COUPLED MAP LATTICES AS MUSICAL INSTRUMENTS

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Abstract. We use one-dimensional coupled lattice maps (CMLs) to generate sounds that reflect spatial organization and temporal evolution of these dynamics started from a random initial configuration corresponding to uncorrelated noise. In many instances, the process approaches an equilibrium, which generates a sustained tone. The pitch of this tone is proportional to the lattice size, so the CML behaves like an instrument that could be tuned. Among exceptional cases, we give an example with a metastable strange attractor, which produces an evolving sound reminiscent of drone music.

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1. Introduction

A common mathematical model of a complex feedback system is iteration of a nonlinear map of one or more variables. The resulting unpredictability and diversity of behavior naturally invite possible connections to music, where such models have indeed found uses, primarily to generate note patterns for use in compositions (e.g., [Bat04]). In this paper we instead adopt this approach to produce a single sound, whose characteristics reflect the evolution induced by a chosen recursive rule.

We focus on *coupled map lattices*, a class of iterative systems built from a large number of identical functions of a single variable linearly coupled to nearest neighbors in a network. As has been known since their inception [Cru84, Kan84, KW84], these are particularly convenient models to study self-organization: from a disordered initialization, many instances exhibit rapid convergence to a steady state with recognizable spatial patterns (see, e.g., [Kan93]) for a survey). (In this article, we use the term "space" to refer to the abstract space in which the evolving array of variables is arranged, not to the physical space of a listening environment or of a musical instrument.) Our main goal, then, is to represent these patterns as sounds.

1.1. **Spatial organization and sound synthesis.**

Suppose we have a system that evolves in time, and we want to use it to generate a sound stream. Such a system possibly has a high-dimensional set of variables, together with one-dimensional time, and for sound generation needs to be reduced to a one-dimensional sequence of numbers. One way to achieve this is to simply produce one signal per time step, as a single function of all variables of the system [SAB15]. There might be a canonical selection of such a function in special cases, but in general the choice is necessarily arbitrary. Moreover, while this method does reveal temporal evolution (at least if the sampling variable is carefully chosen), any spatial organization is not captured. To remedy this, we seek to include *every* variable in our output sequence.

To investigate spatial organization, we must have space, and a particularly useful role is played by the space $\mathbb{Z}^d$ of integer vectors with d coordinates, the d-dimensional *integer* *lattice*. An evolving configuration attaches a variable, in our case a number, to each point in the lattice, and uses a rule to modify the variables over time. Suppose first that $d = 2$ (the same argument applies for higher dimensions). Then there is no natural way to "cut the configuration into one-dimensional ribbons" to generate a sound stream that would incorporate every variable, not even at a single time. In short, the problem is that the time (for the sound sequence) is *a priori* linearly ordered while the two-dimensional lattice is not. Nevertheless, there have been some attempts based on two-dimensional cellular automata such as the Game of Life and voter models ([BE05, Mir07, SM11, SM14]).

We conclude that the most promising are processes on $\mathbb{Z}$, i.e., those whose configuration is a finite array of numbers arranged on an integer interval, say $[0, n-1]$. The number n is the *lattice size*. Referring to the space, we call such processes one-dimensional, although the dimension of the configuration space equals n and could be very large, typically in our examples between 50 and 400.

The configuration of our process at a time t therefore consists of real numbers

$$x_t(0), \dots, x_t(n-1).$$

This process will evolve in discrete time $t = 0, 1, 2, \dots$ according to a rule that will be specified shortly, but first we reveal how we obtain the sequence used to generate sound. Namely, we form the unique sequence of variables that preserves the order of the time, and then the order of the one-dimensional lattice:

$$x_0(0), \dots, x_0(n-1), x_1(0), \dots, x_1(n-1), \dots, x_t(0), \dots, x_t(n-1), \dots \tag{1}$$

1.2. **Coupled map lattices.** Particularly appealing update rules to study self-organization are those with very simple local interaction. We now introduce one such class of rules, which posit that the variable at a location i is first updated using a (possibly nonlinear) function of only its current state; and then the value at the next time step is computed by a weighted average of its value and those of its nearest neighbors in the one-dimensional lattice. (Thus the second step is linear and any perturbation spreads at the speed at most 1, which is often called the "light speed."). That is, x_t is given recursively, for each time $t = 0, 1, 2, \dots$, by the rule

$$x_{t+1}(i) = (1-\epsilon) f(x_t(i)) + \frac{\epsilon}{2}[f(x_t(i+1)) + f(x_t(i-1))]. \tag{2}$$

Here, $\epsilon \in [0, 1]$ is the *coupling strength* that determines the influence of spatial neighbors, and f is a given function. We assume *periodic boundary*, that is, for $i \in [0, n-1]$, its *neighbors* $i-1$ and $i+1$ are computed modulo n. Such a rule is called a one-dimensional *coupled map lattice* (*CML*). We will restrict the variable values $x_t(i)$ to a suitable interval, either $[-1, 1]$ or $[0, 1]$. In the latter case, whenever the updated variable is outside the interval we replace it with its fractional part; equivalently, we could assume that individual variables are points on a (continuous) circle.

Typically, our initial configuration is random. To be precise, $x_0(i)$, $i = 0, \dots, n-1$ are chosen independently and uniformly from the set of possible values. This maximally disordered initialization is arguably the best test of the rule's self-organizing capabilities. However, interesting sounds can be generated with special initial configurations, as we will see later in Sections 5.5 and 5.6.

