An Introduction to the ANALYTIC THEORY OF NUMBERS

By RAYMOND AYOUB

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INTRODUCTION

There exist relatively few books, especially in English, devoted to the analytic theory of numbers and virtually none suitable for use in an introductory course or suitable for a first reading. This is not to imply that there are no excellent books devoted to some of the ideas and theorems of number theory. Mention must certainly be made of the pioneering and monumental work of Landau and in more recent years of the excellent books of Estermann, Ingham, Prachar, Vinogradoff and others. For the most part, however, these works are aimed at the specialist rather than at the general reader. No further apology therefore will be made for adding to the vast and growing list of mathematical treatises.

The subject of analytic number theory is not very clearly defined and while the choice of topics included here is to some extent arbitrary, the topics themselves represent some important problems of number theory to which generations of outstanding mathematicians have contributed.

The book is divided into five chapters.

Chapter I. This is devoted to an old and famous theorem—that of Dirichlet on primes in an arithmetic progression.

The chapter begins with some elementary considerations concerning the infinitude of primes and then lays the basis for the introduction of L-series. Characters are introduced and some of their properties derived and this is followed by some general theorems on ordinary Dirichlet series. A version of the classical proof of Dirichlet's theorem is then given with an analytic proof that $L(1, \chi) \neq 0$. The chapter ends with a definition of Dirichlet density and it is noted that the primes in the progression $kn + m$ have D.D. $1/\varphi(k)$.

Apart from the interest of the theorem itself, the methods and ideas introduced by Dirichlet have had an important influence on number theory as well as other branches of mathematics. The beginning reader would then do well to read this chapter in its entirety.

Chapter II. This chapter is devoted to the prime number theorem and to certain auxiliary arithmetic functions arising in a natural way. The p. n. t. is first proved with a modest error term following the general idea of Riemann's proof as completed by Landau. This requires the development of some properties of the zeta function and the proof leads rather directly to $\pi(x)$ (§ 5). It is then shown that the analysis becomes simpler if mean values and absolutely convergent integrals are introduced and then coupled with a Tauberian argument. At this stage, the error is improved to give the result of de la Vallée Poussin (§ 6B). The next step is to reduce further the analytic requirements and couple the discussion with a deeper Tauberian theorem. This is the Hardy-Littlewood proof (§ 6C). The final proof is that of Wiener, as simplified by Ikehara and Landau (§ 6D). Here the Tauberian element plays the primary role. Wiener's proof completes the equivalence of $\psi(x) \sim x$

with $\zeta(1+it) \neq 0$.

The final section is devoted to other arithmetic functions and applications of the p.n.t. to their asymptotic properties (§ 7).

The chapter is planned so as to give the reader a flexible program. He may wish to read the direct proof of

$$(1) \qquad \pi(x) = \operatorname{li} x + O(xe^{-c(\log x)^{1/10}}) \qquad (\S\, 5)$$

or of

$$(2) \qquad \psi(x) = x + O(xe^{-c(\log x)^{1/10}}) \qquad (\S\, 6\mathrm{A}) .$$

He may read a proof of

$$(3) \qquad \psi_2(x) = \tfrac{1}{2}x^2 + O(x^2e^{-c(\log x)^{1/2}}) \qquad (\S\, 6\mathrm{B})$$

and then deduce that

$$(4) \qquad \pi(x) = \operatorname{li} x + O(xe^{-c(\log x)^{1/2}}) .$$

With a slight rearrangement he may read a proof of (4) directly.

On the other hand, a direct reading of the Hardy-Littlewood or Wiener proof is possible.

The material on arithmetic functions again allows a certain measure of latitude.

Chapter III. This chapter is devoted to the theory of partitions. The chapter begins with proofs of some elementary results and the subsequent material is again arranged to provide options to the reader. It is first proved that

$$(1) \qquad p(n) \sim \frac{1}{4n\sqrt{3}}\, e^{k\sqrt{n}} \qquad (k = \pi\sqrt{(2/3)})$$

with the help of the little known but elegant proof of Uspensky (§ 2). Then Siegel's beautifully simple proof of

$$(2) \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\, \eta(\tau)$$

is given (§ 3).

This is followed by the introduction of the modular transformation and it is proved that the set of modular transformations forms a group with two generators. This allows us to prove that

$$(3) \qquad \eta\left(\frac{a\tau + b}{c\tau + d}\right) = \varepsilon(c\tau + d)^{1/2}\eta(\tau) ,$$

where ε is a 24th root of unity whose nature is as yet undetermined (§ 3).

