

Polynomials are represented by row vectors containing the polynomial coefficients in order of descending degree. For example, the polynomial

$$p(s) = s^3 + 3s^2 + 4$$

is entered as shown in Figure 2.52. Notice that even though the coefficient of the s term is zero, it is included in the input definition of $p(s)$.

If **p** is a row vector containing the coefficients of $p(s)$ in descending degree, then roots(**p**) is a column vector containing the roots of the polynomial. Conversely, if **r** is a column vector containing the roots of the polynomial, then poly(**r**) is a row vector with the polynomial coefficients in descending degree. We can compute the roots of the polynomial $p(s) = s^3 + 3s^2 + 4$ with the roots function as shown in Figure 2.52. In this figure, we show how to reassemble the polynomial with the poly function.

Multiplication of polynomials is accomplished with the conv function. Suppose we want to expand the polynomial

$$n(s) = (3s^2 + 2s + 1)(s + 4).$$

The associated commands using the conv function are shown in Figure 2.53. Thus, the expanded polynomial is

$$n(s) = 3s^3 + 14s^2 + 9s + 4.$$

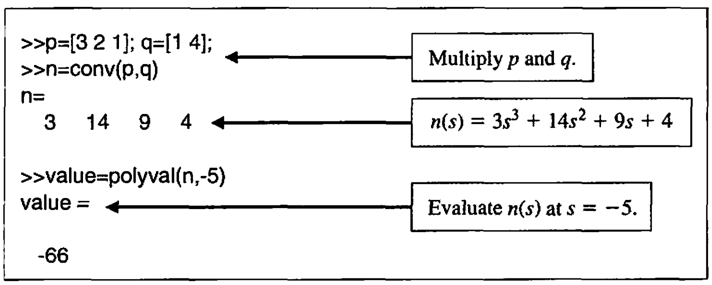


FIGURE 2.53 Using **conv** and **polyval** to multiply and evaluate the polynomials $(3s^2 + 2s + 1)(s + 4)$.

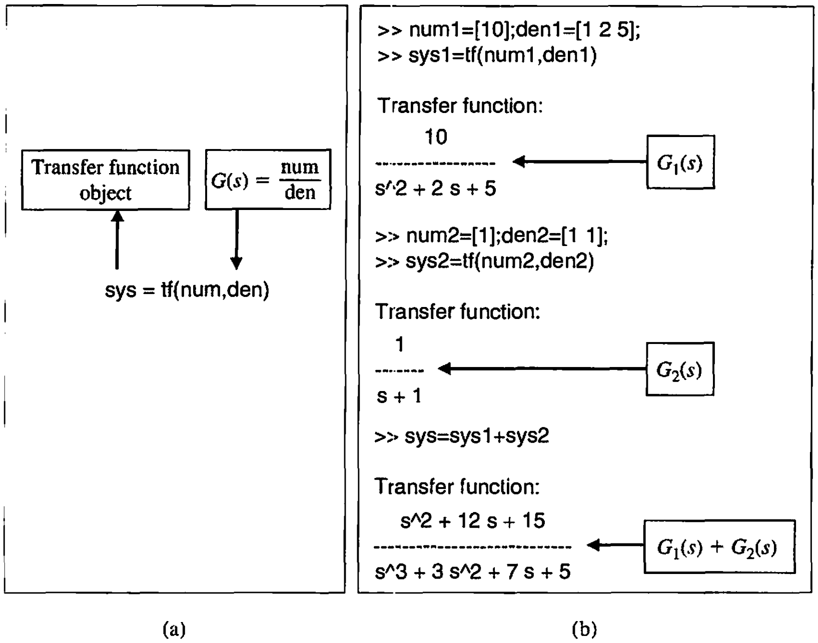


FIGURE 2.54 (a) The **tf** function. (b) Using the **tf** function to create transfer function objects and adding them using the "+" operator.

The function polyval is used to evaluate the value of a polynomial at the given value of the variable. The polynomial $n(s)$ has the value $n(-5) = -66$, as shown in Figure 2.53.

Linear, time-invariant system models can be treated as *objects*, allowing one to manipulate the system models as single entities. In the case of transfer functions, one creates the system models using the tf function; for state variable models one employs the ss function (see Chapter 3). The use of tf is illustrated in Figure 2.54(a). For example, if one has the two system models

$$G_1(s) = \frac{10}{s^2 + 2s + 5} \quad \text{and} \quad G_2(s) = \frac{1}{s + 1},$$

one can add them using the "+" operator to obtain

$$G(s) = G_1(s) - G_2(s) = \frac{s^2 + 12s + 15}{s^3 + 3s^2 + 7s + 5}.$$

The corresponding commands are shown in Figure 2.54(b) where sys1 represents $G_1(s)$ and sys2 represents $G_2(s)$. Computing the poles and zeros associated with a transfer function is accomplished by operating on the system model object with the pole and zero functions, respectively, as illustrated in Figure 2.55.

In the next example, we will obtain a plot of the pole–zero locations in the complex plane. This will be accomplished using the pzmap function, shown in Figure 2.56. On the pole–zero map, zeros are denoted by an "o" and poles are denoted by an "×". If the pzmap function is invoked without left-hand arguments, the plot is generated automatically.

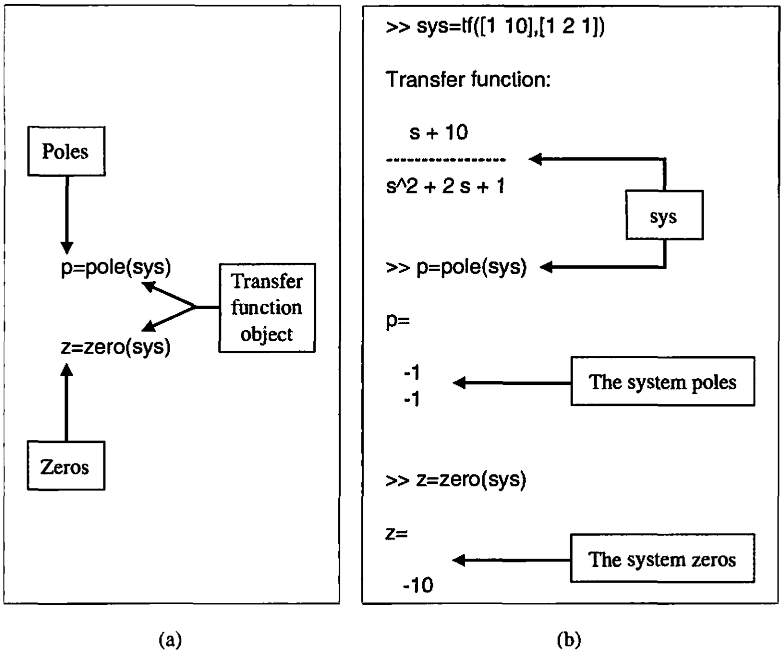


FIGURE 2.55
(a) The **pole** and **zero** functions.
(b) Using the pole and zero functions to compute the **pole** and **zero** locations of a linear system.

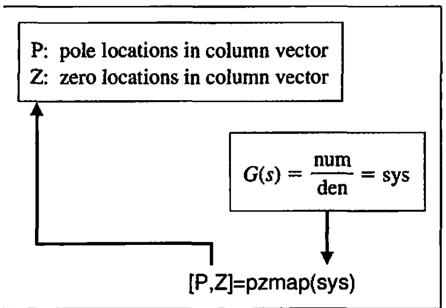


FIGURE 2.56
The **pzmap** function.

EXAMPLE 2.18 **Transfer functions**

Consider the transfer functions

$$G(s) = \frac{6s^2 + 1}{s^3 + 3s^2 + 3s + 1} \quad \text{and} \quad H(s) = \frac{(s + 1)(s + 2)}{(s + 2i)(s - 2i)(s + 3)}.$$

Using an m-file script, we can compute the poles and zeros of $G(s)$, the characteristic equation of $H(s)$, and divide $G(s)$ by $H(s)$. We can also obtain a plot of the pole–zero map of $G(s)/H(s)$ in the complex plane.

The pole–zero map of the transfer function $G(s)/H(s)$ is shown in Figure 2.57, and the associated commands are shown in Figure 2.58. The pole–zero map shows clearly the five zero locations, but it appears that there are only two poles. This

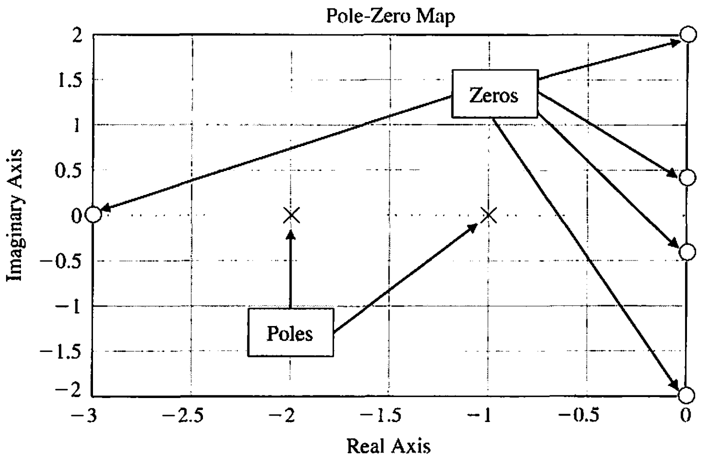


FIGURE 2.57 Pole–zero map for $G(s)/H(s)$.

```
>>numg=[6 0 1]; deng=[1 3 3 1];sysg=tf(numg,deng);
>>z=zero(sysg)

z =
    0 + 0.4082i
    0 - 0.4082i

>>p=pole(sysg)

p =
    -1.0000
    -1.0000 + 0.0000i
    -1.0000 - 0.0000i

>>n1=[1 1]; n2=[1 2]; d1=[1 2*i]; d2=[1 -2*i]; d3=[1 3];
>>numh=conv(n1,n2); denh=conv(d1,conv(d2,d3));
>>sysh=tf(numh,denh)

Transfer function:
     s^2 + 3 s + 2
-----------------------
s^3 + 3 s^2 + 4 s + 12

>>sys=sysg/sysh

Transfer function:
6 s^5 + 18 s^4 + 25 s^3 + 75 s^2 + 4 s +12
-------------------------------------------
  s^5 + 6 s^4 + 14 s^3 + 16 s^2 + 9 s + 2

>>pzmap(sys)
```

Compute poles and zeros of $G(s)$

Expand $H(s)$

$H(s)$

$\frac{G(s)}{H(s)} = \text{sys}$

Pole-zero map

FIGURE 2.58 Transfer function example for $G(s)$ and $H(s)$.

cannot be the case, since we know that for physical systems the number of poles must be greater than or equal to the number of zeros. Using the roots function, we can ascertain that there are in fact four poles at $s = -1$. Hence, multiple poles or multiple zeros at the same location cannot be discerned on the pole–zero map. ■

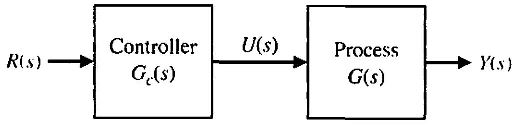


FIGURE 2.59 Open-loop control system (without feedback).

Block Diagram Models. Suppose we have developed mathematical models in the form of transfer functions for a process, represented by $G(s)$, and a controller, represented by $G_c(s)$, and possibly many other system components such as sensors and actuators. Our objective is to interconnect these components to form a control system.

A simple open-loop control system can be obtained by interconnecting a process and a controller in series as illustrated in Figure 2.59. We can compute the transfer function from $R(s)$ to $Y(s)$, as follows.

EXAMPLE 2.19 **Series connection**

Let the process represented by the transfer function $G(s)$ be

$$G(s) = \frac{1}{500s^2},$$

and let the controller represented by the transfer function $G_c(s)$ be

$$G_c(s) = \frac{s + 1}{s + 2}.$$

We can use the series function to cascade two transfer functions $G_1(s)$ and $G_2(s)$, as shown in Figure 2.60.

The transfer function $G_c(s)G(s)$ is computed using the series function as shown in Figure 2.61. The resulting transfer function is

$$G_c(s)G(s) = \frac{s + 1}{500s^3 + 1000s^2} = \text{sys},$$

where sys is the transfer function name in the m-file script. ■

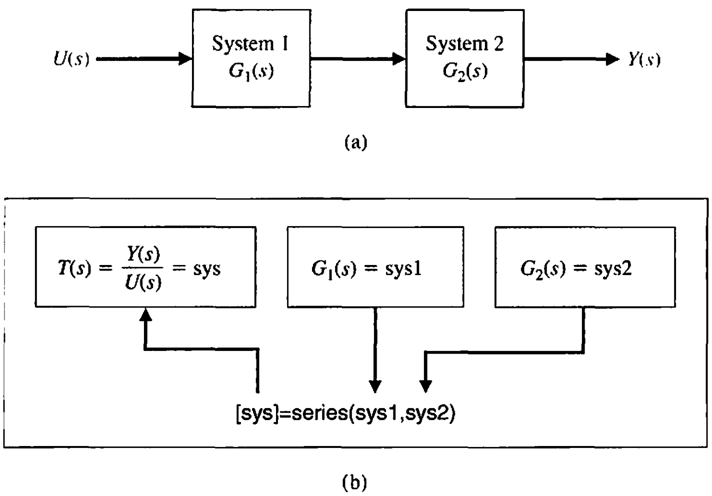
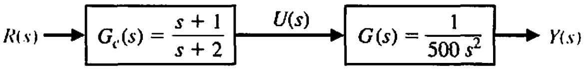


FIGURE 2.60 (a) Block diagram. (b) The **series** function.



(a)

```
>>numg=[1]; deng=[500 0 0]; sysg=tf(numg,deng);
>>numh=[1 1]; denh=[1 2]; sysh=tf(numh,denh);
>>sys=series(sysg,sysh);
>>sys

Transfer function:
        s + 1
-------------------
500 s^3 + 1000 s^2
```

← $G_c(s)G(s)$

(b)

FIGURE 2.61 Application of the **series** function.

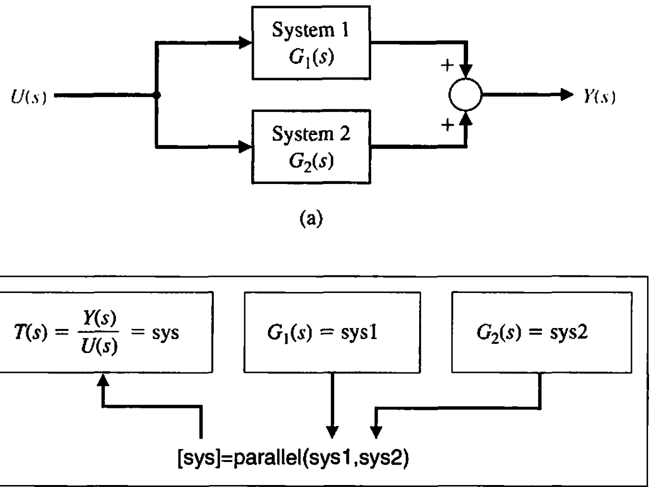


FIGURE 2.62 (a) Block diagram. (b) The **parallel** function.

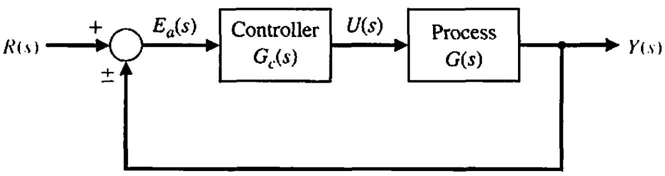
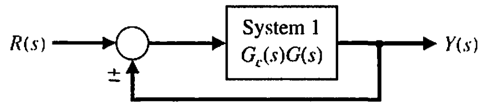


FIGURE 2.63 A basic control system with unity feedback.

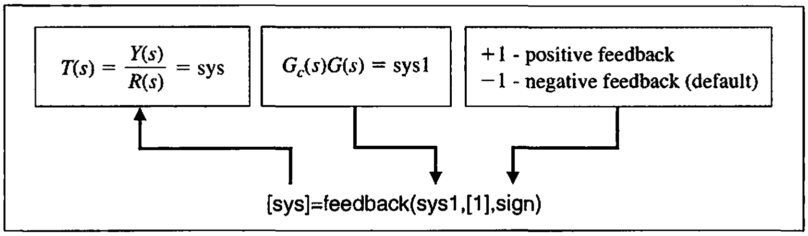
Block diagrams quite often have transfer functions in parallel. In such cases, the function parallel can be quite useful. The parallel function is described in Figure 2.62.

We can introduce a feedback signal into the control system by closing the loop with **unity feedback**, as shown in Figure 2.63. The signal $E_a(s)$ is an **error signal**; the signal $R(s)$ is a **reference input**. In this control system, the controller is in the forward path, and the closed-loop transfer function is

$$T(s) = \frac{G_c(s)G(s)}{1 \mp G_c(s)G(s)}.$$

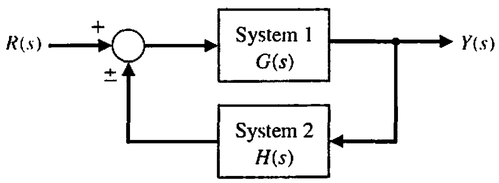


(a)

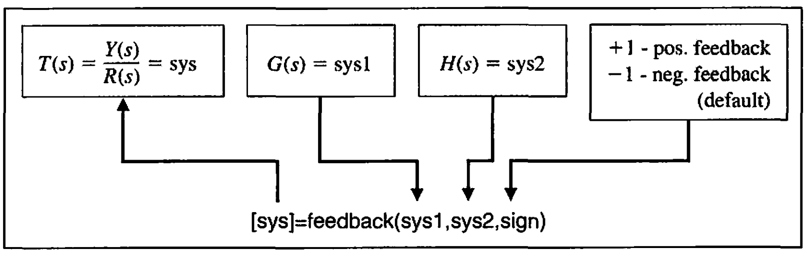


(b)

FIGURE 2.64 (a) Block diagram. (b) The **feedback** function with unity feedback.



(a)



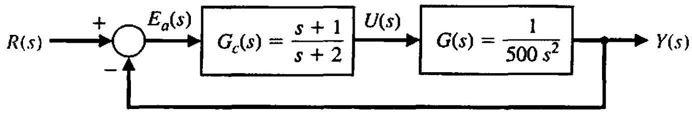
(b)

FIGURE 2.65 (a) Block diagram. (b) The **feedback** function.

We can utilize the feedback function to aid in the block diagram reduction process to compute closed-loop transfer functions for single- and multiple-loop control systems.

It is often the case that the closed-loop control system has unity feedback, as illustrated in Figure 2.63. We can use the feedback function to compute the closed-loop transfer function by setting $H(s) = 1$. The use of the feedback function for unity feedback is depicted in Figure 2.64.

The feedback function is shown in Figure 2.65 with the associated system configuration, which includes $H(s)$ in the feedback path. If the input "sign" is omitted, then negative feedback is assumed.



