Crystals For Dummies

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Abstract

We explain how beautiful combinatorial constructions involving the Robinson-Schensted-Knuth correspondence, evacuation of tableaux, and the Kostka-Foulkes polynomials, arise naturally from the structure of (affine) crystal graphs. The appearance of Kostka-Foulkes polynomials was observed by Nakayashiki and Yamada. Almost all of the constructions presented herein, have analogues for every simple and affine Lie algebra.

Contents

1 Introduction

Fix an integer $n > 1$. Here is a little fine print for experts.

Assumption 1.1. In this primer a "crystal graph" is the crystal base of a finite-dimensional integrable $U_q(sl_n)$-module. An "affine crystal graph" is the affine crystal base of a finite-dimensional integrable $U_q'(\widehat{sl}_n)$-module.

Beyond this, crystal graphs will not be defined precisely nor will they even be characterized combinatorially.[1] Instead we describe some of their properties and give examples. Every once in a while we will invoke the "magic" of representation theory of quantum affine algebras without explanation. Some knowledge is assumed of partitions, tableaux, and the Robinson-Schensted-Knuth correspondence; this can be obtained from [3].

My attribution will be sketchy. Kashiwara invented crystal graphs; see [6] for a real introduction to the subject. The analogue of our "affine crystal graphs", has its origins in the work of Kang, Kashiwara, Misra, Miwa, Nakashima, and Nakayashiki [7]. The most important works for setting up the correct framework for the subject (tensor products of Kirillov-Reshetikhin modules) are [5] [4].

The more familiar crystal graphs of integrable highest weight modules (analogous to our "crystal graphs") can be constructed in Kac-Moody generality using Littelmann paths [14]. For classical simple Lie algebras Cédric Lecouvey has systematically applied the theory of crystal graphs to define a Schensted insertion [12] [13] using the tableaux of Kashiwara and Nakashima [9].

Most of the constructions mentioned here in the context of crystal graphs, were anticipated by Lascoux and Schützenberger (see, for example, [15]), many of whose works on tableau combinatorics preceded the theory of quantum groups and crystal graphs.

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2 Crystal graphs

Example 2.1. Here's a crystal graph for $n = 3$. Its vertex set consists of elements $\boxed{\text{ij}}$ for $i, j \in \{1, 2, 3\}$. Each of its directed edges is labeled by an element of the set $\{1, 2\}$.

$$\begin{tikzcd}
\boxed{11} \arrow[r, "1"] & \boxed{12} \arrow[r, "2"] \arrow[d, "1"'] & \boxed{13} \arrow[d, "1"] \\
\boxed{21} \arrow[d, "2"'] & \boxed{22} \arrow[r, "2"'] & \boxed{23} \arrow[d, "2"] \\
\boxed{31} \arrow[r, "1"'] & \boxed{32} & \boxed{33}
\end{tikzcd}$$

2.1 Basic features

Every crystal graph is a finite directed graph whose edges are labeled by colors in the set $I = \{1, 2, \ldots, n-1\}$. Traditionally a crystal graph is given the same name as its vertex set.

2.1.1 String property

Every crystal graph has the following property. For a fixed $i \in I$, if all of the edges are removed except those colored i, the resulting directed graph consists of a disjoint union of finite directed paths called ***i*-strings**. Given a vertex b, let $f_i(b)$ (resp. $e_i(b)$) be the vertex following (resp. preceding) b in its i-string. If this

[1]For a characterization of crystal graphs see [23].

vertex does not exist then the result is declared to be the special symbol $\emptyset$. Let $\varepsilon_i(b)$ (resp. $\varphi_i(b)$) be the number of steps to the beginning (resp. end) of the i-string of b. f_i and e_i are called the Kashiwara **lowering** and **raising** operators.

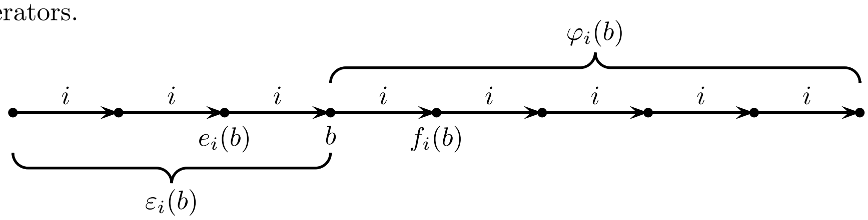


Figure 1: The i-string of b

Let s_i be the involution on a crystal graph that reverses each i-string. In other words, $s_i(b)$ is the element on the i-string of b such that $\varepsilon_i(s_i(b)) = \varphi_i(b)$ or equivalently $\varphi_i(s_i(b)) = \varepsilon_i(b)$. By definition s_i is an involution. It is not obvious, but it can be shown that the s_i define an action of the Weyl group (in our case the symmetric group S_n) on any crystal graph [6]. s_i is the Kashiwara **reflection** operator.

Example 2.2. The 2-strings of the crystal graph of the previous example are

$$\boxed{11}, \quad \boxed{12} \to \boxed{13}, \quad \boxed{22} \to \boxed{23} \to \boxed{33}, \quad \boxed{21} \to \boxed{31}, \quad \boxed{32}.$$

We have $f_2(\boxed{33}) = \emptyset$, $e_2(\boxed{33}) = \boxed{23}$, $\varepsilon_2(\boxed{33}) = 2$, and $\varphi_2(\boxed{33}) = 0$. s_2 fixes $\boxed{11}, \boxed{23}, \boxed{32}$ and exchanges $\boxed{12}$ with $\boxed{13}$, $\boxed{22}$ with $\boxed{33}$, and $\boxed{21}$ with $\boxed{31}$.

2.1.2 Weight function

Every crystal graph B has a **weight function** $\mathrm{wt} : B \to \mathbb{Z}^n$. For $i \in I$ let $\alpha_i = h_i \in \mathbb{Z}^n$ (the simple root and simple coroot) be the i-th standard basis vector minus the $(i+1)$-th. For all $i \in I$ and $b \in B$,

$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i \qquad \text{if } f_i(b) \neq \emptyset. \tag{2.1}$$
$$\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha_i \qquad \text{if } e_i(b) \neq \emptyset. \tag{2.2}$$
$$\mathrm{wt}(s_i(b)) = s_i \mathrm{wt}(b) \tag{2.3}$$
$$\langle h_i \, , \, \mathrm{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b) \tag{2.4}$$

where s_i acts on $\mathbb{Z}^n$ by exchanging the i-th and $(i+1)$-th components.

Example 2.3. In Example 2.1, the weight of $\boxed{\mathrm{ij}}$ is the sum of the i-th and j-th standard basis vectors in $\mathbb{Z}^3$. Let $b = \boxed{33}$. Then $\mathrm{wt}(b) = (0,0,2)$, $\mathrm{wt}(e_2(b)) = \mathrm{wt}(\boxed{23}) = (0,1,1)$, and $(0,1,1) - (0,0,2) = (0,1,-1) = \alpha_2$, verifying (2.2). Also $\langle h_2 \, , \, \mathrm{wt}(b) \rangle = \langle (0,1,-1) \, , \, (0,0,2) \rangle = -2 = \varphi_2(b) - \varepsilon_2(b)$, verifying (2.4).

Remark 2.4. The weight function on B is determined by the colored directed graph structure on B, modulo the "irrelevant" all-ones vector (1^n). We have

$$\langle h_i \, , \, \mathrm{wt}(b) \rangle = \beta_i - \beta_{i+1} \qquad \text{for } \mathrm{wt}(b) = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n. \tag{2.5}$$

The position of b in its i-string specifies the right hand side of (2.4) and hence fixes the values $\beta_i - \beta_{i+1}$ for all $i \in I$. This determines β modulo (1^n). We are using the gl_n-weight lattice $\mathbb{Z}^n$; if we used the (more correct) sl_n-weight lattice is $\mathbb{Z}^n / \mathbb{Z}(1^n)$, then the weight would be determined entirely by the crystal structure.

2.2 Examples

2.2.1 Trivial crystal

The **trivial** crystal graph $B(0)$ consists of a single vertex of weight (0^n) and no directed edges.

2.2.2 Single box

The **single box** crystal graph $B(1)$ is given by

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$$

The weight of $\boxed{i}$ is the i-th standard basis vector in $\mathbb{Z}^n$.

2.2.3 Words

The set $B(1)^k$ of words of length k in the alphabet $B(1)$, is a crystal graph[2]. The weight of a word $b \in B(1)^k$ is just its content, the vector $(\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ where β_i is the number of occurrences of the letter i in b. Let $i \in I$. To go from b to $f_i(b)$ (resp. $e_i(b)$) a single letter i is changed to $i+1$ (resp. $i+1$ to i); this is consistent with (2.1) (resp. (2.2)). To decide which letter to change, view each letter i as a closing parenthesis ")" and each letter $i+1$ as an opening parenthesis "(", with other letters being invisible. Match the parentheses in the usual way; adjacent pairs of parentheses "()" are matched and declared to be invisible until no more matching can be done. The subword of unmatched parentheses has the form $)^{\varphi}(^{\varepsilon}$, some number φ of closing parentheses followed by some number ε of opening parentheses. By definition $\varphi_i(b) = \varphi$ and $\varepsilon_i(b) = \varepsilon$ are the numbers of unmatched letters i and $i+1$ respectively, $f_i(b)$ is obtained by changing the rightmost unmatched i to $i+1$ and $e_i(b)$ is obtained by changing the leftmost unmatched $i+1$ to i. If there is no unmatched letter i then $f_i(b) = \emptyset$ and if there is no unmatched letter $i+1$ then $e_i(b) = \emptyset$.

To indicate the role of i we shall refer to ***i*-matched** and ***i*-unmatched** letters.

Example 2.5. Let $i = 2$. A word b, the associated parentheses with 2-unmatched parentheses underlined, and $f_2(b)$ and $e_2(b)$ are given below.

$$\begin{array}{rccccccccccccccccccccccc}
b & = & 1 & 2 & 4 & 3 & 3 & 1 & 2 & 3 & 2 & 1 & 2 & 4 & 2 & 2 & 3 & 1 & 2 & 3 & 4 & 3 \\
 & & & \underline{)} & & (& (& &) & (&) & &) & & \underline{)} & \underline{)} & (& &) & \underline{(} & & \underline{(} \\
f_2(b) & = & 1 & 2 & 4 & 3 & 3 & 1 & 2 & 3 & 2 & 1 & 2 & 4 & 2 & \mathbf{3} & 3 & 1 & 2 & 3 & 4 & 3 \\
e_2(b) & = & 1 & 2 & 4 & 3 & 3 & 1 & 2 & 3 & 2 & 1 & 2 & 4 & 2 & 2 & 3 & 1 & 2 & \mathbf{2} & 4 & 3
\end{array}$$

In this particular example, $s_2(b) = f_2(b)$ since the 2-unmatched subword is $)^3(^2$.

Example 2.6. Example 2.1 is $B(1)^2$ for $n = 3$.

The operators s_i on words appeared in [15], where they were called **automorphisms of conjugation**. Let u be a word. Define

$$u^{\curvearrowright} = vx \qquad \text{where } u = xv \text{ and } x \text{ is a letter.} \tag{2.6}$$

$u^{\curvearrowright}$ is pronounced "u-crank". It follows easily from the definitions that cranking and automorphisms of conjugation commute:

$$(s_i u)^{\curvearrowright} = s_i(u^{\curvearrowright}). \tag{2.7}$$

2.2.4 Tableaux

Assumption 2.7. All partitions we consider will have at most n parts.

Let λ, μ be partitions with $\mu \subset \lambda$ (that is, $\mu_i \le \lambda_i$ for $1 \le i \le n$) and let $B(\lambda/\mu)$ be the set of semistandard tableaux of the skew shape λ/μ [3].

The **row-reading** (resp. **column-reading**) **word** of a tableau $T \in B(\lambda/\mu)$ is obtained by reading the rows (resp. columns) of T; see the example.

[2]See Warning 2.18 regarding our conventions.

Example 2.8. Let $n = 4$, $\lambda = (5,5,3,0)$ and $\mu = (2,1,0,0)$. A tableau $T \in B(\lambda/\mu)$ and its row- and column-reading words are given below.

$$T = \begin{array}{ccccc} \cline{3-5} & & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \cline{2-5} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \cline{1-5} \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} & & \\ \cline{1-3} \end{array} \qquad \begin{aligned} \mathrm{rowword}(T) &= 233.1245.134 \\ \mathrm{colword}(T) &= 2.31.321.43.54 \end{aligned}$$

Taking the column-reading word of a tableau defines an embedding

$$\begin{aligned} B(\lambda/\mu) &\hookrightarrow B(1)^{|\lambda|-|\mu|} \\ T &\mapsto \mathrm{colword}(T) \end{aligned} \tag{2.8}$$

where $|\lambda| = \sum_i \lambda_i$.

Lemma 2.9. *The set of column-reading words of tableaux in $B(\lambda/\mu)$ is closed under e_i and f_i for all $i \in I$.*

$B(\lambda/\mu)$ is a crystal graph. Given a tableau $T \in B(\lambda/\mu)$, take its column-reading word, apply f_i to that word, and put the result back into the diagram of λ/μ to form the tableau $f_i(T)$, which is semistandard by the Lemma. $e_i(T)$ is defined similarly. The weight of a tableau is the weight of its column-reading word.

Example 2.10. Continuing the previous example, $e_2(T)$ is given below.

$$e_2(\mathrm{colword}(T)) = 2.21.321.43.54 \qquad\qquad e_2(T) = \begin{array}{ccccc} \cline{3-5} & & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \cline{2-5} & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \cline{1-5} \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & & \\ \cline{1-3} \end{array}$$

Example 2.11. For $n = 3$ and $\lambda = (2,1,0)$ the crystal graph $B(\lambda)$ is given below.

$$\begin{tikzcd}
& \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} \arrow[r, "2"] & \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \arrow[r, "2"] & \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} \arrow[dr, "1"] & \\
\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \arrow[ur, "1"] \arrow[dr, "2"'] & & & & \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array} \\
& \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} \arrow[r, "1"'] & \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \arrow[r, "1"'] & \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} \arrow[ur, "2"'] &
\end{tikzcd}$$

Assumption 2.12. From now on a tableau shall be identified with its column-reading word.

Remark 2.13. Using the row-reading word also defines a crystal graph structure on $B(\lambda/\mu)$. The resulting crystal graph structure is the same as that for the column-reading word. More generally [2] any reading order that is a linear extension of the "southwest-to-northeast" partial order, induces the same crystal graph structure on $B(\lambda/\mu)$.

2.2.5 Maximum column

Example 2.14. Consider $B(1^n)$. It has a unique element, the single column tableau $n \cdots 21$. It has weight (1^n) and admits no arrows. It only differs from the trivial crystal $B(0)$ by its weight function, which gives the value (1^n) on its lone element.

2.3 Some general constructions and notions

2.3.1 Connected components

A connected component of a crystal graph is also a crystal graph.

2.3.2 Disjoint union or direct sum

If B and B' are crystal graphs, then their disjoint union is also a crystal graph, traditionally denoted by $B \bigoplus B'$ and called the **direct sum**. Any crystal graph is the direct sum of its connected components.

2.3.3 Morphisms

For us, a **morphism** $\Psi : B \to B'$ of crystal graphs is a map that preserves colored directed edges and weights. More precisely, a morphism is a map $\Psi : B \to B'$ that satifies

$$\begin{aligned} \Psi(f_i(b)) &= f_i(\Psi(b)) \\ \Psi(e_i(b)) &= e_i(\Psi(b)) \\ \mathrm{wt}(\Psi(b)) &= \mathrm{wt}(b). \end{aligned} \tag{2.9}$$

where $\Psi(\emptyset) = \emptyset$ by convention.

Example 2.15. The crystal graph structure on $B(\lambda/\mu)$ was defined by asserting that the map (2.8) is a morphism of crystal graphs.

Composing morphisms yields a morphism. An **isomorphism** of crystal graphs is a bijective morphism of crystal graphs whose inverse function is also a morphism of crystal graphs.

Our definition of morphism preserves strings in the following sense.

Lemma 2.16. *Let $\Psi : B \to B'$ be a morphism that sends b to b' and let $i \in I$. Then b has an outgoing (resp. incoming) i-arrow if and only if b' does. In particular, the i-strings of b and b' are isomorphic, and they lie in same position within their respective i-strings. More formally, (i) $f_i(b) \neq \emptyset$ if and only if $f_i(b') \neq \emptyset$, (ii) $e_i(b) \neq \emptyset$ if and only if $e_i(b') \neq \emptyset$, (iii) $\varphi_i(b) = \varphi_i(b')$, and (iv) $\varepsilon_i(b) = \varepsilon_i(b')$.*

Proof. If $f_i(b) \neq \emptyset$ then $\Psi(f_i(b)) = f_i(\Psi(b)) \neq \emptyset$ since Ψ sends B to B'. Conversely, if $f_i(\Psi(b)) \neq \emptyset$ then $\Psi(f_i(b)) \neq \emptyset$. Since Ψ sends $\emptyset$ to $\emptyset$ and B to B' it follows that $f_i(b) \neq \emptyset$. This proves (i). (iii) follows immediately from (i). (ii) and (iv) are similar. □

Eventually we will see that morphisms have the incredibly strong property that they preserve connected components. The following result is proved as part of Corollary 2.54.

Theorem 2.17. *A morphism sending one element to another, restricts to an isomorphism between their components.*

2.3.4 Tensor product

Warning 2.18. Our convention for tensor products is the left-to-right opposite of that of Kashiwara and most of the literature. Our convention is directly compatible with tableaux, Knuth relations, Robinson-Schensted correspondence, etc.

The tensor product construction is defined below. It may appear to be an unnecessarily complicated version of the construction for words. However this construction is useful conceptually and is necessary later for affine crystal graphs.

Let $B_1, B_2, \dots, B_k$ be crystal graphs. The **tensor product** crystal graph $B_1 \otimes \cdots \otimes B_k$ has vertex set given by the Cartesian product $B_1 \times \cdots \times B_k$. The element $(b_1, b_2, \dots, b_k)$ is denoted $b = b_1 \otimes \cdots \otimes b_k$. The weight function is the sum

$$\mathrm{wt}(b) = \sum_{j=1}^{k} \mathrm{wt}_{B_j}(b_j). \tag{2.10}$$

The Kashiwara operator $f_i(b)$ (resp. $e_i(b)$) is obtained by applying f_i (resp. e_i) to one of the tensor factors of b. Form the word of parentheses

$$)^{\varphi_i(b_1)}(^{\varepsilon_i(b_1)} \otimes \cdots \otimes)^{\varphi_i(b_k)}(^{\varepsilon_i(b_k)}$$

where the tensor symbols indicate to which tensor factor a parenthesis belongs. Match the parentheses as usual. Then define

$$f_i(b) = \cdots \otimes b_{p-1} \otimes f_i(b_p) \otimes b_{p+1} \otimes \cdots \tag{2.11}$$

where b_p is the tensor factor containing the rightmost unmatched ")". If there are no unmatched ")" then $f_i(b) = \emptyset$. Similarly

$$e_i(b) = \cdots \otimes b_{q-1} \otimes e_i(b_q) \otimes b_{q+1} \otimes \cdots \tag{2.12}$$

where b_q is the tensor factor containing the leftmost unmatched "(". If there are no unmatched "(" then $e_i(b) = \emptyset$. As before $\varphi_i(b)$ is the total number of unmatched ")" and $\varepsilon_i(b)$ is the total number of unmatched "(".