The next step is to give Rademacher's adaptation of Siegel's method to another derivation of (3) and an explicit determination of ε in terms of Dedekind sums (§ 4).

Finally, Rademacher's convergent series for $p(n)$ is derived and proved (§ 6).

The reader has 3 options. He may be content with a proof of (1) and (2) (§§ 2 and 3). He may wish to read a proof of (3) and follow this by a proof of Rademacher's formula (§§ 3 and 6), or finally, he may wish to evaluate ε in (3) and then read a proof of Rademacher's formula (§§ 4 and 6).

Chapter IV. This chapter is devoted to Waring's problem for kth powers. The general plan is to discuss first the contribution from the major arc. This is followed by Weyl's estimate for trigonometric sums. No effort is made to present the deeper and much more difficult estimates of Vinogradoff. For those the interested reader may consult the excellent book of Vinogradoff.

The asymptotic formula for the number of representations of n as a sum of s kth powers is proved to hold for

$$(1) \qquad s \geqq k2^k + 1 \qquad (\S\,6,\ \text{Theorem } 6.6)\,.$$

This is then strengthened (§ 6, Theorem 6.7), with the help of a theorem of Hua to

$$(2) \qquad s \geqq 2^k + 1$$

which for small values of k is superior to Vinogradoff's result.

The next section is devoted to a discussion of Vinogradoff's upper bounds for $G(k)$ (§ 7).

With very little additional effort it is shown that (Theorem 7.3)

$$(3) \qquad G(k) = O(k^2 \log k)$$

and with further estimates on the minor arc, that (Theorem 7.6)

$$(4) \qquad G(k) = O(k \log k)\,.$$

The constants are more precisely determined.

The last section is devoted to a discussion of the singular series and kth power Gauss sums.

The reader has several options. He may read the account on the major arcs (§ 4), and then prove either (1), (2), (3) or (4) since they are essentially independent of one another.

Chapter V. The class number of quadratic fields and the related problem of L functions with real characters are discussed here.

The chapter begins by assuming an elementary knowledge of quadratic fields. The concept of class number h is introduced. It is shown that h is finite and that there exists a constant α such that

$$\alpha h = L(1, \chi)$$

for a certain real character χ.

The reader who is unacquainted with the theory of quadratic fields may take this as the definition of h and interpret subsequent results as theorems on $L(1, \chi)$. The next step is to sum the series $L(1, \chi)$ and derive the Gauss-Dirichlet formula for h. A mean value theorem for $h(d)$ is then derived and proved. This necessitates an estimate for sums of characters. The chapter culminates in Siegel's proof that

$$\log h(d)R \sim \tfrac{1}{2} \log |d|\,.$$

The reader may read §§ 1 to 3 and derive the Gauss-Dirichlet formula for $h(d)$, then read § 5 for the mean value of $h(d)$ and proceed to § 6 where Siegel's theorem is proved. He may on the other hand omit §§ 1 to 3 (except for the discussion of the Kronecker symbol) and proceed directly to § 6 or be content to stop after reading §§ 4 and 5.

The mathematical preparation required to read this book is relatively modest. The elements of number theory and algebra, especially group theory, are required. In addition, however, a good working knowledge of the elements of complex function theory and general analytic processes is assumed. The subject matter of the book is of varying difficulty and there is a tendency to leave more to the reader as the book progresses. The first chapter can be read with relative ease, the subsequent chapters require that they be read more and more "with pen in hand."

It is a pleasure at this juncture to acknowledge my indebtedness during the writing of this book. First to the American Mathematical Society who through a contract with the Air Force Office of Scientific Research enabled me to devote a full year to its writing; to Professor R. Webber of the University of Toronto for his careful and critical reading of Chapters I and II. Many of his suggestions have been incorporated. To Professor C. L. Siegël for his generous help in the proof of Theorem 5.4 of Chapter V. Further the author wishes to thank Dr. Gordon Walker for recommending that the book be published in the American Mathematical Society's distinguished Survey Series. As to the mechanics of publication, the author is most grateful to Mrs. Ellen Burns and Mrs. Helen Striedieck for typing and other secretarial help and to Miss Ellen Swanson and her staff at the American Mathematical Society (especially S. Ramanujam) for preparing a chaotic manuscript for the printer.

There is in addition an indebtedness of more abstract character which the author wishes to acknowledge. No devotee of the analytic theory of numbers can help but be influenced by the brilliant writings of Professors H. A. Rademacher, C. L. Siegel, I. M. Vinogradoff, and the late Professor G. H. Hardy. If the reader detects little originality in the present work, it stems merely from the fact that the work of these scholars can hardly be improved upon. It has indeed been the author's hope that some specialists whose knowledge is broader and whose understanding is deeper than his might have undertaken to write a book of the present type. Perhaps the shortcomings of this work will induce them to do so.