(a)

```
>>numg=[1]; deng=[500 0 0]; sys1=tf(numg,deng);
>>numc=[1 1]; denc=[1 2]; sys2=tf(numc,denc);
>>sys3=series(sys1,sys2);
>>sys=feedback(sys3,[1])

Transfer function:
          s + 1
------------------------
500 s^3 + 1000 s^2 + s + 1
```

$$\frac{Y(s)}{R(s)} = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}$$

(b)

FIGURE 2.66 (a) Block diagram. (b) Application of the **feedback** function.

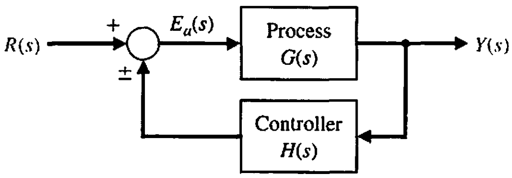


FIGURE 2.67 A basic control system with the controller in the feedback loop.

EXAMPLE 2.20 **The feedback function with unity feedback**

Let the process, $G(s)$, and the controller, $G_c(s)$, be as in Figure 2.66(a). To apply the **feedback** function, we first use the **series** function to compute $G_c(s)G(s)$, followed by the **feedback** function to close the loop. The command sequence is shown in Figure 2.66(b). The closed-loop transfer function, as shown in Figure 2.66(b), is

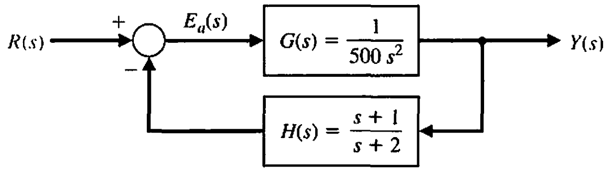
$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{s + 1}{500s^3 + 1000s^2 + s + 1} = \text{sys}. \blacksquare$$

Another basic feedback control configuration is shown in Figure 2.67. In this case, the controller is located in the feedback path. The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1 \mp G(s)H(s)}.$$

EXAMPLE 2.21 **The feedback function**

Let the process, $G(s)$, and the controller, $H(s)$, be as in Figure 2.68(a). To compute the closed-loop transfer function with the controller in the feedback loop, we use



(a)

```
>>numg=[1]; deng=[500 0 0]; sys1=tf(numg,deng);
>>numh=[1 1]; denh=[1 2]; sys2=tf(numh,denh);
>>sys=feedback(sys1,sys2);
>>sys

Transfer function:
          s + 2
-------------------------
500 s^3 + 1000 s^2 + s + 1
```

$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$

(b)

FIGURE 2.68 Application of the **feedback** function: (a) block diagram, (b) m-file script.

the feedback function. The command sequence is shown in Figure 2.68(b). The closed-loop transfer function is

$$T(s) = \frac{s + 2}{500s^3 + 1000s^2 + s + 1} = \text{sys}. \blacksquare$$

The functions series, parallel, and feedback can be used as aids in block diagram manipulations for multiple-loop block diagrams.

EXAMPLE 2.22 **Multiloop reduction**

A multiloop feedback system is shown in Figure 2.26. Our objective is to compute the closed-loop transfer function

$$T(s) = \frac{Y(s)}{R(s)}$$

when

$$G_1(s) = \frac{1}{s + 10}, \quad G_2(s) = \frac{1}{s + 1},$$

$$G_3(s) = \frac{s^2 + 1}{s^2 + 4s + 4}, \quad G_4(s) = \frac{s + 1}{s + 6},$$

and

$$H_1(s) = \frac{s + 1}{s + 2}, \quad H_2(s) = 2, \quad \text{and} \quad H_3(s) = 1.$$

```
>>ng1=[1]; dg1=[1 10]; sysg1=tf(ng1,dg1);
>>ng2=[1]; dg2=[1 1]; sysg2=tf(ng2,dg2);
>>ng3=[1 0 1]; dg3=[1 4 4]; sysg3=tf(ng3,dg3);
>>ng4=[1 1]; dg4=[1 6]; sysg4=tf(ng4,dg4);                Step 1
>>nh1=[1 1]; dh1=[1 2]; sysh1=tf(nh1,dh1);
>>nh2=[2]; dh2=[1]; sysh2=tf(nh2,dh2);
>>nh3=[1]; dh3=[1]; sysh3=tf(nh3,dh3);
>>sys1=sysh2/sysg4;                                       Step 2
>>sys2=series(sysg3,sysg4);
>>sys3=feedback(sys2,sysh1,+1);                           Step 3
>>sys4=series(sysg2,sys3);
>>sys5=feedback(sys4,sys1);                               Step 4
>>sys6=series(sysg1,sys5);
>>sys=feedback(sys6,sysh3);                               Step 5
Transfer function:
                 s^5 + 4 s^4 + 6 s^3 + 6 s^2 + 5 s + 2
---------------------------------------------------------------------
12 s^6 + 205 s^5 + 1066 s^4 + 2517 s^3 + 3128 s^2 + 2196 s + 712
```

FIGURE 2.69 Multiple-loop block reduction.

For this example, a five-step procedure is followed:

- ❑ Step 1. Input the system transfer functions.
- ❑ Step 2. Move H_2 behind G_4.
- ❑ Step 3. Eliminate the $G_3G_4H_1$ loop.
- ❑ Step 4. Eliminate the loop containing H_2.
- ❑ Step 5. Eliminate the remaining loop and calculate $T(s)$.

The five steps are utilized in Figure 2.69, and the corresponding block diagram reduction is shown in Figure 2.27. The result of executing the commands is

$$\text{sys} = \frac{s^5 + 4s^4 + 6s^3 + 6s^2 + 5s + 2}{12s^6 + 205s^5 + 1066s^4 + 2517s^3 + 3128s^2 + 2196s + 712}.$$

We must be careful in calling this the closed-loop transfer function. The transfer function is defined as the input–output relationship after pole–zero cancellations. If we compute the poles and zeros of $T(s)$, we find that the numerator and denominator polynomials have $(s + 1)$ as a common factor. This must be canceled before we can claim we have the closed-loop transfer function. To assist us in the pole–zero cancellation, we will use the minreal function. The minreal function, shown in Figure 2.70, removes common pole–zero factors of a transfer function. The final step in the block reduction process is to cancel out the common factors, as shown in Figure 2.71. After the application of the minreal function, we find that the order of the denominator polynomial has been reduced from six to five, implying one pole–zero cancellation. ■

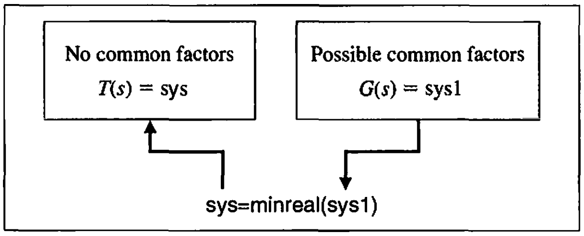


FIGURE 2.70 The **minreal** function.

```
>>num=[1 4 6 6 5 2]; den=[12 205 1066 2517 3128 2196 712];
>>sys1=tf(num,den);
>>sys=minreal(sys1);        <---- Cancel common factors.
Transfer function:
  0.08333 s^4 + 0.25 s^3 + 0.25 s^2 + 0.25 s + 0.1667
-------------------------------------------------------
s^5 + 16.08 s^4 + 72.75 s^3 + 137 s^2 + 123.7 s + 59.33
```

FIGURE 2.71 Application of the **minreal** function.

EXAMPLE 2.23 **Electric traction motor control**

Finally, let us reconsider the electric traction motor system from Example 2.14. The block diagram is shown in Figure 2.44(c). The objective is to compute the closed-loop transfer function and investigate the response of $\omega(s)$ to a commanded $\omega_d(s)$. The first step, as shown in Figure 2.72, is to compute the closed-loop transfer function $\omega(s)/\omega_d(s) = T(s)$. The closed-loop characteristic equation is second order with $\omega_n = 52$ and $\zeta = 0.012$. Since the damping is low, we expect the response to be highly oscillatory. We can investigate the response $\omega(t)$ to a reference input, $\omega_d(t)$, by utilizing the step function. The step function, shown in Figure 2.73, calculates the unit step response of a linear system. The step function is very important, since control system performance specifications are often given in terms of the unit step response.

```
>>num1=[10]; den1=[1 1]; sys1=tf(num1,den1);
>>num2=[1]; den2=[2 0.5]; sys2=tf(num2,den2);
>>num3=[540]; den3=[1]; sys3=tf(num3,den3);
>>num4=[0.1]; den4=[1]; sys4=tf(num4,den4);
>>sys5=series(sys1,sys2);
>>sys6=feedback(sys5,sys4);     <---- Eliminate inner loop
>>sys7=series(sys3,sys6);
>>sys=feedback(sys7,[1])        <---- Compute closed-loop transfer function

Transfer function:
      5400
-----------------          <---- ω(s)/ω_d(s)
2 s^2 + 2.5 s + 5402
```

FIGURE 2.72 Electric traction motor block reduction.

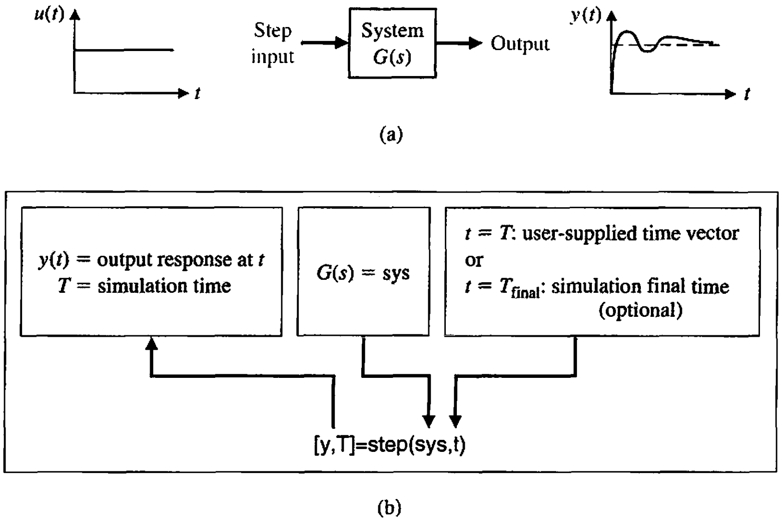
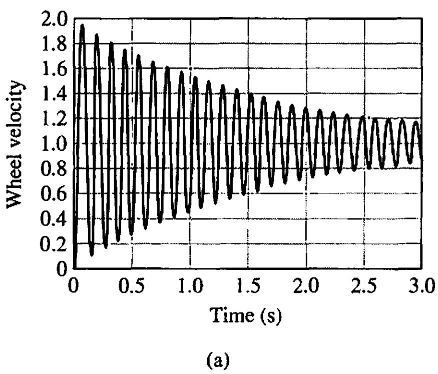


FIGURE 2.73
The **step** function.




```
% This script computes the step
% response of the traction motor
% wheel velocity
%
num=[5400]; den=[2 2.5 5402]; sys=tf(num,den);
t=[0:0.005:3];
[y,t]=step(sys,t);
plot(t,y),grid
xlabel('Time (s)')
ylabel('Wheel velocity')
```

(b)

FIGURE 2.74 (a) Traction motor wheel velocity step response. (b) m-file script.

If the only objective is to plot the output, $y(t)$, we can use the step function without left-hand arguments and obtain the plot automatically with axis labels. If we need $y(t)$ for any purpose other than plotting, we must use the step function with left-hand arguments, followed by the plot function to plot $y(t)$. We define t as a row vector containing the times at which we wish the value of the output variable $y(t)$. We can also select $t = t_{\text{final}}$, which results in a step response from $t = 0$ to $t = t_{\text{final}}$ and the number of intermediate points are selected automatically.

The step response of the electric traction motor is shown in Figure 2.74. As expected, the wheel velocity response, given by $y(t)$, is highly oscillatory. Note that the output is $y(t) \equiv \omega(t)$. ■

2.10 SEQUENTIAL DESIGN EXAMPLE: DISK DRIVE READ SYSTEM

In Section 1.10, we developed an initial goal for the disk drive system: to position the reader head accurately at the desired track and to move from one track to another within 10 ms, if possible. We need to identify the plant, the sensor, and the controller. We will obtain a model of the plant $G(s)$ and the sensor. The disk drive reader uses a permanent magnet DC motor to rotate the reader arm (see Figure 1.29). The DC motor is called a voice coil motor in the disk drive industry. The read head is mounted on a slider device, which is connected to the arm as shown in Figure 2.75. A flexure (spring metal) is used to enable the head to float above the disk at a gap of less than 100 nm. The thin-film head reads the magnetic flux and provides a signal to an amplifier. The error signal of Figure 2.76(a) is provided by reading the error from a prerecorded index track. Assuming an accurate read head, the sensor has a transfer function $H(s) = 1$, as shown in Figure 2.76(b). The model of the permanent magnet DC motor and a linear amplifier is shown in Figure 2.76(b). As a good approximation, we use the model of the armature-controlled DC motor as shown earlier in

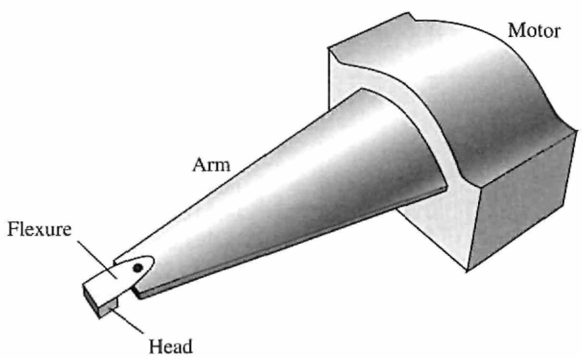


FIGURE 2.75 Head mount for reader, showing flexure.

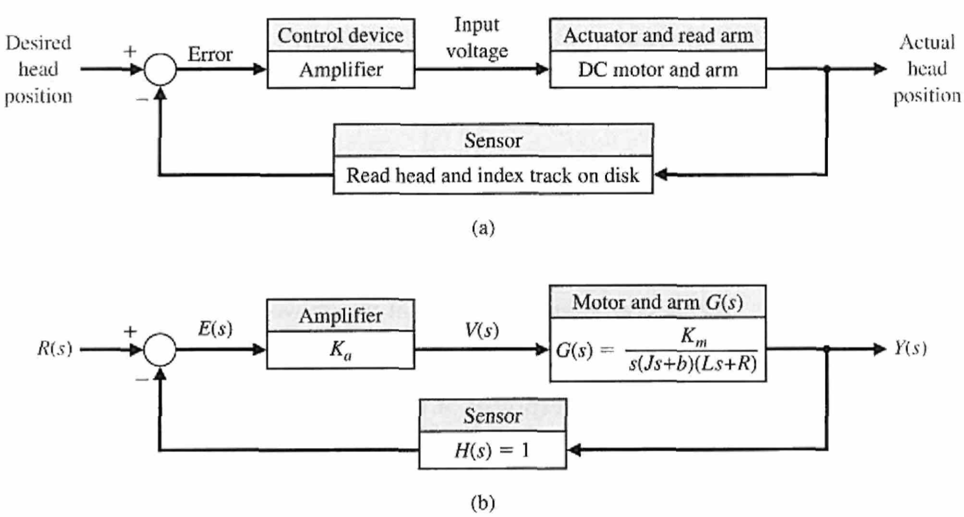


FIGURE 2.76 Block diagram model of disk drive read system.

Table 2.10 Typical Parameters for Disk Drive Reader

Parameter	Symbol	Typical Value
Inertia of arm and read head	J	1 N m s^2/rad
Friction	b	20 N m s/rad
Amplifier	K_a	10–1000
Armature resistance	R	1 Ω
Motor constant	K_m	5 N m/A
Armature inductance	L	1 mH

Figure 2.20 with $K_b = 0$. The model shown in Figure 2.76(b) assumes that the flexure is entirely rigid and does not significantly flex. In Chapter 4, we will consider the model when the flexure cannot be assumed to be completely rigid.

Typical parameters for the disk drive system are given in Table 2.10. Thus, we have

$$G(s) = \frac{K_m}{s(Js + b)(Ls + R)} = \frac{5000}{s(s + 20)(s + 1000)}. \tag{2.138}$$

We can also write

$$G(s) = \frac{K_m/(bR)}{s(\tau_L s + 1)(\tau s + 1)}, \tag{2.139}$$

where $\tau_L = J/b = 50$ ms and $\tau = L/R = 1$ ms. Since $\tau \ll \tau_L$, we often neglect τ. Then, we would have

$$G(s) \approx \frac{K_m/(bR)}{s(\tau_L s + 1)} = \frac{0.25}{s(0.05s + 1)},$$

or

$$G(s) = \frac{5}{s(s + 20)}.$$

The block diagram of the closed-loop system is shown in Figure 2.77. Using the block diagram transformation of Table 2.6, we have

$$\frac{Y(s)}{R(s)} = \frac{K_a G(s)}{1 + K_a G(s)}. \tag{2.140}$$

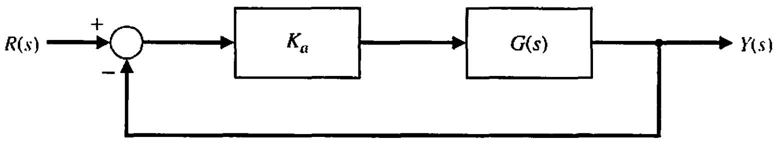


FIGURE 2.77 Block diagram of closed-loop system.

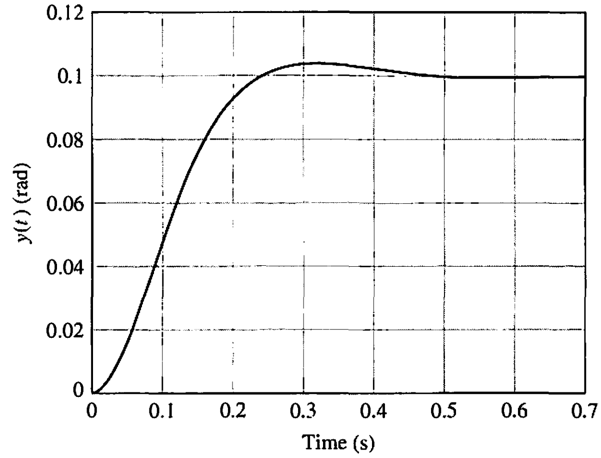


FIGURE 2.78 The system response of the system shown in Figure 2.77 for $R(s) = \frac{0.1}{s}$.

Using the approximate second-order model for $G(s)$, we obtain

$$\frac{Y(s)}{R(s)} = \frac{5K_a}{s^2 + 20s + 5K_a}.$$

When $K_a = 40$, we have

$$Y(s) = \frac{200}{s^2 + 20s + 200}R(s).$$

We obtain the step response for $R(s) = \frac{0.1}{s}$ rad, as shown in Figure 2.78.