Example 2.19. The crystal graph structure defined in section 2.2.3 on the set $B(1)^k$ of words of length k in the alphabet $B(1)$, is the k-fold tensor power $B(1)^{\otimes k}$. The weight function on $B(1)^k$ given by the content of a word, is precisely the sum of the weights of its letters as prescribed by (2.10).

Proposition 2.20. *The tensor product operation on crystal graphs is associative. Any grouping of tensor factors produces an isomorphic crystal graph.*

Proof. It is obvious from the parenthesis constructions that $(B_1 \otimes B_2) \otimes B_3 \cong B_1 \otimes (B_2 \otimes B_3)$ since both are isomorphic to the threefold construction $B_1 \otimes B_2 \otimes B_3$. □

Remark 2.21. The trivial crystal $B(0)$ is the identity for the tensor product: for any crystal graph B, $B \otimes B(0) \cong B(0) \otimes B \cong B$.

Warning 2.22. Representation theory implies that $B \otimes B' \cong B' \otimes B$. In general there is no commutativity in th sense that there is generally no *natural* isomorphism between tensor products in different orders. We shall see an important exception in section 4.8 for certain affine crystal graphs.

The twofold tensor product structure is given explicitly below. Define $\emptyset \otimes b' = \emptyset$ and $b \otimes \emptyset = \emptyset$. Then

$$f_i(b \otimes b') = \begin{cases} b \otimes f_i(b') & \text{if } \varepsilon_i(b) < \varphi_i(b') \\ f_i(b) \otimes b' & \text{otherwise.} \end{cases} \tag{2.13}$$

$$e_i(b \otimes b') = \begin{cases} e_i(b) \otimes b' & \text{if } \varepsilon_i(b) > \varphi_i(b') \\ b \otimes e_i(b') & \text{otherwise} \end{cases} \tag{2.14}$$

and

$$\varphi_i(b \otimes b') = \varphi_i(b) + \max(0, \varphi_i(b') - \varepsilon_i(b)) \tag{2.15}$$

$$\varepsilon_i(b \otimes b') = \varepsilon_i(b') + \max(0, \varepsilon_i(b) - \varphi_i(b')). \tag{2.16}$$

We end this section with a few useful results.

Proposition 2.23. *Let $\Psi_i : B_i \to B_i'$ be morphisms of crystal graphs for $i = 1, 2$. Then $\Psi_1 \otimes \Psi_2 : B_1 \otimes B_2 \to B_1' \otimes B_2'$ given by $b_1 \otimes b_2 \mapsto \Psi_1(b_1) \otimes \Psi_2(b_2)$, is a crystal graph morphism.*

Proof. Follows directly from the definitions. □

Proposition 2.24. *Suppose $b_1 \otimes b_2$ and $c_1 \otimes c_2$ are two elements in the same component of $B_1 \otimes B_2$. Then b_1 and c_1 are in the same component of B_1 and b_2 and c_2 are in the same component of B_2.*

Proof. Since being in the same component is an equivalence relation, we may reduce to the case that $f_i(b_1 \otimes b_2) = c_1 \otimes c_2$. Since f_i acts on one tensor factor or the other by (2.13), the result follows. □

Lemma 2.25. *The concatenation map $B(1)^k \otimes B(1)^l \to B(1)^{k+1}$ given by $u \otimes v \mapsto uv$ is a crystal graph morphism.*

Proof. Example 2.19 and Proposition 2.20. □

Proposition 2.26. *Let $D_1, D_2, \dots, D_k$ be skew partition diagrams with total size N. Then the map $B(D_1) \otimes \cdots \otimes B(D_k) \to B(1)^N$ given by $T_1 \otimes \cdots \otimes T_k \mapsto T_1 T_2 \cdots T_k$ is an injective crystal graph morphism.*

Proof. The map $B(D_i) \to B(1)^{|D_i|}$ given by taking the reading word, is a morphism by definition; see Example 2.19. Tensoring these morphisms together is a morphism by Proposition 2.23. Following that with the concatenation map (a morphism by Lemma 2.25) is the desired map, which is therefore a morphism. □

2.3.5 Dual

This construction comes from the contragredient dual of a module. Given a crystal graph B, there is a crystal graph $B^\vee$ called its dual, obtained by renaming each vertex b by $b^\vee$, reversing each arrow, and taking the negative of the weight function. More precisely, $B^\vee = \{b^\vee \mid b \in B\}$ with

$$\begin{aligned} f_i(b^\vee) &= e_i(b)^\vee \\ e_i(b^\vee) &= f_i(b)^\vee \\ \varphi_i(b^\vee) &= \varepsilon_i(b) \\ \varepsilon_i(b^\vee) &= \varphi_i(b) \\ \mathrm{wt}(b^\vee) &= -\mathrm{wt}(b) \end{aligned} \tag{2.17}$$

where by convention $\emptyset^\vee = \emptyset$.

Example 2.27. The dual $B(1)^\vee$ of the single box crystal is given by reversing the arrows in $B(1)$ (see section 2.2.2) and negating weights:

$$\boxed{1^\vee} \xleftarrow{\ 1\ } \boxed{2^\vee} \xleftarrow{\ 2\ } \boxed{3^\vee} \xleftarrow{\ 3\ } \cdots \xleftarrow{\ n-1\ } \boxed{n^\vee}$$

The weight of $\boxed{i^\vee}$ is the negative of the i-th standard basis vector in $\mathbb{Z}^n$.

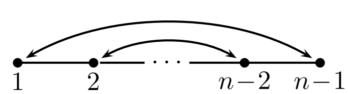
Lemma 2.28. $(B_1 \otimes B_2)^\vee \cong B_2^\vee \otimes B_1^\vee$.

Proof. It follows directly from the definitions that the map $(b_1 \otimes b_2)^\vee \mapsto b_2^\vee \otimes b_1^\vee$ is an isomorphism. □

2.3.6 Dynkin symmetry

In general a crystal graph has an associated Kac-Moody algebra $\mathfrak{g}$ which in our case is sl_n. These algebras can in some sense be completely encoded by a graph called the **Dynkin diagram**, which in our case is the graph A_{n-1} pictured below. Its vertices are labeled by the index set $i \in I$ for the vectors given by the simple roots α_i. It has directed edges labeled by integers. Vertices i and j are adjacent if and only if α_i and α_j are not orthogonal, and the direction and value of the edge depends on the relative lengths of α_i and α_j and the angle between them.

A **Dynkin automorphism** is an automorphism of the Dynkin diagram, a bijective self-map that preserves all of the above graph structure. The Dynkin diagram A_{n-1} has an automorphism of order 2 denoted by $*$, which exchanges the Dynkin vertices i and $n-i$ (or rather, the simple roots α_i and α_{n-i}) for $i \in I$.



Remark 2.29. Every simple Lie algebra has an analogous Dynkin automorphism of order at most 2 (which is the unique nontrivial Dynkin involution in types E_6 and D_n for n odd) given by permuting the simple roots by $\alpha_i \mapsto -w_0 \alpha_i$ where w_0 is the longest element in the Weyl group. For our situation w_0 is the reversing permutation.

Any symmetry τ of the Dynkin diagram gives rise to an induced symmetry for the crystal graphs. By representation theory the crystal graphs and weight lattice have an induced symmetry also denoted τ. For the automorphism $*$ of A_{n-1}, the induced symmetry of the weight lattice is negative reversal:

$$\begin{aligned} * : \mathbb{Z}^n &\to \mathbb{Z}^n \\ (\beta_1, \dots, \beta_n)^* &:= -w_0(\beta) = (-\beta_n, \dots, -\beta_1). \end{aligned} \tag{2.18}$$

Theorem 2.30. *Let τ be any Dynkin automorphism and B any crystal graph. Then there is a crystal graph B^τ that is obtained by renaming each vertex b by $\tau(b)$, relabeling each arrow i by $\tau(i)$, and changing the weight of b by τ. That is, B^τ is defined to be a set with a bijection $\tau : B \to B^\tau$ such that*

$$\begin{aligned} \tau(f_i(b)) &= f_{\tau(i)}(\tau(b)) \\ \tau(e_i(b)) &= e_{\tau(i)}(\tau(b)) \\ \tau(s_i(b)) &= s_{\tau(i)}(\tau(b)) \\ \tau(\mathrm{wt}(b)) &= \mathrm{wt}(\tau(b)) \end{aligned} \tag{2.19}$$

for all $b \in B$ and $i \in I$.

Corollary 2.31. *Given any (type A_{n-1}) crystal graph B there is a crystal graph B^* with vertices b^*, and there is an i-arrow from b to c if and only if there is an $(n-i)$-arrow from b^* to c^*, with $\mathrm{wt}(b^*) = \mathrm{wt}(b)^*$.*

Example 2.32. The crystal graph $B(1)^*$ is given below.

$$\boxed{1^*} \xrightarrow{n-1} \boxed{2^*} \xrightarrow{n-2} \boxed{3^*} \xrightarrow{n-3} \cdots \xrightarrow{1} \boxed{n^*}$$

The weight of i^* is the negative of the $(n+1-i)$-th standard basis vector.

Lemma 2.33. *For any crystal graphs B_1 and B_2, $(B_1 \otimes B_2)^* \cong B_1^* \otimes B_2^*$.*

2.3.7 The # operation, reverse complement, and antitableaux

Combining the Dynkin automorphism $*$ with duality $\vee$ we obtain the # operation. It associates to each crystal graph B a crystal graph $B^\#$ which relabels each vertex $b \in B$ by $b^\#$, reverses each arrow i and relabels it $n-i$. It changes weight by the map $\# : \mathbb{Z}^n \to \mathbb{Z}^n$ given by

$$(\beta_1, \dots, \beta_n)^\# = (\beta_n, \dots, \beta_1). \tag{2.20}$$

Proposition 2.34. *Given any crystal graph B there is a crystal graph $B^\# = \{b^\# \mid b \in B\}$ with crystal structure defined by*

$$\begin{aligned} f_i(b^\#) &= e_{n-i}(b)^\# \\ e_i(b^\#) &= f_{n-i}(b)^\# \\ s_i(b^\#) &= s_{n-i}(b)^\# \\ \mathrm{wt}(b^\#) &= \mathrm{wt}(b)^\# \end{aligned} \tag{2.21}$$

Example 2.35. The crystal graph $B(1)^\#$, obtained from that of $B(1)^*$ by duality, that is, by reversing arrows and negating weights, is given below.

$$\boxed{1^\#} \xleftarrow{n-1} \boxed{2^\#} \xleftarrow{n-2} \boxed{3^\#} \xleftarrow{n-3} \cdots \xleftarrow{1} \boxed{n^\#}$$

The weight of $i^\#$ is the $(n+1-i)$-th standard basis vector.

The following natural isomorphism follows immediately from Lemmata 2.28 and 2.33.

Proposition 2.36. *For any crystal graphs B_1, B_2, the map $(b_2 \otimes b_1)^\# \mapsto b_1^\# \otimes b_2^\#$ is an isomorphism of crystal graphs $(B_2 \otimes B_1)^\# \cong B_1^\# \otimes B_2^\#$.*

Warning 2.37. It turns out that there is an isomorphism $B \cong B^\#$ but it is not canonical in general; it is akin to the fact that the isomorphism $B_1 \otimes B_2 \cong B_2 \otimes B_1$ is not canonical.

Remark 2.38. (i) Comparing $B(1)$ in section 2.2.2 and $B(1)^\#$ in Example 2.35, we see that $B(1)^\# \cong B(1)$ via the identification $x^\# = n + 1 - x$ for $x \in B(1)$.

(ii) By Proposition 2.36 there is a natural bijection $B(1)^k \to B(1)^k$ given by $b = b_k \cdots b_1 \mapsto b^\# := b_1^\# \cdots b_k^\#$ satisfying (2.21). This is the well-known reverse complement map on words. It restricts to a natural bijection $\# : B(r) \to B(r)$ on weakly increasing words of length r.

(iii) For $\beta = (\beta_1, \dots, \beta_k) \in \mathbb{Z}^k_{\geq 0}$, let $B^\beta = B(\beta_k) \otimes \cdots \otimes B(\beta_1)$ and $b = b_k \otimes \cdots \otimes b_1 \in B^\beta$ be a tensor product of weakly increasing words. By Proposition 2.36 there is a natural bijection $\# : B^\beta \to B^{\mathrm{rev}(\beta)}$ given by $b_k \otimes \cdots \otimes b_1 \mapsto b_1^\# \otimes \cdots \otimes b_k^\#$ satisfying (2.21) where $\mathrm{rev}(\beta) = (\beta_k, \dots, \beta_1)$ is the reverse of β.

(iv) The restriction of the map $b \mapsto b^\#$ on $B(1)^k$ to the subset $B(\lambda)$, gives a natural bijection $B(\lambda) \to B(\lambda)^\#$ satisfying (2.21). The latter set, which is already defined by the abstract operation $B \mapsto B^\#$, may be identified (via reading words as usual) with the set of semistandard tableaux of the skew shape obtained by the 180-degree rotation of the diagram of λ. The image $T^\#$ of T is called the antitableau of T.

Example 2.39. Let $n = 4$ and $\lambda = (4, 2, 1, 0)$. Here is a tableau $T \in B(4, 2, 1, 0)$ and its antitableau $T^\#$.

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \qquad T^\# = \begin{array}{cccc} & & & \boxed{2} \\ & & \boxed{1} & \boxed{3} \\ \boxed{3} & \boxed{4} & \boxed{4} & \boxed{4} \end{array}$$

2.4 Connectedness and consequences

2.4.1 $B(\lambda)$ for dominant weights λ

A **dominant weight** is a weakly decreasing sequence of n integers, some of which could be negative. Let $\mathbb{Z}^n_{\geq}$ denote the set of dominant weights. Let $\gamma \in \mathbb{Z}^n_{\geq}$. If γ is a partition then $B(\gamma)$ has already been defined. If $\gamma_n < 0$ define

$$B(\gamma) := (B(1^n)^\vee)^{\otimes -\gamma_n} \otimes B(\gamma - (\gamma_n^n)). \tag{2.22}$$

Example 2.40. Let $n = 3$. Then $B(0, 0, -1) = B(1, 1, 1)^\vee \otimes B(1, 1, 0)$.

Remark 2.41. Since $B(1^n)$ has no arrows, neither does any of its tensor powers nor do those of $B(1^n)^\vee$. Therefore tensoring (on either side) with $B(1^n)^{\otimes k}$ (resp. $(B(1^n)^\vee)^{\otimes k}$) preserves the colored directed graph structure and changes the weight by adding (resp. subtracting) (k^n). Note also that if λ is a partition with $\lambda_n > 0$ then all of the λ_n columns of height n in a tableau of shape λ, must be copies of the single column $n \cdots 21$. Therefore for all $\gamma \in \mathbb{Z}^n_{\geq}$ and $k \geq 0$ we have

$$B(\gamma + (k^n)) = B(1^n)^{\otimes k} \otimes B(\gamma) \tag{2.23}$$
$$B(\gamma - (k^n)) = B(1^n)^{\vee \otimes k} \otimes B(\gamma). \tag{2.24}$$

2.4.2 Highest weight vectors

A **highest weight vector** $b \in B$ is a vertex with no incoming edges, that is, $e_i(b) = \emptyset$ (or equivalently $\varepsilon_i(b) = 0$) for all $i \in I$. Given a crystal graph B, let $\mathbb{Y}(B)$ denote the set of highest weight vectors in B and $\mathbb{Y}(B; \lambda)$ those of weight λ. The weight of a highest weight vector is dominant.

Proposition 2.42. *Let $i \in I$ and $b \in B$ have weight $\beta \in \mathbb{Z}^n$. Then $\varphi_i(b) \geq \beta_i - \beta_{i+1}$ and $\varepsilon_i(b) \geq \beta_{i+1} - \beta_i$. If $b \in \mathbb{Y}(B)$ then $\beta \in \mathbb{Z}^n_{\geq}$.*

Proof. Follows by (2.4), (2.5), and the fact that $\varepsilon_i(b), \varphi_i(b) \geq 0$. $\square$

Lemma 2.43. *For every $\lambda \in \mathbb{Z}^n_{\geq}$, every morphism of crystal graphs sends highest weight vectors of weight λ to highest weight vectors of weight λ.*

Proof. Let $\Psi : B \to B'$ be a morphism and $b \in \mathbb{Y}(B)$. Then for all $i \in I$, $e_i(\Psi(b)) = \Psi(e_i(b)) = \Psi(\emptyset) = \emptyset$. Therefore $\Psi(b) \in \mathbb{Y}(B')$. Finally Ψ, being a morphism, preserves weight. $\square$

Proposition 2.44. *For every $\lambda \in \mathbb{Z}^n_{\geq}$, $B(\lambda)$ is connected. It has a unique highest weight vector denoted y_λ. If λ is a partition y_λ is called the Yamanouchi tableau, the unique tableau of shape λ whose i-th row consists solely of letters i for all i.*

Proof. See the appendix. $\square$

Example 2.45.

$$y_{(4,4,2,1)} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

Remark 2.46. Since $\varepsilon_i(y_\mu) = 0$, by (2.4) and (2.5) we have $\varphi_i(y_\mu) = \mu_i - \mu_{i+1}$ for $i \in I$.

2.4.3 Classification of connected crystal graphs

Lemma 2.47. *Any morphism $\Psi : B \to B'$ whose domain B is connected, is uniquely specified by fixing one particular value of Ψ.*

Proof. Immediate from the definitions. $\square$

The following nontrivial and strong fact is a direct consequence of complete reducibility and highest weight theory for suitable representations of the quantum group $U_q(sl_n)$.

