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A HEURISTIC FOR THE p-CENTER PROBLEM IN GRAPHS

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Generalizing a result of Hochbaum and Shmoys, a polynomial algorithm with a worst-case error ratio of 2 is described for the p-center problem in connected graphs with edge lengths and vertex weights. A slight modification of this algorithm provides ratio 2 also for the absolute p-center problem. Both these heuristics are best possible in the sense that any smaller ratio would imply that $P = NP$.

1. Introduction

Given a connected graph G, the vertex and edge sets and their cardinalities are denoted by $V(G)$, $E(G)$, n and m, respectively. It is assumed that every vertex v is assigned a non-negative real number $w(v)$, called the weight of v, and every edge uv is assigned a positive real number $a(uv)$, the length of uv. The lengths determine the distance $d(u,v)$ between any two vertices u and v as the minimal sum of the edge lengths of a $u-v$ path. The distance between a vertex $v \in V(G)$ and a set $X \subset V(G)$ is $d(v,X) := \min\{d(v,x) \mid x \in X\}$. A p-set is a set of cardinality p. Given G and p, the p-center problem is to find a p-set $X \subset V(G)$ such that the objective function, weighted eccentricity,

$$\eta(X) := \max_{v \in V(G)} \{w(v)\, d(v,X)\}$$

is minimized. The optimal value of $\eta(X)$ is often called the p-radius of G. A p-center is any optimal p-set X. If also any point of a network (either on an edge or at a vertex) is allowed to be an element of X, the corresponding problem is referred to as the absolute p-center problem. (Distances are defined as expected.) Clearly, any edge uv with $d(u,v) < a(uv)$ can be deleted without affecting the optimal eccentricity. Therefore we will assume that $d(u,v) = a(uv)$ for every edge uv. Note that we do not assume any other relations between edge lengths. However, some authors define the p-center problem only for complete graphs. In this case they can put $d(u,v) := a(uv)$ whenever the triangle inequality for the lengths is assumed. Both definitions are clearly equivalent as to the p-center problem but not for the absolute p-center one. Therefore we prefer incomplete graphs. Moreover, sometimes a special structure of G can be exploited (see e.g. [17] for trees).

These problems were first formulated by Hakimi [5,6] in 1964. They are motivated by real problems of locating p emergency facilities (e.g. hospitals or fireman stations) of the same kind along a road network (either at a city or on a road); the importance (e.g. the population) of a city is expressed by the corresponding vertex weight.

The literature on these problems is now very vast. For surveys and further variants see e.g. [1, 2, 11, 14, 16, 17, 18]. Both the problems are NP-hard even in very special cases [4, 9, 11, 12, 13, 15]. Recently Hochbaum and Shmoys [7] considered the special case of the p-center problem where all the vertex weights are equal to 1 (the unweighted problem). They have described an $O(n^2 \log n)$ heuristic which has the worst-case error ratio not exceeding 2, i.e., the heuristic provides a p-set X with $\eta(X)/\eta^* \le 2$, where $\eta^* := \eta(S)$ for any p-center S. Then in [8], they have given polynomial approximation algorithms for a wide variety of NP-hard bottleneck problems in routing, location and network design. A bound of value 2 is, in a sense, a best possible, because the problem to find a p-set X with $\eta(X)/\eta^* \le \varrho$ where $\varrho < 2$, is an NP-hard problem, as proved in [9] and independently in [15]. More recently, Dyer and Frieze [3] have described a simple $O(np)$ heuristic for the p-center problem with ratio $\min\{3, 1+\alpha\}$, where α is the maximum ratio between the weights of vertices.

The aim of this paper is to give a polynomial heuristic which ensures ratio 2 also for the p-center problem in general. Moreover, our approach is very simple and natural. As a consequence, it gives ratio 4 for the absolute p-center problem. However, a slight modification of our heuristic provides ratio 2 also in this case. And again, because of NP-hardness results, there does not exist any polynomial algorithm that has a better performance guarantee (unless P = NP).

1. The p-center problem

Let $n \ge 2$ and $1 \le p \le n$. We assume that the distance matrix (which gives the distance $d(u, v)$ between every pair of vertices u and v) is available. Note that it takes $O(n^3)$ operations to compute the distance matrix in a general graph, but for sparse graphs there are more efficient (at least theoretically) algorithms [10]. Our heuristic is based on the following result.

Theorem 1. *For any real number $r > 0$, if there exists a p-set $X \subset V(G)$ with $\eta(X) \le r$, then the following procedure finds a set $S \subset V(G)$ with $|S| \le p$ and $\eta(S) \le 2r$.*

Procedure RANGE

Step 0. At first all vertices of G are unlabelled; $S := \emptyset$.

Step 1. If all vertices are labelled, then go to Step 2. Else choose an unlabelled vertex u of the maximum weight and put $S := S \cup \{u\}$; label the vertex u and every unlabelled vertex v such that $w(v)d(u, v) \le 2r$; go to Step 1.

Step 2. Output S.