2.11 SUMMARY

In this chapter, we have been concerned with quantitative mathematical models of control components and systems. The differential equations describing the dynamic performance of physical systems were utilized to construct a mathematical model. The physical systems under consideration included mechanical, electrical, fluid, and thermodynamic systems. A linear approximation using a Taylor series expansion about the operating point was utilized to obtain a small-signal linear approximation for nonlinear control components. Then, with the approximation of a linear system, one may utilize the Laplace transformation and its related input–output relationship given by the transfer function. The transfer function approach to linear systems allows the analyst to determine the response of the system to various input signals in terms of the location of the poles and zeros of the transfer function. Using transfer function notations, block diagram models of systems of interconnected components were developed. The block relationships were obtained. Additionally, an alternative use of transfer function models in signal-flow graph form was investigated. Mason's signal-flow gain formula was investigated and was found to be useful for obtaining the relationship between system variables in a complex feedback system. The advantage of the signal-flow graph method was the availability of Mason's signal-flow gain formula, which provides the relationship between system variables without requiring any reduction or manipulation of the flow graph. Thus, in Chapter 2, we have obtained a useful mathematical model for feedback control systems by developing the concept of a transfer function of a linear system and the relationship among system variables using block diagram and signal-flow graph models. We considered the utility of the computer simulation of linear and nonlinear systems to determine the response of a system for several conditions of the system parameters and the environment. Finally, we continued the development of the Disk Drive Read System by obtaining a model in transfer function form of the motor and arm.

SKILLS CHECK

In this section, we provide three sets of problems to test your knowledge: True or False, Multiple Choice, and Word Match. To obtain direct feedback, check your answers with the answer key provided at the conclusion of the end-of-chapter problems. Use the block diagram in Figure 2.79 as specified in the various problem statements.

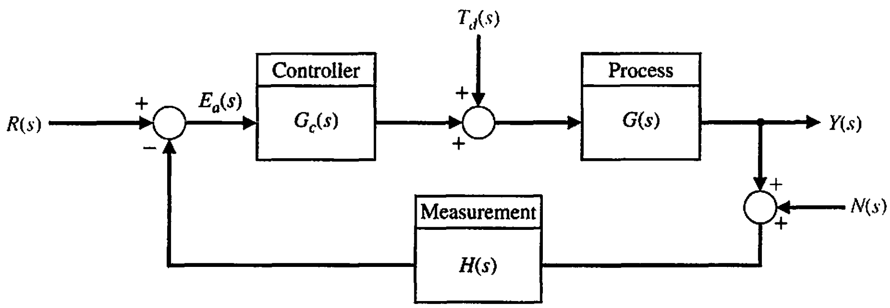


FIGURE 2.79 Block diagram for the Skills Check.

In the following **True or False** and **Multiple Choice** problems, circle the correct answer.

1. Very few physical systems are linear within some range of the variables. *True or False*
2. The s-plane plot of the poles and zeros graphically portrays the character of the natural response of a system. *True or False*
3. The roots of the characteristic equation are the zeros of the closed-loop system. *True or False*
4. A linear system satisfies the properties of superposition and homogeneity. *True or False*
5. The transfer function is the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions equal to zero. *True or False*
6. Consider the system in Figure 2.79 where

$$G_c(s) = 10, \quad H(s) = 1, \quad \text{and} \quad G(s) = \frac{s + 50}{s^2 + 60s + 500}.$$

If the input $R(s)$ is a unit step input, $T_d(s) = 0$, and $N(s) = 0$, the final value of the output $Y(s)$ is:

a. $y_{ss} = \lim_{t \to \infty} y(t) = 100$

b. $y_{ss} = \lim_{t \to \infty} y(t) = 1$

c. $y_{ss} = \lim_{t \to \infty} y(t) = 50$

d. None of the above

7. Consider the system in Figure 2.79 with

$$G_c(s) = 20, \quad H(s) = 1, \quad \text{and} \quad G(s) = \frac{s+4}{s^2 - 12s - 65}.$$

When all initial conditions are zero, the input $R(s)$ is an impulse, the disturbance $T_d(s) = 0$, and the noise $N(s) = 0$, the output $y(t)$ is

a. $y(t) = 10e^{-5t} + 10e^{-3t}$

b. $y(t) = e^{-8t} + 10e^{-t}$

c. $y(t) = 10e^{-3t} - 10e^{-5t}$

d. $y(t) = 20e^{-8t} + 5e^{-15t}$

8. Consider a system represented by the block diagram in Figure 2.80.

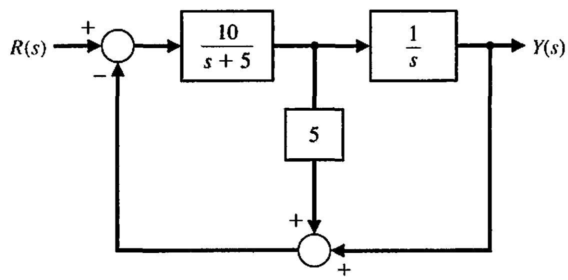


FIGURE 2.80 Block diagram with an internal loop.

The closed-loop transfer function $T(s) = Y(s)/R(s)$ is

a. $T(s) = \dfrac{50}{s^2 + 55s + 50}$

b. $T(s) = \dfrac{10}{s^2 + 55s + 10}$

c. $T(s) = \dfrac{10}{s^2 + 50s + 55}$

d. None of the above

Consider the block diagram in Figure 2.79 for Problems 9 through 11 where

$$G_c(s) = 4, \quad H(s) = 1, \quad \text{and} \quad G(s) = \frac{5}{s^2 + 10s + 5}.$$

9. The closed-loop transfer function $T(s) = Y(s)/R(s)$ is:

a. $T(s) = \dfrac{50}{s^2 + 5s + 50}$

b. $T(s) = \dfrac{20}{s^2 + 10s + 25}$

c. $T(s) = \dfrac{50}{s^2 + 5s + 56}$

d. $T(s) = \dfrac{20}{s^2 + 10s - 15}$

10. The closed-loop unit step response is:

a. $y(t) = \frac{20}{25} + \frac{20}{25}e^{-5t} - t^2e^{-5t}$

b. $y(t) = 1 + 20te^{-5t}$

c. $y(t) = \frac{20}{25} - \frac{20}{25}e^{-5t} - 4te^{-5t}$

d. $y(t) = 1 - 2e^{-5t} - 4te^{-5t}$

11. The final value of $y(t)$ is:

a. $y_{ss} = \lim_{t\to\infty} y(t) = 0.8$

b. $y_{ss} = \lim_{t\to\infty} y(t) = 1.0$

c. $y_{ss} = \lim_{t\to\infty} y(t) = 2.0$

d. $y_{ss} = \lim_{t\to\infty} y(t) = 1.25$

12. Consider the differential equation

$$\ddot{y} + 2\dot{y} + y = u$$

where $y(0) = \dot{y}(0) = 0$ and $u(t)$ is a unit step. The poles of this system are:

a. $s_1 = -1, s_2 = -1$

b. $s_1 = 1j, s_2 = -1j$

c. $s_1 = -1, s_2 = -2$

d. None of the above

13. A cart of mass $m = 1000$ kg is attached to a truck using a spring of stiffness $k = 20{,}000$ N/m and a damper of constant $b = 200$ Ns/m, as shown in Figure 2.81. The truck moves at a constant acceleration of $a = 0.7$ m/s^2.

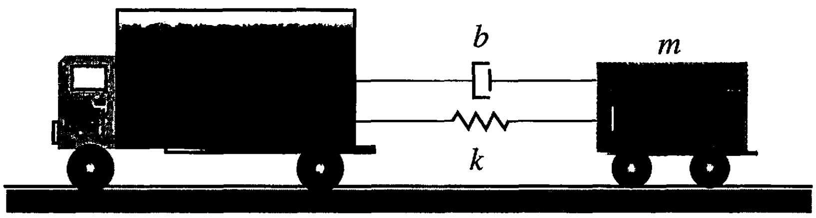


FIGURE 2.81 Truck pulling a cart of mass *m*.

The transfer function between the speed of the truck and the speed of the cart is:

a. $T(s) = \frac{50}{5s^2 + s + 100}$

b. $T(s) = \frac{20 + s}{s^2 + 10s + 25}$

c. $T(s) = \frac{100 + s}{5s^2 + s + 100}$

d. None of the above

14. Consider the closed-loop system in Figure 2.79 with

$$G_c(s) = 15, \quad H(s) = 1, \quad \text{and} \quad G(s) = \frac{1000}{s^3 + 50s^2 + 4500s + 1000}.$$

Compute the closed-loop transfer function and the closed-loop zeros and poles.

a. $T(s) = \dfrac{15000}{s^3 + 50s^2 + 4500s + 16000}$, $s_1 = -3.70, s_{2,3} = -23.15 \pm 61.59j$

b. $T(s) = \dfrac{15000}{50s^2 + 4500s + 16000}$, $s_1 = -3.70, s_2 = -86.29$

c. $T(s) = \dfrac{1}{s^3 + 50s^2 + 4500s + 16000}$, $s_1 = -3.70, s_{2,3} = -23.2 \pm 63.2j$

d. $T(s) = \dfrac{15000}{s^3 + 50s^2 + 4500s + 16000}$, $s_1 = -3.70, s_2 = -23.2, s_3 = -63.2$

15. Consider the feedback system in Figure 2.79 with

$$G_c(s) = \frac{K(s + 0.3)}{s}, \quad H(s) = 2s, \quad \text{and} \quad G(s) = \frac{1}{(s - 2)(s^2 + 10s + 45)}.$$

Assuming $R(s) = 0$ and $N(s) = 0$, the closed-loop transfer function from the disturbance $T_d(s)$ to the output $Y(s)$ is:

a. $\dfrac{Y(s)}{T_d(s)} = \dfrac{1}{s^3 + 8s^2 + (2K + 25)s + (0.6K - 90)}$

b. $\dfrac{Y(s)}{T_d(s)} = \dfrac{100}{s^3 + 8s^2 + (2K + 25)s + (0.6K - 90)}$

c. $\dfrac{Y(s)}{T_d(s)} = \dfrac{1}{8s^2 + (2K + 25)s + (0.6K - 90)}$

d. $\dfrac{Y(s)}{T_d(s)} = \dfrac{K(s + 0.3)}{s^4 + 8s^3 + (2K + 25)s^2 + (0.6K - 90)s}$

In the following **Word Match** problems, match the term with the definition by writing the correct letter in the space provided.

a. Actuator	An oscillation in which the amplitude decreases with time.	_____
b. Block diagrams	A system that satisfies the properties of superposition and homogeneity.	_____
c. Characteristic equation	The case where damping is on the boundary between underdamped and overdamped.	_____
d. Critical damping	A transformation of a function $f(t)$ from the time domain into the complex frequency domain yielding $F(s)$.	_____
e. Damped oscillation	The device that provides the motive power to the process.	_____
f. Damping ratio	A measure of damping. A dimensionless number for the second-order characteristic equation.	_____
g. DC motor	The relation formed by equating to zero the denominator of a transfer function.	_____

h. Laplace transform	Unidirectional, operational blocks that represent the transfer functions of the elements of the system.	______
i. Linear approximation	A rule that enables the user to obtain a transfer function by tracing paths and loops within a system.	______
j. Linear system	An electric actuator that uses an input voltage as a control variable.	______
k. Mason loop rule	The ratio of the Laplace transform of the output variable to the Laplace transform of the input variable.	______
l. Mathematical models	Descriptions of the behavior of a system using mathematics.	______
m. Signal-flow graph	A model of a system that is used to investigate the behavior of a system by utilizing actual input signals.	______
n. Simulation	A diagram that consists of nodes connected by several directed branches and that is a graphical representation of a set of linear relations.	______
o. Transfer function	An approximate model that results in a linear relationship between the output and the input of the device.	______

EXERCISES

Exercises are straightforward applications of the concepts of the chapter.

E2.1 A unity, negative feedback system has a nonlinear function $y = f(e) = e^2$, as shown in Figure E2.1. For an input r in the range of 0 to 4, calculate and plot the open-loop and closed-loop output versus input and show that the feedback system results in a more linear relationship.

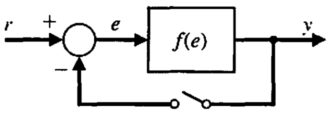


FIGURE E2.1 Open and closed loop.

E2.2 A thermistor has a response to temperature represented by

$$R = R_o e^{-0.1T},$$

where $R_o = 10{,}000\ \Omega$, R = resistance, and T = temperature in degrees Celsius. Find the linear model for the thermistor operating at $T = 20°\text{C}$ and for a small range of variation of temperature.

Answer: $\Delta R = -135\Delta T$

E2.3 The force versus displacement for a spring is shown in Figure E2.3 for the spring-mass-damper system of Figure 2.1. Graphically find the spring constant for the equilibrium point of $y = 0.5$ cm and a range of operation of ± 1.5 cm.

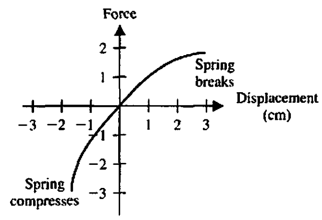


FIGURE E2.3 Spring behavior.

E2.4 A laser printer uses a laser beam to print copy rapidly for a computer. The laser is positioned by a control input $r(t)$, so that we have

$$Y(s) = \frac{4(s + 50)}{s^2 + 30s + 200} R(s).$$

The input $r(t)$ represents the desired position of the laser beam.

(a) If $r(t)$ is a unit step input, find the output $y(t)$.
(b) What is the final value of $y(t)$?

Answer: (a) $y(t) = 1 + 0.6e^{-20t} - 1.6e^{-10t}$, (b) $y_{ss} = 1$

E2.5 A noninverting amplifier uses an op-amp as shown in Figure E2.5. Assume an ideal op-amp model and determine v_o/v_{in}.

Answer: $\dfrac{v_o}{v_{in}} = 1 + \dfrac{R_2}{R_1}$

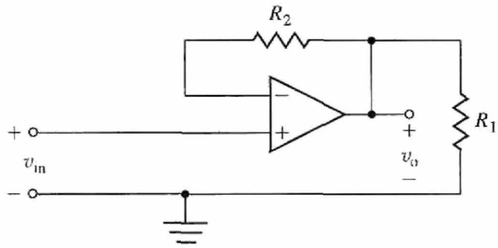


FIGURE E2.5 A noninverting amplifier using an op-amp.

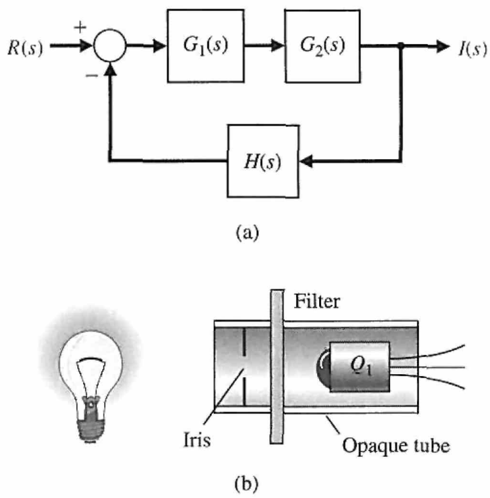


FIGURE E2.7 Lamp controller.

E2.6 A nonlinear device is represented by the function

$$y = f(x) = e^x,$$

where the operating point for the input x is $x_0 = 1$. Determine a linear approximation valid near the operating point.

Answer: $y = ex$

E2.7 A lamp's intensity stays constant when monitored by an optotransistor-controlled feedback loop. When the voltage drops, the lamp's output also drops, and optotransistor Q_1 draws less current. As a result, a power transistor conducts more heavily and charges a capacitor more rapidly [24]. The capacitor voltage controls the lamp voltage directly. A block diagram of the system is shown in Figure E2.7. Find the closed-loop transfer function, $I(s)/R(s)$ where $I(s)$ is the lamp intensity, and $R(s)$ is the command or desired level of light.

E2.8 A control engineer, N. Minorsky, designed an innovative ship steering system in the 1930s for the U.S. Navy. The system is represented by the block diagram shown in Figure E2.8, where $Y(s)$ is the ship's course, $R(s)$ is the desired course, and $A(s)$ is the rudder angle [16]. Find the transfer function $Y(s)/R(s)$.

Answer: $\dfrac{Y(s)}{R(s)} =$

$$\frac{KG_1(s)G_2(s)/s}{1 + G_1(s)H_3(s) + G_1(s)G_2(s)[H_1(s) + H_2(s)] + KG_1(s)G_2(s)/s}$$

E2.9 A four-wheel antilock automobile braking system uses electronic feedback to control automatically the brake force on each wheel [15]. A block diagram model of a brake control system is shown in Figure E2.9, where $F_f(s)$ and $F_R(s)$ are the braking force of the front and rear wheels, respectively, and $R(s)$ is the desired automobile response on an icy road. Find $F_f(s)/R(s)$.

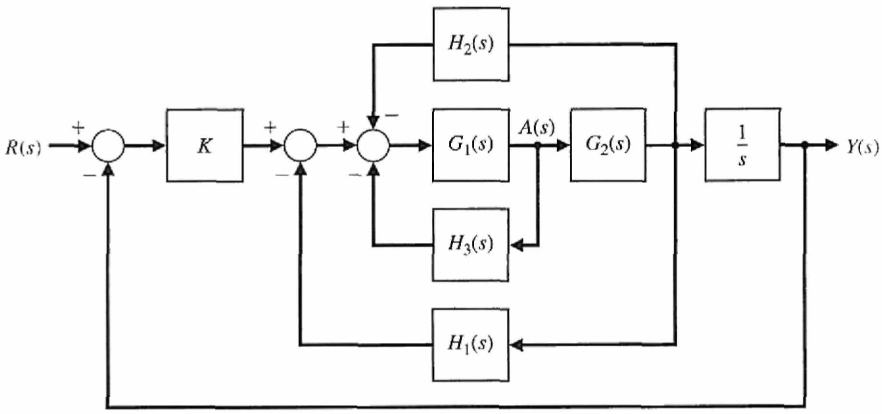


FIGURE E2.8 Ship steering system.

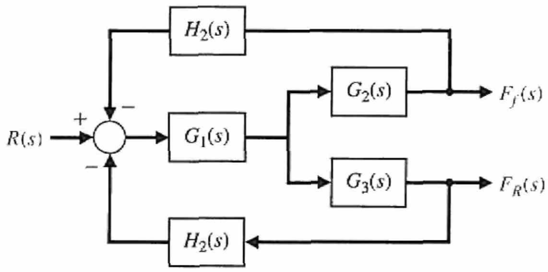


FIGURE E2.9 Brake control system.

E2.10 One of the most potentially beneficial applications of an automotive control system is the active control of the suspension system. One feedback control system uses a shock absorber consisting of a cylinder filled with a compressible fluid that provides both spring and damping forces [17]. The cylinder has a plunger activated by a gear motor, a displacement-measuring sensor, and a piston. Spring force is generated by piston displacement, which compresses the fluid. During piston displacement, the pressure imbalance across the piston is used to control damping. The plunger varies the internal volume of the cylinder. This feedback system is shown in Figure E2.10. Develop a linear model for this device using a block diagram model.

E2.11 A spring exhibits a force-versus-displacement characteristic as shown in Figure E2.11. For small deviations from the operating point x_o, find the spring constant when x_o is (a) -1.4; (b) 0; (c) 3.5.