CMLs were utilized by Kunihiko Kaneko [Kan93] to model spatiotemporal chaos with discrete-time, discrete-space, and continuous-state dynamics. We will only consider three examples of the map f: the linear map (Section 3), the logistic map (Section 4) used by Kaneko, and the circle map (Section 5). For the latter two choices, CMLs are, in a sense, coupled nonlinear oscillators. The linear map facilitates the exposition through its simplicity, and demonstrates an interesting connection between CMLs and the Karplus-Strong algorithm. The logistic map is considered because of the role it has played in the existing literature on CMLs, and because it provides good examples of periodic, intermittent, and chaotic behavior. The most interesting sounds, however, are generated by the circle map, which exhibits a number of interesting evolutions. Examples we give include chaotic, periodic, and almost periodic behavior; synchronization; interacting particles; and metastable strange attractors. Often, the mixture of noise and order in the CML is reminiscent of chimera states in neural network dynamics [HKZL$^+$16]; see for example the CML given by (6).

There are several natural ways in which (2) can be generalized. For example, one may change the neighborhood of i from nearest neighbor $i \pm 1$ to a larger range r: $i \pm 1, \ldots, i \pm r$. This does not seem to make a qualitative difference. Another possibility is to forgo translation invariance and to assign either a different ϵ or a different f to different lattice points i. A *random* such assignment is an analogue of spin-glasses [SN13] and is an intriguing direction that is largely beyond the scope of this paper, although we do provide one example in Section 3. Finally, generalizations to other lattices are mathematically interesting, but, as discussed, it is unclear if this could be done without losing the natural correspondence to digital audio signals.

1.3. **Previous work.** Of all existing synthesis techniques, our approach most closely resembles the Linear Automata Synthesis (LASy) of Jacques Cheryron [Cha90]. There are two main differences between LASy and CML synthesis. The first is that the former uses cellular automata with a large set of discrete variable values, whereas the latter uses coupled map lattices, which have continuous variables. In this sense, our approach is a natural extension of LASy. The second difference is that the LASy method of [Cha90] focuses on linear cellular automata, mentioning other possibilities only in passing, while we view linear rules as a starting point (Section 3) into more general CMLs. Notice again that both of these schemes employ one dimensional lattices; higher dimensional cases require mapping the output of the system in a more creative way. For example, Histogram Mapping Synthesis [SM11, SM14] uses densities of variables in evolving cellular automata dynamics to control the parameters of a synthesizer. Another approach, Chaosynth, uses a cellular automaton inspired by neural reverberatory circuits to control the pitch and duration of sinusoidal grains in a granular synthesizer [Mir95].

A recent paper [SAB15] uses ordinary differential equations for digital synthesis. This method bears some similarity to ours in that it uses dynamical systems. However, it generates only one signal per time step, as a function of all variables, to generate a time series that produces the sound. By contrast, each of our variables generates a signal in the lattice order, so the resulting sound reflects spatial organization as well as temporal evolution. Furthermore, discrete space and time have the practical advantage in that computations are simple and very fast. By contrast, coupled ordinary differential equations (discrete space and continuous time) or partial differential equations (both continuous) would require potentially sophisticated solvers [Dzj15]. (There is also a theoretical advantage to CMLs: no existence and uniqueness results are needed.)

1.4. **Key features of CML synthesis.** In our experiments, most CML dynamics rapidly converge to a stable equilibrium, which results in a steady tone. In almost all cases, the pitch of the tone is inversely proportional to the lattice size. The proportionality constant depends on the sampling rate and the temporal behavior of the CML. While we do not have a formal definition, we refer to these cases as *regular*. Two exceptions are discussed in some detail: a remark in Section 4 covers highly chaotic behavior (producing white noise), and Section 5.2 highly organized behavior (resulting in a high frequency tone); both of these require very specific parameter settings.

In the regular cases, the timbre of a CML's tone is determined by the chosen parameters and the initial configuration. The random initialization results in slight variations between different runs of the dynamics, akin to physical instruments, which also produce subtly different tones even when the same pitch is played with the same loudness. The effect of the parameters on the timbre of a CML is dependent on the definition of the map, but some general observations can be made. The degree to which the map f itself is chaotic has the expected effect. Namely, the nonlinearity parameter (a in the logistic map (4), K in the circle map (5)) governs how simple the sound is: large nonlinearity results in chaotic noise, and other values generate tones with varied degrees of complexity. The coupling strength ϵ has an unpredictable effect on evolution, and often needs fine tuning.

1.5. **Outline and highlights.** Our main purpose is to use CMLs as instruments in the following fashion: generate a random initial configuration, choose the dynamic parameters and observe the resulting sound. Different choices produce starkly different sounds, from pitched tones, to noise, to something in between. In the following sections, we concentrate on examples, both to illustrate the regular dynamics and to provide interesting unusual cases. First, however, we give an overview of our approach in Section 2. Section 3 is a brief exposition on the basic case of the linear map, and Section 4 gives three examples with logistic map which demonstrates how CMLs generate sound that range between order and noise. Perhaps the most interesting is Section 5, which uses the circle map f. We provide four examples: the first demonstrates that CMLs are capable of producing rich and varied textures due to slow convergence to the steady state, the second is an example of a non-unique equilibrium with unusual lack of dependence on the lattice size, the third produces slowly changing timbre, and the last CML makes a frequency which is lower than expected. Finally, we give a few examples with special initializations. We demonstrate the use of CMLs as musical instruments by tuning them to play a few melodies, some well-known and some computer generated. Some examples can be found online.[1], with the choice of parameters described in the Appendix.

2. Interpreting a CML evolution as a sound

As explained in Section 1, we interpret each lattice variable in a CML evolution as a digital audio signal, and order these signals as in (1). We have chosen a sample rate of 44.1 kHz (the compact disc standard) in all our examples. After a short burst of white noise (due to the initial randomness), the resulting audio stream is a sound with evolving timbre. Because the dynamics tend to rapidly evolve into a stable or metastable steady state, the

[1]https://soundcloud.com/user-138099222/sets/melodies

transient period is almost imperceptible and the result is close a single sound. The nature of the sound depends upon the behavior of the CML. If the equilibrium is spatially simple, dominated by temporarily periodic regions, our synthesizer produces a pitched tone.