State College, Pennsylvania
November, 1962

Notation

We make extensive use of the order notation (O, o, $\sim$) in this book, and for the benefit of those readers who have not encountered it before, we give a brief summary of the definition and principal properties. The notation was first introduced by Bachmann in analytic theory of numbers and has by now made its way into general analytic processes.

A. Big O. Let a be any real number including the possibilities $\pm\infty$. Let $f(x)$ and $g(x)$ be two functions defined in some neighborhood of a and suppose that $g(x) > 0$. We say that $f(x)$ is "big O of $g(x)$" and we write

$$f(x) = O(g(x)) ,$$

if there exists a constant $K > 0$ and a neighborhood $N(a)$ of a such that

$$|f(x)| \leqq Kg(x)$$

for all x in $N(a)$.

In particular, the notation

$$f(x) = O(1)$$

means that $f(x)$ is bounded in absolute value in a suitable neighborhood of a.

Examples. (i) Suppose that $a = 0$. Then

$$\sin x = O(x) , \qquad x^3 = O(x^2) .$$

(ii) If $a = \infty$, then

$$\sin x = O(1) , \qquad x = O(x^2) .$$

Some simple properties follow at once.

I. If $f_i(x) = O(g_i(x))$, $i = (1, 2)$, then

$$f_1(x) + f_2(x) = O(g_1(x) + g_2(x)) ,$$
$$f_1(x)f_2(x) = O(g_1(x)g_2(x)) .$$

II. If c is a constant and

$$f(x) = O(g(x)) ,$$

then

$$cf(x) = O(g(x)) .$$

The notation is frequently used with functions of more than one variable and here some care must be exercised in its use and interpretation. For example, we frequently encounter a function $f(s)$ of the complex variable $s = \sigma + it$ and write

$$f(s) = O(g(t)) \qquad (t \to \infty) .$$

The constant K whose existence is implied by the O is dependent upon σ, and the dependence may be such that $K = K(\sigma)$ is unbounded for σ in some neighborhood. Sometimes the dependence of K on the auxiliary variables or parameters is explicitly stated and sometimes it is implied by the context.

We use the notation for sequences as well—the sequences may be sequences of functions or sequences of real or complex numbers. For example,

$$f(n) = O(g(n))$$

means that there exists a constant K and an integer N_0 such that if $n > N_0$, then

$$|f(n)| \leqq Kg(n) .$$

To allow for greater flexibility and to use the O symbolism as effectively as possible, it is convenient to define $O(g(x))$ standing by itself. By $O(g(x))$, we shall mean the class of functions $C(g)$ such that $f \in C(g)$ if and only if

$$f(x) = O(g(x)) .$$

Thus in particular, $O(1)$ is the class of bounded functions. If

$$C(g) \subset C(h) ,$$

we write

$$O(g) = O(h) .$$

The reader will readily adapt himself to the mathematical anarchy in which the symbol of equality is used for a relation which is not symmetric. Surprisingly enough, this almost never leads to confusion! We define the sum and product of two O's. By

$$O(g) + O(h)$$

we mean the class of functions C consisting of sums $f_1 + f_2$ where $f_1 \in C(g)$ and $f_2 \in C(h)$. Similarly with $O(g)O(h)$. In addition to a finite sum, we often take an infinite sum of O's.

The following examples will illustrate some of the points.

(i) If

$$f(x) = x \sin (1/x) ,$$

then, as $x \to \infty$,

$$f(x) = O(x)O(1) = O(x) = O(x \log x) .$$

Note carefully that although

$$O(x) = O(x \log x) ,$$
$$O(x \log x) \neq O(x)!$$

(ii) If $f(x) = x \cos x \, e^{\sqrt{(\log x)}} + x \sin x \log^{-9} x$, then, as $x \to \infty$,

$$f(x) = O(xe^{-\sqrt{(\log x)}}) + O(x \log^{-9} x)$$
$$= O(x \log^{-9} x) = O(x) .$$

(iii) If $s = \sigma + it$, and

$$f(s) = \sum_{n=1}^{\infty} \frac{\sin n}{n^s},$$

then, as $t \to \infty$,

$$\begin{aligned} f(s) &= \sum_{n=1}^{\infty} O(n^{-\sigma}) \\ &= O\left(\sum_{n=1}^{\infty} n^{-\sigma}\right) \\ &= O(1), \end{aligned}$$

if $\sigma > 1$. However the constant implied by the O depends on σ in a critical manner.

