INTERPRETING SERGEYEV'S NUMERICAL METHODOLOGY WITHIN A HYPERREAL NUMBER SYSTEM

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Abstract. In this paper we show the consistency of the essential part of Sergeyev's numerical methodology ([1], [2]) by constructing a model of it within the framework of an ultrapower of the ordinary real number system.

1. Sergeyev's Numerical System

In [1] and [2], Ya. D. Sergeyev extends the ordinary positive integers by adjoining a new symbol Ⓢ that he calls *grossone*.[1] Three axioms and three methodological postulates govern the role of grossone in this number system. For example, the number Ⓢ is regarded to be infinitely large in the sense that it satisfies the following axiom.

- *Infinity.* For all the ordinary positive integers n, we have $n < Ⓢ$.

The symbol $\mathbb{N}$ is defined to be the set

$$\mathbb{N} = \{1, 2, 3, \ \ldots \ Ⓢ - 2, Ⓢ - 1, Ⓢ\}, \tag{1.0.1}$$

and the symbol $\hat{\mathbb{N}}$ is defined to be the set

$$\hat{\mathbb{N}} = \{1, 2, 3, \ \ldots \ Ⓢ - 2, Ⓢ - 1, Ⓢ, Ⓢ + 1, Ⓢ + 2, \ldots\}. \tag{1.0.2}$$

Based on (1.0.1), the number Ⓢ is regarded as the cardinality of $\mathbb{N}$ with Ⓢ serving as its largest element. There is a second axiom that is called *identity* and lists the following relations.

- *Identity.* $0 \cdot Ⓢ = 0 = Ⓢ \cdot 0$; $Ⓢ - Ⓢ = 0$; $\frac{Ⓢ}{Ⓢ} = 1$; $Ⓢ^0 = 1$; $1^{Ⓢ} = 1$; $0^{Ⓢ} = 0$.

Finally, there is a divisibility axiom that requires that the number Ⓢ be divisible by all the ordinary positive integers. This axiom is stated as follows.

- *Divisibility.* For each ordinary positive integers n and each positive integer k with $1 \leq k \leq n$, the sets $\mathbb{N}_{k;n}$, being the nth parts of the set $\mathbb{N}$ have the same number of elements indicated by the numeral $\frac{Ⓢ}{n}$, where

$$\mathbb{N}_{k;n} = \{k, k+n, k+2n, k+3n, \ldots\} \quad \text{and} \quad \mathbb{N} = \bigcup_{k=1}^{n} \mathbb{N}_{k;n}.$$

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[1] Sergeyev's grossone is in fact a circled 1 rather than a circled S. But that does not matter. I would have used his symbol if I knew how to produce it in LaTex.

2. Ultrapowers

In this section, we recall the notion of an ultrapower of an infinite set in general and the real number system in particular.

2.1. **Definition** (Ultrafilters)**.** A family $\mathcal{U}$ of subsets of $\boldsymbol{Z}^+$ is called a *free ultrafilter* on $\boldsymbol{Z}^+$ if it satisfies the following conditions:

(1) $\emptyset \notin \mathcal{U}$.
(2) If $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.
(3) If $A \in \mathcal{U}$ and B is a subset of $\boldsymbol{Z}^+$ that contains A, then $B \in \mathcal{U}$.
(4) If $S \subseteq \boldsymbol{Z}^+$, then $S \in \mathcal{U}$ or $S' = \{x \in \boldsymbol{Z}^+ : x \notin S\} \in \mathcal{U}$.
(5) No finite subset of $\boldsymbol{Z}^+$ belongs to $\mathcal{U}$.

Now let $\boldsymbol{X}$ be any infinite set, and let $\boldsymbol{X}^{\boldsymbol{Z}^+}$ be the set of all the sequences (a_n) in $\boldsymbol{X}$. We define an equivalence relation $\equiv$ on $\boldsymbol{X}^{\boldsymbol{Z}^+}$ by writing $(a_n) \equiv (b_n)$ if and only if $\{n \in \boldsymbol{Z}^+ : a_n = b_n\} \in \mathcal{U}$. It is easy to see that $\equiv$ is indeed an equivalence relation. For example, to prove reflexivity (i.e., $(a_n) \equiv (a_n)$ for all $(a_n) \in \boldsymbol{X}^{\boldsymbol{Z}^+}$), we must show that $\{n \in \boldsymbol{Z}^+ : a_n = a_n\} \in \mathcal{U}$. This is an immediate consequence of conditions (1) and (4) of Definition (2.1) since the set $\{n \in \boldsymbol{Z}^+ : a_n = a_n\}$ is none other than the entire $\boldsymbol{Z}^+$.