However, spatial simplicity alone does not suffice for a pitched tone: as we will see later, there is an example with nearly constant equilibrium whose values change chaotically and hence result in noise, even with constant initialization (see Section 5.6). On the other hand, it is very easy to generate very simple temporal behavior (namely, by choosing f to be the identity map with $\epsilon = 0$), which leaves the spatial chaos invariant and hence also produces noise.

We reiterate that two sounds, generated with the same map, parameters, and lattice size, but different random initial samples, will both sound similar, as they are generated by the same process, but they are subtly distinct from one another. This balance of similarity and variation is analogous to the way sounds produced by a physical instrument will have a recognizable timbre, but will reveal subtle differences when sounding the same note. Another parallel is that the dominant frequency, and therefore the tuning procedure, cannot be determined a priori for all parameter choices, but only after running the particular dynamics — by "playing the instrument," as it were. We elaborate on this point next.

With the exception of a few special cases, the pitch of the tone produced by a CML is inversely proportional to the lattice size n. Given a CML with a lattice of size n and a sample rate of s samples per second, the fundamental frequency of the tone produced by the CML synthesizer will be $s/(nT)$ Hz for some $T \in \{1, 2, 3, ...\}$. If a CML has a steady state that is (nearly) periodic with a sufficiently small temporal period, then T will be determined by the temporal period of the CML. This is because the sequence of samples generated by the consecutive lattice points will be (nearly) periodic with period nT and so will produce a tone of frequency $s/(nT)$. If the temporal period is too large, then $s/(nT)$ will be below the range of human hearing and we will perceive the strongest upper partial of this frequency as the fundamental. In this case, the dominating frequency tends to be s/n.

The only exceptions to the linear dependence on n we are aware of involve rules that produce a very close approximation to white noise and rare spatially periodic examples. Even chaotic sounds that do not produce a tone having a clear, definite musical pitch do have an audible change of pitch as a result of a change in the lattice size. Additionally, the discontinuity between temporal updates provides a periodic signal also with frequency inversely proportional to the lattice size. These two contributions are fundamentally different; for example, if n is large the second effect amounts to a sequence of "pops." To reiterate, the inverse of n will be present in the frequency domain, except when the lattice configuration is spatially periodic and temporarily constant. The latter level of regularity is very difficult to achieve with a CML starting from a random initialization. We do, however, provide an example in which this happens at least with a positive probability; see Section 5.2, especially Figure 5. Trivially, we can also achieve this by the identity map f, $\epsilon = 0$, and with a periodic initial configuration.

The diffusion caused by the local averaging acts, in the frequency domain, as a low pass filter on the signal after the possibly chaotic map f has been applied to each lattice point. This filtering operation mitigates the extent to which higher partial tones are present in the sound produced by the CML, and thus helps to reinforce the fundamental frequency proportional to the inverse of n. The strong and predictable relationship between the lattice size and the frequency of the sound enables CMLs to be used as a natural method of digital synthesis.

For example, a tone generated by a CML with temporal period 2 and lattice size 50 at 44.1 kHz will have a fundamental frequency of 441 Hz, which is about 4 cents sharper than the middle A tuned at 440 Hz. Notice that with the standard 44.1 kHz sample rate, we cannot get closer to A440. The relationship between n and frequency allows us to easily control the pitch generated by a CML, but it also causes the precision of our tuning to be limited by the sample rate.

3. A Simple example: the linear case

Consider a linear map $f : [-1, 1] \to [-1, 1]$, defined by $f(x) = ax$ for all $x \in [-1, 1]$. Here, the coefficient a is in $[-1, 1]$ (values of a outside of this range can be used, but then we have to use a fractional part operation to map back into the interval $[-1, 1]$). If we start from a random initialization and take $\epsilon \in (0, 1]$, this gives us a CML which is, effectively, an implementation of the Karplus-Strong algorithm [KS83] for string synthesis. Indeed, the transition rule of our CML takes a weighted average of adjacent samples in the sound stream generated by equation (1), and so acts as a low-pass filter on the signal.

This linear case is particularly attractive for the following reason: when it starts from randomness, it emulates a plucked string of length proportional to the length of lattice. In fact, this case has appeared in the literature [Cha90] in the guise of a linear cellular automaton with a large number of variable values.

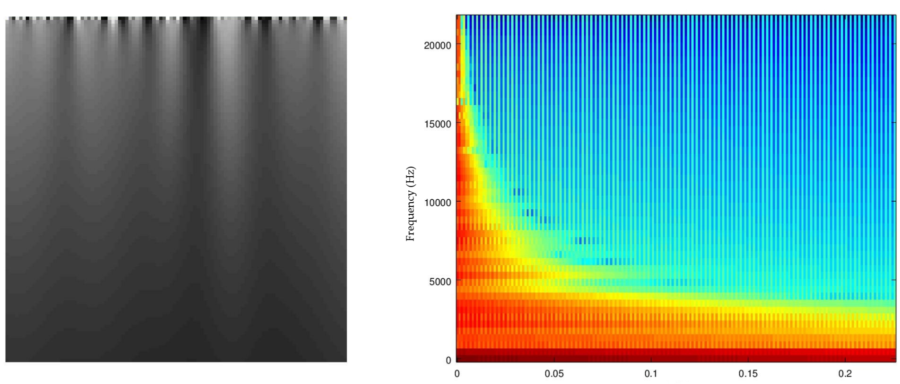

Figure 1. The evolution of the linear CML with lattice size $n = 100$, $a = 0.99$, and $\epsilon = 0.5$, from a random initial configuration, in the time-domain (left, with time on the vertical axis, oriented downwards, and space on the horizontal axis) and frequency-domain (right).

A few examples generated using $a = 0.99$ and $\epsilon = 0.5$ can be heard online [2] and a visual illustration is in Figure 1.