Theorem 2.48. *For every connected crystal graph B, there is a unique $\lambda \in \mathbb{Z}^n_{\geq}$ such that $B \cong B(\lambda)$. Moreover the isomorphism is unique.*

Proposition 2.49. *Every connected crystal graph B unique highest weight vector; if the weight of that vector is λ then $B \cong B(\lambda)$.*

Proof. Let B be connected. By Theorem 2.48 there is an isomorphism $P : B \to B(\lambda)$ for some $\lambda \in \mathbb{Z}^n_{\geq}$. By Proposition 2.44 $B(\lambda)$ has a unique highest weight vector y_λ. By Lemma 2.43 $P^{-1}(y_\lambda)$ is the unique highest weight vector of B and it has weight λ by Lemma 2.43. $\square$

Lemma 2.50. *Let $\Psi : B \cong B'$ be an isomorphism of connected crystal graphs. Then the highest weight vectors of B and B' have the same weight.*

Proof. By Proposition 2.49 B and B' have unique highest weight vectors, say y and y'. By Lemma 2.43 $\Psi(y) = y'$, so y and y' have the same weight since morphisms preserve weight. $\square$

Remark 2.51. The crystal graphs $B(\lambda)$ for $\lambda \in \mathbb{Z}^n_{\geq}$ form a set of representatives of the isomorphism classes of connected crystal graphs. This holds by Theorem 2.48, Lemma 2.50, and Proposition 2.44.

2.4.4 Decomposition

By Proposition 2.49 we may define the map $Y : B \to \mathbb{Y}(B)$ that sends $b \in B$ to the unique highest weight vector in its component. The following is the abstract canonical decomposition of a crystal graph into its components.

Corollary 2.52. *Let B be any crystal graph. For each component C of B, let $\lambda = \lambda(C) \in \mathbb{Z}^n_{\geq}$ and $P = P_C : C \cong B(\lambda)$ be the isomorphism of Theorem 2.48. Then there is an isomorphism*

$$\begin{aligned} B &\cong \bigoplus_{\lambda} B(\lambda) \times \mathbb{Y}(B;\lambda) \\ b &\mapsto (P(b), Y(b)) \end{aligned} \tag{2.25}$$

such that

$$P(F(b)) = F(P(b)) \tag{2.26}$$

$$Y(F(b)) = Y(b) \tag{2.27}$$

for all $b \in B$, where F is one of the operators f_i, e_i, s_i for $i \in I$.

Proof. See the appendix. □

Example 2.53. The crystal graph in Example 2.1 has components which are actually equal to $B(2)$ and $B(1,1)$ (under the identification of a tableau with its column reading word) with respective highest weight vectors 11 and 21.

Corollary 2.54. *Let $\Psi : B \to B'$ be any crystal graph morphism. Then (i) for all $b \in B$, Ψ restricts to an isomorphism of the component of b with that of $\Psi(b)$. (ii) $P(\Psi(b)) = P(b)$. (iii) $b \in \mathbb{Y}(B)$ if and only if $\Psi(b) \in \mathbb{Y}(B')$.*

Proof. See the appendix. □

Corollary 2.55. *Let B and B' be crystal graphs.*

1. *Let $\Psi : B \to B'$ be a morphism of crystal graphs. Then for each partition λ, there is a map $\Psi_\lambda : \mathbb{Y}(B,\lambda) \to \mathbb{Y}(B',\lambda)$ given by restriction of Ψ to $\mathbb{Y}(B,\lambda)$. If Ψ is an isomorphism then the Ψ_λ are bijections.*

2. *Suppose for each λ there is prescribed map of sets $\Psi_\lambda : \mathbb{Y}(B,\lambda) \to \mathbb{Y}(B',\lambda)$. Then there is a unique morphism $\Psi : B \cong B'$ whose restriction to $\mathbb{Y}(B,\lambda)$ is Ψ_λ for all λ. If each of the Ψ_λ is a bijection then Ψ is an isomorphism.*

Proof. See the appendix. □

A crystal graph is **multiplicity-free** if its connected components are nonisomorphic.

Corollary 2.56. *Let B and B' be multiplicity-free and $\Psi : B \to B'$ a morphism. Then Ψ is an isomorphism and is the unique isomorphism $B \to B'$.*

Proof. Immediate from Corollary 2.55 as any function between singleton sets gives the unique bijection between them. □

2.5 Multiplicities in tensor products

Let B be a crystal graph. Define the multiplicity of $B(\lambda)$ in B by the number of highest weight vectors of B of weight λ:

$$[B : B(\lambda)] = |\mathbb{Y}(B, \lambda)|. \tag{2.28}$$

We shall compute this when B is a tensor product of various kinds. There are two kinds of formulae. The first comes from the definition: just count highest weight vectors. The other kind of formula is dual to the first; one applies the bijection (2.34) in some form and counts certain tableaux. We will see them later as Q-tableaux from the Robinson-Schensted-Knuth correspondence.

Proposition 2.57. *$b \otimes b' \in \mathbb{Y}(B \otimes B')$ if and only if $b' \in \mathbb{Y}(B')$ and $\varepsilon_i(b) \leq \varphi_i(b')$ for all $i \in I$.*

Proof. Follows from (2.16). □

2.5.1 Words, Yamanouchi property, and Robinson-Schensted

Say that the word $b = b_k \cdots b_2 b_1 \in B(1)^k$ is **Yamanouchi** if the weight of each of its right factors $b_j \cdots b_2 b_1$ is a partition.

Example 2.58. Let $n = 3$. $y = 2131121$ is a Yamanouchi word. Its right factors (in increasing length) are the empty word $\varnothing$, 1, 21, 121, 1121, 31121, 131121, and 2131121, whose weights are $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(2,1,0)$, $(3,1,0)$, $(3,1,1)$, $(4,1,1)$, $(4,2,1)$, which are all partitions.

Lemma 2.59. *There is a bijection from the set of Yamanouchi words of weight λ, to the set $\mathrm{ST}(\lambda)$ of standard tableaux of shape λ. It sends $y = y_k \cdots y_1 \mapsto Q \in \mathrm{ST}(\lambda)$ where the letter j is placed in the y_j-th row of Q.*

Example 2.60. The standard tableau Q for y in the previous example is

$$\begin{array}{cccccccc} j & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ y_j & 2 & 1 & 3 & 1 & 1 & 2 & 1 \end{array} \qquad Q = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}\,. \tag{2.29}$$

Proposition 2.61. *A word is a highest weight vector if and only if it is Yamanouchi.*

Proof. See the appendix. □

Here the direct count of highest weight vectors of weight λ in $B(1)^k$ is the number of Yamanouchi words of weight λ. The multiplicity space tableaux are in this case the set $\mathrm{ST}(\lambda)$.

By Corollary 2.52 and Lemma 2.59 we obtain the decomposition

$$B(1)^k \cong \bigoplus_{\lambda} B(\lambda) \times \mathrm{ST}(\lambda). \tag{2.30}$$

The explicit computation of this bijection, the Robinson-Schensted correspondence, is developed later in section 3.2.

2.5.2 Skew shape and partition; the LR rule

For a partition μ, say that a word u is **μ-Yamanouchi** if and only if uy_μ is Yamanouchi. Equivalently, by Proposition 2.57 and Remark 2.46,

$$u \text{ is } \mu\text{-Yamanouchi if and only if } \varepsilon_i(u) \leq \mu_i - \mu_{i+1} \text{ for all } i \in I. \tag{2.31}$$

Let $\mathrm{LR}_\mu(\delta/\gamma, \beta)$ the set of semistandard tableaux of shape δ/γ that are μ-Yamanouchi of weight β.

Proposition 2.62. *There is a bijection from the set* $\mathrm{LR}_\mu(\delta/\gamma, \lambda-\mu)$ *to* $\mathbb{Y}(B(\delta/\gamma)\otimes B(\mu), \lambda)$ *given by* $T \mapsto T\otimes y_\mu$. *In particular*

$$[B(\delta/\gamma)\otimes B(\mu) : B(\lambda)] = |\mathrm{LR}_\mu(\delta/\gamma, \lambda-\mu)|. \tag{2.32}$$

Proof. Let $T\otimes T' \in \mathbb{Y}(B(\delta/\gamma)\otimes B(\mu), \lambda)$. Then $T' = y_\mu$ by Propositions 2.57 and 2.44. By Propositions 2.26 and 2.61, Ty_μ is Yamanouchi of weight λ, that is, T is μ-Yamanouchi of weight $\lambda-\mu$. □

Given partitions λ,μ,ν define the Littlewood-Richardson (LR) coefficient $c^\lambda_{\mu\nu}$ by

$$c^\lambda_{\mu\nu} = [B(\mu)\otimes B(\nu) : B(\lambda)]. \tag{2.33}$$

Corollary 2.63. $c^\lambda_{\mu\nu} = |\mathrm{LR}_\mu(\nu, \lambda-\mu)|$.

We shall recover the usual Littlewood-Richardson rule.

Let $W(\alpha,\beta)$ be the set of sequences $u = (\dots, u_2, u_1)$ of weakly increasing words such that u_i has length α_i for all i and $\sum_i \mathrm{wt}(u_i) = \beta$.

Lemma 2.64. *There is a bijection*

$$W(\alpha,\beta) \leftrightarrow W(\beta,\alpha) \tag{2.34}$$

sending u *to* v*, such that the number of letters* i *in* v_j*, is the number of letters* j *in* u_i.

Proof. Obvious. □

Define the **overlap** $\mathrm{ov}(c,b)$ of the weakly increasing words c and b, to be the maximum r such that there is a bijection from the letters of a subword b' of b of length r, to the letters of a subword c' of c of length r, such that every letter is sent to a larger one. This measures by how many columns one may slide the row c under the row b to obtain a two-row skew tableau.

Example 2.65. Let $c = 12245$ and $b = 334$. Then $\mathrm{ov}(c,b) = 2$, since

$$\begin{array}{|c|c|c|c|c|c|} \hline & & & 3 & 3 & 4 \\ \hline 1 & 2 & 2 & 4 & 5 & \\ \hline \end{array}$$ is a semistandard skew tableau but $$\begin{array}{|c|c|c|c|c|} \hline & & 3 & 3 & 4 \\ \hline 1 & 2 & 2 & 4 & 5 \\ \hline \end{array}$$ is not.

Lemma 2.66. *Let* u *and* v *correspond under the bijection* (2.34)*. Then* $\mathrm{ov}(u_{i+1}, u_i)$ *is the number of* i*-matched pairs in the concatenated word* $\cdots v_2 v_1$.

Proof. This follows from the fact that the number of i-matched pairs in a word is the same as the maximum r such that there is a bijection from a subset of r of the letters i to a subset of r of the letters $i+1$ such that each i maps to an $i+1$ to its left. □

Lemma 2.67. *Let* u *and* v *correspond under the bijection* (2.34)*. Then* u *gives the rows of a tableau of skew shape* λ/μ *if and only if* v *is* μ*-Yamanouchi of weight* $\lambda-\mu$.

Proof. Clearly the rows of u have the right size if and only if v has weight $\lambda-\mu$. We may assume that these conditions hold. The following are equivalent: (i) The rows u_{i+1} and u_i fit as a semistandard tableau into the i-th and $(i+1)$-th rows of λ/μ. (ii) $\mathrm{ov}(u_{i+1}, u_i) \ge \lambda_{i+1} - \mu_i$. (iii) v has at least $\lambda_{i+1}-\mu_i$ i-matched pairs. (iv) v has at most $\lambda_{i+1} - \mu_{i+1} - (\lambda_{i+1}-\mu_i) = \mu_i - \mu_{i+1}$ i-unmatched letters $i+1$. (v) v is μ-Yamanouchi. (i) and (ii) are clearly equivalent. (ii) and (iii) are equivalent by Lemma 2.66. (iii) and (iv) are equivalent since an $i+1$ is either i-matched or not. (iv) and (v) are equivalent by (2.31). □

Corollary 2.68. *The bijection* (2.34) *restricts to a bijection* $\mathrm{LR}_\mu(\delta/\gamma, \lambda-\mu) \cong \mathrm{LR}_\gamma(\lambda/\mu, \delta-\gamma)$.

Proof. Lemma 2.67 applied twice. □

We recover the classical LR rule.

Corollary 2.69. $c^\lambda_{\mu\nu} = |\mathrm{LR}(\lambda/\mu, \nu)|$.

Proof. Proposition 2.62 and Corollary 2.68. □

2.5.3 Row and partition; the Pieri rule

Let us consider the special case $B(r)\otimes B(\mu)$ in particular. We want to know exactly when $[B(r)\otimes B(\mu) : B(\lambda)]$ is nonzero and what the answer is.

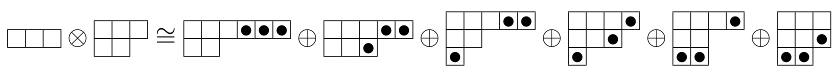
Given a partition $\mu = (\mu_1, \dots, \mu_n)$ and $r \in \mathbb{Z}_{\geq 0}$, let $(r) \otimes \mu$ be the set of partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ such that the skew diagram λ/μ is a **horizontal strip** of size r, meaning that the diagram of λ contains that of μ and their difference consists of exactly r cells, at most one in each column. Such skew shapes are in obvious bijection with the set $\mathcal{T}(\lambda/\mu, (r))$ of semistandard tableaux of shape λ/μ of weight (r).

Proposition 2.70. *Let $\mu = (\mu_1, \dots, \mu_n)$ be a partition and $r \in \mathbb{Z}_{\geq 0}$. There is a unique isomorphism*

$$B(r) \otimes B(\mu) \cong \bigoplus_{\lambda \in (r)\otimes\mu} B(\lambda). \tag{2.35}$$

Proof. By Corollary 2.69, $c^{\lambda}_{(r),\mu}$ is the number of semistandard tableaux of shape λ/μ that are Yamanouchi of weight (r). But any tableau of weight (r) is Yamanouchi. The letters 1 in any semistandard tableau must form a horizontal strip. Therefore $c^{\lambda}_{(r),\mu} = 1$ if λ/μ is a horizontal strip of size r and 0 otherwise. This also shows that $B(r) \otimes B(\mu)$ is multiplicity-free, so the isomorphism is unique by Corollary 2.56. $\square$

Example 2.71. Let $n = 3$ and $\mu = (3, 2, 0)$. We have



where $B(\lambda)$ is represented by the diagram of λ and the cells in the added horizontal strips are marked with a $\bullet$. The highest weight vectors in $B(3)\otimes B(3,2)$, in order according to the expansion on the right, are given by $u \otimes y_{(3,2)}$ where u runs over the set $111, 112, 113, 123, 133, 233$. Note that each u indicates the row indices of new cells to be added to $(3, 2)$ to obtain the corresponding $\lambda \in (3) \otimes (3, 2)$.

2.5.4 Rows and partition; Robinson-Schensted-Knuth correspondence

We iterate the above case. Let $\beta = (\beta_1, \beta_2, \dots, \beta_k) \in \mathbb{Z}^k_{\geq 0}$ and let

$$B^\beta = B(\beta_k) \otimes \cdots \otimes B(\beta_1) \tag{2.36}$$

be the tensor product of crystal graphs indexed by single-rowed partitions. The elements of B^β are just lists $b = (b_k, \dots, b_2, b_1)$ where b_j is a weakly increasing word of length β_j.

Let $\mathcal{T}(\lambda/\mu; \beta)$ is the set of semistandard tableaux Q of shape λ/μ and weight β.

Proposition 2.72. *Let $\beta \in \mathbb{Z}^k_{\geq 0}$ and μ a partition. There is a bijection $\mathbb{Y}(B^\beta \otimes B(\mu), \lambda) \to \mathcal{T}(\lambda/\mu; \beta)$ sending $y \otimes y_\mu = y_k \otimes \cdots \otimes y_1 \otimes y_\mu$ to Q such that y and the rows of Q listed in decreasing order by row index, correspond under the bijection* (2.34).

Proof. The bijection is obtained by iterating Proposition 2.70. Let $y \otimes y_\mu \in \mathbb{Y}(B^\beta \otimes B(\mu), \lambda)$. For any j, $y_j \otimes \cdots \otimes y_1 \otimes y_\mu$ is a highest weight vector, having weight $\mu^{(j)}$ say. We have a chain $\mu = \mu^{(0)} \subset \cdots \subset \mu^{(k)} = \lambda$ where each $\mu^{(j)}/\mu^{(j-1)}$ is a horizontal strip. Q is obtained by placing β_j letters j in the j-th horizontal strip. By Proposition 2.70 this defines the desired bijection. It easily satisfies (2.34). $\square$

Example 2.73. Let $\beta = (3, 3, 2)$, $\lambda = (4, 3, 1)$, and $\mu = \varnothing$. A highest weight element $y \in B^\beta$ of weight λ and the tableau $Q \in \mathcal{T}(\lambda; \beta)$ are given below.

$$y = \begin{array}{cccccccc} &&&&&1&1&1\\ &&1&2&2&&&\\ 2&3&&&&&& \end{array} \mapsto \begin{array}{cccc} 1&1&1&2\\ 2&2&3&\\ 3&&& \end{array} = Q.$$

By Proposition 2.72 with $\mu = \varnothing$ we obtain the isomorphism

$$B^\beta \cong \bigoplus_{\lambda} B(\lambda) \times \mathcal{T}(\lambda; \beta). \tag{2.37}$$

The map is computed explicity in section 3.3.

2.6 Schensted's P-tableau

We now derive Schensted's column-insertion algorithm [3] [19] to compute the P-tableau, directly from crystal graph constructions.

2.6.1 Knuth relations

Proposition 2.74. *The antitableau crystal graph is isomorphic to the tableau crystal graph:*

$$B(\lambda)^{\#} \cong B(\lambda). \tag{2.38}$$

Proof. See the appendix. □

Computing the isomorphism (2.38) explicitly for $\lambda = (2, 1)$ recovers the Knuth relations.

Proposition 2.75. *For* $\lambda = (2, 1)$ *the isomorphism* (2.38) *is given explicitly by*

$$\begin{array}{cc} & \boxed{y} \\ \boxed{x} & \boxed{z} \end{array} \mapsto \begin{array}{cc} \boxed{x} & \boxed{y} \\ \boxed{z} & \end{array} \qquad \text{if } x \leq y < z \tag{2.39}$$

$$\begin{array}{cc} & \boxed{x} \\ \boxed{y} & \boxed{z} \end{array} \mapsto \begin{array}{cc} \boxed{x} & \boxed{z} \\ \boxed{y} & \end{array} \qquad \text{if } x < y \leq z. \tag{2.40}$$

Proof. Let $J : B(2,1)^{\#} \to B(2,1)$ be defined by (2.39) and (2.40). J is well-defined and bijective by definition. It remains to check that $J(f_i(b)) = f_i(J(b))$ and $J(e_i(b)) = e_i(J(b))$ for all $b \in B(2,1)^{\#}$ and $i \in I$. The map J commutes with f_i in an obvious fashion except when $b = i(i+1)i$, in which case we have $J(f_i(b)) = J((i+1)(i+1)i) = (i+1)i(i+1) = f_i((i+1)ii) = f_i(J(i(i+1)i)) = f_i(J(b))$. The commutation of J and e is similar. □

The **Knuth equivalence relation** $\equiv$ on words (on $B(1)$) is that which is generated by elementary Knuth transpositions, which are relations of the form

$$\begin{aligned} uxzyv &\equiv uzxyv \qquad \text{where } x \leq y < z \\ uyxzv &\equiv uyzxv \qquad \text{where } x < y \leq z \end{aligned} \tag{2.41}$$

with letters $x, y, z \in B(1)$ and words u and v.