B. Little o. Suppose that $f(x)$ and $g(x)$ are defined in a neighborhood of a, and suppose that $g(x) > 0$. Then we say that $f(x)$ is "little o of $g(x)$" and we write

$$f(x) = o(g(x))$$

if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

In a similar manner, we define "little o" for sequences. We write

$$f(n) = o(g(n))$$

if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$

It is easily seen that if

$$f_i = o(g_i) \qquad (i = 1, 2),$$

then

$$f_1 f_2 = o(g_1 g_2).$$

As for "big O," we define $o(g)$ as the class of functions $D(g)$ with the property that $f \in D$ if and only if $f = o(g)$. Then we can define

$$o(g) + o(h) \quad \text{and} \quad o(g)o(h).$$

If $D(g) \subset D(h)$, we write

$$o(g) = o(h).$$

If $C(g)$ is the class of functions which are $O(g)$, and $C(g) \subset D(h)$, we write

$$O(g) = o(h).$$

Thus we encounter statements of the following types:

$$\begin{aligned} f &= g_1 + g_2 \\ &= O(g_3) + O(g_4) \\ &= O(g_5) = o(g_6) \end{aligned}$$

and

$$\begin{aligned} f &= O(g_1) + O(g_2) \\ &= o(g_3) + o(g_4) \\ &= o(g_5) \;. \end{aligned}$$

C. **Asymptotic equality**. Finally we define $\sim$. If f and g are two functions defined in a neighborhood of a, we say that f is asymptotic to g and write

$$f \sim g$$

if

$$\lim_{x \to a} \frac{f}{g} = 1 \;.$$

The definition applies to both functions of real or complex variables and to sequences. The relation is evidently symmetric and transitive.

APPENDIX A

The gamma function and the Mellin transform. Though there are many equivalent definitions of the gamma function, one of the most convenient starting points is the Weierstrass product formula. For all s, we define

$$\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}, \tag{1}$$

where γ is Euler's constant, $\gamma = \lim_{N\to\infty} (\sum_{n=1}^{N} 1/n - \log N)$. We show that this is analytic for all s.

THEOREM A.1. *The product*

$$se^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}$$

represents an analytic function of s for all values of s.

PROOF. Let k be arbitrary and suppose that $|s| < k/2$. Then for $n > k$,

$$\begin{aligned} \left|\log\left(1 + \frac{s}{n}\right) - \frac{s}{n}\right| &\leqq \left|-\frac{1}{2}\frac{s^2}{n^2} + \frac{1}{3}\frac{s^3}{n^3} - \cdots\right| \\ &\leqq \frac{|s|^2}{n^2}\left(1 + \left|\frac{s}{n}\right| + \left|\frac{s}{n}\right|^2 + \cdots\right) \\ &\leqq \frac{|s|^2}{n^2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) \leqq \frac{k^2}{4n^2}\cdot 2 = \frac{1}{2}\frac{k^2}{n^2}. \end{aligned}$$

It follows that

$$\sum_{n=k+1}^{\infty} \left|\log\left(1 + \frac{s}{n}\right) - \frac{s}{n}\right| \leqq \frac{1}{2}\sum_{n=k+1}^{\infty} \frac{k^2}{n^2} = O(1),$$

and therefore

$$\sum_{n=k+1}^{\infty} \left(\log\left(1 + \frac{s}{n}\right) - \frac{s}{n}\right) \tag{2}$$

is an absolutely and uniformly convergent series of analytic functions which is therefore itself analytic. Consequently its exponential

$$\prod_{n=k+1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}$$

is analytic; hence

$$se^{s\gamma} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n} \tag{3}$$

is analytic for $|s| < \frac{1}{2}k$. However, k was arbitrarily chosen and therefore (3) is analytic for all s.

From this definition of $\Gamma(s)$, we see that $1/\Gamma(s)$ has zeros at $s = 0, -1, -2,$

$\cdots$, and therefore that $\Gamma(s)$ itself is analytic everywhere except for poles at $0, -1, -2, \cdots$.

THEOREM A.2.

$$\Gamma(s) = \frac{1}{s} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^s \left(1 + \frac{s}{n}\right)^{-1}, \tag{4}$$

the formula being valid except for $s = 0, -1, -2, \cdots$.