E2.12 Off-road vehicles experience many disturbance inputs as they traverse over rough roads. An active suspension system can be controlled by a sensor that looks "ahead" at the road conditions. An example of a simple suspension system that can accommodate the bumps is shown in Figure E2.12. Find the appropriate gain K_1 so that the vehicle does not bounce when the desired deflection is $R(s) = 0$ and the disturbance is $T_d(s)$.

Answer: $K_1K_2 = 1$

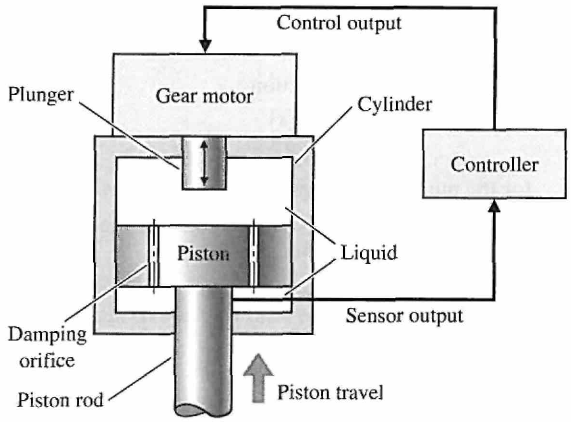


FIGURE E2.10 Shock absorber.

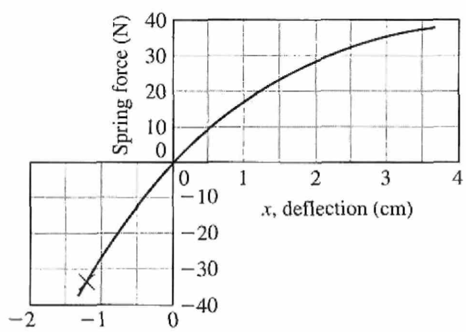


FIGURE E2.11 Spring characteristic.

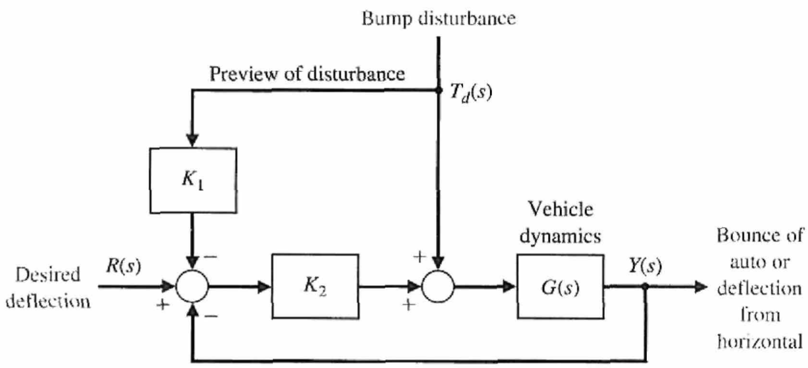


FIGURE E2.12 Active suspension system.

E2.13 Consider the feedback system in Figure E2.13. Compute the transfer functions $Y(s)/T_d(s)$ and $Y(s)/N(s)$.

E2.14 Find the transfer function

$$\frac{Y_1(s)}{R_2(s)}$$

for the multivariable system in Figure E2.14.

E2.15 Obtain the differential equations for the circuit in Figure E2.15 in terms of i_1 and i_2.

E2.16 The position control system for a spacecraft platform is governed by the following equations:

$$\frac{d^2p}{dt^2} + 2\frac{dp}{dt} + 4p = \theta$$

$$v_1 = r - p$$

$$\frac{d\theta}{dt} = 0.6v_2$$

$$v_2 = 7v_1.$$

The variables involved are as follows:

$r(t)$ = desired platform position
$p(t)$ = actual platform position
$v_1(t)$ = amplifier input voltage
$v_2(t)$ = amplifier output voltage
$\theta(t)$ = motor shaft position

Sketch a signal-flow diagram or a block diagram of the system, identifying the component parts and determine the system transfer function $P(s)/R(s)$.

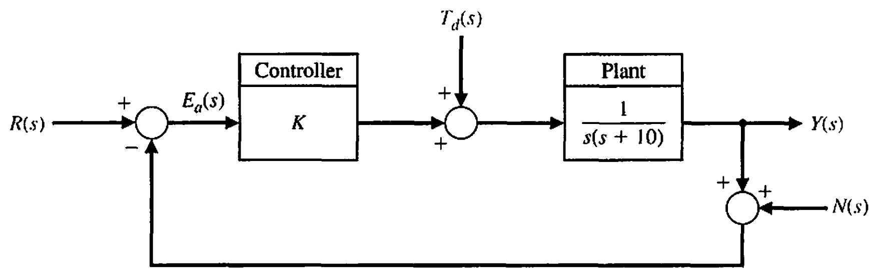


FIGURE E2.13 Feedback system with measurement noise, $N(s)$, and plant disturbances, $T_d(s)$.

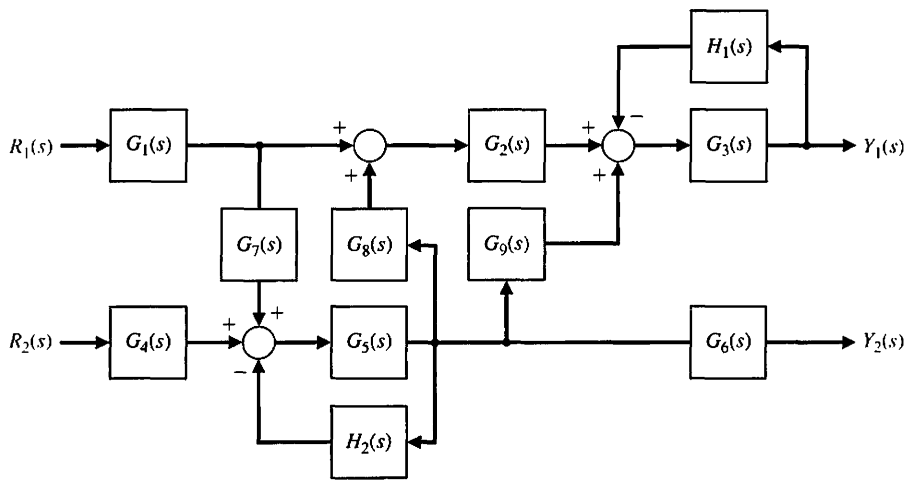


FIGURE E2.14 Multivariable system.

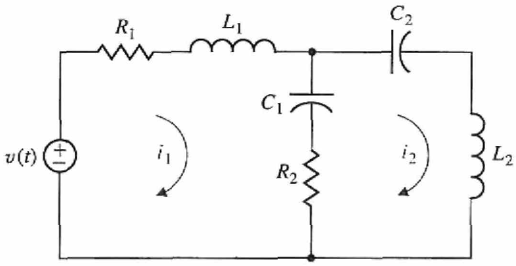


FIGURE E2.15 Electric circuit.

E2.17 A spring develops a force f represented by the relation

$$f = kx^2,$$

where x is the displacement of the spring. Determine a linear model for the spring when $x_o = \frac{1}{2}$.

E2.18 The output y and input x of a device are related by

$$y = x + 1.4x^3.$$

(a) Find the values of the output for steady-state operation at the two operating points $x_o = 1$ and $x_o = 2$. (b) Obtain a linearized model for both operating points and compare them.

E2.19 The transfer function of a system is

$$\frac{Y(s)}{R(s)} = \frac{15(s+1)}{s^2 + 9s + 14}.$$

Determine $y(t)$ when $r(t)$ is a unit step input.

Answer: $y(t) = 1.07 + 1.5e^{-2t} - 2.57e^{-7t}, t \geq 0$

E2.20 Determine the transfer function $V_0(s)/V(s)$ of the operational amplifier circuit shown in Figure E2.20. Assume an ideal operational amplifier. Determine the transfer function when $R_1 = R_2 = 100\ \text{k}\Omega$, $C_1 = 10\ \mu\text{F}$, and $C_2 = 5\ \mu\text{F}$.

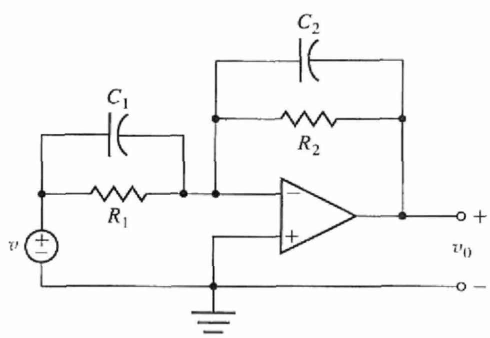


FIGURE E2.20 Op-amp circuit.

E2.21 A high-precision positioning slide is shown in Figure E2.21. Determine the transfer function $X_p(s)/X_{in}(s)$ when the drive shaft friction is $b_d = 0.7$, the drive shaft spring constant is $k_d = 2$, $m_c = 1$, and the sliding friction is $b_s = 0.8$.

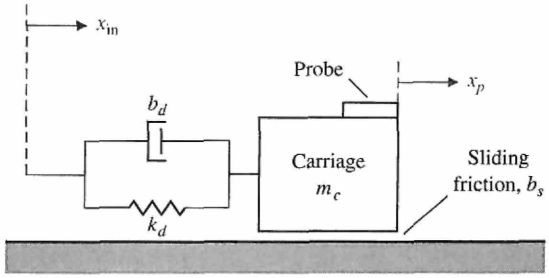


FIGURE E2.21 Precision slide.

E2.22 The rotational velocity ω of the satellite shown in Figure E2.22 is adjusted by changing the length of the beam L. The transfer function between $\omega(s)$ and the incremental change in beam length $\Delta L(s)$ is

$$\frac{\omega(s)}{\Delta L(s)} = \frac{2(s+4)}{(s+5)(s+1)^2}.$$

The beam length change is $\Delta L(s) = 1/s$. Determine the response of the rotation $\omega(t)$.

Answer: $\omega(t) = 1.6 + 0.025e^{-5t} - 1.625e^{-t} - 1.5te^{-t}$

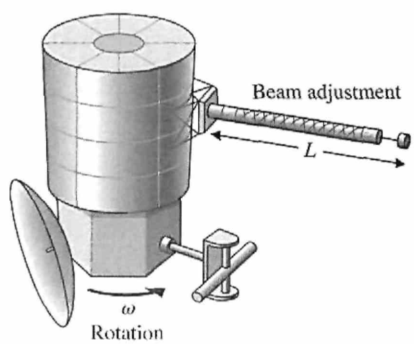


FIGURE E2.22 Satellite with adjustable rotational velocity.

E2.23 Determine the closed-loop transfer function $T(s) = Y(s)/R(s)$ for the system of Figure E2.23.

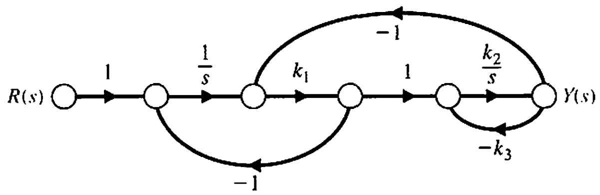


FIGURE E2.23 Control system with three feedback loops.

E2.24 The block diagram of a system is shown in Figure E2.24. Determine the transfer function $T(s) = Y(s)/R(s)$.

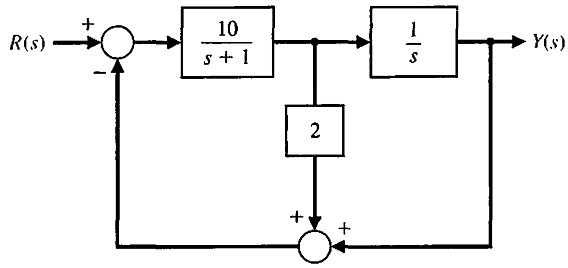


FIGURE E2.24 Multiloop feedback system.

E2.25 An amplifier may have a region of deadband as shown in Figure E2.25. Use an approximation that uses a cubic equation $y = ax^3$ in the approximately linear region. Select a and determine a linear approximation for the amplifier when the operating point is $x = 0.6$.

E2.26 Determine the transfer function $X_2(s)/F(s)$ for the system shown in Figure E2.26. Both masses slide on a frictionless surface, and $k = 1$ N/m.

Answer: $\dfrac{X_2(s)}{F(s)} = \dfrac{1}{s^2(s^2 + 2)}$

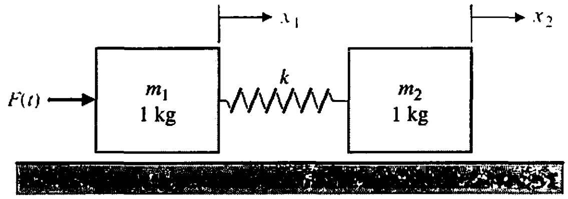


FIGURE E2.26 Two connected masses on a frictionless surface.

E2.27 Find the transfer function $Y(s)/T_d(s)$ for the system shown in Figure E2.27.

Answer: $\dfrac{Y(s)}{T_d(s)} = \dfrac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}$

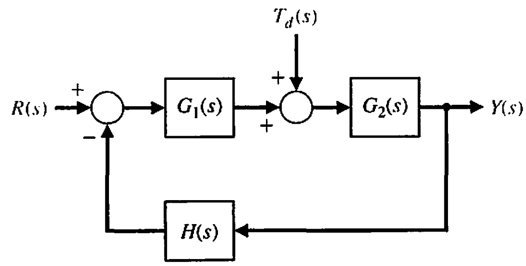


FIGURE E2.27 System with disturbance.

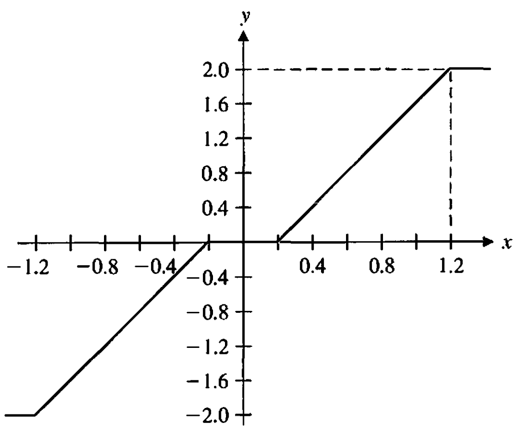


FIGURE E2.25
An amplifier with a deadband region.

E2.28 Determine the transfer function $V_0(s)/V(s)$ for the op-amp circuit shown in Figure E2.28 [1]. Let $R_1 = 167\ \text{k}\Omega$, $R_2 = 240\ \text{k}\Omega$, $R_3 = 1\ \text{k}\Omega$, $R_4 = 100\ \text{k}\Omega$, and $C = 1\ \mu\text{F}$. Assume an ideal op-amp.

E2.29 A system is shown in Fig. E2.29(a).

(a) Determine $G(s)$ and $H(s)$ of the block diagram shown in Figure E2.29(b) that are equivalent to those of the block diagram of Figure E2.29(a).

(b) Determine $Y(s)/R(s)$ for Figure E2.29(b).

E2.30 A system is shown in Figure E2.30.

(a) Find the closed-loop transfer function $Y(s)/R(s)$ when $G(s) = \dfrac{10}{s^2 + 2s + 10}$.

(b) Determine $Y(s)$ when the input $R(s)$ is a unit step.

(c) Compute $y(t)$.

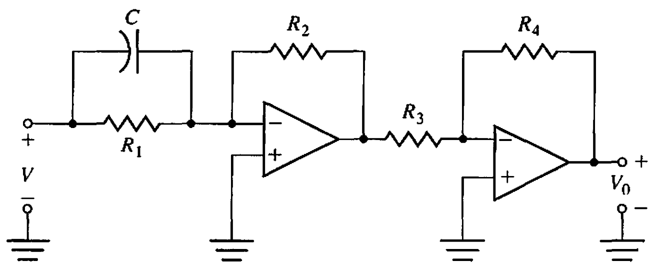


FIGURE E2.28 Op-amp circuit.

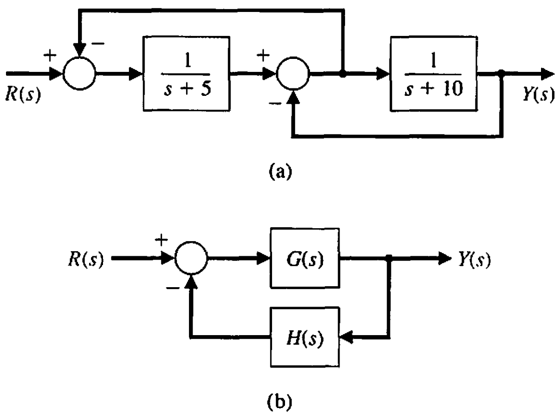


FIGURE E2.29 Block diagram equivalence.

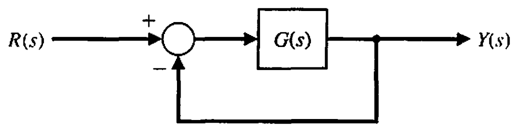


FIGURE E2.30 Unity feedback control system.

E2.31 Determine the partial fraction expansion for $V(s)$ and compute the inverse Laplace transform. The transfer function $V(s)$ is given by:

$$V(s) = \frac{400}{s^2 + 8s + 400}.$$

PROBLEMS

Problems require an extension of the concepts of the chapter to new situations.

P2.1 An electric circuit is shown in Figure P2.1. Obtain a set of simultaneous integrodifferential equations representing the network.

P2.2 A dynamic vibration absorber is shown in Figure P2.2. This system is representative of many situations involving the vibration of machines containing unbalanced components. The parameters M_2 and k_{12} may be chosen so that the main mass M_1 does not vibrate in the steady state when $F(t) = a\sin(\omega_0 t)$. Obtain the differential equations describing the system.

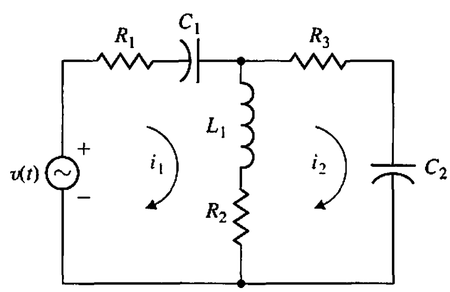


FIGURE P2.1 Electric circuit.

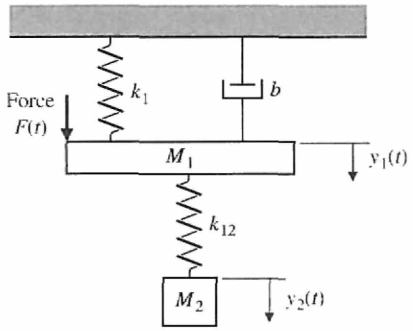


FIGURE P2.2 Vibration absorber.

P2.3 A coupled spring–mass system is shown in Figure P2.3. The masses and springs are assumed to be equal. Obtain the differential equations describing the system.