Proposition 2.76. *Let $b \equiv b'$ for some words b, b' in the alphabet $B(1)$. Then there is a unique isomorphism from the connected component of b to that of b' sending b to b'; moreover it sends words to Knuth-equivalent words. In particular, for all $i \in I$,*

(i) $\varphi_i(b) = \varphi_i(b')$ *and* $f_i(b) \equiv f_i(b')$.

(ii) $\varepsilon_i(b) = \varepsilon_i(b')$ *and* $e_i(b) \equiv e_i(b')$.

(iii) $s_i(b) \equiv s_i(b')$.

(iv) The number of i-matched pairs are the same in b and b'.

Proof. See the appendix. □

Lemma 2.77. *Let u be a strictly decreasing word and x a letter in u. Then $xu \equiv ux$.*

Example 2.78. Let $x = 3$ and $u = 54321$. We have $354321 \equiv 534321 \equiv 543321 \equiv 543231 \equiv 543213$.

Lemma 2.79. *For every Yamanouchi word y of weight λ, $y \equiv y_\lambda$.*

Proof. See the appendix. □

The Knuth relations furnish a concrete characterization of the P map of Theorem 2.48 for the crystal graph $B(1)^k$.

Theorem 2.80. *(i) For every word $b \in B(1)^k$, $b \equiv P(b)$, and $P(b)$ is the unique tableau (of partition shape) that is Knuth equivalent to b. (ii) $b \equiv b'$ if and only if $P(b) = P(b')$. (iii) $P(F(b)) = F(P(b))$ for all b, where F is one of e_i, f_i, s_i for $i \in I$.*

Proof. See the appendix. □

Given a word b we define the P-**tableau** $P(b)$ to be the unique tableau of partition shape such that $b \equiv P(b)$.

2.6.2 Robinson's computation of the P-tableau

We give Robinson's method to compute the P-tableau [18], restated in crystal graph terms.

Start with the word b. Repeatedly apply raising operators until no longer possible, reaching a Yamanouchi word y. Let $E(b) = y$ where $E = e_{i_N} \cdots e_{i_2} e_{i_1}$ is the sequence of raising operators used to reach y from b. Let $\lambda = \mathrm{wt}(y)$. Let $F = f_{i_1} f_{i_2} \cdots f_{i_N}$ be the sequence of lowering operators such that $F(E(b)) = b$. Apply F to y_λ. The result is $P(b)$; see the proof of Theorem 2.80.

Example 2.81. Let $n = 4$ and $b = 3141221$. First, we apply raising operators to b until we reach a Yamanouchi word y.

$$b = 3141221 \overset{e_1}{\to} 3141121 \overset{e_3}{\to} 3131121 \overset{e_2}{\to} 2131121 = y.$$

$\mathrm{wt}(y) = \lambda = (4, 2, 1, 0)$. We apply the reverse sequence of lowering operators to $y_{(4,2,1,0)}$.

$$P(b) = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&4\\\cline{1-2} 3\\\cline{1-1}\end{array} \overset{f_1}{\leftarrow} \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&4\\\cline{1-2} 3\\\cline{1-1}\end{array} \overset{f_3}{\leftarrow} \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array} \overset{f_2}{\leftarrow} \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array} = y_{(4,2,1,0)}$$

This algorithm has the drawback that the distance from b to y may be very large even if the objects b and y are small.

2.6.3 Schensted's column insertion algorithm

We derive Schensted's column insertion algorithm [19] to compute $P(b)$, directly from considering crystal graphs.

Let $b = b_k \cdots b_2 b_1$ with $b_i \in B(1)$. Schensted's algorithm computes the sequence of tableaux $\varnothing = P(\varnothing)$, $P(b_1)$, $P(b_2 b_1)$, $\dots$, $P(b_k \cdots b_1) = P(b)$. We have $P(b_i \cdots b_1) = P(b_i P(b_{i-1} \cdots b_1))$ by Theorem 2.80. So to compute the above sequence of tableaux, it suffices to compute the P tableau for words of the form xT, where T is a tableau and x is a letter. Say T has shape μ. Write

$$(x \to T) := P(xT). \tag{2.42}$$

The element $(x \to T)$ is the image of $x \otimes T$ under the map (2.35) in the case $r = 1$. The shape λ of the resulting tableau is in the set $(1) \otimes \mu$, the set of partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ obtained by adding a single cell to μ.

We derive an algorithm for computing $(x \to T)$: it is called the column insertion of the letter x into the tableau T. Our derivation proceeds by cases of increasing difficulty.

Let T have shape μ.

- **$\mu = \varnothing$ is the empty partition:**

Here (2.35) is the isomorphism $B(1) \otimes B(\varnothing) \cong B(1)$ (see Remark 2.21) and

$$(x \to \varnothing) = \boxed{x}\,. \tag{2.43}$$

The column insertion of x into the empty tableau $\varnothing$ is by definition $P(x) = x$, the tableau with singleton entry x.

- **$\mu = (1^r)$ is a single column for $1 \le r \le n-1$:**

Then (2.35) becomes $B(1) \otimes B(1^r) \cong B(1^{r+1}) \oplus B(2, 1^{r-1})$. Let $x \otimes b \in B(1) \otimes B(1^r)$. Write $b = b_r \cdots b_2 b_1$ with $b_j \in B(1)$ and $b_r > \cdots > b_2 > b_1$.

Suppose first that $x > b_r$. Then xb is strictly decreasing, that is, $xb \in B(1^{r+1})$. Therefore $P(xb) = xb$ since the latter is already a tableau of partition shape. Otherwise $x \le b_r$ and $xb \in B(2, 1^{r-1})^{\#}$ is an antitableau. By Proposition 2.74 it follows that $P(xb) \in B(2, 1^{r-1})$. Let i be smallest such that $x \le b_i$. Let c be obtained from b by replacing $y = b_i$ with x. It is not hard to show that $xb \equiv cy$ and that $cy \in B(2, 1^{r-1})$ is a tableau of partition shape; see Example 2.82. Therefore $P(xb) = cy$.

Example 2.82. Let $x = 3$ and $b = 5421$. Then $y = 4$, $c = 5321$, and

$$x \otimes b = \begin{array}{cc} & \boxed{1} \\ & \boxed{2} \\ & \boxed{4} \\ & \boxed{5} \\ \boxed{3} & \end{array} \qquad xb = \begin{array}{cc} & \boxed{1} \\ & \boxed{2} \\ & \boxed{4} \\ \boxed{3} & \boxed{5} \end{array} \equiv \begin{array}{cc} & \boxed{1} \\ & \boxed{2} \\ \boxed{3} & \boxed{4} \\ \boxed{5} & \end{array} \equiv \begin{array}{cc} & \boxed{1} \\ \boxed{2} & \boxed{4} \\ \boxed{3} & \\ \boxed{5} & \end{array} \equiv \begin{array}{cc} \boxed{1} & \boxed{4} \\ \boxed{2} & \\ \boxed{3} & \\ \boxed{5} & \end{array} = cy = (x \to b).$$

The rule for computing $(x \to b)$ is summarized below.

$$\left(x \to \begin{array}{c} \boxed{b_1} \\ \boxed{\cdot} \\ \boxed{\cdot} \\ \boxed{b_r} \end{array} \right) = \begin{array}{c} \boxed{b_1} \\ \boxed{\cdot} \\ \boxed{\cdot} \\ \boxed{b_r} \\ \boxed{x} \end{array} \qquad \text{if } x > b_r. \tag{2.44}$$

$$\left(x \to \begin{array}{c} \boxed{b_1} \\ \boxed{\cdot} \\ \boxed{b_i} \\ \boxed{\cdot} \\ \boxed{b_r} \end{array} \right) = \begin{array}{cc} \boxed{b_1} & \boxed{b_i} \\ \boxed{\cdot} & \\ \boxed{x} & \\ \boxed{\cdot} & \\ \boxed{b_r} & \end{array} \qquad \text{if } i \text{ is minimum with } x \le b_i. \tag{2.45}$$

- **$\mu = (1^n)$ is a column partition of maximum height:**

$B(\mu) = B(1^n)$ has the lone element $T \in B(1^n)$ and no arrows where T is unique column tableau $n \cdots 21$ of height n. It is easy to check that $B(1) \otimes B(1^n)$ has the lone highest weight vector $1 \otimes T$. Therefore $B(1) \otimes B(1^n) \cong B(2, 1^{n-1})$. We have $P(xT) = Tx \in B(2, 1^{n-1})$ since x commutes with T by Lemma 2.77. This rule agrees with (2.45) if we allow $r = n$.

$$\left(x \to \begin{array}{c} \boxed{1} \\ \boxed{2} \\ \boxed{\cdot} \\ \boxed{x} \\ \boxed{\cdot} \\ \boxed{n} \end{array} \right) = \begin{array}{cc} \boxed{1} & \boxed{x} \\ \boxed{2} & \\ \boxed{\cdot} & \\ \boxed{x} & \\ \boxed{\cdot} & \\ \boxed{n} & \end{array}$$

- **μ has more than one column:**

Let r be the size of the first column of μ and $\widehat{\mu}$ the partition obtained by removing the first column from μ. For $T \in B(\mu)$ write $T = T_1 T_2$ where $T_1 \in B(1^r)$ is the first column of T and $T_2 \in B(\widehat{\mu})$ is the rest of T. Compute $(x \to T_1)$ using the single-column case. Let S_1 be the first column of $(x \to T_1)$ and y the second column.

(i) Suppose $y = \varnothing$. Then $(x \to T) = xT$ is the tableau obtained by placing x at the bottom of the first column of T.

(ii) Suppose $y \in B(1)$. Then $(x \to T) = S_1S_2$ (the tableau with first column S_1 and the rest S_2) where $S_2 = (y \to T_2)$ has been computed by induction.

This agrees with the definition (2.42). If $y = \varnothing$ then it is clear that xT is a tableau, in which case $(x \to T) = P(xT) = xT$ by (2.42) and Theorem 2.80(i). If $y \in B(1)$ then by employing standard arguments [19] it may be shown that S_1S_2 is a tableau. This given, we have $xT = xT_1T_2 \equiv S_1yT_2 \equiv S_1S_2$ since the smaller column insertions preserve Knuth equivalence due to the fact that $b \equiv P(b)$ (Theorem 2.80(i)). Then $P(xT) = P(S_1S_2) = S_1S_2$ by Theorem 2.80.

Example 2.83. Let's compute the insertion of $x = 3$ into the tableau T below. T has more than one column. Split it into its first column T_1 and the rest T_2.

$$T = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2\\ \cline{1-1} 4\\ \cline{1-1}\end{array} \qquad T_1 = \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 4\\ \hline\end{array} \qquad T_2 = \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline\end{array}.$$

Insert x into T_1. The single-column recipe produces the following results.

$$\left(3 \to \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 4\\ \hline\end{array}\right) = \begin{array}{|c|c|}\hline 1&4\\ \hline 2\\ \cline{1-1} 3\\ \cline{1-1}\end{array} \qquad S_1 = \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array} \qquad y = 4$$

Inductively we need to know how to insert $y = 4$ into T_2, which we split into its first column T_{21} and the remainder T_{22}.

$$T_{21} = \begin{array}{|c|}\hline 1\\ \hline\end{array} \qquad T_{22} = \begin{array}{|c|c|}\hline 1&2\\ \hline\end{array}.$$

Inserting 4 into T_{21} yields

$$\left(4 \to \begin{array}{|c|}\hline 1\\ \hline\end{array}\right) = \begin{array}{|c|}\hline 1\\ \hline 4\\ \hline\end{array}.$$

Therefore

$$\left(4 \to \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline\end{array}\right) = \begin{array}{|c|c|c|}\hline 1&1&2\\ \hline 4\\ \cline{1-1}\end{array} = S_2.$$

Finally

$$\left(3 \to \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2\\ \cline{1-1} 4\\ \cline{1-1}\end{array}\right) = S_1S_2 = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&4\\ \cline{1-2} 3\\ \cline{1-1}\end{array}.$$

2.6.4 Pieri property

Proposition 2.84. *Let $T \in B(\mu)$ be a tableau and $u = u_r \cdots u_1 \in B(r)$ be a weakly increasing word with letters $u_i \in B(1)$. Then successive column insertions into T of the letters u_1, then u_2, and so on, change the shape of the tableau by adding cells from left to right in a horizontal strip.*

Proof. See the appendix. □

2.7 Reverse complement and evacuation

By Proposition 2.74 there is an isomorphism $B(\lambda)^\# \cong B(\lambda)$ given by $S \mapsto P(S)$. Define the **evacuation** T^{ev} of T by

$$\begin{aligned} \mathrm{ev} : B(\lambda) &\to B(\lambda) \\ T^{\mathrm{ev}} &= P(T^\#). \end{aligned} \tag{2.46}$$

Equivalently, T^{ev} is the unique tableau of partition shape such that

$$T^{\mathrm{ev}} \equiv T^\#. \tag{2.47}$$

Example 2.85. Let $n = 4$ and $\lambda = (4, 2, 1, 0)$. We give a tableau T, its word, colword(T), the reverse complement of its word, and its antitableau $T^{\#}$, and T^{ev}.

$$T = \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 & 4 \\ 3 \end{ytableau} \qquad\qquad \text{colword}(T) = 321.41.1.2$$

$$T^{\#} = \begin{ytableau} \none & \none & \none & 2 \\ \none & \none & 1 & 3 \\ 3 & 4 & 4 & 4 \end{ytableau} \qquad\qquad \text{colword}(T)^{\#} = 3.4.41.432$$

$$\begin{ytableau} \none & \none & \none & 2 \\ \none & \none & 1 & 3 \\ 3 & 4 & 4 & 4 \end{ytableau} \equiv \begin{ytableau} \none & \none & 1 & 2 \\ \none & \none & 3 & 4 \\ 3 & 4 & 4 \end{ytableau} \equiv \begin{ytableau} \none & 1 & 2 & 4 \\ \none & 3 & 4 \\ 3 & 4 \end{ytableau} \equiv \begin{ytableau} 1 & 2 & 4 & 4 \\ 3 & 3 \\ 4 \end{ytableau} = T^{\text{ev}}.$$

Directly from the definitions, one sees that the operation $\#$ on words, sends Knuth classes to Knuth classes.

Proposition 2.86. *For $b, b' \in B(1)^k$, $b \equiv b'$ if and only if $b^{\#} \equiv b'^{\#}$.*

Proposition 2.87. *(i) For every word $b \in B(1)^k$, $P(b^{\#}) = P(b)^{\text{ev}}$.*

(ii) $T \mapsto T^{\text{ev}}$ is an involution on $B(\lambda)$.

(iii) $f_i(T^{\text{ev}}) = e_{n-i}(T)^{\text{ev}}$, $e_i(T^{\text{ev}}) = f_{n-i}(T)^{\text{ev}}$, and $s_i(T^{\text{ev}}) = s_{n-i}(T)^{\text{ev}}$ for all $T \in B(\lambda)$ and $i \in I$.

Proof. See the appendix. $\square$

Example 2.88. Let $n = 4$ and $b = 24 \otimes 134 \otimes 123$. We have $b^{\#} = 234 \otimes 124 \otimes 13$. We may compute $P(b)$ and $P(b^{\#})$ using column insertion.

$$P(b) = \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 \end{ytableau} \qquad P(b^{\#}) = \begin{ytableau} 1 & 1 & 3 & 4 \\ 2 & 2 & 4 \\ 3 \end{ytableau}.$$

We have

$$P(b)^{\#} = \begin{ytableau} \none & \none & \none & 1 \\ \none & 1 & 2 & 3 \\ 2 & 3 & 4 & 4 \end{ytableau} \equiv \begin{ytableau} 1 & 1 & 3 & 4 \\ 2 & 2 & 4 \\ 3 \end{ytableau} = P(b)^{\text{ev}}$$

which verifies Proposition 2.87(i).

2.8 Schensted row insertion

Schensted row insertion may be defined in a similar manner beginning with the $r = 1$ case of the following right hand analogue of Proposition 2.70, but we shall not pursue this.

Proposition 2.89. *There is a unique crystal graph isomorphism*

$$B(\mu) \otimes B(r) \cong \bigoplus_{\lambda \in (r) \otimes \mu} B(\lambda)$$
$$T \otimes u \mapsto P(Tu).$$

3 Recording tableaux

From the viewpoint of the theory of crystal graphs, the recording tableaux of the Robinson-Schensted-Knuth correspondence are merely combinatorial objects which label the connected components of certain tensor product crystal graphs. Therefore we shouldn't think of the recording tableaux as naturally living inside crystal graphs. At the same time, we will end up apply some crystal graph operations to them since they are tableaux (see [10] for a description of the full-fledged crystal structure on Q-tableaux and the duality of raising and lowering operators with jeu-de-taquin). We develop some properties of recording tableaux, based on the properties of the crystal graphs whose components they label.

Warning 3.1. For simple Lie algebras other than sl_n, the analogue of recording tableau will look nothing like the kind of tableau which naturally label vertices of a crystal graph.

3.1 Two definitions of the standard Q-tableau

Let $b \in B(1)^k$ be a word. Define $Q'(b)$ to be the standard tableau associated with the unique Yamanouchi word $y = Y(b)$ (see Lemma 2.59) in the component of b. This is essentially Robinson's definition [18].

Example 3.2. Example 2.81 computes the Yamanouchi word y in the component of the word b. $Q'(b)$, which is the associated standard tableau of y, is given in Example 2.60.

Let $b = b_k \cdots b_1 \in B(1)^k$ be a word. Schensted defines the (column insertion) **Q-tableau** $Q(b)$ to be the standard tableau of the same shape as $P(b)$, such that j appears in the cell that must be added to the shape of $P(b_{j-1} \cdots b_1)$ to get to the shape of $P(b_j \cdots b_1)$, for $1 \leq j \leq k$.

Example 3.3. Let $b = 3141221$ as in Example 2.81. We compute the P-tableaux of each right factor of b by successive column insertions.

$$\varnothing, \quad \begin{array}{|c|}\hline 1 \\ \hline\end{array}, \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 4 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} = P(b).$$

Therefore

$$Q(b) = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}.$$

This agrees with the Robinson recording tableau $Q'(b)$ of the previous example.