PROOF. The proof is a straightforward consequence of (1):

$$\begin{aligned}
\frac{1}{\Gamma(s)} &= s \lim_{m\to\infty} \left[\exp s\left(\sum_{n=1}^{m} \frac{1}{n} - \log m\right)\right] \prod_{n=1}^{m} \left(1 + \frac{s}{n}\right) e^{-s/n} \\
&= s \lim_{m\to\infty} m^{-s} \prod_{n=1}^{m} \left(1 + \frac{s}{n}\right) = s \lim_{m\to\infty} \prod_{n=1}^{m-1} \left(1 + \frac{1}{n}\right)^{-s} \prod_{n=1}^{m} \left(1 + \frac{s}{n}\right) \\
&= s \lim_{m\to\infty} \prod_{n=1}^{m} \left(1 + \frac{1}{n}\right)^{-s} \left(1 + \frac{s}{n}\right) \left(1 + \frac{1}{m}\right)^{s}.
\end{aligned}$$

Since $(1 + 1/m)^s \to 1$, the proof is complete.
Two important corollaries follow.

THEOREM A.3.

$$\Gamma(s) = \lim_{n\to\infty} \frac{(n-1)!}{s(s+1)\cdots(s+n-1)}\, n^s. \tag{5}$$

PROOF. From (4),

$$\begin{aligned}
\Gamma(s) &= \frac{1}{s} \lim_{n\to\infty} \prod_{k=1}^{n-1} \left(\frac{k+1}{k}\right)^s \left(\frac{k}{k+s}\right) = \frac{1}{s} \lim_{n\to\infty} n^s \prod_{k=1}^{n-1} \left(\frac{k}{k+s}\right) \\
&= \frac{1}{s} \lim_{n\to\infty} n^s \frac{(n-1)!}{(s+1)(s+2)\cdots(s+n-1)}.
\end{aligned}$$

The next corollary exhibits $\Gamma(s)$ as an interpolation formula for $s!$.

THEOREM A.4.

$$\Gamma(s+1) = s\Gamma(s). \tag{6}$$

In particular, if s *is a positive integer,*

$$\Gamma(s+1) = s!. \tag{7}$$

PROOF. Again from (4),

$$\begin{aligned}
\frac{\Gamma(s+1)}{\Gamma(s)} &= \frac{s}{s+1} \lim_{m\to\infty} \prod_{n=1}^{m} \left(1 + \frac{1}{n}\right)^{s+1} \left(1 + \frac{s+1}{n}\right)^{-1} \left(1 + \frac{1}{n}\right)^{-s} \left(1 + \frac{s}{n}\right) \\
&= \frac{s}{s+1} \lim_{m\to\infty} \prod_{n=1}^{m} \left(1 + \frac{1}{n}\right) \left(\frac{n+s}{n+s+1}\right) \\
&= \frac{s}{s+1} \lim_{m\to\infty} (m+1) \prod_{n=1}^{m} \left(\frac{n+s}{n+s+1}\right) = s \lim_{m\to\infty} \frac{m+1}{m+s+1} = s.
\end{aligned}$$

The next result is a functional relation which establishes a connection with the circular functions.

THEOREM A.5.

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}\,. \tag{8}$$

PROOF. From the definition (1),

$$\Gamma(s)\Gamma(-s) = -\frac{1}{s^2}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)^{-1}e^{s/n}\prod_{n=1}^{\infty}\left(1-\frac{s}{n}\right)^{-1}e^{-s/n} = -\frac{1}{s^2}\prod_{n=1}^{\infty}\left(1-\frac{s^2}{n^2}\right)^{-1}.$$

On the other hand, the Weierstrass product for $(\sin \pi s)/\pi s$ is $\prod_{n=1}^{\infty}(1-s^2/n^2)$, and therefore

$$\Gamma(s)\Gamma(-s) = -\frac{1}{s^2}\,\frac{\pi s}{\sin \pi s} = \frac{-\pi}{s \sin \pi s}\,. \tag{9}$$

From (6), however,

$$\Gamma(1-s) = -s\,\Gamma(-s)\,,$$

and the theorem follows from (9).

In particular, if $s = \frac{1}{2}$,

$$\Gamma\left(\tfrac{1}{2}\right)^2 = \pi\,, \qquad \Gamma\left(\tfrac{1}{2}\right) = \pm\sqrt{\pi}\,,$$

but from the definition, $\Gamma(\frac{1}{2}) > 0$, and therefore

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\,. \tag{10}$$

We prove Legendre's duplication formula in the following:

THEOREM A.6.

$$\Gamma(2s) = \pi^{-1/2}\,2^{2s-1}\,\Gamma(s)\,\Gamma\left(s+\tfrac{1}{2}\right)\,. \tag{11}$$

PROOF. The proof starts from (5) of Theorem A.3.