2.2. **Definition.** The set $\mathbb{X}$ of all the equivalence classes of $\boldsymbol{X}^{\boldsymbol{Z}^+}$ that are induced by $\equiv$ is called an *ultrapower* of $\boldsymbol{X}$. For each $x \in \boldsymbol{X}$ the equivalence class $[(x, x, x, \cdots)]$ of the constant sequence $(x, x, x, \cdots)$ is denoted by ${}^*x$. An element $\boldsymbol{x} \in \mathbb{X}$ is called *standard* if there is an $x \in \boldsymbol{X}$ such that $\boldsymbol{x} = {}^*x$. The rest of the elements of $\mathbb{X}$ are called *nonstandard.* The collection of all the standard elements of $\mathbb{X}$ is denoted by ${}^\sigma\mathbb{X}$. The next theorem gives the condition that guarantees the existence of nonstandard elements in $\mathbb{X}$.

2.3. **Remark.** The symbol ${}^*\boldsymbol{X}$ is an alternative notation for $\mathbb{X}$. We shall use this symbol particularly to denote an ultrapower ${}^*\boldsymbol{R}$ of the set of the ordinary real numbers $\boldsymbol{R}$.

2.4. **Theorem.** *Let $\mathbb{X}$ be an ultrapower of the set $\boldsymbol{X}$. If $\boldsymbol{X}$ is infinite, then $\mathbb{X}$ has nonstandard elements.*

Proof. Since $\boldsymbol{X}$ is infinite, there is a sequence (a_n) in it whose terms are distinct. Let $\boldsymbol{\alpha} = [(a_n)]$. We claim that $\boldsymbol{\alpha} \notin {}^\sigma\mathbb{X}$. That is, (a_n) does not belong to the equivalence class of any constant sequence $(x, x, x, \cdots)$. To see this, fix $x \in \boldsymbol{X}$. Since the set $S = \{n \in \boldsymbol{Z}^+ : x = a_n\}$ has at most one element, by condition (5) of Definition (2.1), we have $S \notin \mathcal{U}$. This means that $(a_1, a_2, a_3, \cdots)$ and $(x, x, x, \cdots)$ are not related by the relation $\equiv$. Hence $\boldsymbol{\alpha} \neq {}^*\boldsymbol{x}$. □

Now fix a free ultrafilter $\mathcal{U}$ on $\boldsymbol{Z}^+$, and let ${}^*\boldsymbol{R}$ denote the ultrapower of $\boldsymbol{R}$ that is obtained by means of $\mathcal{U}$. For convenience, for each sequence (a_n) in $\boldsymbol{R}$, let its equivalence class $[(a_n)]$ be denoted by the bold symbol $\boldsymbol{a}$.

2.5. **Definition.** Given $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in {}^*\boldsymbol{R}$, we write

$\boldsymbol{a} = \boldsymbol{b}$ if and only if $(a_n) \equiv (b_n)$.

$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$ if and only if $(a_n + b_n) \equiv (c_n)$.

$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{c}$ if and only if $(a_n \cdot b_n) \equiv (c_n)$.

$\boldsymbol{a} < \boldsymbol{b}$ if and only if $\{n \in \boldsymbol{Z}^+ : a_n < b_n\} \in \mathcal{U}$.

2.6. **Hyperreal Number System.** The system $({}^*\boldsymbol{R}, +, \cdot, <)$ that we just defined is referred to as a *hyperreal number system.* It is not difficult to prove (see [3], page 5) that the system $({}^*\boldsymbol{R}, +, \cdot, <)$ is a linearly ordered field, and contains an isomorphic copy of the system $(\boldsymbol{R}, +, \cdot, <)$. This isomorphism assigns to each $a \in \boldsymbol{R}$ the equivalence class $\boldsymbol{a} = [(a, a, a, \cdots)]$. The set ${}^*\boldsymbol{R}$ has *unlimited* elements (or *elements with infinitely large magnitudes*). These are elements whose absolute values are greater than every positive element of ${}^\sigma[{}^*\boldsymbol{R}]$. For example, if $\omega = [(1, 2, 3, \cdots)]$, then we have $\omega > \boldsymbol{a}$ for all $\boldsymbol{a} \in {}^\sigma[{}^*\boldsymbol{R}]$ since

$$\{n \in \boldsymbol{Z}^+ : n > a\} \in \mathcal{U} \qquad \text{for each } a \in \boldsymbol{R}.$$

The reciprocals of unlimited numbers are the *infinitesimals.* Thus for example,

$$\frac{1}{\omega} = \left[\left(1, \frac{1}{2}, \frac{1}{3}, \cdots\right)\right]$$

is an infinitesimal — it is smaller than every standard positive number in ${}^*\boldsymbol{R}$. Two numbers $\boldsymbol{a}, \boldsymbol{b}$ in $\mathbb{R}$ are infinitely close, written $\boldsymbol{a} \simeq \boldsymbol{b}$, if $|\boldsymbol{a} - \boldsymbol{b}|$ is an infinitesimal.

The real number system $({}^*\boldsymbol{R}, +, \cdot, <)$ can be visualized as in Figure (1).