Proposition 3.4. *$Q(b) = Q'(b)$.*

Proof. See the appendix. ☐

3.2 Robinson-Schensted correspondence

Let $\mathrm{ST}(\lambda)$ be the set of standard tableaux of shape λ. The Robinson-Schensted correspondence (given below) is the explicit decomposition of $B(1)^k$ whose existence was given by Corollary 2.52. The Q tableau is the one in the previous section, and the P tableau is the one computed explicitly by column insertion.

Theorem 3.5. *The map*

$$\begin{aligned} B(1)^k &\cong \bigoplus_{|\lambda|=k} B(\lambda) \times \mathrm{ST}(\lambda) \\ b &\mapsto (P(b), Q(b)) \end{aligned} \tag{3.1}$$

is a crystal graph isomorphism, where for all $i \in I$ and $b \in B(1)^k$,

$$\begin{aligned} P(f_i(b)) &= f_i(P(b)) \qquad & Q(f_i(b)) &= Q(b) \\ P(e_i(b)) &= e_i(P(b)) \qquad & Q(e_i(b)) &= Q(b) \\ P(s_i(b)) &= s_i(P(b)) \qquad & Q(s_i(b)) &= Q(b). \end{aligned} \tag{3.2}$$

Proof. See the appendix. □

3.3 Robinson-Schensted-Knuth correspondence

The bijection of Theorem 3.5 can be generalized from $B(1)^k$ to B^β. For $b = b_k \otimes \cdots \otimes b_1 \in B^\beta$, define $P(b) = P(b_k \cdots b_1)$ to be the P tableau of the word $b_k \cdots b_2 b_1$ given by concatenating the weakly increasing words b_i. Define $Q(b)$ to be the tableau of the same shape as $P(b)$, such that the restriction of $Q(b)$ to the alphabet $[j]$ is equal to the shape of $P(b_j \cdots b_1)$ for all $1 \le j \le k$. That is, put letters j in the cells that are in the shape of $P(b_j \cdots b_1)$ that are not in the shape of $P(b_{j-1} \cdots b_1)$. It follows from Proposition 2.70 that $Q(b)$ is a semistandard tableau of weight β and the same shape as $P(b)$. Let $\mathcal{T}(\lambda;\beta)$ denote the set of semistandard tableaux of shape λ and weight β.

The following isomorphism is called the (column insertion) Robinson-Schensted-Knuth correspondence.

Theorem 3.6. *There is a crystal graph isomorphism*

$$\begin{aligned} B^\beta &\cong \bigoplus_\lambda B(\lambda) \times \mathcal{T}(\lambda;\beta) \\ b &\mapsto (P(b), Q(b)) \end{aligned} \tag{3.3}$$

where λ runs over the partitions of $|\beta|$ having at most n parts. It satisfies (3.2).

Proof. See the appendix. □

Example 3.7. Let $n = 4$, $\beta = (3,3,2)$ and $b = 24 \otimes 134 \otimes 123$. We have

$$P(123) = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \qquad P(134123) = \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 3&4\\\cline{1-2}\end{array} \qquad P(24134123) = \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3&4\\\cline{1-3} 4\\\cline{1-1}\end{array}$$

and the sequence of shapes of these tableaux is

$$\varnothing \subset \begin{array}{|c|c|c|}\hline \ &\ &\ \\\hline\end{array} \subset \begin{array}{|c|c|c|c|}\hline \ &\ &\ &\ \\\hline \ &\ \\\cline{1-2}\end{array} \subset \begin{array}{|c|c|c|c|}\hline \ &\ &\ &\ \\\hline \ &\ &\ \\\cline{1-3} \ \\\cline{1-1}\end{array}.$$

So

$$b = \begin{array}{cccccccc} &&&&&\boxed{1}&\boxed{2}&\boxed{3}\\ &&\boxed{1}&\boxed{3}&\boxed{4}&&&\\ \boxed{2}&\boxed{4}&&&&&& \end{array} \mapsto \left(\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3&4\\\cline{1-3} 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&2&3\\\cline{1-3} 3\\\cline{1-1}\end{array} \right) = (P(b), Q(b)).$$

3.4 Skew RSK

We find it convenient to define a skew version of RSK. One may do the same thing as in usual RSK but just insert into an existing tableau.

For $b \otimes T \in B^\beta \otimes B(\mu)$ and $b = b_k \otimes \cdots \otimes b_1 \in B^\beta$, let λ be the shape of $P(bT) = P(b_k \cdots b_1 T)$. Starting with the pair $(T, \varnothing_\mu)$ where $\varnothing_\mu$ is the empty skew tableau of shape μ/μ, for $j = 1, 2, \ldots$ let us column insert b_j, which adds cells in a horizontal strip from left to right by Proposition 2.84. Let us adjoin letters j to the right hand tableau in this newly created horizontal strip. Denote the result by $(P(bT), Q(b \otimes T))$. If λ is the shape of $P(bT)$ we have seen that $Q(b \otimes T) \in \mathcal{T}(\lambda/\mu, \beta)$.

Theorem 3.8. *There is a crystal graph isomorphism*

$$\begin{aligned} B^\beta \otimes B(\mu) &\cong \bigoplus_\lambda B(\lambda) \times \mathcal{T}(\lambda/\mu;\beta) \\ b \otimes T &\mapsto (P(bT), Q(b \otimes T)) \end{aligned} \tag{3.4}$$

such that $Q(F(b \otimes T)) = Q(b \otimes T)$ where F is one of e_i, f_i, s_i for $i \in I$.

Proposition 3.9. *In the bijection* (3.4), *with $b = b_k \otimes \cdots \otimes b_1$, $\mathrm{ov}(b_{r+1}, b_r)$ is the number of r-matched pairs in $Q(b \otimes T)$.*

Proof. See the appendix. □

3.5 Reverse complement and recording tableaux

Theorem 3.10. *Let $\beta = (\beta_1, \dots, \beta_k)$ and $\mathrm{rev}(\beta) = (\beta_k, \dots, \beta_1)$. Recall the map $\# : B^\beta \to B^{\mathrm{rev}(\beta)}$ from Remark 2.38(iv). Then for all $b \in B^\beta$,*

$$\begin{aligned} P(b^\#) &= P(b)^{\mathrm{ev}_n} \\ Q(b^\#) &= Q(b)^{\mathrm{ev}_k} \end{aligned} \tag{3.5}$$

where ev_k uses complementation with respect to the alphabet $[k] = \{1, 2, \dots, k\}$.

Proof. See the appendix. $\square$

Example 3.11. With b and β as in the previous example, we have $k = 3$ and $b^\# = 234 \otimes 124 \otimes 13 = c_3 \otimes c_2 \otimes c_1$. We have

$$P(c_1) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \end{array} \qquad P(c_2 c_1) = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 & \\ \cline{1-2} \end{array} \qquad P(c_3 c_2 c_1) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 4 \\ \hline 2 & 2 & 4 & \\ \cline{1-3} 3 & & & \\ \cline{1-1} \end{array} = P(b^\#)$$

so that

$$Q(b^\#) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 & \\ \cline{1-3} 3 & & & \\ \cline{1-1} \end{array} .$$

We compute $Q(b)^{\mathrm{ev}}$ with respect to $k = 3$, taking $Q(b)$ from the previous example.

$$Q(b) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & \\ \cline{1-3} 3 & & & \\ \cline{1-1} \end{array} \qquad Q(b)^\# = \begin{array}{cccc} \cline{4-4} & & & \multicolumn{1}{|c|}{1} \\ \cline{2-4} & \multicolumn{1}{|c|}{1} & 2 & \multicolumn{1}{c|}{2} \\ \hline \multicolumn{1}{|c|}{2} & 3 & 3 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \equiv \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 & \\ \cline{1-3} 3 & & & \\ \cline{1-1} \end{array} = Q(b)^{\mathrm{ev}}.$$

This agrees with $Q(b^\#)$.

4 Affine crystal graphs

In this section we discuss the additional rich structure of affine crystal graphs (of type $A^{(1)}_{n-1}$) that exists on certain of the crystal graphs we have examined in the previous section.

4.1 Basic features

Affine crystal graphs are crystal graphs in the sense of section 2. However affine crystal graphs also have extra directed edges labeled with the new color 0. So let $\widehat{I} = I \cup \{0\}$ be the set of colors of directed edges for affine crystal graphs. We can view $\widehat{I}$ as the set $\mathbb{Z}/n\mathbb{Z}$. Due to the circular symmetry of the affine Dynkin diagram $A^{(1)}_{n-1}$ (which looks like a cycle with vertices labeled by $\mathbb{Z}/n\mathbb{Z}$) every construction will have rotational symmetry.

There is a zero-th simple root, which for our purposes will be given as $\alpha_0 = (-1, 0^{n-2}, 1) \in \mathbb{Z}^n$. Note that $\sum_{i \in \widehat{I}} \alpha_i = 0$ in $\mathbb{Z}^n$. We also write $h_0 = \alpha_0$ for the zero-th simple coroot. Every affine crystal B has the 0-string property (see section 2.1.1) so that $\varphi_0(b)$, $\varepsilon_0(b)$, $f_0(b)$, and $e_0(b)$ are defined for $b \in B$. In addition equations (2.1) through (2.4) hold for $b \in B$ and $i = 0$. The analogue of (2.5) for $i = 0$ is

$$\langle h_0 \,, \mathrm{wt}(b) \rangle = \beta_n - \beta_1 \qquad \text{for } \mathrm{wt}(b) = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n. \tag{4.1}$$

We should vaguely think that f_0 will change a letter n into a letter 1 and e_0 will do the opposite, although it turns out that often other letters must move.

4.2 Examples

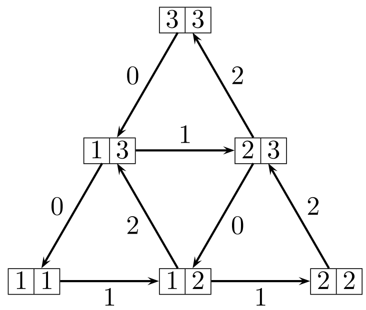
4.2.1 Single box $B^{1,1}$

The affine crystal graph $B^{1,1}$ is the single box crystal graph $B(1)$ (see section 2.2.2) together with a 0 arrow going from $\boxed{n}$ to $\boxed{1}$.

4.2.2 Single row $B^{1,s}$

For any positive integer s the affine crystal graph $B^{1,s}$ is the crystal graph $B(s)$ given by weakly increasing words of length s, plus some extra 0 arrows. To apply f_0 to $b \in B^{1,s}$, remove a letter n from the end of b and put a letter 1 at the beginning; if there is no n in b then $f_0(b) = \emptyset$. Similarly $e_0(b)$ is obtained from b by removing a letter 1 from the beginning and adding a letter n to the end; if there is no 1 in b then $e_0(b) = \emptyset$. So $\varphi_0(b)$ is the number of letters n in b and $\varepsilon_0(b)$ is the number of letters 1 in b. In the crystal graph $B^{1,s}$ all the elements have different weights, so that the entire affine crystal graph structure is determined by (2.1).

Example 4.1. For $n = 3$ the affine crystal graph $B^{1,2}$ is pictured below.



4.2.3 Rectangle $B^{r,s}$

For $r \in I$ and $s \geq 1$ the affine crystal graph $B^{r,s}$ is the ordinary crystal graph $B(s^r)$ for the $r \times s$ rectangular partition (s^r), with additional zero arrows. An explicit rule for the 0-arrows on $B^{r,s}$ is given in [21]. The affine crystal graphs $B^{r,s}$ are called Kirillov-Reshetikhin (KR) crystals.

4.3 Tensor products

The tensor product construction for crystals in section 2.3.4 also works for affine crystals: the usual rule applies for $i = 0$.

Example 4.2. $B^{1,1} \otimes B^{1,1}$ is given in Figure 2. Note that $B^{1,2}$ does not embed into $B^{1,1} \otimes B^{1,1}$ as an affine crystal graph. However if we forget the zero arrows then the resulting crystal graph $B(2)$ embeds naturally into $B(1) \otimes B(1)$.

4.4 Connectedness

Tensor products of connected crystal graphs are usually disconnected. Affine crystal graphs behave in the opposite fashion. The following result was proved using representation theory [1].

Theorem 4.3. *Any tensor product of KR crystals is connected.*

Example 4.4. See Figure 2 for the connected affine crystal graph $B^{1,1} \otimes B^{1,1}$.

Conjecture 4.5. *(Kashiwara) Every connected affine crystal graph is isomorphic to a tensor product of KR crystals.*

If this conjecture holds, then the world of connected affine crystal graphs consists of tensor products of rectangles.

Assumption 4.6. For simplicity we shall only consider the set $\mathcal{C}$ of tensor products of single row KR crystals $B^s := B^{1,s}$, but everything mentioned here can be extended to tensor products of arbitrary rectangular KR crystals $B^{r,s}$; see [20] [21] [22].

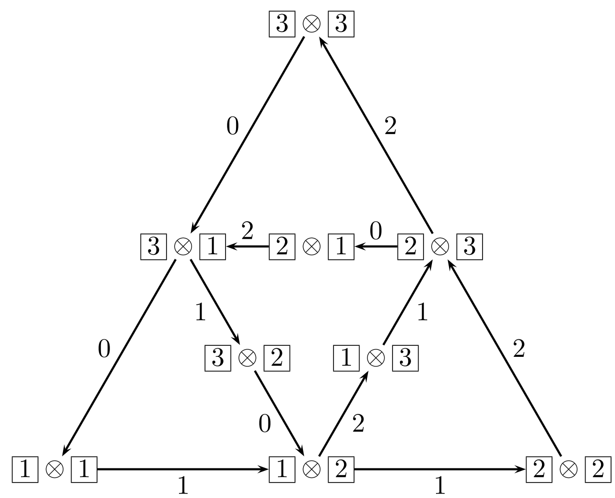


Figure 2: The affine crystal graph $B^{1,1} \otimes B^{1,1}$ for $n = 3$

4.5 Classical structure

Given an affine crystal graph B, we can forget about its 0-arrows; the resulting crystal graph is called the classical crystal graph structure on B. We can then refer to classical components and classical highest weight vectors of B.

4.6 The leading vector

Given $B \in \mathcal{C}$, define the **leading vector** (or dominant extremal vector) $u(B) \in B$ to be the tensor product whose factors are all Yamanouchi tableaux. The leading vector $u(B)$ has partition weight. There is no other element of B with the same weight as $u(B)$. It is also true that $u(B)$ is an extremal weight vector in B: any other element in B is in the convex hull of the S_n-orbit of the weight of $u(B)$. For $B, B' \in \mathcal{C}$ we have $u(B \otimes B') = u(B) \otimes u(B')$.

4.7 Uniqueness of isomorphisms

Proposition 4.7. *Let $B, B' \in \mathcal{C}$. If there is an affine crystal graph isomorphism $B \cong B'$ then it is unique.*

Proof. Since any isomorphism preserves weights, any isomorphism $B \cong B'$ must send $u(B)$ to $u(B')$. By Theorem 4.3, $B \in \mathcal{C}$ is connected. By Lemma 2.47 $B \to B'$ is uniquely specified. $\square$

4.8 R-matrix

Here is another feature peculiar to affine crystal graphs.

Theorem 4.8. *Given any $B, B' \in \mathcal{C}$, there is a unique isomorphism of affine crystal graphs $R_{B,B'} : B \otimes B' \to B' \otimes B$ called the* ***combinatorial R-matrix****. They satisfy the identities*

$$R_{B,B} = 1_{B\otimes B} \qquad \text{for all } B \in \mathcal{C} \tag{4.2}$$

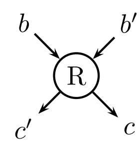
$$R_{B',B} \circ R_{B,B'} = 1_{B\otimes B'} \qquad \text{for all } B, B' \in \mathcal{C}. \tag{4.3}$$

and the Yang-Baxter equation, which is the commutation of the diagram

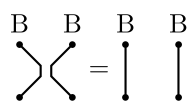
$$\begin{tikzcd}
 & B \otimes B' \otimes B'' \arrow[dl, "R_{B,B'}\otimes 1"'] \arrow[dr, "1\otimes R_{B',B''}"] & \\
B' \otimes B \otimes B'' \arrow[d, "1\otimes R_{B,B''}"'] & & B \otimes B'' \otimes B' \arrow[d, "R_{B,B''}\otimes 1"] \\
B' \otimes B'' \otimes B \arrow[dr, "R_{B',B''}\otimes 1"'] & & B'' \otimes B \otimes B' \arrow[dl, "1\otimes R_{B,B'}"] \\
 & B'' \otimes B' \otimes B &
\end{tikzcd} \tag{4.4}$$

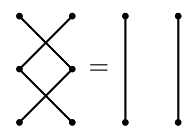
Proof. The proof of existence uses representation theory (the crystal limit of the universal R-matrix) [7]. Proposition 4.7 implies uniqueness and (4.2), (4.3), and (4.4). $\square$

The physics interpretation is that b and b' are two particles which collide and scatter according to R with output particles c' and c. Time evolves from the top of the picture to the bottom.

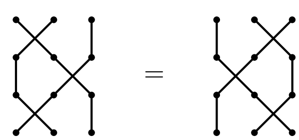


We shall use pictures to represent identities involving R-matrices. Each strand represents a particle or tensor factor. Each crossing represents the action of an R-matrix. A diagram represents an isomorphism from a tensor product to another using R-matrices. Equations (4.2) and (4.3) can be respectively pictured by





The Yang-Baxter equation (4.4) asserts that



(4.5)

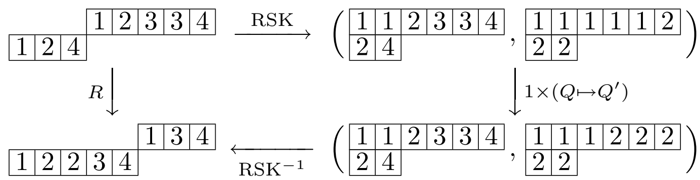
Proposition 4.9. *Suppose $B = B^r$ and $B' = B^s$ are single rows. Then the R-matrix R_{B^r,B^s} can be computed by the composition of the (classical) crystal graph isomorphisms given by Proposition 2.70:*

$$B(r) \otimes B(s) \cong \bigoplus_{\lambda \in (r)\otimes(s)} B(\lambda) = \bigoplus_{\lambda \in (s)\otimes(r)} B(\lambda) \cong B(s) \otimes B(r). \tag{4.6}$$

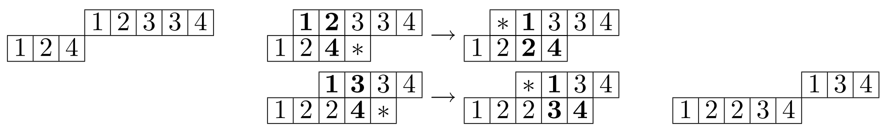
Proof. See the appendix. $\square$

Example 4.10. The bijection in Proposition 2.70 is computed using column insertion. Let $n = 4$, $r = 3$, $s = 5$. We take a typical element of $b \otimes b' \in B^r \otimes B^s$, column insert to obtain a tableau pair (P, Q) where Q is semistandard of shape λ, say, and weight (s, r). Let Q' be the unique semistandard tableau of shape λ

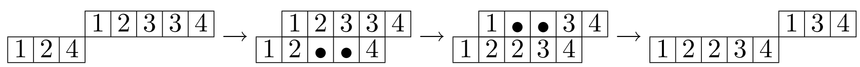
and weight (r, s). Let $c' \otimes c \in B^s \otimes B^r$ be such that their tableau pair is (P, Q'). Then $R(b \otimes b') = c' \otimes c$.