$$\Gamma(2s) = \lim_{n\to\infty}\frac{(2n-1)!\,(2n)^{2s}}{2s(2s+1)\cdots(2s+2n-1)}\,,$$

and therefore

$$\begin{aligned}
&\frac{2^{2s-1}\Gamma(s)\Gamma(s+\frac{1}{2})}{\Gamma(2s)}\\
&= \lim_{n\to\infty}\frac{2^{2s-1}((n-1)!)^2 n^{2s+1/2}(2s)(2s+1)\cdots(2s+2n-1)}{(2n)^{2s}(2n-1)!\,s(s+1)\cdots(s+n-1)(s+\frac{1}{2})(s+\frac{3}{2})\cdots(s+\frac{1}{2}+n-1)}\\
&= \lim_{n\to\infty}\frac{2^{2n-1}((n-1)!)^2 n^{1/2}(2s)(2s+1)\cdots(2s+2n-1)}{2s(2s+2)\cdots(2s+2n-2)(2s+1)(2s+3)\cdots(2s+2n-1)(2n-1)!}\\
&= \lim_{n\to\infty}\frac{2^{2n-1}((n-1)!)^2 n^{1/2}}{(2n-1)!} = \lim_{n\to\infty}\varphi(n) \qquad \text{(say)}.
\end{aligned}$$

We notice that the right-hand side is independent of s. Hence its value may be determined by giving s some convenient value. For example, we let $s = \frac{1}{2}$, then

$$\lim_{n\to\infty} \varphi(n) = \frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(1)} = \Gamma(\tfrac{1}{2}) = \sqrt{\pi}\ ,$$

by (10). This observation completes the proof.

We can convert $\Gamma(s)$ into what is, perhaps, a more familiar integral formula.

THEOREM A.7. *If $s = \sigma + it$, and $\sigma > 0$, then*

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx\ . \tag{12}$$

PROOF. Because

$$e^{-x} = \lim_{n\to\infty}\left(1 - \frac{x}{n}\right)^n ,$$

we can expect that

$$\gamma(s, n) = \int_0^n \left(1 - \frac{x}{n}\right)^n x^{s-1}\, dx$$

will converge to the integral in (12). On the other hand, we evaluate $\gamma(s, n)$ explicitly. In fact, if $u = x/n$, then

$$\gamma(s, n) = n^s \int_0^1 (1 - u)^n u^{s-1}\, du\ .$$

If n is an integer > 0, we integrate by parts n times and an easy calculation gives

$$\begin{aligned} \gamma(s, n) &= n^s \cdot \frac{n}{s} \cdot \frac{n-1}{s+1} \cdots \frac{1}{s+n-1} \int_0^1 u^{s+n-1}\, du \\ &= \frac{n^s n!}{s(s+1)\cdots(s+n-1)(s+n)}\ . \end{aligned} \tag{13}$$

Thus on the one hand, the right-hand side of (13) converges to $\Gamma(s)$ by Theorem A.3. On the other hand, it remains to show that $\gamma(s, n)$ converges to the integral in (12). This is seen as follows:

$$\begin{aligned} \lim_{n\to\infty}\left\{\int_0^\infty e^{-x}x^{s-1}dx - \gamma(s, n)\right\} &= \lim_{n\to\infty}\left\{\int_0^n \left(e^{-x} - \left(1 - \frac{x}{n}\right)^n\right) x^{s-1}dx + \int_n^\infty e^{-x}x^{s-1}dx\right\} \\ &= \lim_{n\to\infty}(j_1 + j_2)\ . \end{aligned}$$

Since $\sigma > 0$, the integral in (12) converges and therefore $\lim_{n\to\infty} j_2 = 0$. To show that j_1 tends to 0, we notice that the sequence $(1 - x/n)^n$ converges to e^{-x} from below while $(1 + x/n)^n$ converges to e^x also from below; therefore

$$0 \leqq e^{-x} - \left(1 - \frac{x}{n}\right)^n \leqq e^{-x}\left\{1 - e^x\left(1 - \frac{x}{n}\right)^n\right\}$$

$$\leqq e^{-x}\left\{1 - \left(1 - \frac{x}{n}\right)^n\left(1 + \frac{x}{n}\right)^n\right\} \leqq e^{-x}\left\{1 - \left(1 - \frac{x^2}{n^2}\right)^n\right\}$$

$$\leqq e^{-x}\frac{x^2}{n^2}\cdot n = \frac{x^2 e^{-x}}{n} .$$

Consequently,

$$j_1 = O\left(\frac{1}{n}\right)\int_0^n e^{-x}x^{\sigma+1}\,dx = O\left(\frac{1}{n}\right) = o(1) .$$

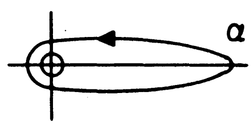
This completes the proof.