In the left frames of Figures 1–3, and in Figures 5–7, we exhibit space-time pictures of the corresponding CML evolutions. As is standard in the CML literature, the entire spatial configurations $x_t(0), \ldots, x_t(n-1)$ are depicted horizontally, with the configuration at time $t+1$ below the one at time t. Thus, the initial state (at time $t = 0$) is at the top and time runs downwards. The variable at each lattice point is linearly mapped into a shade of gray, so that the white color corresponds to the right endpoint of the state space and the black color to the left endpoint; for example, in Figures 1–3 the variables are in $[-1, 1]$, so the lighter and darker shades depict values close to 1 and -1, respectively.

Also, in Figures 1–3 the right frames are spectrograms of the sound generated by the same CMLs, with red corresponding to larger amplitudes and blue to smaller, using the same time period as the space-time pictures. In these frames, time runs horizontally, as is standard for spectrograms.

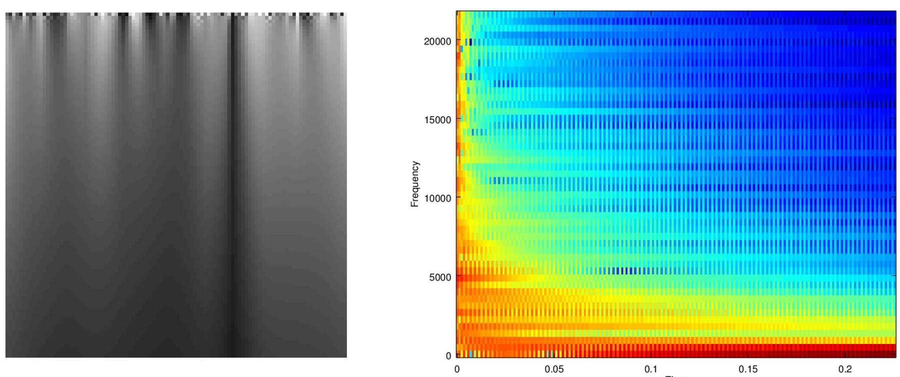

FIGURE 2. The evolution of the disordered CML (3) with lattice size $n = 100$, $a = 0.99$, and ϵ_i chosen independently and uniformly on $[0, 1]$, from a random initial configuration, in the time-domain (left) and frequency-domain (right).

We also give a preliminary example of the disordered version of the rule (2), which is given by

$$x_{t+1}(i) = (1 - \epsilon_i) f(x_t(i)) + \frac{\epsilon_i}{2}[f(x_t(i+1)) + f(x_t(i-1))], \tag{3}$$

where ϵ_i are random, in our case chosen independently and uniformly on $[0, 1]$. This produces a sound with more pronounced upper partials which correspond to enhanced horizontal lines in the spectrogram in Figure 2.