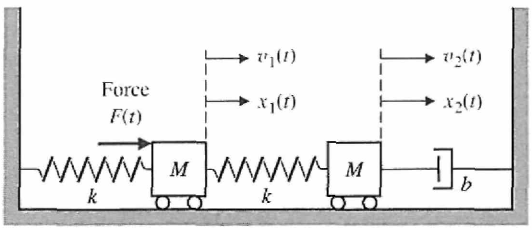


FIGURE P2.3 Two-mass system.

P2.4 A nonlinear amplifier can be described by the following characteristic:

$$v_0(t) = \begin{cases} v_{in}^2 & v_{in} \geq 0 \\ -v_{in}^2 & v_{in} < 0 \end{cases}.$$

The amplifier will be operated over a range of ± 0.5 volts around the operating point for v_{in}. Describe the amplifier by a linear approximation (a) when the operating point is $v_{in} = 0$ and (b) when the operating point is $v_{in} = 1$ volt. Obtain a sketch of the nonlinear function and the approximation for each case.

P2.5 Fluid flowing through an orifice can be represented by the nonlinear equation

$$Q = K(P_1 - P_2)^{1/2},$$

where the variables are shown in Figure P2.5 and K is a constant [2]. (a) Determine a linear approximation

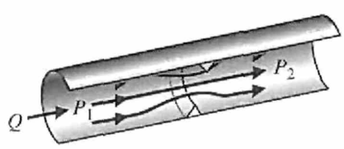


FIGURE P2.5 Flow through an orifice.

for the fluid-flow equation. (b) What happens to the approximation obtained in part (a) if the operating point is $P_1 - P_2 = 0$?

P2.6 Using the Laplace transformation, obtain the current $I_2(s)$ of Problem P2.1. Assume that all the initial currents are zero, the initial voltage across capacitor C_1 is zero, $v(t)$ is zero, and the initial voltage across C_2 is 10 volts.

P2.7 Obtain the transfer function of the differentiating circuit shown in Figure P2.7.

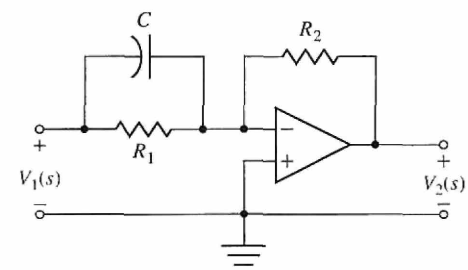


FIGURE P2.7 A differentiating circuit.

P2.8 A bridged-T network is often used in AC control systems as a filter network [8]. The circuit of one bridged-T network is shown in Figure P2.8. Show that the transfer function of the network is

$$\frac{V_o(s)}{V_{in}(s)} = \frac{1 + 2R_1Cs + R_1R_2C^2s^2}{1 + (2R_1 + R_2)Cs + R_1R_2C^2s^2}.$$

Sketch the pole–zero diagram when $R_1 = 0.5$, $R_2 = 1$, and $C = 0.5$.

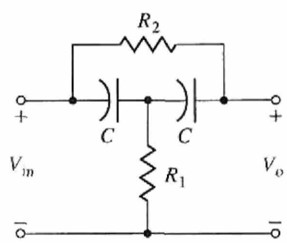


FIGURE P2.8 Bridged-T network.

P2.9 Determine the transfer function $X_1(s)/F(s)$ for the coupled spring–mass system of Problem P2.3. Sketch the s-plane pole–zero diagram for low damping when $M = 1$, $b/k = 1$, and

$$\zeta = \frac{1}{2}\frac{b}{\sqrt{kM}} = 0.1.$$

P2.10 Determine the transfer function $Y_1(s)/F(s)$ for the vibration absorber system of Problem P2.2. Determine the necessary parameters M_2 and k_{12} so that the mass M_1 does not vibrate in the steady state when $F(t) = a\sin(\omega_0 t)$.

P2.11 For electromechanical systems that require large power amplification, rotary amplifiers are often used [8, 19]. An amplidyne is a power amplifying rotary amplifier. An amplidyne and a servomotor are shown in Figure P2.11. Obtain the transfer function $\theta(s)/V_c(s)$, and draw the block diagram of the system. Assume $v_d = k_2 i_q$ and $v_q = k_1 i_c$.

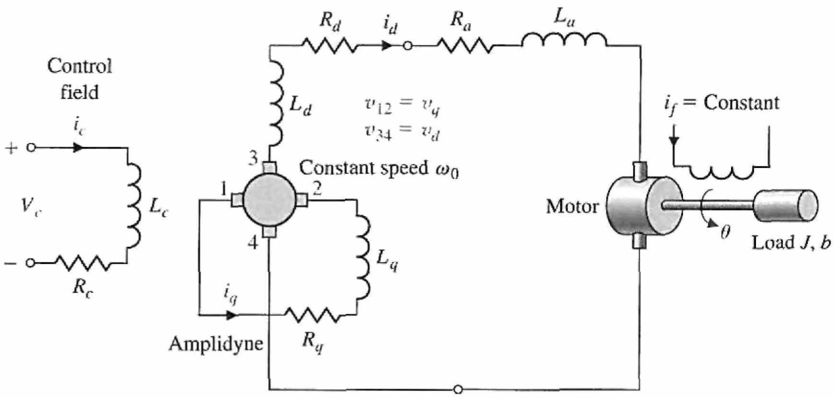


FIGURE P2.11 Amplidyne and armature-controlled motor.

P2.12 For the open-loop control system described by the block diagram shown in Figure P2.12, determine the value of K such that $y(t) \to 1$ as $t \to \infty$ when $r(t)$ is a unit step input. Assume zero initial conditions.

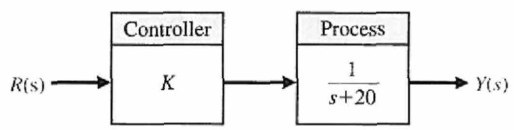


FIGURE P2.12 Open-loop control system.

P2.13 An electromechanical open-loop control system is shown in Figure P2.13. The generator, driven at a constant speed, provides the field voltage for the motor. The motor has an inertia J_m and bearing friction b_m. Obtain the transfer function $\theta_L(s)/V_f(s)$ and draw a block diagram of the system. The generator voltage v_g can be assumed to be proportional to the field current i_f.

P2.14 A rotating load is connected to a field-controlled DC electric motor through a gear system. The motor is assumed to be linear. A test results in the output load reaching a speed of 1 rad/s within 0.5 s when a constant 80 V is applied to the motor terminals. The output steady-state speed is 2.4 rad/s. Determine the transfer function $\theta(s)/V_f(s)$ of the motor, in rad/V. The inductance of the field may be assumed to be negligible (see Figure 2.18). Also, note that the application of 80 V to the motor terminals is a step input of 80 V in magnitude.

P2.15 Consider the spring-mass system depicted in Figure P2.15. Determine a differential equation to describe the motion of the mass m. Obtain the system response $x(t)$ with the initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$.

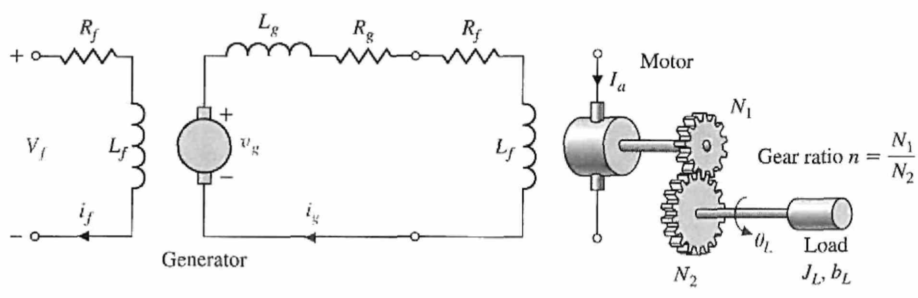


FIGURE P2.13 Motor and generator.

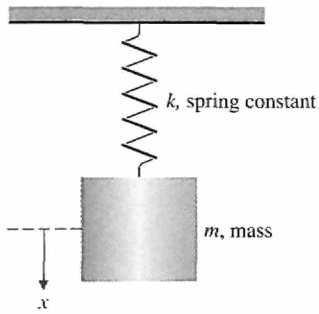


FIGURE P2.15 Suspended spring–mass system.

P2.16 Obtain a signal-flow graph to represent the following set of algebraic equations where x_1 and x_2 are to be considered the dependent variables and 6 and 11 are the inputs:

$$x_1 + 1.5x_2 = 6, \qquad 2x_1 + 4x_2 = 11.$$

Determine the value of each dependent variable by using the gain formula. After solving for x_1 by Mason's signal-flow gain formula, verify the solution by using Cramer's rule.

P2.17 A mechanical system is shown in Figure P2.17, which is subjected to a known displacement $x_3(t)$ with respect to the reference. (a) Determine the two independent equations of motion. (b) Obtain the equations of motion in terms of the Laplace transform, assuming that the initial conditions are zero. (c) Sketch a signal-flow graph representing the system of equations. (d) Obtain the relationship $T_{13}(s)$ between $X_1(s)$ and $X_3(s)$ by using Mason's signal-flow gain formula. Compare the work necessary to obtain $T_{13}(s)$ by matrix methods to that using Mason's signal-flow gain formula.

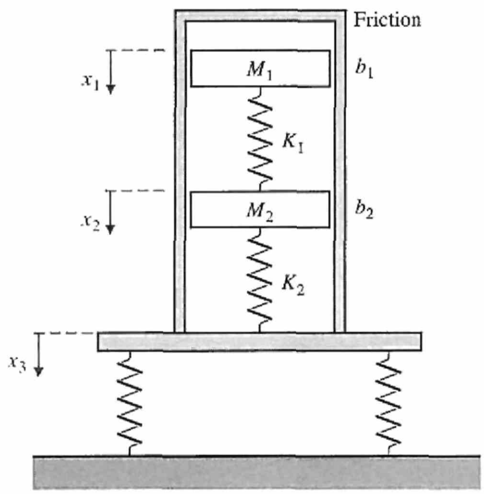


FIGURE P2.17 Mechanical system.

P2.18 An LC ladder network is shown in Figure P2.18. One may write the equations describing the network as follows:

$$I_1 = (V_1 - V_a)Y_1, \qquad V_a = (I_1 - I_a)Z_2,$$
$$I_a = (V_a - V_2)Y_3, \qquad V_2 = I_aZ_4.$$

Construct a flow graph from the equations and determine the transfer function $V_2(s)/V_1(s)$.

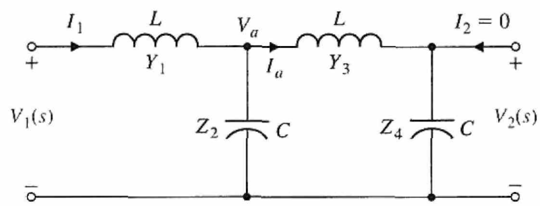


FIGURE P2.18 LC ladder network.

P2.19 A voltage follower (buffer amplifier) is shown in Figure P2.19. Show that $T = v_0/v_{\text{in}} = 1$. Assume an ideal op-amp.

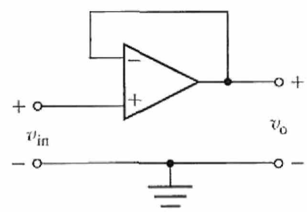


FIGURE P2.19 A buffer amplifier.

P2.20 The source follower amplifier provides lower output impedance and essentially unity gain. The circuit diagram is shown in Figure P2.20(a), and the small-signal model is shown in Figure P2.20(b). This circuit uses an FET and provides a gain of approximately unity. Assume that $R_2 \gg R_1$ for biasing purposes and that $R_g \gg R_2$. (a) Solve for the amplifier gain. (b) Solve for the gain when $g_m = 2000\ \mu\Omega$ and $R_s = 10\ \text{k}\Omega$ where $R_s = R_1 + R_2$. (c) Sketch a block diagram that represents the circuit equations.

P2.21 A hydraulic servomechanism with mechanical feedback is shown in Figure P2.21 [18]. The power piston has an area equal to A. When the valve is moved a small amount Δz, the oil will flow through to the cylinder at a rate $p \cdot \Delta z$, where p is the port coefficient. The

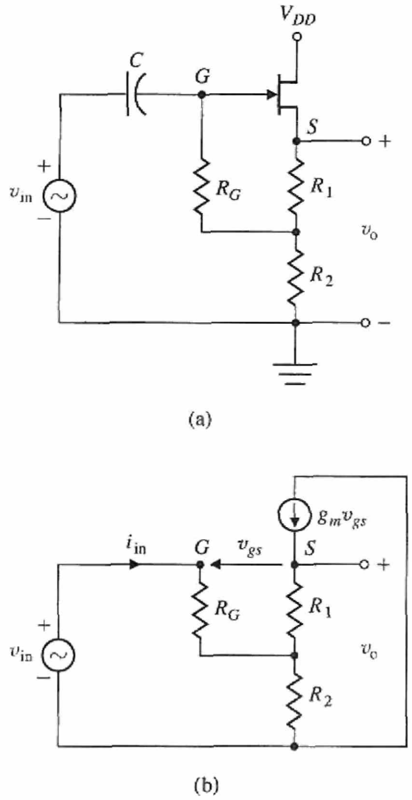


FIGURE P2.20 The source follower or common drain amplifier using an FET.

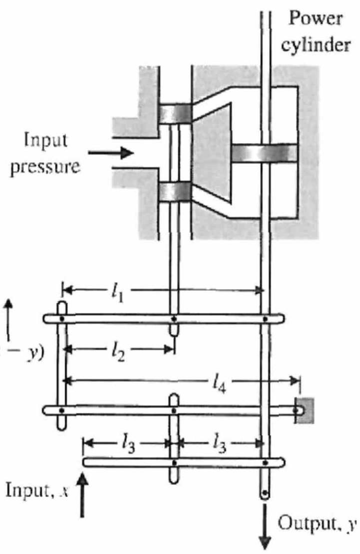


FIGURE P2.21 Hydraulic servomechanism.

input oil pressure is assumed to be constant. From the geometry, we find that $\Delta z = k\dfrac{l_1 - l_2}{l_1}(x - y) - \dfrac{l_2}{l_1}y$. (a) Determine the closed-loop signal-flow graph or block diagram for this mechanical system. (b) Obtain the closed-loop transfer function $Y(s)/X(s)$.

P2.22 Figure P2.22 shows two pendulums suspended from frictionless pivots and connected at their midpoints by a spring [1]. Assume that each pendulum can be represented by a mass M at the end of a massless bar of length L. Also assume that the displacement is small and linear approximations can be used for $\sin\theta$ and $\cos\theta$. The spring located in the middle of the bars is unstretched when $\theta_1 = \theta_2$. The input force is represented by $f(t)$, which influences the left-hand bar only. (a) Obtain the equations of motion, and sketch a block diagram for them. (b) Determine the transfer function $T(s) = \theta_1(s)/F(s)$. (c) Sketch the location of the poles and zeros of $T(s)$ on the s-plane.

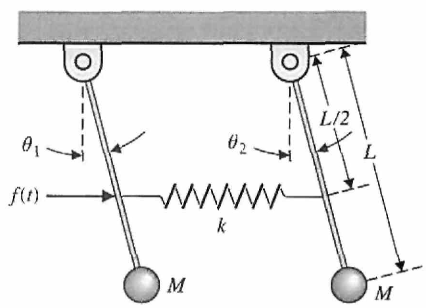


FIGURE P2.22 The bars are each of length L and the spring is located at $L/2$.

P2.23 The small-signal circuit equivalent to a common-emitter transistor amplifier is shown in Figure P2.23. The transistor amplifier includes a feedback resistor R_f. Determine the input–output ratio v_{ce}/v_{in}.

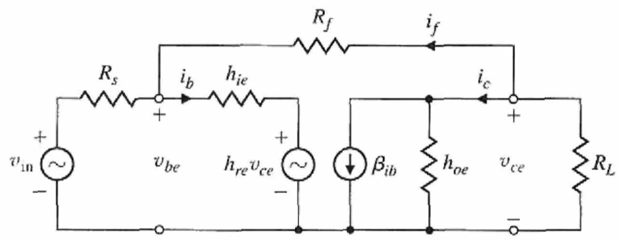
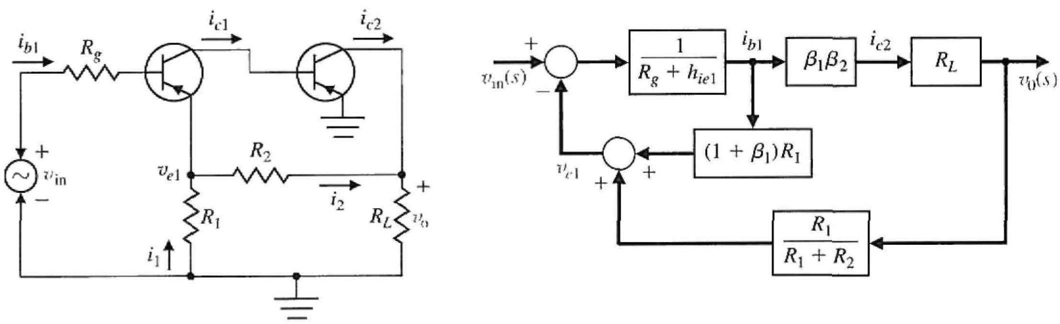


FIGURE P2.23 *CE* amplifier.

P2.24 A two-transistor series voltage feedback amplifier is shown in Figure P2.24(a). This AC equivalent circuit neglects the bias resistors and the shunt capacitors. A block diagram representing the circuit is shown in Figure P2.24(b). This block diagram neglects the effect of h_{re}, which is usually an accurate approximation, and assumes that $R_2 + R_L \gg R_1$. (a) Determine the voltage gain v_o/v_{in}. (b) Determine the current gain i_{c2}/i_{b1}. (c) Determine the input impedance v_{in}/i_{b1}.

P2.25 H. S. Black is noted for developing a negative feedback amplifier in 1927. Often overlooked is the fact that three years earlier he had invented a circuit design technique known as feedforward correction [19]. Recent experiments have shown that this technique offers the potential for yielding excellent amplifier stabilization. Black's amplifier is shown in Figure P2.25(a) in the form recorded in 1924. The block diagram is shown in Figure P2.25(b). Determine the transfer function between the output $Y(s)$ and the input $R(s)$ and between the output and the disturbance $T_d(s)$. $G(s)$ is used to denote the amplifier represented by μ in Figure P2.25(a).



(a) (b)

FIGURE P2.24 Feedback amplifier.

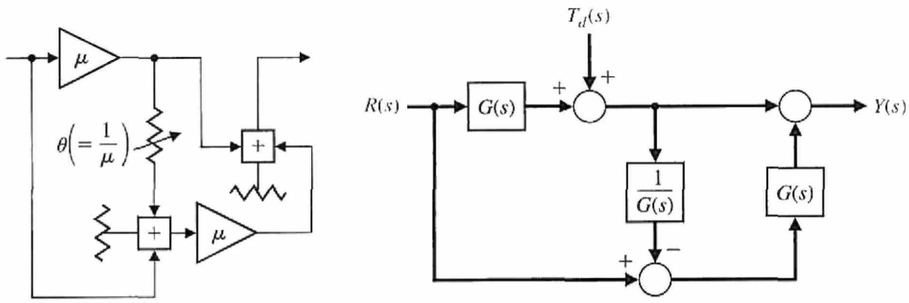


FIGURE P2.25 H. S. Black's amplifier.