The integral of Theorem A.6 is valid only for $\sigma > 0$; we derive a contination of the integral of (12) which is valid for all s (we bypass the singularities of $\Gamma(s)$).

Theorem A.8. *If $\mathscr{C}$ denotes a path which starts at ∞, circles the origin in a counter-clockwise direction and returns to ∞, then*

$$\Gamma(s) = -\frac{1}{2i\,\sin \pi s}\int_{\mathscr{C}}(-t)^{s-1}e^{-t}\,dt . \tag{14}$$

Proof. The proof incorporates the principle of the so-called Hankel transform. Let D be a contour which starts at α on the real axis, circles



the origin in a counter-clockwise direction and returns to α. We consider the integral

$$\int_D (-u)^{s-1}e^{-u}\,du ,$$

with $\sigma > 0$ and s not an integer. The many-valued function $(-u)^{s-1} = \exp[(s-1)\log(-u)]$ is made precise by choosing that branch of the logarithm which is real when $u < 0$; that is to say, on D, $-\pi \leqq \arg(-u) \leqq \pi$. We transform D itself into a path which starts at α, proceeds along the real axis to a point δ, circles the origin counter-clockwise by a circle of radius δ and returns to α along the lower part of the real axis. On the upper part of the real axis, we have

$$\arg(-u) = -\pi ,$$

so that

$$(-u)^{s-1} = \exp[(s-1)\log(-u)] = \exp[(s-1)(-\pi i + \log u)] = u^{s-1}e^{-i\pi(s-1)} \tag{15}$$

and on the lower part, by the same reasoning,

$$(-u)^{s-1} = u^{s-1}e^{i\pi(s-1)} . \tag{16}$$

On the circle, write

$$-u = \delta e^{i\theta} ,$$

and then by (15) and (16)

$$\begin{aligned}\int_D (-u)^{s-1}e^{-u}\,du &= \int_\alpha^\delta e^{-i\pi(s-1)}u^{s-1}e^{-u}\,du + \int_\delta^\alpha e^{i\pi(s-1)}u^{s-1}e^{-u}\,du \\ &\quad + \int_{-\pi}^{\pi} (\delta e^{i\theta})^{s-1}e^{\delta(\cos\theta + i\sin\theta)}\delta e^{i\theta} i\,d\theta .\end{aligned} \tag{17}$$

The first and second integrals combine to give

$$-2i\sin\pi s \int_\delta^\alpha u^{s-1}e^{-u}du ,$$

while the third integral clearly tends to 0 as $\delta \to 0$. Consequently, from (17)

$$\int_D (-u)^{s-1}e^{-u}du = -2i\ \sin\pi s \int_0^\alpha u^{s-1}e^{-u}du .$$

This relation holds for all $\alpha > 0$. We let $\alpha \to \infty$ and we let $\mathscr{C}$ be the "limit" of the path D, then

$$\int_{\mathscr{C}} (-u)^{s-1}e^{-u}du = -2i\ \sin\pi s \int_0^\infty u^{s-1}e^{-u}du .$$

In other words,

$$\Gamma(s) = -\frac{1}{2i\ \sin\pi s}\int_{\mathscr{C}} (-u)^{s-1}e^{-u}du ,$$

as was to be proved.

The importance of this representation stems from the fact that since $\mathscr{C}$ does not pass through the origin, the integral is a single-valued and analytic function of s for all s. The restriction $\sigma > 0$ is no longer necessary. The formula (14) holds for all s except for $s = 0, \pm 1, \pm 2, \cdots$.

The next theorems concern the asymptotic behavior of $\Gamma(s)$. We prove first a somewhat debased form of Stirling's formula.

Theorem A.9. *If N is an integer, then there exists a constant c such that*

$$\log N! = \sum_{n\le N} \log n = \left(N + \frac{1}{2}\right)\log N - N + c + O\left(\frac{1}{N}\right) . \tag{18}$$

Proof. We use the Euler-MacLaurin formula,

$$\begin{aligned}\sum_{n\le N} \log n &= \frac{1}{2}\log N + \int_1^N \log x\,dx + \int_1^N \frac{x - [x] - \frac{1}{2}}{x}\,dx \\ &= \frac{1}{2}\log N + N\log N - N + \int_1^N \frac{x - [x] - \frac{1}{2}}{x}\,dx .\end{aligned} \tag{19}$$