[2] https://soundcloud.com/user-138099222/sets/linear-case

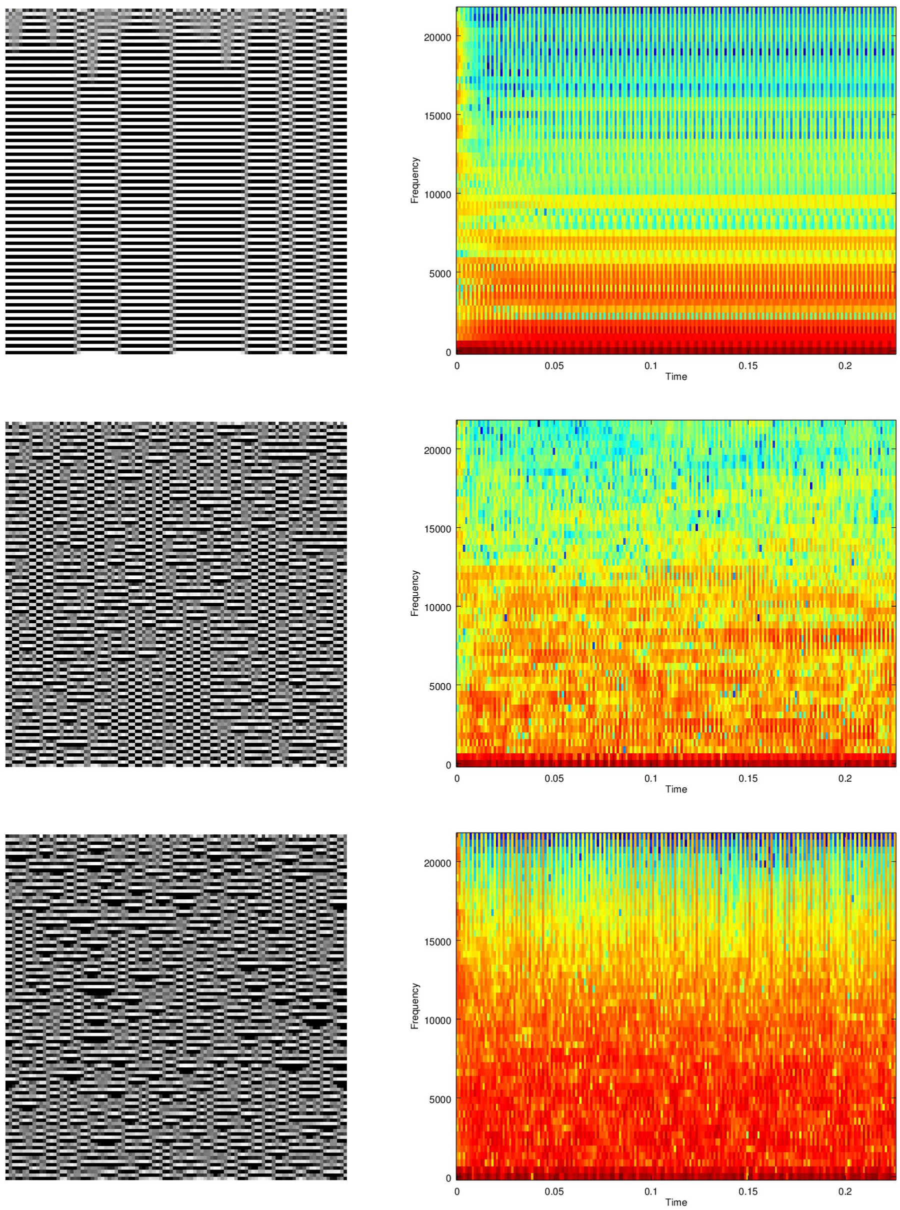

Figure 3. Three examples of the logistic map, each with 100 iterations and lattice size $n = 100$. Time domain plots are given on the left with spectrograms of the signal generated by the same CML on the right. From top to bottom, parameter values used are: $a = 1.1, \epsilon = 0.3$; $a = 1.75, \epsilon = 0.3$; and $a = 2, \epsilon = 0.5$. Audio examples generated using first, second, and third pairs of parameter values listed above can be found online.

4. The logistic map

We use the following version of the logistic map

$$f(x) = 1 - ax^2, \tag{4}$$

We will assume that the nonlinearity parameter a is in the interval $[1, 2]$, so that f maps the interval $[-1, 1]$ into itself. The choice (4) was made by Kaneko in his pioneering work on CMLs [Kan93]. Other versions of the logistic map are equally natural and produce similar results (e.g., $f(x) = ax(1 - x)$).

Values of a near 1 result in a sound that rapidly converges to a steady tone. As a increases, the CML behaves more chaotically, resulting in sounds that become closer to noise than to a pitched tone. See Figure 3 for time domain and spectrogram plots of logistic map CMLs with a few different parameter values. Audio examples can be found online.[3]

We remark that when $a = 2$ and $\epsilon = 0$, each lattice point behaves chaotically and independently of others, resulting in a close approximation to white noise. This is a simple instance in which the lattice size has no effect on the sound produced by the CML.

5. The circle map

In this case, the variables have values in the interval $[0, 1]$ and

$$f(x) = x + \omega - \frac{K}{2\pi}\sin(2\pi x) \mod 1. \tag{5}$$

Here, ω represents a phase shift and K the strength of nonlinearity. Taking the fractional part mod 1 is needed because without it f does not necessarily map $[0, 1]$ into $[0, 1]$. In the following subsections, we examine the behavior in the case of four different parameter settings with random initial configurations, and explore special initializations.

Kuramoto model is an ordinary differential equation that has been widely studied as a prototype for synchronization, whereby the evolution leads to nearly equal variable values across space in spite of the possibly chaotic temporal evolution (see [ABPV$^+$05] for a comprehensive survey and Section 5.1 and 5.4 for examples). With the circle map f as in (5), the CML could be interpreted as a version of the Kuramoto model with discrete time and local coupling. Thus it is no surprise that we observe synchronization in several of our examples (see remarks after (6) and 9), although conditions on the parameter values that result in this scenario are rather unclear.

5.1. **Evolution of CML with the circle map: metastable strange attractor.** First, we consider the circle map with parameters

$$\omega = 5/9, K = 1, \epsilon = 0.3. \tag{6}$$

Due to the complexity and longevity of its evolution to the steady state, we need to develop some statistical tools to understand this CML. Figure 4 depicts the behavior of three statistics. The simplest is the unscaled *variance*

$$\delta_t = \sum_{i=0}^{n-1} x_t(i)^2 - \frac{1}{n}\left(\sum_{i=0}^{n-1} x_t(i)\right)^2,$$

[3] https://soundcloud.com/user-138099222/sets/logistic-map

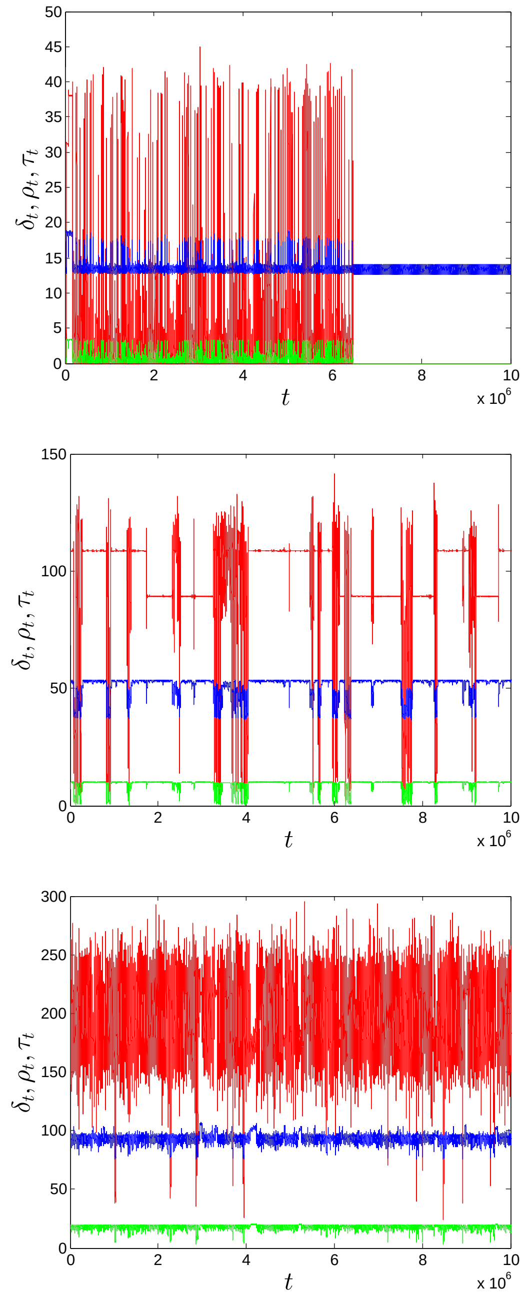

FIGURE 4. Evolution of the CML given by (6) with (top to bottom) $n = 70$, $n = 200$, and $n = 400$. In each frame, the bottom (green) curve depicts δ_t, the middle (blue) ρ_t and the top (red) τ_t. See text for full explanation.