One may also compute this R-matrix using a jeu de taquin on a two-row skew shape that slides the correct number of entries from one row to the other:



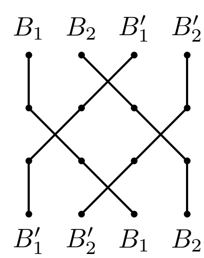
Or, move elements of the lower row as far as possible to the right to make semistandard columns, and then move the first several upper letters that lie in singleton columns, to the lower row:



Proposition 4.11. *If B and B' each consist of several tensor factors, then $R_{B,B'}$ may be computed using any composition of "smaller" R-matrices that exchange the factors of B to the right past those of B'.*

Proof. Proposition 4.7 and Theorem 4.8. □

Example 4.12. If $B = B_1 \otimes B_2$ and $B' = B'_1 \otimes B'_2$ then the R-matrix $R_{B,B'}$ may be computed pictorially by



The R-matrices R_{B_1,B'_1} and R_{B_2,B'_2} in the middle time step can be computed in either order since they don't involve common tensor factors.

Proposition 4.13. *Let $\Psi_i : B_i \to B'_i$ be isomorphisms of affine crystal graphs $B_i, B'_i \in \mathcal{C}$ for $i = 1, 2$. Then $(\Psi_2 \otimes \Psi_1) \circ R_{B_1,B_2} = R_{B'_1,B'_2} \circ (\Psi_1 \otimes \Psi_2)$.*

Proof. This is an immediate consequence of Proposition 4.7 and the existence of the R-matrix in Theorem 4.8. □

4.9 Local coenergy function

Another amazing feature of affine crystal graphs is the coenergy function. Given any $B, B' \in \mathcal{C}$, there is a function $\overline{H}_{B,B'} : B \otimes B' \to \mathbb{Z}$ called the local coenergy function. This function measures the interaction between a pair of neighboring particles.

Theorem 4.14. *[7] Let $B, B' \in \mathcal{C}$. There is a unique function $\overline{H}_{B,B'} : B \otimes B' \to \mathbb{Z}$ called the* ***local coenergy function****, that satisfies the following properties.*

(i) $\overline{H}_{B,B'}(u(B) \otimes u(B')) = 0$.

(ii) $\overline{H}_{B,B'}$ *is constant on classical components.*

(iii) Let $b \otimes b' \in B \otimes B'$ *and* $R_{B,B'}(b \otimes b') = c' \otimes c$. *Then*

$$\overline{H}(e_0(b \otimes b')) = \overline{H}(b \otimes b') + \begin{cases} 1 & \text{if } e_0(b \otimes b') = e_0(b) \otimes b' \text{ and} \\ & \quad e_0(c' \otimes c) = e_0(c') \otimes c \\ -1 & \text{if } e_0(b \otimes b') = b \otimes e_0(b') \text{ and} \\ & \quad e_0(c' \otimes c) = c' \otimes e_0(c) \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

In particular

$$\overline{H}_{B',B} = \overline{H}_{B,B'} \circ R_{B',B}. \tag{4.8}$$

Proof. The existence of $\overline{H}$ again follows from representation theory [7]. The uniqueness of $\overline{H}$ follows immediately from the connectedness of $B \otimes B' \in \mathcal{C}$ given by Theorem 4.3. Equation (4.8) is a direct consequence of the definitions and (4.3). $\square$

Equation 4.8 says that the local coenergy between a pair of neighboring particles doesn't change across a collision.

If $B = B'$ then by (4.2) we have the simpler rule

$$\overline{H}(e_0(b \otimes b')) = \overline{H}(b \otimes b') + \begin{cases} 1 & \text{if } e_0(b \otimes b') = e_0(b) \otimes b' \\ -1 & \text{if } e_0(b \otimes b') = b \otimes e_0(b'). \end{cases} \tag{4.9}$$

Proposition 4.15. *Suppose* $B = B^r$ *and* $B' = B^s$ *are single rows and* $b \otimes b' \in B^r \otimes B^s$. *Then* $\overline{H}(b \otimes b') = \mathrm{ov}(b, b')$, *the overlap of* b *and* b'.

Proof. See the appendix. $\square$

Example 4.16. The coenergy of $b \otimes b'$ (as well as $c' \otimes c$) in Example 4.10 is 2, since

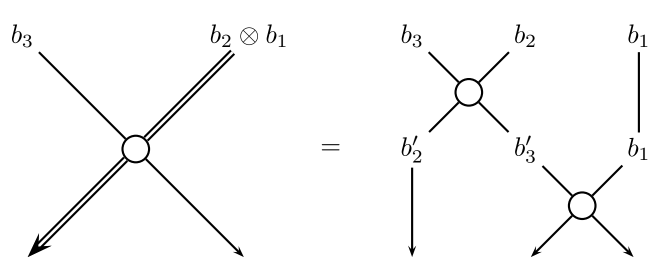
$$\begin{array}{|c|c|c|c|c|c|} \hline & 1 & 2 & 3 & 3 & 4 \\ \hline 1 & 2 & 4 & & & \\ \hline \end{array}\,.$$

Proposition 4.17. *If* B *and* B' *consist of several tensor factors, then the value of* $\overline{H}_{B,B'}(b \otimes b')$ *is given by the sum of the local coenergy functions evaluated at the neighboring tensor factors that must be exchanged in the computation of the R-matrix* $R_{B,B'}$.

We shall state the result precisely for the tensor product of a single factor with a twofold tensor product. Consider $\overline{H}_{B_3,B_2\otimes B_1}$ and $b_i \in B_i$ for $1 \le i \le 3$. Then $R_{B_3,B_2\otimes B_1}$ is given by applying $R_{B_3,B_2} \otimes 1$ and then $1 \otimes R_{B_3,B_1}$ and

$$\begin{aligned} &\overline{H}_{B_3,B_2\otimes B_1}(b_3 \otimes (b_2 \otimes b_1)) = \overline{H}_{B_3,B_2}(b_3 \otimes b_2) + \overline{H}_{B_3,B_1}(b_3' \otimes b_1) \\ &\text{where} \qquad R_{B_3,B_2}(b_3 \otimes b_2) = b_2' \otimes b_3'. \end{aligned} \tag{4.10}$$

We will picture such identities with diagrams in which certain crossings are circled. A diagram represents a sum of local coenergy function evaluations, one for each circled crossing, where the local coenergy function is evaluated at the two neighboring tensor factors that are entering the collision. Then (4.10) is depicted by

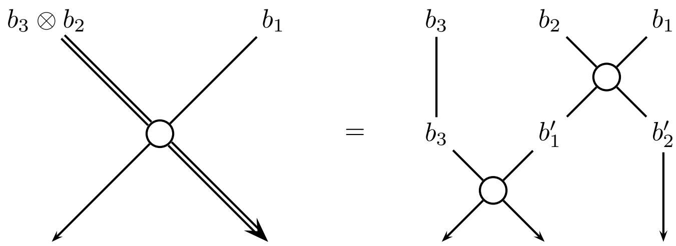


(4.11)

Similarly, we have

$$\begin{aligned}\overline{H}_{(B_3\otimes B_2)\otimes B_1}((b_3\otimes b_2)\otimes b_1) &= \overline{H}_{B_2,B_1}(b_2\otimes b_1)+\overline{H}_{B_3,B_1}(b_3\otimes b_1') \\ \text{where} \qquad R_{B_2,B_1}(b_2\otimes b_1) &= b_1'\otimes b_2'.\end{aligned} \tag{4.12}$$

The associated picture is



(4.13)

Proof of Proposition 4.17. By induction and Proposition 4.11, the proof reduces to proving the special cases (4.10) and (4.12). These cases can be verified directly by considering all the possible ways that e_0 could act on three tensor factors and their images under the appropriate R-matrices [17, Prop. 2.11]. □

The local coenergy function only depends on the factors up to isomorphism.

Proposition 4.18. *Let $B_1, B_2, B_1', B_2' \in \mathcal{C}$ and suppose there are affine crystal graph isomorphisms $\Psi_i : B_i \to B_i'$ for $i = 1, 2$. Then*

$$\overline{H}_{B_2,B_1} = \overline{H}_{B_2',B_1'} \circ (\Psi_2 \otimes \Psi_1). \tag{4.14}$$

In particular, if $B, B', B'' \in \mathcal{C}$ then

$$\overline{H}_{B'',B'\otimes B} = \overline{H}_{B'',B\otimes B'} \circ (1\otimes R_{B',B}) \tag{4.15}$$

$$\overline{H}_{B'\otimes B,B''} = \overline{H}_{B\otimes B',B''} \circ (R_{B',B}\otimes 1) \tag{4.16}$$

Proof. This follows directly from Proposition 4.13 and the definition of coenergy in Theorem 4.14. □

4.10 Intrinsic energy function

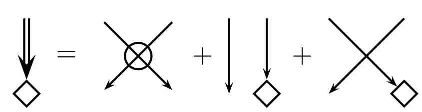
The definitions in this subsection, taken from [17], were inspired by [4]. Every affine crystal graph $B \in \mathcal{C}$ has an **intrinsic coenergy function** $\overline{D}_B : B \to \mathbb{Z}$. It is defined inductively as follows. The intrinsic coenergy of a KR crystal (of type $A_{n-1}^{(1)}$) is zero:

$$\overline{D}_{B^s} = 0. \tag{4.17}$$

Second, suppose the intrinsic coenergy $\overline{D}_{B_1}$ and $\overline{D}_{B_2}$ have been defined for some $B_1, B_2 \in \mathcal{C}$. Then define

$$\begin{aligned}\overline{D}_{B_2\otimes B_1}(b_2\otimes b_1) &= \overline{H}_{B_2,B_1}(b_2\otimes b_1)+\overline{D}_{B_1}(b_1)+\overline{D}_{B_2}(b_2') \\ \text{where} \qquad R_{B_2,B_1}(b_2\otimes b_1) &= b_1'\otimes b_2'.\end{aligned} \tag{4.18}$$

Let us picture the evaluation of $\overline{D}$ using a diamond. Then the definition (4.18) can be depicted by



(4.19)

Proposition 4.19. *For $B_1, B_2 \in \mathcal{C}$,*

$$\overline{D}_{B_2\otimes B_1} = \overline{D}_{B_1\otimes B_2} \circ R_{B_2,B_1}. \tag{4.20}$$

Proof. This follows immediately from (4.18), (4.8), and (4.3). □

Proposition 4.20. $\overline{D}_{(B_3\otimes B_2)\otimes B_1} = \overline{D}_{B_3\otimes(B_2\otimes B_1)}$ *as functions* $B_3 \otimes B_2 \otimes B_1 \to \mathbb{Z}$.

Proof. See the appendix. $\square$

This result implies that the intrinsic coenergy of $B \in \mathcal{C}$ is well-defined, that is, it doesn't depend on the grouping of a tensor product involving several factors, into two-fold factors. Iterating the above twofold tensor construction for $\overline{D}$ we have the following formula, which follows by directly by induction and Proposition 4.17 using the grouping $B_k \otimes (B_{k-1} \otimes \cdots \otimes B_1)$.

Proposition 4.21. *[17, Prop. 2.14] Consider the k-fold tensor product* $B = B_k \otimes \cdots \otimes B_2 \otimes B_1$ *with* $B_i \in \mathcal{C}$*. Let* $b = b_k \otimes \cdots \otimes b_1 \in B$*. Then*

$$\overline{D}_B(b) = \sum_{1\le i<j\le k} \overline{H}_{B_j,B_i}(b_j^{(i+1)} \otimes b_i) + \sum_{j=1}^{k} \overline{D}_{B_j}(b_j^{(1)}), \tag{4.21}$$

where $b_j^{(j)} = b_j$ *and for* $j > i$*, the element* $b_j^{(i)} \in B_j$ *is the right hand factor in the result of applying to* $b_j \otimes b_{j-1} \otimes \cdots \otimes b_i$ *the composition of R-matrices that exchanges* b_j *to the right past each of the tensor factors* b_{j-1} *through* b_i*.*

If each B_i is a KR crystal B^{μ_i} for $1 \le i \le k$, then writing $B^\mu = B^{\mu_k} \otimes \cdots \otimes B^{\mu_1}$, we have

$$\overline{D}_{B^\mu}(b) = \sum_{1\le i<j\le k} \overline{H}_{B^{\mu_j},B^{\mu_i}}(b_j^{(i+1)} \otimes b_i). \tag{4.22}$$

Example 4.22. For $n = 4$ let $b \in B^2 \otimes B^3 \otimes B^3$ be the element in Example 3.7.

$$b = b_3 \otimes b_2 \otimes b_1 = \begin{array}{cccccccc} & & & & & 1 & 2 & 3 \\ & & 1 & 3 & 4 & & & \\ 2 & 4 & & & & & & \end{array}.$$

We have

$$\begin{aligned}
\overline{H}(b_2 \otimes b_1) &= 2 = \text{overlap} \begin{array}{cccc} & 1 & 2 & 3 \\ 1 & 3 & 4 & \end{array}, \\
\overline{H}(b_3 \otimes b_2) &= 2 = \text{overlap} \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 4 & \end{array}, \\
R(b_3 \otimes b_2) &= \begin{array}{ccccc} & & & 1 & 3 \\ 2 & 4 & 4 & & \end{array} \text{ since } \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 4 & \end{array} \equiv \begin{array}{ccc} & 1 & 3 \\ 2 & 4 & 4 \end{array}, \\
\overline{H}(b_3^{(2)} \otimes b_1) &= 1 = \text{overlap} \begin{array}{cccc} & 1 & 3 & 4 \\ 1 & 3 & & \end{array},
\end{aligned}$$

so that $\overline{D}(b) = 2 + 2 + 1 = 5$.

An important property of the intrinsic energy is that it is unchanged by permuting tensor factors by R-matrices.

Proposition 4.23. *Let* $B, B' \in \mathcal{C}$ *and let* $\Psi : B \to B'$ *be an affine crystal graph isomorphism given by a composition of R-matrices. Then*

$$\overline{D}_B = \overline{D}_{B'} \circ \Psi. \tag{4.23}$$

Proof. See the appendix. $\square$

4.11 One-dimensional sums and Kostka-Foulkes polynomials

For $\beta \in \mathbb{Z}_{\geq 0}^k$ and a partition λ, define the one-dimensional sum

$$\overline{X}_{\lambda;\beta}(q) = \sum_{y \in \mathbb{Y}(B^\beta,\lambda)} q^{\overline{D}(y)}. \tag{4.24}$$

Define the cocharge Kostka-Foulkes polynomial $\overline{K}_{\lambda;\beta}(q)$ by

$$\overline{K}_{\lambda;\beta}(q) = \sum_{Q \in \mathcal{T}(\lambda;\beta)} q^{\overline{c}(Q)} \tag{4.25}$$

where $\overline{c}$ is the cocharge (defined in section 5.4). The cocharge Kostka-Foulkes polynomial may be realized as a one-dimensional sum.

Theorem 4.24. *[16] For $\beta \in \mathbb{Z}_{\geq 0}^k$ and a partition λ, $\overline{X}_{\lambda;\beta}(q) = \overline{K}_{\lambda;\beta}(q)$.*

Proof. By Proposition 2.72 there is a bijection from the set of highest weight vectors y in B^β of weight λ, and semistandard tableaux Q of shape λ and weight β. It is given by $Q = Q(y)$, the semistandard RSK recording tableau for the list of words y. By the definitions (4.24) and (4.25) it suffices to prove Proposition 4.25. □

Proposition 4.25. *For all $b \in B^\beta$,*

$$\overline{D}(b) = \overline{c}(Q(b)). \tag{4.26}$$

We shall discuss the proof of this result in the next section.

5 Statistics on recording tableaux

Our first goal is to transfer the coenergy statistic $\overline{D}$ on elements $b \in B^\beta$, to a statistic on the recording tableaux $Q(b)$. The coenergy statistic is computed using the combinatorial R-matrices and the local coenergy function. We translate these in terms of the recording tableaux. Then we show that this "tableau energy" statistic coincides with cocharge.

5.1 R-matrices and the recording tableau

For any composition $\beta \in \mathbb{Z}_{\geq 0}^k$, let $B^\beta = B^{\beta_k} \otimes \cdots \otimes B^{\beta_1}$ be the tensor product of single row KR crystals. Since $B^s \cong B(s)$ as classical crystal graphs the notation for B^β is consistent with that in (2.36). Let $s_i\beta = (\dots, \beta_{i+1}, \beta_i, \dots)$ be β with its i-th and $(i+1)$-th parts exchanged. By definition, the combinatorial R-matrix that switches these two tensor factors gives an affine crystal graph isomorphism

$$R_i : B^\beta \cong B^{s_i\beta}. \tag{5.1}$$

Since this is also a classical crystal graph isomorphism and since the P-tableau doesn't change when applying a morphism of crystal graphs between sets of words (Corollary 2.54), we have

$$P(R_i(b)) = P(b) \qquad \text{for all } b \in B^\beta. \tag{5.2}$$

We want to see what R_i does to the Q-tableau of RSK. Suppose $b, b' \in B^\beta$ are such that $Q(b) = Q(b')$. Then b and b' are in the same component. Since R_i is an isomorphism, $R_i(b)$ and $R_i(b')$ are in the same component, that is, $Q(R_i(b)) = Q(R_i(b'))$. Therefore there is a well-defined bijection

$$\begin{aligned} \sigma_i : \mathcal{T}(\lambda,\beta) &\to \mathcal{T}(\lambda, s_i\beta) \\ Q(b) &\mapsto Q(R_i(b)). \end{aligned} \tag{5.3}$$

Since the R-matrices satisfy (4.2), (4.3), and (4.4) (and obviously commute if they act on disjoint pairs of tensor positions) it follows that the σ_i define an action of the symmetric group S_k on the set $\bigcup_{w \in S_k} \mathcal{T}(\lambda; w\beta)$ for each λ.