(a) (b)

P2.26 A robot includes significant flexibility in the arm members with a heavy load in the gripper [6, 20]. A two-mass model of the robot is shown in Figure. P2.26. Find the transfer function $Y(s)/F(s)$.

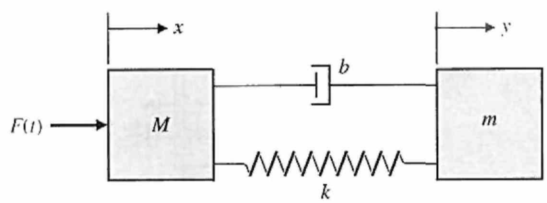


FIGURE P2.26 The spring-mass-damper model of a robot arm.

P2.27 Magnetic levitation trains provide a high-speed, very low friction alternative to steel wheels on steel rails. The train floats on an air gap as shown in Figure P2.27 [25]. The levitation force F_L is controlled by the coil current i in the levitation coils and may be approximated by

$$F_L = k\frac{i^2}{z^2},$$

where z is the air gap. This force is opposed by the downward force $F = mg$. Determine the linearized relationship between the air gap z and the controlling current near the equilibrium condition.

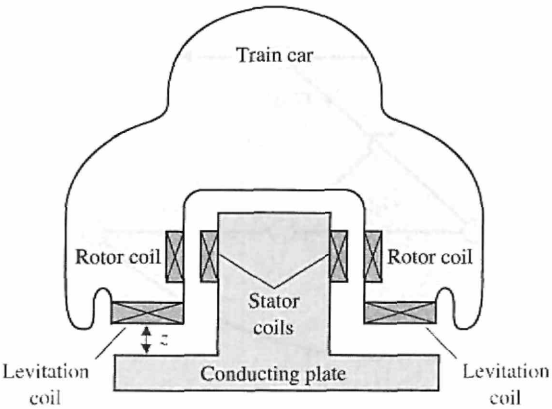


FIGURE P2.27 Cutaway view of train.

P2.28 A multiple-loop model of an urban ecological system might include the following variables: number of people in the city (P), modernization (M), migration into the city (C), sanitation facilities (S), number of diseases (D), bacteria/area (B), and amount of garbage/area (G), where the symbol for the variable is given in parentheses. The following causal loops are hypothesized:

1. $P \rightarrow G \rightarrow B \rightarrow D \rightarrow P$
2. $P \rightarrow M \rightarrow C \rightarrow P$
3. $P \rightarrow M \rightarrow S \rightarrow D \rightarrow P$
4. $P \rightarrow M \rightarrow S \rightarrow B \rightarrow D \rightarrow P$

Sketch a signal-flow graph for these causal relationships, using appropriate gain symbols. Indicate whether you believe each gain transmission is positive or negative. For example, the causal link S to B is negative because improved sanitation facilities lead to reduced bacteria/area. Which of the four loops are positive feedback loops and which are negative feedback loops?

P2.29 We desire to balance a rolling ball on a tilting beam as shown in Figure P2.29. We will assume the motor

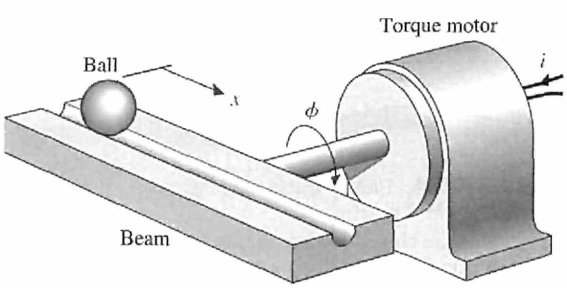


FIGURE P2.29 Tilting beam and ball.

input current i controls the torque with negligible friction. Assume the beam may be balanced near the horizontal ($\phi = 0$); therefore, we have a small deviation of ϕ. Find the transfer function $X(s)/I(s)$, and draw a block diagram illustrating the transfer function showing $\phi(s)$, $X(s)$, and $I(s)$.

P2.30 The measurement or sensor element in a feedback system is important to the accuracy of the system [6]. The dynamic response of the sensor is important. Most sensor elements possess a transfer function

$$H(s) = \frac{k}{\tau s + 1}$$

Suppose that a position-sensing photo detector has $\tau = 4\ \mu s$ and $0.999 < k < 1.001$. Obtain the step response of the system, and find the k resulting in the fastest response—that is, the fastest time to reach 98% of the final value.

P2.31 An interacting control system with two inputs and two outputs is shown in Figure P2.31. Solve for $Y_1(s)/R_1(s)$ and $Y_2(s)/R_1(s)$ when $R_2 = 0$.

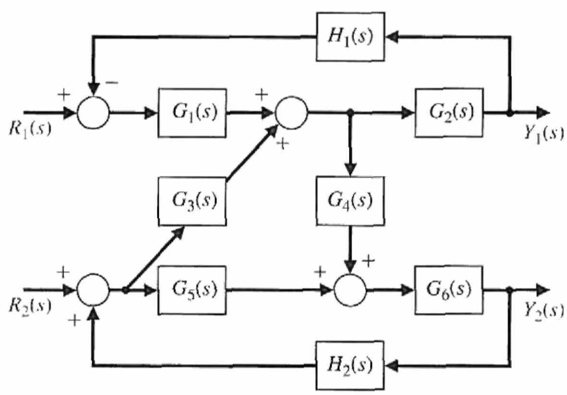


FIGURE P2.31 Interacting System.

P2.32 A system consists of two electric motors that are coupled by a continuous flexible belt. The belt also passes over a swinging arm that is instrumented to allow measurement of the belt speed and tension. The basic control problem is to regulate the belt speed and tension by varying the motor torques.

An example of a practical system similar to that shown occurs in textile fiber manufacturing processes when yarn is wound from one spool to another at high speed. Between the two spools, the yarn is processed in a way that may require the yarn speed and tension to be controlled within defined limits. A model of the system is shown in Figure P2.32. Find $Y_2(s)/R_1(s)$. Determine a relationship for the system that will make Y_2 independent of R_1.

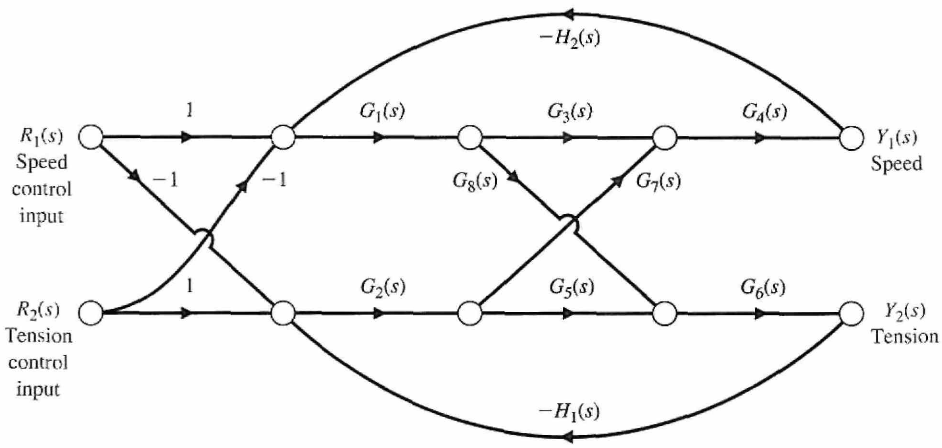


FIGURE P2.32 A model of the coupled motor drives.

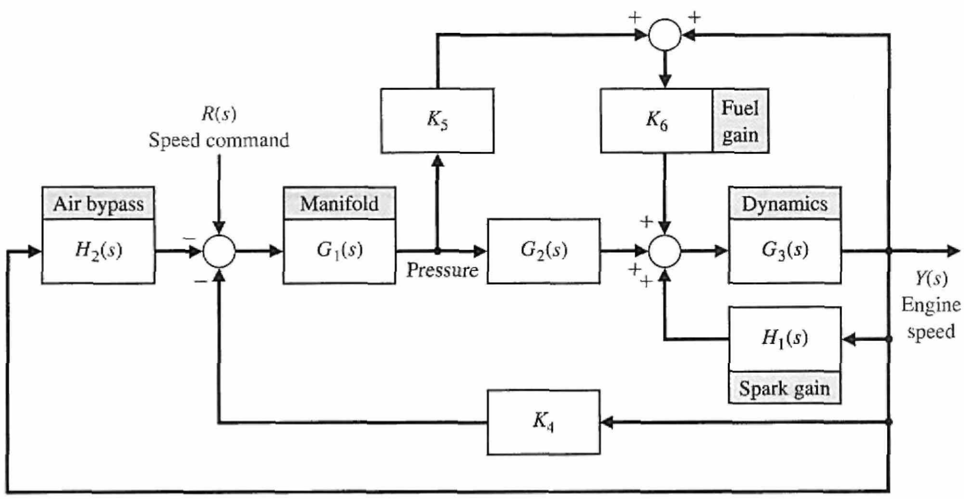


FIGURE P2.33 Idle speed control system.

P2.33 Find the transfer function for $Y(s)/R(s)$ for the idle-speed control system for a fuel-injected engine as shown in Figure P2.33.

P2.34 The suspension system for one wheel of an old-fashioned pickup truck is illustrated in Figure P2.34. The mass of the vehicle is m_1 and the mass of the wheel is m_2. The suspension spring has a spring constant k_1 and the tire has a spring constant k_2. The damping constant of the shock absorber is b. Obtain the transfer function $Y_1(s)/X(s)$, which represents the vehicle response to bumps in the road.

P2.35 A feedback control system has the structure shown in Figure P2.35. Determine the closed-loop transfer function $Y(s)/R(s)$ (a) by block diagram manipulation and (b) by using a signal-flow graph and Mason's signal-flow gain formula. (c) Select the gains K_1 and K_2 so that the closed-loop response to a step input is critically damped with two equal roots at $s = -10$. (d) Plot the critically damped response for a unit step

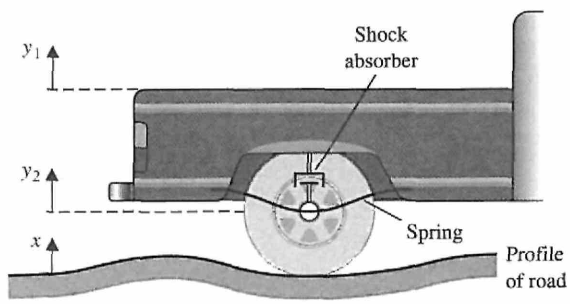


FIGURE P2.34 Pickup truck suspension.

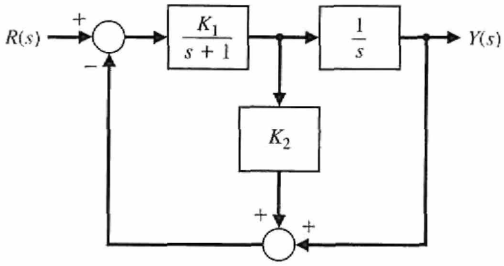


FIGURE P2.35 Multiloop feedback system.

input. What is the time required for the step response to reach 90% of its final value?

P2.36 A system is represented by Figure P2.36. (a) Determine the partial fraction expansion and $y(t)$ for a ramp input, $r(t) = t, t \geq 0$. (b) Obtain a plot of $y(t)$ for part (a), and find $y(t)$ for $t = 1.0$ s. (c) Determine the impulse response of the system $y(t)$ for $t \geq 0$. (d) Obtain a plot of $y(t)$ for part (c) and find $y(t)$ for $t = 1.0$ s.

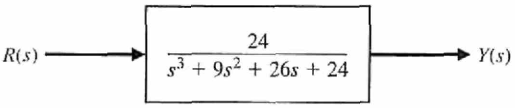


FIGURE P2.36 A third-order system.

P2.37 A two-mass system is shown in Figure P2.37 with an input force $u(t)$. When $m_1 = m_2 = 1$ and $K_1 = K_2 = 1$, find the set of differential equations describing the system.

P2.38 A winding oscillator consists of two steel spheres on each end of a long slender rod, as shown in Figure P2.38. The rod is hung on a thin wire that can be twisted many revolutions without breaking. The device will be wound up 4000 degrees. How long will it take until the motion decays to a swing of only 10 degrees? Assume that the thin wire has a rotational spring constant of 2×10^{-4} N m/rad and that the viscous friction coefficient for the sphere in air is 2×10^{-4} N m s/rad. The sphere has a mass of 1 kg.

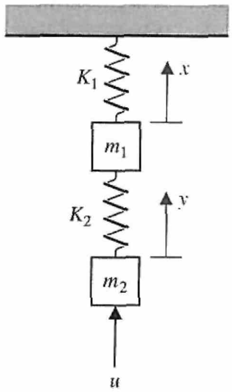


FIGURE P2.37 Two-mass system.

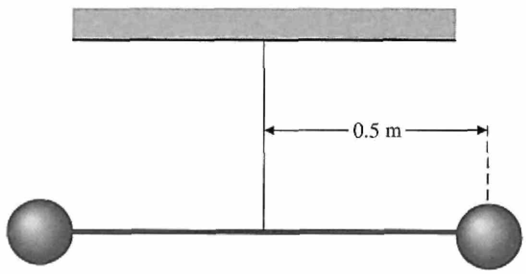


FIGURE P2.38 Winding oscillator.

P2.39 For the circuit of Figure P2.39, determine the transform of the output voltage $V_0(s)$. Assume that the circuit is in steady state when $t < 0$. Assume that the switch moves instantaneously from contact 1 to contact 2 at $t = 0$.

P2.40 A damping device is used to reduce the undesired vibrations of machines. A viscous fluid, such as a heavy oil, is placed between the wheels, as shown in

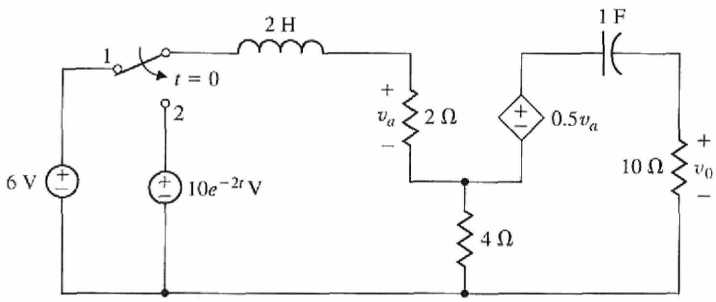


FIGURE P2.39 Model of an electronic circuit.

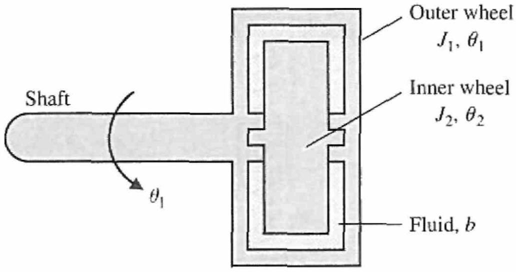


FIGURE P2.40 Cutaway view of damping device.

Figure P2.40. When vibration becomes excessive, the relative motion of the two wheels creates damping. When the device is rotating without vibration, there is no relative motion and no damping occurs. Find $\theta_1(s)$ and $\theta_2(s)$. Assume that the shaft has a spring constant K and that b is the damping constant of the fluid. The load torque is T.

P2.41 The lateral control of a rocket with a gimbaled engine is shown in Figure P2.41. The lateral deviation from the desired trajectory is h and the forward rocket speed is V. The control torque of the engine is T_c and the disturbance torque is T_d. Derive the describing equations of a linear model of the system, and draw the block diagram with the appropriate transfer functions.

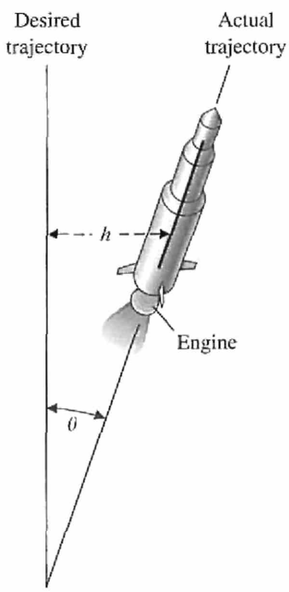


FIGURE P2.41 *Rocket with gimbaled engine.*

P2.42 In many applications, such as reading product codes in supermarkets and in printing and manufacturing, an optical scanner is utilized to read codes, as shown in Figure P2.42. As the mirror rotates, a friction force is developed that is proportional to its angular speed. The friction constant is equal to 0.06 N s/rad, and the moment of inertia is equal to 0.1 kg m^2. The output variable is the velocity $\omega(t)$. (a) Obtain the differential equation for the motor. (b) Find the response of the system when the input motor torque is a unit step and the initial velocity at $t = 0$ is equal to 0.7.

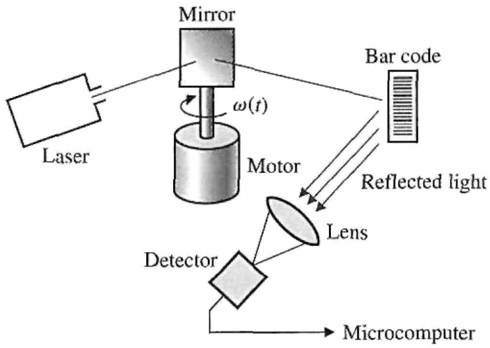


FIGURE P2.42 Optical scanner.

P2.43 An ideal set of gears is shown in Table 2.5, item 10. Neglect the inertia and friction of the gears and assume that the work done by one gear is equal to that of the other. Derive the relationships given in item 10 of Table 2.5. Also, determine the relationship between the torques T_m and T_L.

P2.44 An ideal set of gears is connected to a solid cylinder load as shown in Figure P2.44. The inertia of the motor shaft and gear G_2 is J_m. Determine (a) the inertia of the load J_L and (b) the torque T at the motor shaft. Assume the friction at the load is b_L and the friction at the motor shaft is b_m. Also assume the density of the load disk is ρ and the gear ratio is n. Hint: The torque at the motorshaft is given by $T = T_1 + T_m$.

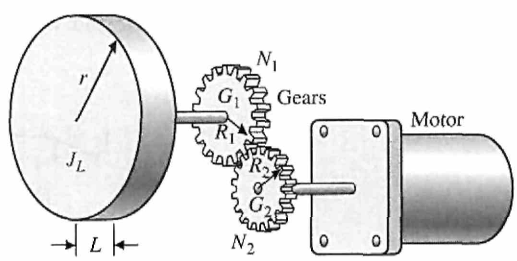


FIGURE P2.44 Motor, gears, and load.