which simply measures how far x_t is from a constant. A more involved tool is the *cross-correlation* [SM05], defined for $t = 1, 2, \ldots$ by

$$\texttt{cross\_corr}_t(k) = \sum_{i=0}^{n-1} x_{t-1}(i-k)\, x_t(i).$$

(When negative, $i-k$ is interpreted as $i-k+n$ due to the periodic boundary conditions.) This quantity compares a configuration with its spatial shift by k, and leads to the other two statistics:

$$\rho_t = \max_k \texttt{cross\_corr}_t(k),$$
$$\tau_t = \min\{k : \texttt{cross\_corr}_t(k) = \rho_t\}.$$

The quantity ρ_t measures the extent to which x_t is a translation of x_{t-1}, while τ_t is the optimal translation, that is, one with largest cross-correlation. See Figure 4 for the resulting evolution for three lattices sizes, up to time 10^7. By this time, the dynamics with $n = 70$ reached a stable synchronized configuration, with δ_t very close to 0. Therefore, at any fixed time, all the variables are nearly equal. Their (approximately) common value, however, changes chaotically over time due to nonlinearity of f. The resulting sound evolves into chaotic taps due to the discontinuities between iterates. A tone with frequency proportional to $1/n$ is also heard (see the discussion in Section 2). We conjecture that this is the fate of CML with parameters (6) for every lattice size. However, the waiting time for $n \geq 100$ is so large that synchronicity is in practice never reached. Instead, for n between about 100 and 300 we see the dynamics largely oscillating between two metastable *polarity* configurations, characterized by waves traveling in two opposite directions, with brief periods of polarity reversal between them. This is most clearly seen in the middle frame of Figure 4, where polarity periods are distinguished by flat lines in the translation statistic τ_t. The evolution is thus dominated by a kind of strange attractor that creates an evolving sound reminiscent of drone music (particularly of popular German electronic music of the 1970's, e.g., [Tan72]), punctuated by rhythmic cracking during reversals. For larger n, the temporal evolution appears chaotic in Figure 4, as the polarity is only rarely reached; however, the spatial organization is similar to that of the polarity configurations and thus the droning sound persists with rare or no reversals.

Although the above properties make these parameter settings inappropriate for use in a melodic or harmonic context, they result in a rich evolving timbre that is well-suited for avant-garde compositions. Example sounds can be found online.[4]

5.2. **Evolution of CML with the circle map: annihilating diffusions.** We now consider the case with parameters

$$\omega = 0.5, K = 1, \epsilon = 1. \tag{7}$$

These generate two competing equilibria, one with horizontal stripes (with temporal period 2 and spatial period 1) and one with vertical stripes (with temporal period 1 and spatial period 2) with diffusing boundaries between them. See Figure 5 for an illustration. There are also stationary defects (generating pronounced vertical lines) that interact with the boundaries in an unpredictable fashion. The invariant state is therefore not unique: one or the other equilibrium is reached, each with probability 1/2. When the lattice size is small, it is very likely that the stationary defects disappear and so, with probability close to 1/2, the

[4]https://soundcloud.com/user-138099222/sets/metastable-strange-attracter

generated sound is a square wave with frequency equal to half the sample rate; this sound is very high-pitched and independent of the lattice size. Audio examples can be found online.[5]

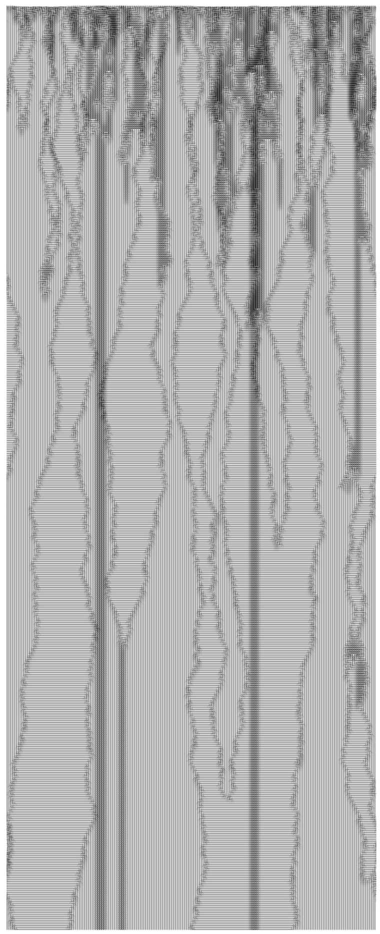

FIGURE 5. First 1000 time steps of the evolution of the CML given by (7) with $n = 400$.

5.3. **Evolution of CML with the circle map: repelling particles.** Next is the case with parameters

$$\omega = 0.5, K = 0.1, \epsilon = 0.6. \tag{8}$$

After a considerable time, the evolution self-organizes into a system of repelling particles on a slowly evolving background. Figure 6 provides an illustration.