Proposition 5.1. *The bijections σ_i of* (5.3) *are given by the crystal reflection operators s_i.*

Proof. See the appendix. $\square$

Example 5.2. Let b and $Q(b)$ be as in Example 3.7;

$$Q(b) = \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 \end{ytableau} \qquad s_2 Q(b) = \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 \end{ytableau}.$$

With $R_2(b) = 244 \otimes 13 \otimes 123$ from Example 4.22 we have

$$P(123) = \begin{ytableau} 1 & 2 & 3 \end{ytableau} \qquad P(13.123) = \begin{ytableau} 1 & 1 & 2 & 3 \\ 3 \end{ytableau} \qquad P(244.13.123) = \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 \end{ytableau}$$

so that

$$Q(R_2(b)) = \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 \end{ytableau} = s_2 Q(b).$$

5.2 Local coenergy on the recording tableau

We express the local coenergy in terms of the recording tableau $Q(b)$.

Given $b = b_k \otimes \cdots \otimes b_1 \in B^\beta$, by abuse of notation similar to that in (5.1), for $1 \leq i \leq k-1$ define $\overline{H}_i(b) = \overline{H}(b_{i+1} \otimes b_i)$.

Proposition 5.3. *For $b \in B^\beta$, $\overline{H}_i(b)$ is the number of i-matched pairs in $Q(b)$.*

Proof. Propositions 4.15 and 3.9. $\square$

Let's define $\overline{H}_i(Q)$ to be the number of i-matched pairs in Q.

Example 5.4. In Example 4.22 we had an element b and computed $\overline{H}_1(b) = 2$, $\overline{H}_2(b) = 2$, $\overline{H}_1(R_2(b)) = 1$. For this b in Example 5.2 we had

$$Q(b) = \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 \end{ytableau} \qquad Q(R_2(b)) = s_2 Q(b) = \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 \end{ytableau}.$$

Counting matching pairs, $\overline{H}_1(Q(b)) = 2$, $\overline{H}_2(Q(b)) = 2$, and $\overline{H}_1(s_2(Q(b))) = 1$.

5.3 Coenergy on recording tableaux and words

We may now define the coenergy directly on a tableau:

$$\overline{D}(Q) = \sum_{1 \leq i < j \leq k} \overline{H}_i(s_i s_{i+1} \cdots s_{j-1} Q). \tag{5.4}$$

By Propositions 5.1 and 5.3 we have

$$\overline{D}(Q(b)) = \overline{D}(b) \qquad \text{for all } b \in B^\beta. \tag{5.5}$$

Example 5.5. With $Q = Q(b)$ and the $\overline{H}_i$ computations from the previous example and the value of $\overline{D}(b)$ from (4.22), we have $\overline{D}(Q) = \overline{H}_1(Q) + \overline{H}_2(Q) + \overline{H}_1(s_2 Q) = 2 + 2 + 1 = \overline{D}(b)$.

Therefore to prove Proposition 4.25 it suffices to show that

$$\overline{D}(Q) = \overline{c}(Q). \tag{5.6}$$

Since the definition of $\overline{D}(Q)$ and that of cocharge (not yet given) apply equally well to words, we shall prove

$$\overline{D}(u) = \overline{c}(u) \tag{5.7}$$

for any word u in the alphabet $[k]$.

5.4 Cocharge

All of the results in this section are due to Lascoux and Schützenberger [15] [11].

The cocharge $\overline{c}$ is the statistic on words defined as follows.

1. $\overline{c}(s_i u) = \overline{c}(u)$ for all i. Using this we may reduce to defining $\overline{c}$ for words of partition weight.

2. $\overline{c}(\varnothing) = 0$.

3.] Suppose u has weight (1^k). Define $c_1 = 0$ and $c_i = c_{i-1}$ if i is right of $i-1$ in u and $c_i = c_{i-1}+1$ if i is left of $i-1$ in u. Define $\overline{c}(u) = \sum_i c_i$.

4. Suppose u has partition weight. Underline the rightmost 1 in u. Given an underlined $i-1$, underline the rightmost i to its left, if it exists; otherwise underline the rightmost i in u; if it exists, and otherwise stop. Let u_1 be the underlined subword of u. Erase the underlined subword from u and repeat the process, extracting a standard word u_2. Continue until u has been exhausted. Define $\overline{c}(u) = \sum_i \overline{c}(u_i)$.

Example 5.6. We compute the cocharge of a word u. The extracted permutations are indicated as u_i. For each permutation u_i, the quantities c_i are indicated as subscripts.

$$\begin{array}{cccccccccccccc}
u & = & 2 & 1 & 3 & 5 & 1 & 4 & 1 & 2 & 4 & 3 & 2 & 3 \\
u_1 & = & 2_1 & & & 5_3 & & & 1_0 & & 4_2 & & & 3_1 \\
u_2 & = & & & & & 1_0 & 4_2 & & & & 3_1 & 2_0 & \\
u_3 & = & & 1_0 & 3_1 & & & & & 2_0 & & & &
\end{array}$$

$\overline{c}(u_1) = 1+3+0+2+1 = 7$, $\overline{c}(u_2) = 0+2+1+0 = 3$, $\overline{c}(u_3) = 0+1+0 = 1$, $\overline{c}(u) = 7+3+1 = 11$.

Theorem 5.7. *The cocharge is the unique function $\overline{c}$ on words such that*

(C0) $\overline{c}(s_i u) = \overline{c}(u)$ for all words u and all i.

(C1) $\overline{c}$ is zero for any weakly increasing word.

(C2) $\overline{c}$ is constant on Knuth classes.

(C3) If u has partition weight and $u = xv$ with $x \neq 1$ a letter then $\overline{c}(xv) = \overline{c}(vx)+1$.

Proof. This follows by Propositions 5.8 and 5.12 below. ☐

Proposition 5.8. *The cocharge as defined above, satisfies the properties (C0) through (C3).*

Proof. See the appendix. ☐

For the following construction we shall identify a tableau (of partition shape) with its row-reading word. Let T be a tableau. Write $T = T'T_1$ where T_1 is the first row of T and T' is the rest of T. Define the katabolism of T by

$$\mathbb{K}(T) = P(T_1 T'). \tag{5.8}$$

Lemma 5.9. *For any tableau T, all of the numbers of value 1 through $i+1$ are in the first row in $\mathbb{K}^i(T)$. Therefore $\mathbb{K}^N(T)$ is a single row tableau for N large.*

Proof. Suppose it is true for $i-1$. We may remove all letters that are greater than $i+1$. By induction all letters of value at most i are in the first row. Therefore the tableau only has two rows and the second row has all values equal to $i+1$. It is then clear that taking $\mathbb{K}$ of this tableau results in a single-row tableau, and we are done. ☐

Lemma 5.10. *For any tableau T and any i, $s_i\mathbb{K}(T) = \mathbb{K}(s_i T)$.*

Proof. This holds since T_1T' is obtained from T by iterated cranking and since the s_i commute with both cranking (by (2.7)) and taking the P-tableau (Theorem 2.80(iii)). ☐

Lemma 5.11. *For any tableau T of shape λ,*

$$\overline{c}(T) = \overline{c}(\mathbb{K}(T)) + |\lambda| - \lambda_1. \tag{5.9}$$

Proof. See the appendix. ◻

Proposition 5.12. *There is at most one function satisfying (C0) through (C3).*

Proof. See the appendix. ◻

Example 5.13. Let's compute $\overline{c}(u)$ using Proposition 5.12. In this computation we will identify a tableau with its row-reading word. We first take the P-tableau

$$\overline{c}(u) = \overline{c}(P(u)) \qquad \text{where} \qquad P(u) = \begin{ytableau} 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & 4 \\ 4 \\ 5 \end{ytableau}$$

Now we apply iterated katabolisms.

$$\begin{ytableau} 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & 4 \\ 4 \\ 5 \end{ytableau} \xrightarrow{\mathbb{K}} \begin{ytableau} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 3 & 4 \\ 5 \end{ytableau} \xrightarrow{\mathbb{K}} \begin{ytableau} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\ 4 & 5 \end{ytableau}$$
$$\xrightarrow{\mathbb{K}} \begin{ytableau} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 5 \end{ytableau}$$

so that $\overline{c}(u) = \overline{c}(P(u)) = 6 + 3 + 2 = 11$.

5.5 Equality of coenergy and cocharge on words

We shall prove (5.7). It suffices to show that the statistic $\overline{D}$ on words defined by (5.4), satisfies the defining properties of $\overline{c}$ given in Theorem 5.7.

$\overline{D}$ satisfies (C0) by (5.5) and Propositions 4.23 and 5.1. For (C1), the set of weakly increasing words of a given length M forms the crystal graph $B(M)$ of tableaux of the single-row shape (M), and as such is stabilized by the s_i by Lemma 2.9. For every $i \in I$, $\overline{H}_i$ is zero on $B(M)$ since a single-row tableau cannot have any matched pairs. So $\overline{D}$ is zero on weakly increasing words. The statistic $\overline{D}$ satisfies (C2) by Proposition 2.76, since applying s_i to Knuth equivalent words yields Knuth equivalent words and Knuth equivalence preserves the number of i-matched letters for any i. It remains to show that $\overline{D}$ satisfies (C3). The following proof is analogous to one sketched in [8].

Let $\mathrm{first}(u)$ denote the first letter of the word u. Recall the definition of $u^{\frown}$ from (2.6). Define

$$\Delta_i(u) = \overline{H}_i(u) - \overline{H}_i(u^{\frown}) \tag{5.10}$$

Say that u has ***i*-dominant** weight if u has at least as many letters i as letters $i+1$.

Lemma 5.14. *Suppose u has i-dominant weight. Then*

$$\Delta_i(u) = \begin{cases} 1 & \textit{if } \mathrm{first}(u) = i+1 \\ -1 & \textit{if } \mathrm{first}(u) = i \textit{ and } \mathrm{first}(s_i(u)) = i \\ 0 & \textit{otherwise.} \end{cases} \tag{5.11}$$

Proof. We may assume that $\mathrm{first}(u) \in \{i, i+1\}$ for otherwise the formula clearly gives 0 as desired. It is also clear that we may ignore all letters not in $\{i, i+1\}$. So we may assume that $i = 1$ and u consists of letters in $\{1, 2\}$. If $\mathrm{first}(u) = 2$ then the 2 is matched since $\mathrm{wt}(u)$ is a partition. It is unmatched after cranking. In this case cranking destroys exactly one matching pair. Suppose $\mathrm{first}(u) = 1$. This 1 is unmatched. It is matched in $u^{\frown}$ if and only if u has an unmatched 2, if and only if $\mathrm{first}(s_1(u)) = 1$. The result follows. ◻

Let u be a word in the alphabet $[k]$. We have

$$\begin{aligned}\overline{D}(u) - \overline{D}(u^{\frown}) &= \sum_{1\le i<j\le k} \left(\overline{H}_i(s_{i+1}\cdots s_{j-1}u) - \overline{H}_i(s_{i+1}\cdots s_{j-1}(u^{\frown}))\right) \\ &= \sum_{1\le i<j\le k} \left(\overline{H}_i(s_{i+1}\cdots s_{j-1}u) - \overline{H}_i((s_{i+1}\cdots s_{j-1}u)^{\frown})\right) \\ &= \sum_{1\le i<j\le k} \Delta_i(s_{i+1}\cdots s_{j-1}u)\end{aligned} \tag{5.12}$$

by the definition (5.4) of $\overline{D}(u)$, the commutation of the s_i with cranking (2.7), and the definition (5.10) of Δ. Let

$$\Delta^{(j)}(u) = \sum_{1\le i<j} \Delta_i(s_{i+1}\cdots s_{j-1}u). \tag{5.13}$$

denote the sum over i in the right hand side of (5.12) with fixed j.

Let u be a word of partition content and $\mathrm{first}(u) = x > 1$. Observe that for all $i < j$, the word $s_{i+1}s_{i+2}\cdots s_{j-1}u$ has i-dominant weight. It suffices to show that

$$\Delta^{(j)}(u) = \delta_{xj}. \tag{5.14}$$

For $1 \le i < j$, define $F_i = \mathrm{first}(s_{i+1}\cdots s_{j-1}u)$. Rewriting the result of Lemma 5.14 we have

$$\Delta_i(s_{i+1}\cdots s_{j-1}u) = \begin{cases} 1 & \text{if } F_i = i+1 \\ -1 & \text{if } F_i = F_{i-1} = i \\ 0 & \text{otherwise.}\end{cases} \tag{5.15}$$

Suppose first that $j < x$. Then $F_{j-1} = F_{j-2} = \cdots = x > j > i$ for all $1 \le i \le j-1$; in particular $x \notin \{i, i+1\}$. By (5.15) $\Delta_i(s_{i+1}\cdots s_{j-1}u) = 0$ for all i and $\Delta^{(j)}(u) = 0$ as desired.

Next suppose $j = x$. The first letter x of u is $(x-1)$-matched since u has $(x-1)$-dominant weight. Therefore $F_i = x$ for all $1 \le i \le j-1$. By (5.15) we have $\Delta^{(j)}(u) = 1$ as desired; for $\Delta_{j-1}(u) = 1$ and $\Delta_i(s_{i+1}\cdots s_{j-1}u) = 0$ for $i < j-1$.

Finally let $j > x$. It is clear that $F_{j-1} = \cdots = F_x = x$. By Lemma 5.14, $\Delta_i(s_{i+1}\cdots s_{j-1}u) = 0$ for $x+1 \le i \le j-1$.

Let $u' = s_{x+1}\cdots s_{j-1}u$; it has $\mathrm{first}(u') = x$. Consider $F_{x-1} = \mathrm{first}(s_x u')$; it can either be x or $x+1$. Suppose $F_{x-1} = x+1$. Then $\Delta_x(s_{x+1}\cdots s_{j-1}u) = 0$ and $F_{x-1} = F_{x-2} = \cdots = F_2 = x+1$, so that $\Delta_i(s_{i+1}\cdots s_{j-1}u) = 0$ for $i < x$ as well. In this case we have $\Delta^{(j)}(u) = 0$. Otherwise $F_{x-1} = x$, whence $\Delta_x(s_{x+1}\cdots s_{j-1}u) = -1$ and $\Delta_{x-1}(s_x\cdots s_{j-1}u) = 1$ by (5.15). When we apply s_{x-1} to $s_x u'$ which has $\mathrm{first}(s_x u') = F_{x-1} = x$, since $s_x u'$ has $(x-1)$-dominant weight, this leftmost x is $(x-1)$-matched, and cannot change. So $F_{x-2} = F_{x-3} = \cdots = F_2 = x$. Therefore $\Delta_i(s_{i+1}\cdots s_{j-1}u) = 0$ for $i \le x-2$. We have $\Delta^{(j)}(u) = 0$ again as desired.

This completes the proof of (5.14) and the equality of $\overline{D}$ and $\overline{c}$ on tableaux.

6 Appendix: Proofs

Proof of Proposition 2.44. By Remark 2.41, to prove that $\mathbb{Y}(B(\lambda))$ is a singleton we may reduce to the case that λ is a partition. It is easy to check directly that y_λ is a highest weight vector. Suppose $T \in B(\lambda)$ with $T \neq y_\lambda$. Let r be the smallest row index such that the r-th row of T does not consist entirely of letters r. Let $i+1$ be the last letter of this row. Within T this letter $i+1$ is i-unmatched. Hence T admits e_i and is not a highest weight vector. □

Proof of Corollary 2.52. B decomposes into a direct sum of its components. By Proposition 2.49 C has a unique highest weight vector y. Let $\mathrm{wt}(y) = \lambda$. Then there is a unique isomorphism $P_C : C \cong B(\lambda)$. Putting these maps P_C together over all components C we obtain the desired isomorphism. Equation 2.26 holds since each P_C is a morphism. Equation 2.27 is obvious. □

Proof of Corollary 2.54. Let $b \in B$ and let C and C' be the components of b and $\Psi(b)$. By the definition of a morphism, Ψ restricts to a morphism $\Psi : C \to C'$. By Proposition 2.49 C and C' have unique highest weight vectors y and y', of the same weight (say λ) by Lemma 2.50. Let $P_C : C \cong B(\lambda)$ and $P_{C'} : C' \cong B(\lambda)$ be the isomorphisms of Theorem 2.48. $P_{C'} \circ \Psi \circ P_C^{-1}$ is a morphism from $B(\lambda)$ to itself, which is the identity by the uniqueness in Theorem 2.48. Therefore Ψ is an isomorphism and $P_C = P_{C'} \circ \Psi$, proving (i) and (ii). (iii) follows by (i) and Lemma 2.43. □

Proof of Corollary 2.55. The first part of (i) holds by Lemma 2.43. For the second part, suppose Ψ is also an isomorphism. Then its inverse Φ restricts to maps $\Phi_\lambda : \mathbb{Y}(B', \lambda) \to \mathbb{Y}(B, \lambda)$ for all λ. But Ψ and Φ are inverse, so it follows that Ψ_λ and Φ_λ are inverse for all λ, that is, Ψ_λ is a bijection for all λ.

For (ii) let $\Psi_\lambda : \mathbb{Y}(B, \lambda) \to \mathbb{Y}(B', \lambda)$ be maps. Define $\Psi : B \to B'$ as follows. Using the isomorphism $b \mapsto (P(b), Y(b))$ and $b' \mapsto (P(b'), Y(b'))$ of Corollary 2.54, for $b \in B$ define $\Psi(b)$ by $P(\Psi(b)) = P(b)$ and $Y(\Psi(b)) = \Psi_\lambda(Y(b))$ for all b that are in a component isomorphic to $B(\lambda)$. This is clearly a morphism with the desired properties. It is unique since its restriction to each component of B is uniquely specified. If in addition each Ψ_λ is a bijection then let Φ_λ be the inverse bijection: we obtain a crystal graph morphism Φ extending the Φ_λ and it is the inverse of Ψ because the same is true for the maps Φ_λ and Ψ_λ. □

Proof of Proposition 2.61. Suppose b is highest weight. Let $b = uv$ be any factorization so that v is an arbitrary right factor of b. The word v is a highest weight vector and therefore has partition weight by Propositions 2.57 and 2.42. So b is Yamanouchi.