P2.45 To exploit the strength advantage of robot manipulators and the intellectual advantage of humans, a class of manipulators called **extenders** has been examined

[22]. The extender is defined as an active manipulator worn by a human to augment the human's strength. The human provides an input $U(s)$, as shown in Figure P2.45. The endpoint of the extender is $P(s)$. Determine the output $P(s)$ for both $U(s)$ and $F(s)$ in the form

$$P(s) = T_1(s)U(s) + T_2(s)F(s).$$

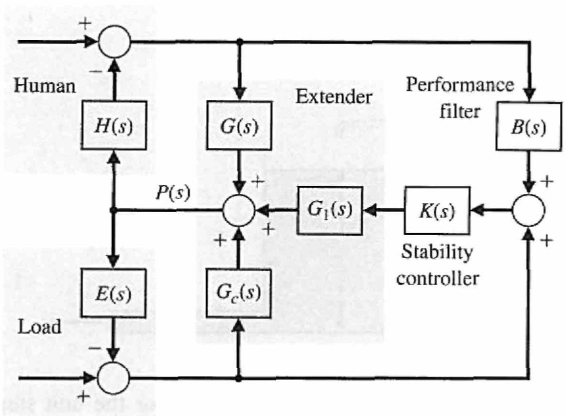


FIGURE P2.45 Model of extender.

P2.46 A load added to a truck results in a force F on the support spring, and the tire flexes as shown in Figure P2.46(a). The model for the tire movement is shown in Figure P2.46(b). Determine the transfer function $X_1(s)/F(s)$.

P2.47 The water level $h(t)$ in a tank is controlled by an open-loop system, as shown in Figure P2.47. A DC motor controlled by an armature current i_a turns a shaft, opening a valve. The inductance of the DC motor is negligible, that is, $L_a = 0$. Also, the rotational friction of the motor shaft and valve is negligible, that is, $b = 0$. The height of the water in the tank is

$$h(t) = \int [1.6\theta(t) - h(t)]\, dt,$$

the motor constant is $K_m = 10$, and the inertia of the motor shaft and valve is $J = 6 \times 10^{-3}$ kg m^2. Determine (a) the differential equation for $h(t)$ and $v(t)$ and (b) the transfer function $H(s)/V(s)$.

P2.48 The circuit shown in Figure P2.48 is called a lead-lag filter.

(a) Find the transfer function $V_2(s)/V_1(s)$. Assume an ideal op-amp.
(b) Determine $V_2(s)/V_1(s)$ when $R_1 = 100\ k\Omega$, $R_2 = 200\ k\Omega$, $C_1 = 1\ \mu F$, and $C_2 = 0.1\ \mu F$.
(c) Determine the partial fraction expansion for $V_2(s)/V_1(s)$.

P2.49 A closed-loop control system is shown in Figure P2.49.

(a) Determine the transfer function

$$T(s) = Y(s)/R(s).$$

(b) Determine the poles and zeros of $T(s)$.
(c) Use a unit step input, $R(s) = 1/s$, and obtain the partial fraction expansion for $Y(s)$ and the value of the residues.

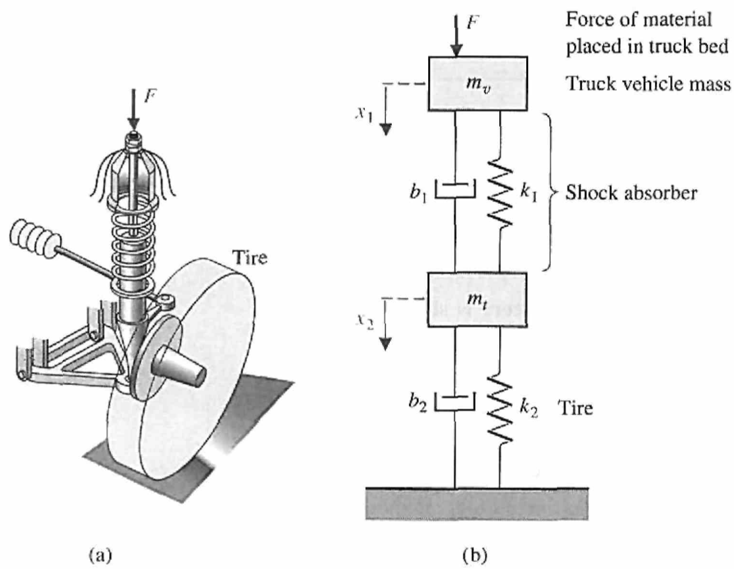


FIGURE P2.46 Truck support model.

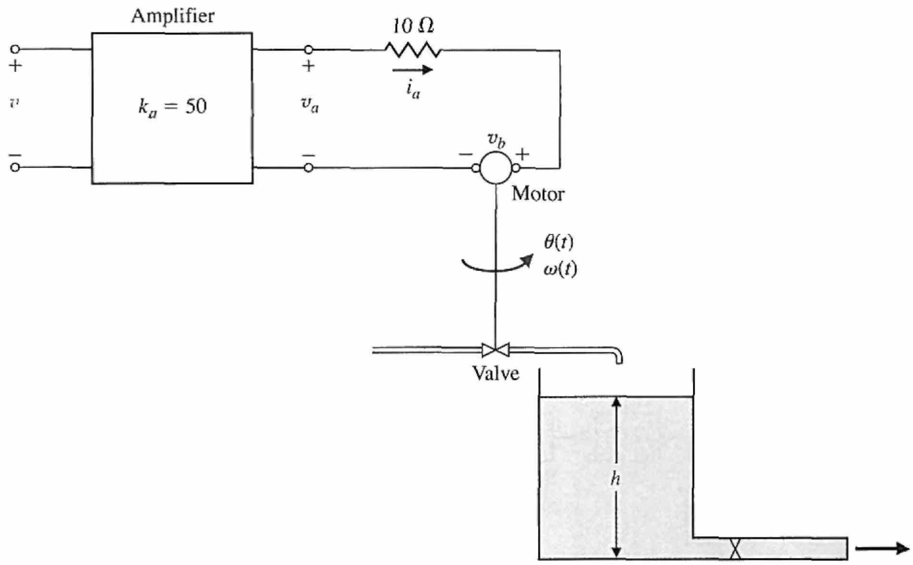


FIGURE P2.47 Open-loop control system for the water level of a tank.

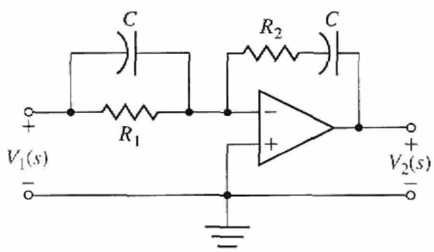


FIGURE P2.48 Lead-lag filter.

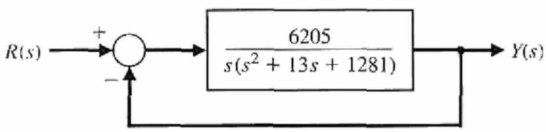


FIGURE P2.49 Unity feedback control system.

(d) Plot $y(t)$ and discuss the effect of the real and complex poles of $T(s)$. Do the complex poles or the real poles dominate the response?

P2.50 A closed-loop control system is shown in Figure P2.50.

(a) Determine the transfer function $T(s) = Y(s)/R(s)$.
(b) Determine the poles and zeros of $T(s)$.
(c) Use a unit step input, $R(s) = 1/s$, and obtain the partial fraction expansion for $Y(s)$ and the value of the residues.
(d) Plot $y(t)$ and discuss the effect of the real and complex poles of $T(s)$. Do the complex poles or the real poles dominate the response?
(e) Predict the final value of $y(t)$ for the unit step input.

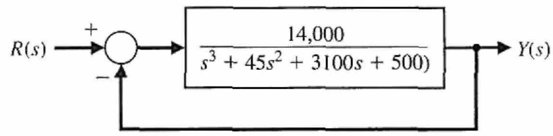


FIGURE P2.50 Third-order feedback system.

P2.51 Consider the two-mass system in Figure P2.51. Find the set of differential equations describing the system.

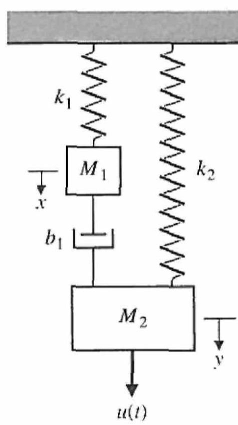


FIGURE P2.51 Two-mass system with two springs and one damper.

ADVANCED PROBLEMS

AP2.1 An armature-controlled DC motor is driving a load. The input voltage is 5 V. The speed at $t = 2$ seconds is 30 rad/s, and the steady speed is 70 rad/s when $t \to \infty$. Determine the transfer function $\omega(s)/V(s)$.

AP2.2 A system has a block diagram as shown in Figure AP2.2. Determine the transfer function

$$T(s) = \frac{Y_2(s)}{R_1(s)}.$$

It is desired to decouple $Y_2(s)$ from $R_1(s)$ by obtaining $T(s) = 0$. Select $G_5(s)$ in terms of the other $G_i(s)$ to achieve decoupling.

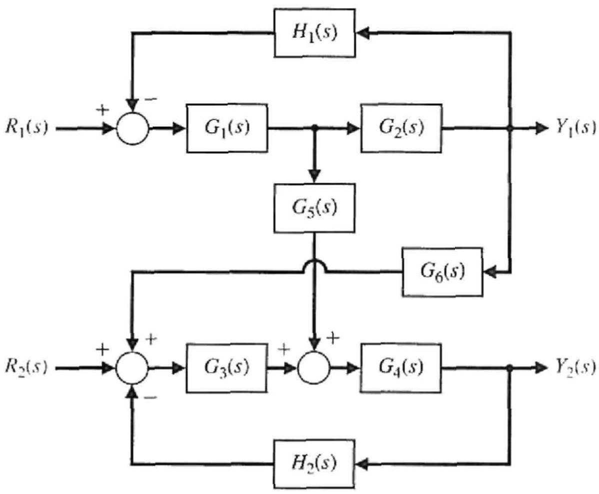


FIGURE AP2.2 Interacting control system.

AP2.3 Consider the feedback control system in Figure AP2.3. Define the tracking error as

$$E(s) = R(s) - Y(s).$$

(a) Determine a suitable $H(s)$ such that the tracking error is zero for any input $R(s)$ in the absence of a disturbance input (that is, when $T_d(s) = 0$). (b) Using $H(s)$ determined in part (a), determine the response $Y(s)$ for a disturbance $T_d(s)$ when the input $R(s) = 0$. (c) Is it possible to obtain $Y(s) = 0$ for an arbitrary disturbance $T_d(s)$ when $G_d(s) \neq 0$? Explain your answer.

AP2.4 Consider a thermal heating system given by

$$\frac{\mathcal{T}(s)}{q(s)} = \frac{1}{C_t s + (QS + 1/R_t)},$$

where the output $\mathcal{T}(s)$ is the temperature difference due to the thermal process, the input $q(s)$ is the rate of

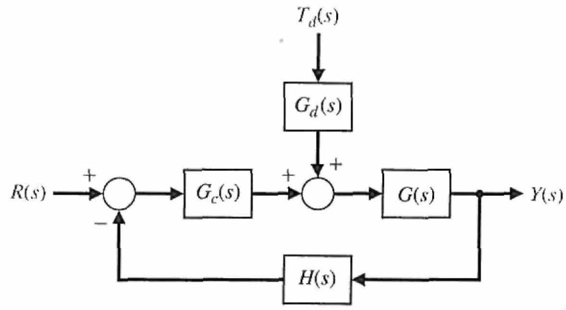


FIGURE AP2.3 Feedback system with a disturbance input.

heat flow of the heating element. The system parameters are C_t, Q, S, and R_t. The thermal heating system is illustrated in Table 2.5. (a) Determine the response of the system to a unit step $q(s) = 1/s$. (b) As $t \to \infty$, what value does the step response determined in part (a) approach? This is known as the steady-state response. (c) Describe how you would select the system parameters C_t, Q, S, and R_t to increase the speed of response of the system to a step input.

AP2.5 For the three-cart system illustrated in Figure AP2.5, obtain the equations of motion. The system has three inputs u_1, u_2, and u_3 and three outputs x_1, x_2, and x_3. Obtain three second-order ordinary differential equations with constant coefficients. If possible, write the equations of motion in matrix form.

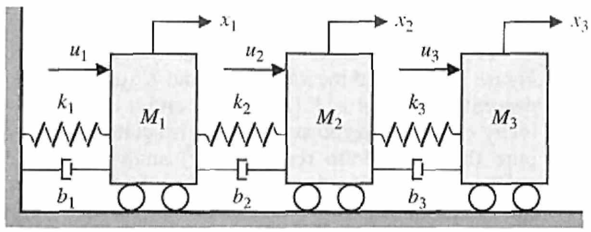
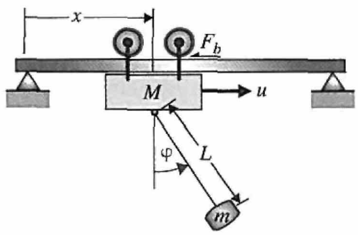


FIGURE AP2.5 Three-cart system with three inputs and three outputs.

AP2.6 Consider the hanging crane structure in Figure AP2.6. Write the equations of motion describing the motion of the cart and the payload. The mass of the cart is M, the mass of the payload is m, the massless rigid connector has length L, and the friction is modeled as $F_b = -b\dot{x}$ where x is the distance traveled by the cart.

AP2.7 Consider the unity feedback system described in the block diagram in Figure AP2.7. Compute analytically the response of the system to an impulse disturbance.

(a)



(b)

FIGURE AP2.6 (a) Hanging crane supporting the Space Shuttle Atlantis (Image Credit: NASA/Jack Pfaller) and (b) schematic representation of the hanging crane structure.

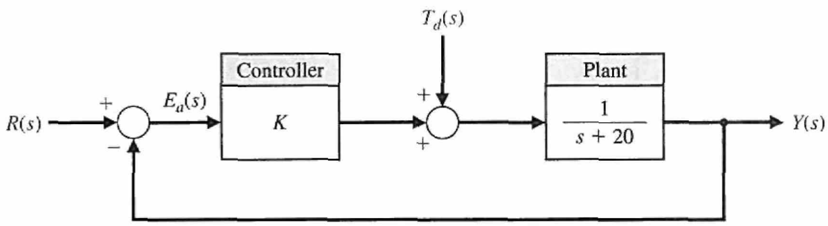


FIGURE AP2.7 Unity feedback control system with controller $G_c(s) = K$.

Determine a relationship between the gain K and the minimum time it takes the impulse disturbance response of the system to reach $y(t) < 0.1$. Assume that $K > 0$. For what value of K does the disturbance response first reach at $y(t) = 0.1$ at $t = 0.05$?

AP2.8 Consider the cable reel control system given in Figure AP2.8. Find the value of A and K such that the percent overshoot is $P.O. \leq 10\%$ and a desired velocity of 50 m/s in the steady state is achieved. Compute the closed-loop response $y(t)$ analytically and confirm that the steady-state response and $P.O.$ meet the specifications.

AP2.9 Consider the inverting operational amplifier in Figure AP2.9. Find the transfer function $V_o(s)/V_i(s)$. Show that the transfer function can be expressed as

$$G(s) = \frac{V_o(s)}{V_i(s)} = K_P + \frac{K_I}{s} + K_D s,$$

where the gains K_P, K_I, and K_D are functions of C_1, C_2, R_1, and R_2. This circuit is a proportional-integral-derivative (PID) controller (more on PID controllers in Chapter 7).

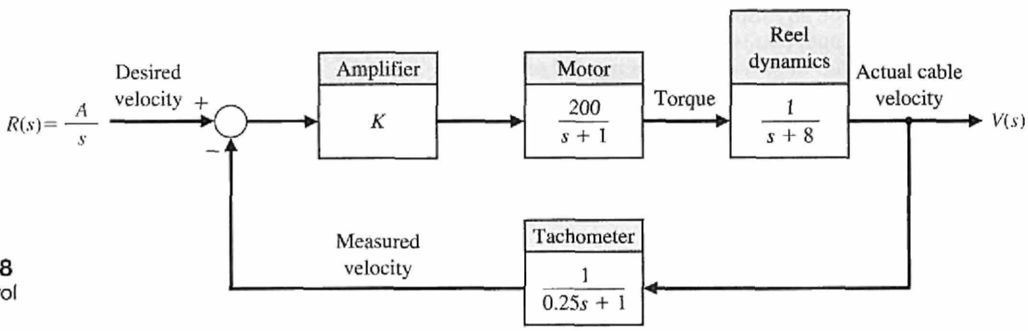


FIGURE AP2.8 Cable reel control system.

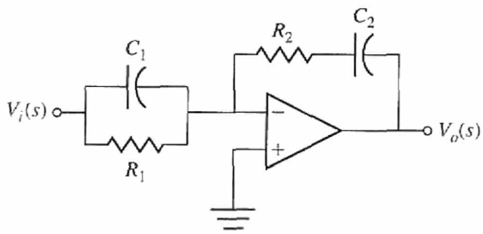


FIGURE AP2.9 An inverting operational amplifier circuit representing a PID controller.

DESIGN PROBLEMS

CDP2.1 We want to accurately position a table for a machine as shown in Figure CDP2.1. A traction-drive motor with a capstan roller possesses several desirable characteristics compared to the more popular ball screw. The traction drive exhibits low friction and no backlash. However, it is susceptible to disturbances. Develop a model of the traction drive shown in Figure CDP2.1(a) for the parameters given in Table CDP2.1. The drive uses a DC armature-controlled motor with a capstan roller attached to the shaft. The drive bar moves the linear slide-table. The slide uses an air bearing, so its friction is negligible. We are considering the open-loop model, Figure CDP2.1(b), and its transfer function in this problem. Feedback will be introduced later.

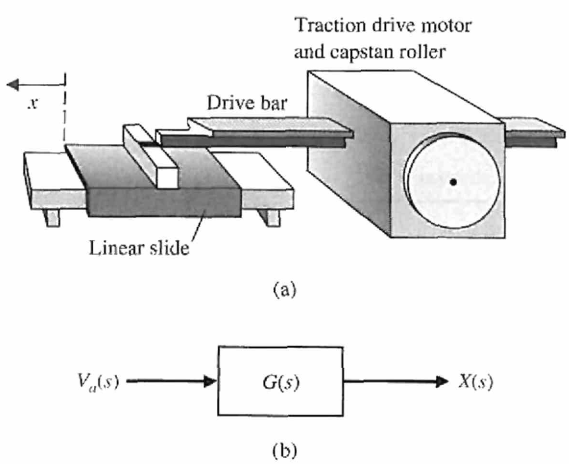


FIGURE CDP2.1 (a) Traction drive, capstan roller, and linear slide. (b) The block diagram model.

Table CDP2.1 Typical Parameters for the Armature-Controlled DC Motor and the Capstan and Slide

M_s	Mass of slide	5.693 kg
M_b	Mass of drive bar	6.96 kg
J_m	Inertia of roller, shaft, motor and tachometer	$10.91 \cdot 10^{-3}$ kg m^2
r	Roller radius	$31.75 \cdot 10^{-3}$ m
b_m	Motor damping	0.268 N ms/rad
K_m	Torque constant	0.8379 N m/amp
K_b	Back emf constant	0.838 V s/rad
R_m	Motor resistance	1.36 Ω
L_m	Motor inductance	3.6 mH

DP2.1 A control system is shown in Figure DP2.1. The transfer functions $G_2(s)$ and $H_2(s)$ are fixed. Determine the transfer functions $G_1(s)$ and $H_1(s)$ so that the closed-loop transfer function $Y(s)/R(s)$ is exactly equal to 1.