As a result, the tone produced by the CML with these parameters will have steadily changing timbre. Because the repelling particles travel slowly, a high degree of structural similarity is preserved between consecutive lattice variables and so the tone evolves smoothly. After an initial period, during which the particles are created and destroyed, their number

[5]https://soundcloud.com/user-138099222/sets/annihilating-diffusions

stabilizes, which ensures an upper bound on the difference between two timbres at different times. This can be heard in examples online.[6]

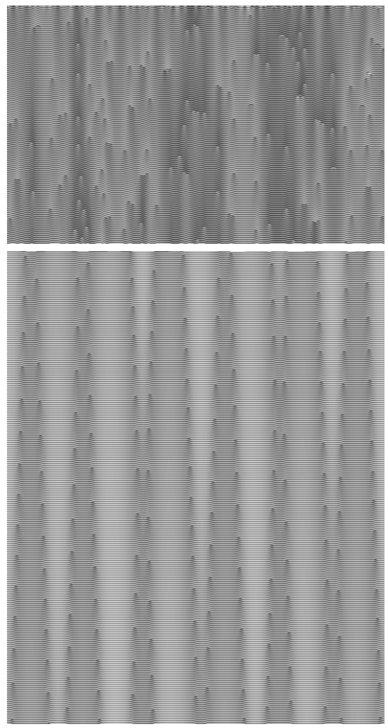

FIGURE 6. First 250 time steps, followed by 500 time steps starting later around time step 40,000, of the evolution of the CML given by (8), with $n = 400$.

5.4. **Evolution of CML with the circle map: near-periodicity.** Finally, we consider the case with parameters

$$\omega = 0.2, K = 0.2, \epsilon = 0.12. \tag{9}$$

This selection leads, after a few thousand iterates, to a stable nearly periodic equilibrium with spatial period about 80; see Figure 7. We remark that ϵ requires some fine-tuning here: a significantly different ϵ leads to a chaotic state (if larger), or to synchronization (if smaller). However, unlike in the case (6), the approach to synchronicity is gradual. The sound produced by these parameters has a timbre that varies somewhat more than the repelling particle case (8), but considerably less than the metastable strange attractor case (6). In addition to being nearly periodic spatially, these dynamics are also close to being temporally periodic, with a temporal period of five. As a result, the fundamental frequency F of a tone produced at a sample rate s with a lattice of size n is given by $F = s/(5n)$. The fifth upper partial (frequency s/n) is particularly strong, and can be heard distinctly above

[6]https://soundcloud.com/user-138099222/sets/repelling-particles-2

the fundamental. Some examples are online.[7]

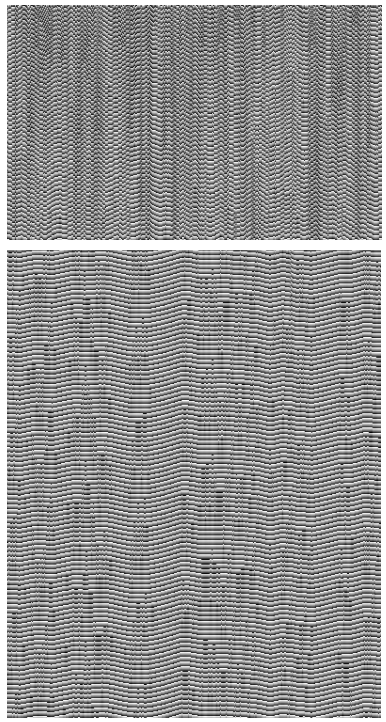

FIGURE 7. First 250 time steps, followed by 500 time steps starting later around time step 40,000, of the evolution of the CML given by (9), with $n = 400$.

5.5. **Sine wave initialization.** Interesting results can be obtained by setting the initial configuration as follows:

$$x_i(0) = \frac{1}{2}(\sin(2\pi mi/n) + 1), \quad i = 0, \dots, n-1,$$

where m is a positive integer parameter. Note that with identity map f and $\epsilon = 0$, this initial configuration would result in a sine wave with frequency sm/n. By contrast with a random initial configuration, which approximates a range of frequencies at equal strength, this initialization uses a single frequency.

Using the circle map (5), and a sufficiently large lattice size (around $n = 10,000$), this setting creates a rhythmically pulsing tone. With a given sample rate s, and m and n, the tone has frequency $f = sm/n$. The rhythm is determined by the dynamic parameters ω, K, and ϵ. As a rule of thumb, if $\omega = p/q$ is a rational number represented by a reduced fraction, then the rhythm can be interpreted as having q beats per measure, with tempo s/n. Due to the rounding errors, the pulsing is slowly overcome by noise. This "descent into chaos" is particularly striking when $m = 100$ and f is the circle map with parameters (6). Again, several examples are online[8], with parameters given in Table 1.

[7]https://soundcloud.com/user-138099222/sets/nearly-periodic

[8]https://soundcloud.com/user-138099222/sets/sine-wave-initialization

	n	m	ω	K	ϵ
Example 1	10,000	10	5/9	1	0.3
Example 2	10,000	10	3/7	1	0.3
Example 3	200	1	5/9	1	0.3
Example 4	10,000	100	5/9	1	0.3

TABLE 1. Parameter values for sine wave initializations.

5.6. **Zero initialization.** If the system is synchronized initially, as when $x_i(0) = 0$ for all i, then it remains synchronized forever. Therefore, the system reduces to the iteration of the map f, and the coupling strength ϵ has no influence. This allows us to provide a simple example of a sound generated by CML that has spatial simplicity but still produces noise. This is illustrated by the circle map with parameters $\omega = 5/9$ and $K = 2.5$. The generated sounds make it clear that the temporal chaos "drowns out" the spatial order. This is especially pronounced with smaller lattice sizes, as shown by audio examples online.[9]

6. Conclusions and future directions

We explore the use of complex nonlinear dynamics as a sound generator. Specifically, we use one-dimensional coupled map lattices, which combine iteration of a nonlinear map with linear local averaging. We present a method to synthesize sound from the resulting dynamics, by using variables at all spatial locations, in the lattice order (and also in the temporal order). In a typical case, rapid convergence to an equilibrium produces a steady tone. The spatial organization of the steady state determines a characteristic timbre of this tone, with subtle variations due to different initial conditions. Moreover, the size (i.e., the length) of the network determines the pitch. From this perspective, such a mathematical object is an abstract form of a musical instrument.