The proof of the converse proceeds by induction on the length of b. The empty word $b = \varnothing$ is Yamanouchi and also highest weight. A nonempty Yamanouchi word can be written $b = xc$ with $x \in B(1)$; c is Yamanouchi and by induction is highest weight. By Proposition 2.57 and the fact that $\varepsilon_i(x) = \delta_{i,x-1}$, we need only show that $\varphi_{x-1}(c) > 0$. Since the weights of b and c are partitions and b is obtained from c by adding one to the x-th coordinate, it follows that c has strictly more letters $x - 1$ than letters x. Then $\varphi_{x-1}(c) > 0$ by Proposition 2.42. □

Proof of Proposition 2.74. By Remark 2.41 we may assume λ is a partition. Let $y \in \mathbb{Y}(B(\lambda)^\#)$ be an antitableau; it is Yamanouchi by Proposition 2.61. We prove by induction that each column of y of height r must be equal to $r \cdots 21$. Suppose the rightmost $j - 1$ columns have the desired form. Consider the j-th column c; say it has height r and the column of y immediately to its right has height $r' \geq r$. By semistandardness the elements of c are all $\leq r'$. If c has some letter $i + 1$ and no i, then the letter $i + 1$ is i-unmatched since all columns to the right have both i and $i + 1$. Therefore y admits e_i, a contradiction. It follows that c and therefore y have the desired form. Thus the columns of y are the columns of y_λ except they occur in reverse order from left to right. It follows that $B(\lambda)^\#$ has a unique highest weight vector y of weight λ. Therefore $B(\lambda)^\# \cong B(\lambda)$ by Proposition 2.49. □

Proof of Proposition 2.76. We may assume that $b = ucv$ and $b' = uJ(c)v$ for u and v words of lengths k and l, say, and $c \in B(2,1)^\#$. The map $B(1)^k \otimes B(2,1)^\# \otimes B(1)^l \to B(1)^k \otimes B(2,1) \otimes B(1)^l$ given by $u' \otimes c' \otimes v' \mapsto u' \otimes J(c') \otimes v'$, is a morphism by Proposition 2.23, and it sends words to Knuth-equivalent words. It restricts to a an isomorphism from the component of $u \otimes c \otimes v$ to that of $u \otimes J(c) \otimes v$ by Corollary 2.54. Since the number of i-matched pairs in a word is constant on its i-string and is equal to the number of letters $i + 1$ in the word at the beginning of its i-string, properties (i) through (iv) hold. □

Proof of Lemma 2.79. The result holds if λ is the zero weight: in this case $y = \varnothing = y_\lambda$ are empty. Let y be Yamanouchi of a nonzero weight λ and write $xv = y$ for $x \in B(1)$. Then v is Yamanouchi, of weight μ, say, such that λ is obtained from μ by adding one cell in row x and column $c = \lambda_x$. By induction $y = xv \equiv xy_\mu$. The first $c - 1$ columns of y_μ all contain x and therefore Knuth-commute with x by Lemma 2.77. Commuting x past these columns yields y_λ. That is, $xy_\mu \equiv y_\lambda$ as desired. □

Proof of Theorem 2.80. Consider the crystal graph isomorphism $P : C \cong B(\lambda)$ of Theorem 2.48 where C is the component of b and λ is a partition. It must send the lone highest weight vector y of C to the lone highest weight vector y_λ of $B(\lambda)$. By Proposition 2.61 y is a Yamanouchi word, necessarily of weight λ. By Lemma 2.79 $y \equiv y_\lambda$. Let F be a sequence of crystal graph operators such that $F(y_\lambda) = P(b)$; it exists by

Proposition 2.44. By Proposition 2.76 $F(y) \equiv F(y_\lambda) = P(b)$. Now $P(b) = F(y_\lambda) = F(P(y)) = P(F(y))$. Since P is an isomorphism $b = F(y)$. So $b \equiv P(b)$.

Uniqueness in (i) follows from Theorem 2.48. (ii) follows from immediately from (i). (iii) just says that P is a morphism of crystal graphs, which it is by its definition in Theorem 2.48. $\square$

Proof of Proposition 2.84. We first reduce to the case that $u \otimes T \in B(r) \otimes B(\mu)$ is a highest weight vector. Let $u' \otimes T'$ be the highest weight vector in the component of $u \otimes T$. Write $u' = u'_r \cdots u'_1$ with letters $u'_i \in B(1)$. Since the map $u \otimes T \mapsto uT$ is a crystal morphism, $u'T'$ is the highest weight vector in the component of uT. Any right factor $u'_i \cdots u'_1 T'$ is highest weight by Proposition 2.61. Since uT and $u'T'$ are in the same component, by Proposition 2.24 so are $u_i \cdots u_1 T$ and $u'_i \cdots u'_1 T'$. By Theorem 2.80(iii), $P(u_i \cdots u_1 T)$ and $P(u'_i \cdots u'_1 T')$ are tableaux in the same component, and therefore have the same shape. We may therefore assume that $u \otimes T \in \mathbb{Y}(B(r) \otimes B(\mu), \lambda)$. Let $u = u_r \cdots u_1$ and let $\mu^{(j)}$ be the weight of $u_j \cdots u_1 y_\mu$, which is the shape of the Yamanouchi tableau $P(u_j \cdots u_1 y_\mu)$. It is evident that the insertion of u_j just adds a cell to the u_j-th row for all j. Since u is weakly increasing, the cells are being added in rows of smaller and smaller index. Therefore the cells, which lie in a horizontal strip, are being filled in from left to right. $\square$

Proof of Proposition 2.87. For (i), for any word b we have $b \equiv P(b)$ by Theorem 2.80. Applying $\#$ we have $b^\# \equiv P(b)^\#$ by Proposition 2.86. Taking P, we have $P(b^\#) = P(P(b)^\#) = P(b)^{\mathrm{ev}}$ by Theorem 2.80(ii) and (2.46), proving (i). For (ii),

$$T^{\mathrm{evev}} = P(P(T^\#)^\#) = P(T^{\#\#}) = P(T) = T.$$

The first equality holds by (2.46). The second equality holds by the proof of part (i) with $b = T^\#$. The third equality holds since $\#$ is obviously an involution. The last equality holds by Theorem 2.80(i). For (iii),

$$f_i(T^{\mathrm{ev}}) \equiv f_i(T^\#) = e_{n-i}(T)^\# \equiv e_{n-i}(T)^{\mathrm{ev}}.$$

The first equivalence holds by applying Proposition 2.76 to (2.47). The equality holds by (2.21). The last equivalence holds by (2.47). The Knuth-equivalent tableaux $f_i(T^{\mathrm{ev}})$ and $e_{n-i}(T)^{\mathrm{ev}}$ must coincide by Theorem 2.80(i). The statements for e_i and s_i have similar proofs. $\square$

Proof of Proposition 3.4. If $b = \varnothing$ is empty then by definition both $Q(b)$ and $Q'(b)$ are empty. Suppose $b = xc$ is not empty with $x \in B(1)$. By induction $Q(c) = Q'(c)$. Both $Q(b)$ and $Q'(b)$ share the same shape as $P(b)$. It suffices to show that $Q'(c)$ is obtained from $Q'(b)$ by removing the largest entry. Equivalently we must show that if y is the Yamanouchi word in the component of b, and $y = x'y'$ with $x' \in B(1)$ then y' (automatically being Yamanouchi) is the component of c. Since $b = xc$ and $y = x'y'$ with b and y in the same component, by Proposition 2.24 c and y' are in the same component as desired. $\square$

Proof of Theorem 3.5. The isomorphism (3.1) follows just by iterating the isomorphism of Proposition 2.70 k times for $r = 1$. That is the Schensted approach. The Robinson approach is to use Corollary 2.52 and then use the bijection between Yamanouchi words and standard tableaux.

The commutation of f_i, e_i, s_i and P is already given in Theorem 2.80(iii). $f_i(b)$, $e_i(b)$, and $s_i(b)$ are all in the same component as b and therefore have the same associated Yamanouchi word y. So the Robinson recording tableau Q' is the same for all these words, and by Proposition 3.4 so is the Schensted recording tableau Q. $\square$

Proof of Theorem 3.6. The isomorphism (3.1) follows just by iterating the isomorphism of Proposition 2.70. Or Corollary 2.52 and then use Proposition 2.72. The other properties hold for the same reasons as they do in Theorem 3.5. $\square$

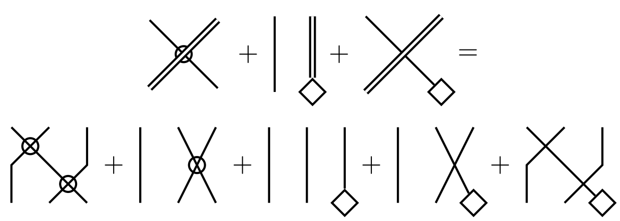
Proof of Proposition 3.9. We first reduce to the case that $b \otimes T$ is highest weight. The condition (i) $\mathrm{ov}(b_{r+1}, b_r) \geq m$, is equivalent to saying that $b_{r+1} b_r$ is the row-reading word of a semistandard tableau of the two-row skew shape D_m that has rows of sizes β_r and β_{r+1} with exactly m columns of height two. Since $B(D_m)$ is a crystal graph, it follows that condition (i) is preserved on the component of $b_{r+1} b_r$. Condition (i) still preserved on the component of $b \otimes T$, by Proposition 2.24. This argument shows that the condition of (i) not holding for $m+1$, is also preserved on the component of $b \otimes T$. Therefore $\mathrm{ov}(b_{r+1}, b_r)$ is invariant on the component of $b \otimes T$. However $Q(b \otimes T)$ is also invariant on the component of $b \otimes T$ by Theorem 3.8. Therefore we may assume that $b \otimes T$ is highest weight, so that $T = y_\mu$ as usual. We are done by Proposition 2.72 and Lemma 2.66. $\square$

Proof of Theorem 3.10. The statement for P-tableaux is in Proposition 2.87(i). One may prove the statement about Q-tableaux by applying the symmetry of the RSK map that exchanges the P- and Q-tableaux [3] and then using the statement for P-tableaux. $\square$

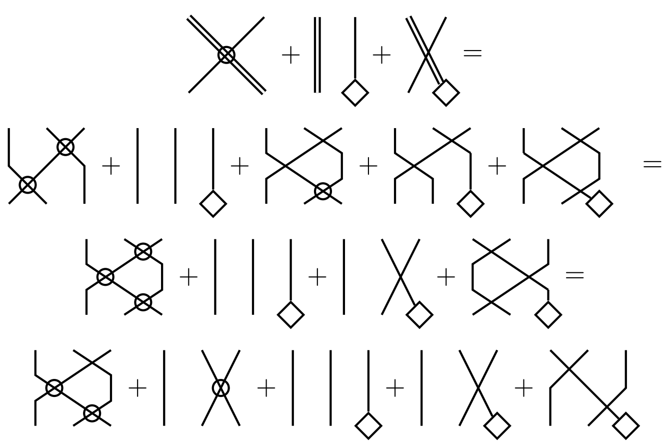
Proof of Proposition 4.9. The sets of partitions $(s) \otimes (r)$ and $(r) \otimes (s)$ of Proposition 2.70 coincide since both consist of the partitions $\lambda = (\lambda_1, \lambda_2)$ where $\lambda_1 + \lambda_2 = r + s$ and $0 \le \lambda_2 \le \min(r, s)$. Since the middle terms of (4.6) are multiplicity-free as classical crystal graphs, it follows that there is a unique isomorphism $B(r) \otimes B(s) \cong B(s) \otimes B(r)$ of classical crystal graphs. As a classical crystal graph, $B^r \otimes B^s$ is isomorphic to $B(r) \otimes B(s)$, and an isomorphism of affine crystal graphs is also an isomorphism of classical crystal graphs. It follows that the map (4.6) must be the R-matrix. $\square$

Proof of Proposition 4.15. By Theorem 4.14 the value of $\overline{H}(b \otimes b')$ depends only on the classical component of $b \otimes b'$, which is computed by the classical crystal graph isomorphism (4.6). These classical components are indexed by the partitions $\lambda \in (r) \otimes (s)$. We first prove that $\overline{H}(b, b') = \lambda_2$ by induction on $m = \lambda_2 \le \min(r, s)$ where λ is the shape of $P(bb')$. For $m = 0$, we have $u(B^r) \otimes u(B^s) = y_{(r)} \otimes y_{(s)} \mapsto y_{(s+r)}$ in $B(s + r)$, so $\overline{H}$ must be zero on the component $\lambda = (s + r)$ as desired. Suppose the result holds for $0 \le m < \min(r, s)$. We show it holds for $m + 1$. Let $b \otimes b' = 1^{r-m}2^m \otimes 1^s \in B^r \otimes B^s$. This element is highest weight of weight $\lambda = (r + s - m, m)$. We have $R(b \otimes b') = c' \otimes c = 1^{s-m}2^m \otimes 1^r$. We consider the 0-string of $b \otimes b'$ and $c' \otimes c$. Since $m < r$ and $m < s$ it follows that $e_0(b \otimes b') = e_0(b) \otimes b'$ and $e_0(c' \otimes c) = e_0(c') \otimes c$. Therefore $\overline{H}(e_0(b) \otimes b') = \overline{H}(e_0(b \otimes b')) = \overline{H}(b \otimes b') + 1 = m + 1$. Now $e_0(b) \otimes b' = 1^{r-m-1}2^m n \otimes 1^s$ is in the same classical component as $1^{r-m-1}2^{m+1} \otimes 1^s$, which is highest weight of weight $(r+s-(m+1), m+1)$. Therefore $\overline{H}(b, b') = \lambda_2$ by induction. The equality with overhang follows from Proposition 3.9, since, for a tableau of two-row partition shape $\lambda = (\lambda_1, \lambda_2)$ in the alphabet $\{1, 2\}$, it is clear that λ is the number of 1-matched pairs. $\square$

Proof of Proposition 4.20. $\overline{D}_{B_3 \otimes (B_2 \otimes B_1)}$ can be computed pictorially by:

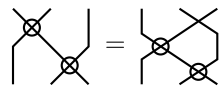


using (4.11) and (4.19). $\overline{D}_{(B_3 \otimes B_2) \otimes B_1}$ is computed pictorially below.



The first equality holds by (4.13) and (4.19). The only significant change in the second equality is in the last diagram, which uses (4.5). The third equality is trivial.

Comparing the final lines of the two computations, we need only show that



But this follows from (4.15). $\square$

Proof of Proposition 4.23. Say that a triple (B, B', Ψ) is good if $B, B' \in \mathcal{C}$, $\Psi : B \cong B'$ is an affine crystal graph isomorphism and (4.23) holds. We observe that good triples are closed under tensor product, that is, if (B_1, B_1', Ψ_1) and (B_2, B_2', Ψ_2) are good, then so is $(B_2 \otimes B_1, B_2' \otimes B_1', \Psi_2 \otimes \Psi_1)$; this follows immediately from the definition (4.18) and Proposition 4.13.

Without loss of generality we may assume that B and B' differ by exchanging two adjacent tensor factors by an R-matrix. Since $(B'', B'', 1)$ is a good triple for all $B'' \in \mathcal{C}$, we may reduce to the case that $B = B_2 \otimes B_1$, $B' = B_1 \otimes B_2$, and $\Psi = R_{B_2,B_1}$. But that is Proposition 4.19. $\square$

Proof of Proposition 5.1. Since neither the map $Q = Q(b) \mapsto Q(R_i(b))$ nor the map $Q \mapsto s_i Q$ touches letters greater than $i + 1$, we may assume that R_i exchanges the leftmost two tensor factors and $i = k - 1$. Since evacuation is an involution (Proposition 2.87) it suffices to show that

$$(s_i Q(b))^{\mathrm{ev}} = Q(R_i(b))^{\mathrm{ev}}. \tag{6.1}$$

We have

$$\begin{aligned}(s_i Q(b))^{\mathrm{ev}} &= P((s_i Q(b))^{\#}) = P(s_1 Q(b)^{\#}) \\ &= s_1 P(Q(b)^{\#}) = s_1 Q(b)^{\mathrm{ev}} = s_1 Q(b^{\#}).\end{aligned} \tag{6.2}$$

This follows by (2.46), Proposition 2.87(iii), Theorem 2.80(iii), and Theorem 3.10.

On the other hand, with respect to the alphabet $[k]$,

$$Q(R_i(b))^{\mathrm{ev}} = Q(R_i(b)^{\#}) = Q(R_1(b^{\#})) \tag{6.3}$$

by Theorem 3.10 and Proposition 6.1 (below).

Comparing (6.1), (6.2), and (6.3), we have reduced to the case $i = 1$ and $k = 2$. But in that case both $s_1(Q(b))$ and $Q(R_1(b))$ are semistandard tableax of the same partition shape (namely that of $P(b) = P(R_1(b))$), and the same weight (β_2, β_1). But there is only one such semistandard tableau, so the two must agree. $\square$

Proposition 6.1. *Let $\beta \in \mathbb{Z}^k_{\geq 0}$, $1 \leq i \leq k - 1$, and $b \in B^\beta$. Let R_i denote the R-matrix that exchanges the i-th and $(i + 1)$-th tensor factors, indexed from the right. Then*

$$R_i(b)^{\#} = R_{k-i}(b^{\#}). \tag{6.4}$$

Proof. Directly from the definitions we may reduce to the case of two tensor factors $\beta = (r, s)$, in which case we must show that the diagram commutes.

$$\begin{tikzcd} B^s \otimes B^r \arrow[r, "\#"] \arrow[d, "R_{B^s,B^r}"'] & B^r \otimes B^s \arrow[d, "R_{B^r,B^s}"] \\ B^r \otimes B^s \arrow[r, "\#"'] & B^s \otimes B^r \end{tikzcd}$$

All of the maps are bijections. If we consider the composite map from $B^s \otimes B^r$ to itself by going around the square, we obtain a crystal graph isomorphism by (2.21) and the fact that the R-matrix is a crystal graph isomorphism. Since $B^s \otimes B^r$ is multiplicity-free as a classical crystal graph there is only one such isomorphism (Corollary 2.56), namely, the identity. The commutation of the diagram follows. $\square$

Proof of Proposition 5.8. (C0) holds by definition. (C1) is also obvious since any extracted permutation u_i will be the increasing permutation, which has zero cocharge. For (C2) one can show that if two words of partition content differ by a Knuth transposition, then all the extracted subwords are the same except that one has a transposition of two values that are adjacent in position but nonadjacent in value. Therefore the cocharges of the extracted subwords remain the same. For (C3) one observes that all extracted subwords are the same for xv as for vx except that a single extracted subword has been cranked with x being the cranked letter. It is easy to check that the cocharge drops by one for that standard subword. $\square$

Proof of Lemma 5.11. Since the s_i preserve the shape of a tableau, by Lemma 5.10 we may assume that the weight of T is a partition. Certainly all the letters 1 are in the first row of T, so in the computation of $\mathbb{K}(T)$ the cranked letters are never equal to 1. $\mathbb{K}(T)$ requires $|\lambda| - \lambda_1$ cranks, the number of letters not in the first row of T. The result follows by properties (C3) and (C2). $\square$

Proof of Proposition 5.12. Let u be a word. By (C0) we may assume u has partition content. By (C2) we may assume u is a tableau. We compute the sequence of tableaux $u, \mathbb{K}(u), \mathbb{K}^2(u), \dots$. By Lemma 5.9 this sequence is eventually constant, given by the single-row tableau of the same weight as u. By Lemma 5.11 $\overline{c}(u)$ is determined by adding up the number of cells in all of these tableaux that are not in the first row. Therefore we have evaluated $\overline{c}(u)$ using the rules (C0) through (C3) and are done. $\square$

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