Proof. Let X consist of vertices $v_1, \dots, v_p$ and let the 'regions' corresponding to these vertices be $S_1, \dots, S_p$, respectively (i.e., $S_1 \cup \cdots \cup S_p = V(G)$ and for every $i = 1, \dots, p$, $w(v)d(v_i, v) \le r$ whenever $v \in S_i$). By the algorithm, we have $w(v)d(S, v) \le 2r$ for any $v \in V(G)$ and hence $\eta(S) \le 2r$. To prove that $|S| \le p$ we will show that at most one vertex of each S_i belongs to S. Consider an iteration of Step 1. Let u be the chosen vertex and let S_i be the set containing u. Then for every other unlabelled vertex v of S_i we have $w(v) \le w(u)$ and by the triangle inequality we get

$$\begin{aligned} w(v)\, d(u, v) &\le w(v)[d(u, v_i) + d(v_i, v)] \\ &\le w(u)\, d(v_i, u) + w(v)\, d(v_i, v) \le 2r. \end{aligned}$$

Therefore one must label all the unlabelled vertices of S_i, i.e., no further vertex of S_i will be added to S. □

One can see that our procedure RANGE is a dual approximation algorithm in the sense of Hochbaum and Shmoys [7,8] and hence all their results apply. In fact, the following heuristic for the p-center problem goes in line with [7]. It is based on the simple observation that the optimal weighted eccentricity (p-radius) is one of the weighted distances.

Heuristic CENTER

Step 1. Arrange the $n(n-1)$-multiset of weighted distances $d(u, w)\, w(v)$ with $u, v \in V(G)$ into a non-decreasing sequence and deleting duplicates reduce it to an increasing sequence

$$f_1 < f_2 < \cdots < f_q. \tag{1}$$

Step 2. Find r^*, the least value of $r \in \{f_1, f_2, \dots, f_q\}$ for which RANGE yields an output S with $|S| \le p$.

Step 3. Augment S arbitrarily to a set S' of p vertices. Output S' and stop.

Step 1 can be performed by a sorting procedure in time $O(n^2 \log n)$. Using a binary search running on indices $1, 2, \dots, q$ (see [7]), it is sufficient to apply RANGE only $O(\log n)$ times to find r^*. It is convenient to arrange the n vertices according to their weights (in time $O(n \log n)$). Then each iteration of Step 1 in RANGE can be done in time $O(n)$ and thus the total complexity of RANGE is $O(n^2)$. Therefore the complexity of Step 2 in CENTER is $O(n^2 \log n)$. As Step 3 is trivial, the complexity of CENTER is $O(n^2 \log n)$.

Since $\eta(S') \le \eta(S) \le 2r^*$, we see (by Theorem 1) that CENTER is a 2-approximation algorithm. As the weighted problem includes the unweighted one, the NP-hardness result from [9,15], mentioned in Introduction, works here too. Consequently, CENTER is a best possible polynomial heuristic as to the worst-case ratio.

Note that the p-radius is bounded from below by r^* (and also by certain intermediate results from binary search), which can be used in a branch-and-bound heuristic, as observed in [7].

The following example shows that in CENTER ratio 2 can be attained. Let $V(G) := \{1,2,3\}$, $E(G) := \{12,23\}$, $p := 1$ and all lengths and weights be equal to 1. Then $\eta^* = 1$ but CENTER can give $S' = \{1\}$ with $\eta(S') = 2$.

2. Absolute p-centers

In this section we deal with finding approximate solutions of the absolute p-center problem (in this problem the points-facilities can be located also on edges which are considered as curves of the corresponding lengths). We will present a slight modification of CENTER.

It is well known (see e.g. [11]) that without loss of generality we can restrict the points of an absolute p-center to be chosen out of a set of no more than $O(m\,n^2)$ points. Hence, there are at most $O(m\,n^2)$ possible values r for the weighted eccentricity of an absolute p-center.

We will use the following assertion which has essentially the same proof as Theorem 1 (now $v_1, \ldots, v_p$ are not necessarily vertices but generally points).

Theorem 2. *For any real number $r > 0$, if there exists a p-set X of points of G with $\eta(X) \le r$, then the procedure* RANGE *finds a set $S \subset V(G)$ with $|S| \le p$ and $\eta(S) \le 2r$.*

Our heuristic for the absolute p-center problem is called ABCENTER and proceeds as follows. ABCENTER first finds all possible values r in time $O(m\,n^2)$ see [11] for such an algorithm). Then it arranges these values into an increasing sequence (1) which takes $O(m\,n^2 \log n)$ operations. This is Step 1 and the rest of ABCENTER consists of Steps 2 and 3 which are the same as in CENTER. Thus the overall complexity of ABCENTER is $O(m\,n^2 \log n)$.

By Theorem 2, ABCENTER is a 2-approximation algorithm. Since the ϱ-approximation absolute p-center problem is NP-hard whenever $\varrho < 2$ [15], ABCENTER is a best possible polynomial heuristic as to the worst-case ratio (unless P = NP).

As a corollary of Theorem 2, we have the following result which can be of interest in its own right. However, the result can be proved also directly. Moreover, one of the referees believes that people in the field know it.

Theorem 3. *For any absolute p-center A and any p-center C of G,*

$$\eta(A) \le \eta(C) \le 2\eta(A).$$

Moreover, multiplier 2 is the least possible.

Proof. The left inequality is trivial. To prove the right inequality apply Theorem 2 with $r := \eta(A)$, $X := A$ and observe that $\eta(C) \leq \eta(S)$. To show that multiplier 2 cannot be decreased, consider a graph G consisting of a single edge of length 2 with unit vertex weights. Then for $p=1$ we have $\eta(A)=1$ and $\eta(C)=2$. □

Note that by this result CENTER is a 4-approximation $O(n^2 \log n)$ algorithm for the absolute p-center problem.

Lastly, we raise the following question: Is there a polynomial ϱ-approximation algorithm for the unweighted absolute p-center problem for some $\varrho < 2$? Note that the ϱ-approximation absolute unweighted p-center problem is known to be NP-hard only for $\varrho < 3/2$ [9,15].

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