We have limited our exploration to several case studies with a linear, logistic, or circle map. The linear case emulates the sound of a plucked string, with detectable alteration when the coupling strengths are chosen at random. The behavior induced by our first nonlinear function, the logistic map, is as expected: as the nonlinearity increases, the resulting sound interpolates between a pitched tone and noise.

By far, the most versatile is the circle map, for which we provide four particularly interesting choices for its three parameters. We give two examples with embedded particle systems, one with annihilating diffusions with two competing equilibria, and another with repelling particles that produces a steadily changing timbre. Another circle map case exhibits nearly periodic behavior with a resulting combination of frequencies. Our leading circle map example converges to its equilibrium very slowly, providing an opportunity to hear a sonic representation of a metastable strange attractor, which turns out to resemble drone music. The same parameter choice also produces, when initialized by a sine wave, a melody that is gradually overcome by noise.

Our results open several avenues for further exploration. Clearly, one could look for new features using different functions f from those that make the appearance here. Perhaps a greater priority is to fully understand the potential of the circle map with its three parameters ϵ, ω, and K. Indeed, our most interesting examples were obtained by little more than educated guessing. An "atlas" of parameter space with a catalog of behaviors would be expedient if one wanted to contemplate possible applications to, say, sound effects in computer

[9]https://soundcloud.com/user-138099222/sets/zero-initialization

games. Particularly the role of the coupling strength ϵ is mysterious and counter-intuitive, and deserves additional investigation. Furthermore, we only touched on the effects of *spatial* heterogeneity: either ϵ or a parameter in the function f could vary from location to location (e.g., by a random choice). *Temporal* heterogeneity is another possibility: the update rule may change abruptly or gradually — for example, to create a tone decay. Finally, adding random noise to the update rule may test the stability of some of the observed behaviors.

7. Appendix: parameters in examples of melodies

We provide three examples of melodies. To play "In the Hall of the Mountain King," we use the circle map with $\omega = 0.5$ and $K = 0.1$ throughout the excerpt with three different values of ϵ, starting with $\epsilon = 0.6$, switching a third of the way through to $\epsilon = 0.65$, and in the final third to $\epsilon = 0.7$. "Opening to Beethoven's 5th Symphony" is played using the logistic map with different parameter values for each note, hence there are too many values to list. Finally, the sequence of notes for the "Generated Melody" was produced by a computer algorithm unrelated to the topic of this article, and is played using the circle map with parameters $\omega = 0.13$, $K = 0.9$, and $\epsilon = 0.3$.

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References

[ABPV+05] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort, and R. Spigler. The Kuramoto model: A simple paradigm for synchronization phenomena. *Reviews of Modern Physics*, 77:137–185, 2005.

[Bat04] B. Battey. Musical pattern generation with variable-coupled iterated map networks. *Organised Sound*, 2004:137–150, 2004.

[BE05] D. Burraston and E. Edmonds. Cellular automata in generative electronic music and sonic art: a historical and technical review. *Digital Creativity*, 16:165–185, 2005.

[Cha90] J. P. Chareyron. Digital synthesis of self-modifying waveforms by means of linear automata. *Computer Music Journal*, 14:25–41, 1990.

[Cru84] J. Crutchfield. Space-time dynamics in video feedback. *Physica D*, 10:229–245, 1984.

[Dzj15] M. Dzjaparidze. *Exploring the creative potential of physically inspired sound synthesis*. PhD thesis, Queen's University Belfast, 2015.

[HKZL+16] J. Hizanidis, N. E. Kouvaris, G. Zamora-López, A. Díaz-Guilera, and C. G. Antonopoulos. Chimera-like states in modular neural networks. *Scientific Reports*, 6:19845, 2016.

[Kan84] K. Kaneko. Period-doubling of kink-antikink patterns, quasiperiodicity in antiferro-like structures and spatial intermittency in coupled logistic lattice: towards a prelude of a "field theory of chaos". *Progress of Theoretical Physics*, 72:480–486, 1984.

[Kan93] K. Kaneko. The coupled map lattice. In K. Kaneko, editor, *Theory and Applications of Coupled Map Lattices*, pages 1–49. Wiley, New York, 1993.

[KS83] K. Karplus and A. Strong. Digital synthesis of plucked string and drum timbres. *Computer Music Journal*, 7:43–55, 1983.

[KW84] R. Kapral and I. Weller. Spatial and temporal structure in systems of coupled nonlinear oscillators. *Physical Review A*, 30:2047–2055, 1984.

[Mir95] E. R. Miranda. Granular synthesis of sounds by means of a cellular automaton. *Leonardo*, 28:297–300, 1995.

[Mir07] E. R. Miranda. Cellular automata music: from sound synthesis to musical form. In E. R. Miranda and J. A. Biles, editors, *Evolutionary Computer Music*, pages 170–193. Springer-Verlag, London, 2007.

[SAB15] N. Stefanakis, M. Abel, and A. Bergner. Sound synthesis based on ordinary differential equations. *Computer Music Journal*, 39:25–43, 2015.

[SM05] P. Stoica and R. Moses. *Spectral Analysis of Signals*. Prentice Hall, New Jersey, 2005.

[SM11] J. Serquera and E. R. Miranda. Cellular automata dynamic control for sound design with histogram mapping synthesis and the multitype voter model. In *Proceedings of the International Computer Music Conference*, pages 139–142, San Francisco, 2011. International Computer Music Association.

[SM14] J. Serquera and E. R. Miranda. Histogram mapping synthesis: A cellular automata-based technique for flexible sound design. *Computer Music Journal*, 38:38–52, 2014.

[SN13] D. L. Stein and C. M. Newman. *Spin Glasses and Complexity*. Princeton University Press, Princeton, 2013.

[Tan72] Tangerine Dream. *Zeit*. Ohr, 1972. LP.

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