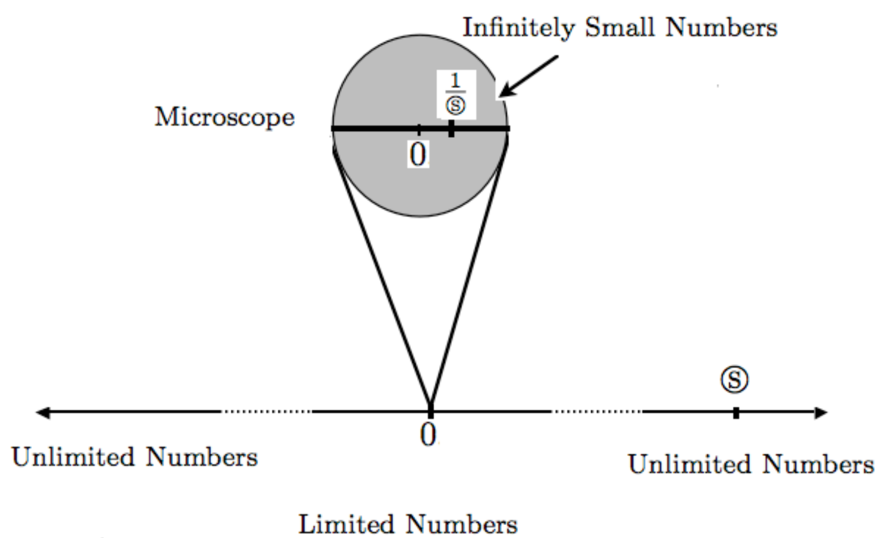

FIGURE 1. Infinitesimals viewed through a microscope

3. INTERPRETATION OF GROSSONE IN $({}^*\boldsymbol{R}, +, \cdot, <)$

Let $({}^*\boldsymbol{R}, +, \cdot, <)$ be the hyperreal number system discussed in the previous section. Recall that the internal properties of the system $({}^*\boldsymbol{R}, +, \cdot, <)$ are formally identical with the properties of the ordinary real number system. This means, in particular, that ${}^*\boldsymbol{R}$ has subsystems such as $({}^*\boldsymbol{Z}, +, \cdot, <)$ and $({}^*\boldsymbol{Q}, +, \cdot, <)$ whose internal properties are formally identical with the properties of the ordinary system $(\boldsymbol{Z}, +, \cdot, <)$ of integers and the ordinary system $(\boldsymbol{Q}, +, \cdot, <)$ of rational numbers.

In other words, the system $({}^*\boldsymbol{Z}, +, \cdot, <)$ is a linearly ordered ring and the system $({}^*\boldsymbol{Q}, +, \cdot, <)$ is a linearly ordered field.

Now if we define the symbol Ⓢ by the equation Ⓢ $= [(n!)]$, we see that Ⓢ is a nonstandard element of ${}^*\boldsymbol{Z}^+$ that satisfies the infinity axiom of Sergeyev's grossone. This follows from the fact that

$$\{n \in \boldsymbol{Z}^+ : n! > a\} \in \mathcal{U} \qquad \text{for each } a \in \boldsymbol{R}.$$

Now define the symbols $\mathbb{N}$ and $\hat{\mathbb{N}}$ as follows:

$$\mathbb{N} = \{\boldsymbol{m} \in {}^*\boldsymbol{Z}^+ : \boldsymbol{m} \leq \text{Ⓢ}\} \qquad \text{and} \qquad \hat{\mathbb{N}} = {}^*\boldsymbol{Z}^+.$$

Then $\mathbb{N}$ is an internal (nonstandard) finite set whose internal cardinality is Ⓢ. The field properties of the system $({}^*\boldsymbol{R}, +, \cdot, <)$ guarantee that the number Ⓢ satisfies all of the statements listed in Sergeyev's identity axiom.

It remains to prove Sergeyev's divisibility axiom for grossone, which follows easily from the next theorem. In this theorem we use the conventional notation for divisibility of integers. That is, we write $p|q$ to indicate that q is divisible by p.

3.1. **Theorem.** *For each* $\boldsymbol{p} \in {}^\sigma[{}^*\boldsymbol{Z}^+]$, *we have* $\boldsymbol{p}|$Ⓢ.

Proof. Fix $\boldsymbol{p} \in {}^\sigma[{}^*\boldsymbol{Z}^+]$. Then we have $\boldsymbol{p} = {}^*p$ for some $p \in \boldsymbol{Z}^+$. The conclusion of the theorem is an obvious consequence of the fact that

$$\{n \in \boldsymbol{Z}^+ : p|n!\} \in \mathcal{U} \qquad \text{for each } p \in \boldsymbol{R}.$$

□

References

[1] Ya. D. Sergeyev, *A new applied approach for executing computations with infinite and infinitesimal quantities*, Informatica 19 (2008).

[2] Ya. D. Sergeyev, *Methodology of Numerical Computations with Infinite and Infinitesimals*, Rend. Sem. Mat. Univ. Politec. Torino

[3] Albert E. Hurd and Peter A. Loeb, *An Introduction to Nonstandard Analysis*, Academic Press, 1985

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