DP2.2 The television beam circuit of a television is represented by the model in Figure DP2.2. Select the unknown conductance G so that the voltage v is 24 V. Each conductance is given in siemens (S).

DP2.3 An input $r(t) = t, t \geq 0$, is applied to a black box with a transfer function $G(s)$. The resulting output response, when the initial conditions are zero, is

$$y(t) = e^{-t} - \frac{1}{4}e^{-2t} - \frac{3}{4} + \frac{1}{2}t, \; t \geq 0.$$

Determine $G(s)$ for this system.

DP2.4 An operational amplifier circuit that can serve as a filter circuit is shown in Figure DP2.4. Determine the transfer function of the circuit, assuming an ideal op-amp. Find $v_0(t)$ when the input is $v_1(t) = At$, $t \geq 0$.

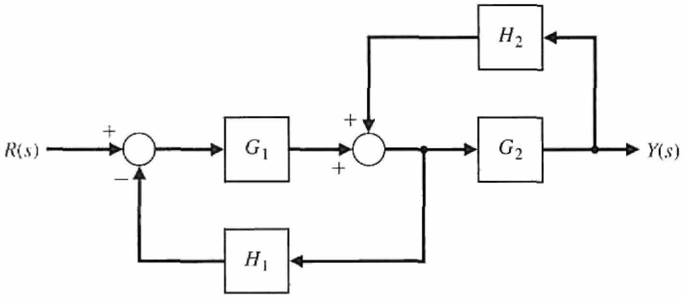


FIGURE DP2.1 Selection of transfer functions.

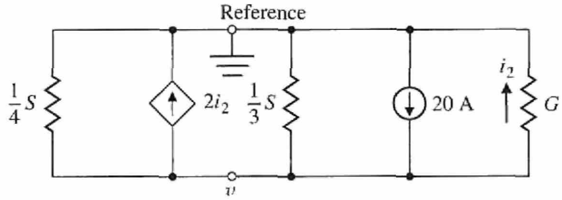


FIGURE DP2.2 Television beam circuit.

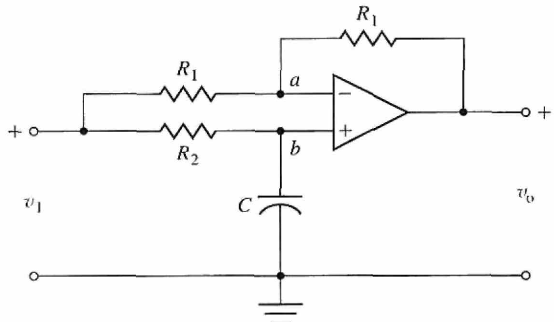


FIGURE DP2.4 Operational amplifier circuit.

DP2.5 Consider the clock shown in Figure DP2.5. The pendulum rod of length L supports a pendulum disk. Assume that the pendulum rod is a massless rigid thin rod and the pendulum disc has mass m. Design the length of the pendulum, L, so that the period of motion is 2 seconds. Note that with a period of 2 seconds each "tick" and each "tock" of the clock represents 1 second, as desired. Assume small angles, φ, in the analysis so that $\sin\varphi \approx \varphi$. Can you explain why most grandfather clocks are about 1.5 m or taller?

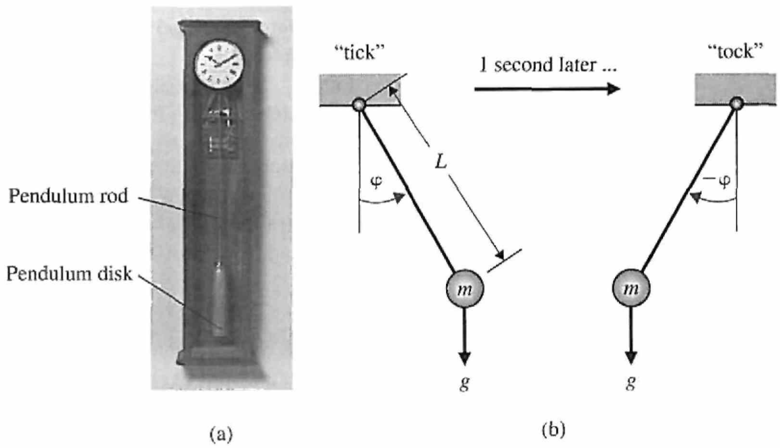


FIGURE DP2.5 (a) Typical clock (photo courtesy of SuperStock) and (b) schematic representation of the pendulum.

COMPUTER PROBLEMS

CP2.1 Consider the two polynomials

$$p(s) = s^2 + 7s + 10$$

and

$$q(s) = s + 2.$$

Compute the following

(a) $p(s)q(s)$

(b) poles and zeros of $G(s) = \dfrac{q(s)}{p(s)}$

(c) $p(-1)$

CP2.2 Consider the feedback system depicted in Figure CP2.2.

(a) Compute the closed-loop transfer function using the series and feedback functions.

(b) Obtain the closed-loop system unit step response with the step function, and verify that final value of the output is 2/5.

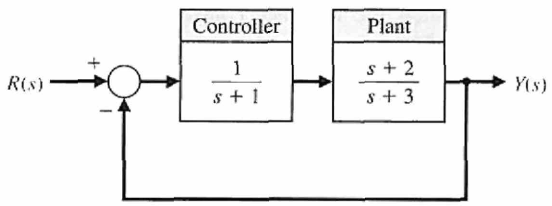


FIGURE CP2.2 A negative feedback control system.

CP2.3 Consider the differential equation

$$\ddot{y} + 4\dot{y} + 3y = u,$$

where $y(0) = \dot{y}(0) = 0$ and $u(t)$ is a unit step. Determine the solution $y(t)$ analytically and verify by co-plotting the analytic solution and the step response obtained with the step function.

CP2.4 Consider the mechanical system depicted in Figure CP2.4. The input is given by $f(t)$, and the output is $y(t)$. Determine the transfer function from $f(t)$ to $y(t)$ and, using an m-file, plot the system response to a unit step input. Let $m = 10$, $k = 1$, and $b = 0.5$. Show that the peak amplitude of the output is about 1.8.

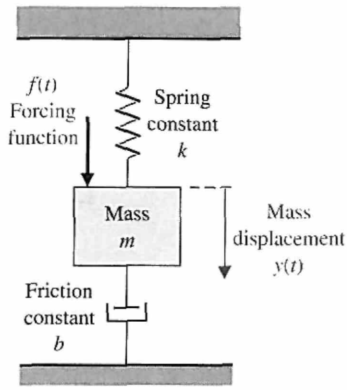


FIGURE CP2.4 A mechanical spring-mass-damper system.

CP2.5 A satellite single-axis attitude control system can be represented by the block diagram in Figure CP2.5. The variables k, a, and b are controller parameters, and J is the spacecraft moment of inertia. Suppose the nominal moment of inertia is $J = 10.8\text{E}8\,(\text{slug ft}^2)$, and the controller parameters are $k = 10.8\text{E}8$, $a = 1$, and $b = 8$.

(a) Develop an m-file script to compute the closed-loop transfer function $T(s) = \theta(s)/\theta_d(s)$.

(b) Compute and plot the step response to a 10° step input.

(c) The exact moment of inertia is generally unknown and may change slowly with time. Compare the step response performance of the spacecraft when J is reduced by 20% and 50%. Use the controller parameters $k = 10.8\text{E}8$, $a = 1$, and $b = 8$ and a 10° step input. Discuss your results.

CP2.6 Consider the block diagram in Figure CP2.6.

(a) Use an m-file to reduce the block diagram in Figure CP2.6, and compute the closed-loop transfer function.

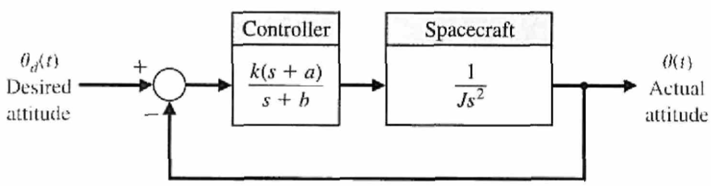


FIGURE CP2.5 A spacecraft single-axis attitude control block diagram.

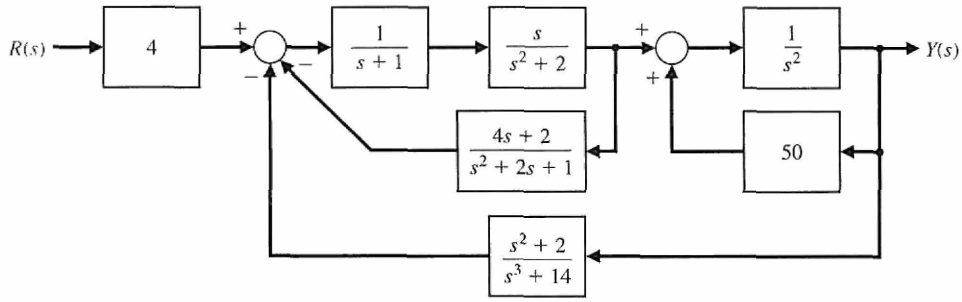


FIGURE CP2.6 A multiple-loop feedback control system block diagram.

(b) Generate a pole–zero map of the closed-loop transfer function in graphical form using the pzmap function.

(c) Determine explicitly the poles and zeros of the closed-loop transfer function using the pole and zero functions and correlate the results with the pole–zero map in part (b).

CP2.7 For the simple pendulum shown in Figure CP2.7, the nonlinear equation of motion is given by

$$\ddot{\theta}(t) + \frac{g}{L}\sin\theta = 0,$$

where $L = 0.5$ m, $m = 1$ kg, and $g = 9.8$ m/s^2. When the nonlinear equation is linearized about the equilibrium point $\theta = 0$, we obtain the linear time-invariant model,

$$\ddot{\theta} + \frac{g}{L}\theta = 0.$$

Create an m-file to plot both the nonlinear and the linear response of the simple pendulum when the initial angle of the pendulum is $\theta(0) = 30°$ and explain any differences.

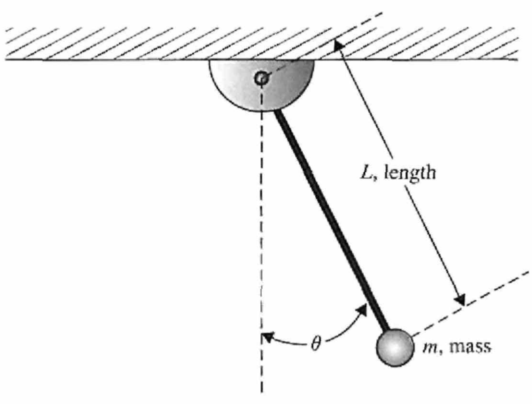


FIGURE CP2.7 Simple pendulum.

CP2.8 A system has a transfer function

$$\frac{X(s)}{R(s)} = \frac{(20/z)(s + z)}{s^2 + 3s + 20}.$$

Plot the response of the system when $R(s)$ is a unit step for the parameter $z = 5, 10,$ and 15.

CP2.9 Consider the feedback control system in Figure CP2.9, where

$$G(s) = \frac{s + 1}{s + 2} \quad \text{and} \quad H(s) = \frac{1}{s + 1}.$$

(a) Using an m-file, determine the closed-loop transfer function.

(b) Obtain the pole–zero map using the pzmap function. Where are the closed-loop system poles and zeros?

(c) Are there any pole–zero cancellations? If so, use the minreal function to cancel common poles and zeros in the closed-loop transfer function.

(d) Why is it important to cancel common poles and zeros in the transfer function?

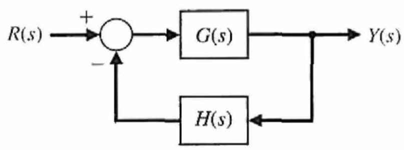


FIGURE CP2.9 Control system with nonunity feedback.

CP2.10 Consider the block diagram in Figure CP2.10. Create an m-file to complete the following tasks:

(a) Compute the step response of the closed-loop system (that is, $R(s) = 1/s$ and $T_d(s) = 0$) and plot the steady-state value of the output $Y(s)$ as a function of the controller gain $0 < K \leq 10$.

(b) Compute the disturbance step response of the closed-loop system (that is, $R(s) = 0$ and

$T_d(s) = 1/s$) and co-plot the steady-state value of the output $Y(s)$ as a function of the controller gain $0 < K \le 10$ on the same plot as in (a) above.

(c) Determine the value of K such that the steady-state value of the output is equal for both the input response and the disturbance response.

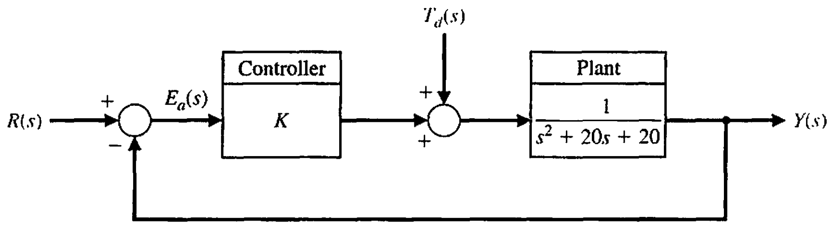


FIGURE CP2.10 Block diagram of a unity feedback system with a reference input $R(s)$ and a disturbance input $T_d(s)$.

ANSWERS TO SKILLS CHECK

True or False: (1) False; (2) True; (3) False; (4) True; (5) True

Multiple Choice: (6) b; (7) a; (8) b; (9) b; (10) c; (11) a; (12) a; (13) c; (14) a; (15) a

Word Match (in order, top to bottom): e, j, d, h, a, f, c, b, k, g, o, l, n, m, i

TERMS AND CONCEPTS

Across-Variable A variable determined by measuring the difference of the values at the two ends of an element.

Actuator The device that causes the process to provide the output. The device that provides the motive power to the process.

Analogous variables Variables associated with electrical, mechanical, thermal, and fluid systems possessing similar solutions providing the analyst with the ability to extend the solution of one system to all analogous systems with the same describing differential equations.

Assumptions Statements that reflect situations and conditions that are taken for granted and without proof. In control systems, assumptions are often employed to simplify the physical dynamical models of systems under consideration to make the control design problem more tractable.

Block diagrams Unidirectional, operational blocks that represent the transfer functions of the elements of the system.

Branch A unidirectional path segment in a signal-flow graph that relates the dependency of an input and an output variable.

Characteristic equation The relation formed by equating to zero the denominator of a transfer function.

Closed-loop transfer function A ratio of the output signal to the input signal for an interconnection of systems when all the feedback or feedfoward loops have been closed or otherwise accounted for. Generally obtained by block diagram or signal-flow graph reduction.

Coulomb damper A type of mechanical damper where the model of the friction force is a nonlinear function of the mass velocity and possesses a discontinuity around zero velocity. Also know as dry friction.

Critical damping The case where damping is on the boundary between underdamped and overdamped.

Damped oscillation An oscillation in which the amplitude decreases with time.

Damping ratio A measure of damping. A dimensionless number for the second-order characteristic equation.

DC motor An electric actuator that uses an input voltage as a control variable.

Differential equation An equation including differentials of a function.

Error signal The difference between the desired output $R(s)$ and the actual output $Y(s)$; therefore $E(s) = R(s) - Y(s)$.

Final value The value that the output achieves after all the transient constituents of the response have faded. Also referred to as the steady-state value.

Final value theorem The theorem that states that $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$, where $Y(s)$ is the Laplace transform of $y(t)$.

Homogeneity The property of a linear system in which the system response, $y(t)$, to an input $u(t)$ leads to the response $\beta y(t)$ when the input is $\beta u(t)$.

Inverse Laplace transform A transformation of a function $F(s)$ from the complex frequency domain into the time domain yielding $f(t)$.

Laplace transform A transformation of a function $f(t)$ from the time domain into the complex frequency domain yielding $F(s)$.

Linear approximation An approximate model that results in a linear relationship between the output and the input of the device.

Linear system A system that satisfies the properties of superposition and homogeneity.

Linearized Made linear or placed in a linear form. Taylor series approximations are commonly employed to obtain linear models of physical systems.

Loop A closed path that originates and terminates on the same node of a signal-flow graph with no node being met twice along the path.

Mason loop rule A rule that enables the user to obtain a transfer function by tracing paths and loops within a system.

Mathematical models Descriptions of the behavior of a system using mathematics.

Natural frequency The frequency of natural oscillation that would occur for two complex poles if the damping were equal to zero.

Necessary condition A condition or statement that must be satisfied to achieve a desired effect or result. For example, for a linear system it is necessary that the input $u_1(t) + u_2(t)$ results in the response $y_1(t) + y_2(t)$, where the input $u_1(t)$ results in the response $y_1(t)$ and the input $u_2(t)$ results in the response $y_2(t)$.

Node The input and output points or junctions in a signal-flow graph.

Nontouching Two loops in a signal-flow graph that do not have a common node.

Overdamped The case where the damping ratio is $\zeta > 1$.

Path A branch or a continuous sequence of branches that can be traversed from one signal (node) to another signal (node) in a signal-flow graph.

Poles The roots of the denominator polynomial (i.e., the roots of the characteristic equation) of the transfer function.

Positive feedback loop Feedback loop wherein the output signal is fed back so that it adds to the input signal.

Principle of superposition The law that states that if two inputs are scaled and summed and routed through a linear, time-invariant system, then the output will be identical to the sum of outputs due to the individual scaled inputs when routed through the same system.

Reference input The input to a control system often representing the desired output, denoted by $R(s)$.

Residues The constants k_i associated with the partial fraction expansion of the output $Y(s)$, when the output is written in a residue-pole format.

Signal-flow graph A diagram that consists of nodes connected by several directed branches and that is a graphical representation of a set of linear relations.

Simulation A model of a system that is used to investigate the behavior of a system by utilizing actual input signals.

Steady state The value that the output achieves after all the transient constituents of the response have faded. Also referred to as the final value.

s-plane The complex plane where, given the complex number $s = s + jw$, the x-axis (or horizontal axis) is the s-axis, and the y-axis (or vertical axis) is the jw-axis.

Taylor series A power series defined by $g(x) = \sum_{m=0}^{\infty} \frac{g^{(m)}(x_0)}{m!}(x - x_0)^m$. For $m < \infty$, the series is an approximation which is used to linearize functions and system models.

Through-variable A variable that has the same value at both ends of an element.

Time constant The time interval necessary for a system to change from one state to another by a specified percentage. For a first order system, the time constant is the time it takes the output to manifest a 63.2% change due to a step input.

Transfer function The ratio of the Laplace transform of the output variable to the Laplace transform of the input variable.

Underdamped The case where the damping ratio is $\zeta < 1$.

Unity feedback A feedback control system wherein the gain of the feedback loop is one.

Viscous damper A type of mechanical damper where the model of the friction force is linearly proportional to the velocity of the mass.

Zeros The roots of the numerator polynomial of the transfer function.