On the other hand, if we put

$$\varphi(x) = \int_1^x \left(u - [u] - \frac{1}{2}\right) du ,$$

then because the integrand has period 1 and $\varphi(2) = \varphi(1) = 0$, it follows that $\varphi(x)$ is bounded, in fact,

$$|\varphi(x)| \leqq \tfrac{1}{2} .$$

If now we integrate by parts the integral in (19), we get

$$\begin{aligned}\int_1^N \frac{u - [u] - \frac{1}{2}}{u} du &= \frac{\varphi(N)}{N} + \int_1^N \frac{\varphi(x)}{x^2} dx \\ &= \frac{\varphi(N)}{N} + \int_1^\infty \frac{\varphi(x)}{x^2} dx - \int_N^\infty \frac{\varphi(x)}{x^2} dx \\ &= O\left(\frac{1}{N}\right) + c + O\left(\int_N^\infty \frac{dx}{x^2}\right) \\ &= O\left(\frac{1}{N}\right) + c .\end{aligned}$$

We have used the fact that $\int_1^\infty \varphi(x)/x^2$ converges and have denoted its value by c. This proves the theorem.

We pass to the general case.

THEOREM A.10. *There exists an absolute constant a such that if s is not on the negative real axis, i. e.,*

$$-\pi + \delta \leqq \arg s \leqq \pi - \delta , \tag{20}$$

for $\delta > 0$, then

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + a + O\left(\frac{1}{|s|}\right) . \qquad (s \neq 0) \tag{21}$$

PROOF. By definition,

$$\begin{aligned}\log \Gamma(s) &= \lim_{N\to\infty} \left\{\sum_{n=1}^N \left(\frac{s}{n} - \log\left(1 + \frac{s}{n}\right)\right)\right\} - \gamma s - \log s \\ &= \lim_{N\to\infty} \left\{\sum_{n=1}^N \frac{s}{n} - \sum_{n=0}^N \log(n+s) + \sum_{n=1}^N \log n\right\} - \gamma s .\end{aligned} \tag{22}$$

We apply the Euler-MacLaurin formula to the second sum:

$$\begin{aligned}\sum_{n=0}^N \log(n+s) &= \frac{1}{2}\log(N+s) + \frac{1}{2}\log s + \int_0^N \log(x+s)\,dx + \int_0^N \frac{x-[x]-\frac{1}{2}}{x+s} dx \\ &= \frac{1}{2}\log(N+s) + \left(\frac{1}{2} - s\right)\log s + s + (N+s)\log(N+s) \\ &\quad - (N+s) + \int_0^N \frac{x-[x]-\frac{1}{2}}{x+s} dx .\end{aligned} \tag{23}$$

Accordingly, if we use (18) and the fact, proved previously Chapter II, Theorem 2.4, that

$$\sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma + O\left(\frac{1}{N}\right),$$

we get from (22) and (23)

$$\begin{aligned}
\log \Gamma(s) &= \left(s - \frac{1}{2}\right)\log s + c \\
&\quad + \lim_{N\to\infty} \Big\{ s(\log N - \log(N+s)) + N(\log N - \log(N+s)) \\
&\quad + \frac{1}{2}(\log N - \log(N+s)) - \int_0^N \frac{x - [x] - \frac{1}{2}}{x+s}\, dx \Big\} \qquad (24) \\
&= \left(s - \frac{1}{2}\right)\log s - s + a - \int_0^\infty \frac{x - [x] - \frac{1}{2}}{x+s}\, dx .
\end{aligned}$$

As in the previous theorem, we integrate the integral in (24) by parts:

$$\begin{aligned}
\int_0^\infty \frac{x - [x] - \frac{1}{2}}{x+s}\, dx = \int_0^\infty \frac{\varphi(x)}{(x+s)^2}\, dx &= O\left(\int_0^\infty \frac{dx}{x^2 + 2x\sigma + |s|^2}\right) \\
&= O\left(\int_0^\infty \frac{dx}{x^2 + 2x|s|\cos\arg s + |s|^2}\right) \\
&= O\left(\int_0^\infty \frac{dx}{x^2 - 2x|s|\cos\delta + |s|^2}\right),
\end{aligned}$$

where we have used the fact (which follows from (20)) that $\cos\arg s \geqq -\cos\delta$. The substitution $x/|s| = u$ gives

$$\int_0^\infty \frac{dx}{x^2 - 2x|s|\cos\delta + |s|^2} = O\left(\frac{1}{|s|}\right)\int_0^\infty \frac{du}{u^2 - 2u\cos\delta + 1} = O\left(\frac{1}{|s|}\right),$$

as required.

As a corollary, we deduce an important result concerning the behavior of $\Gamma(\sigma + it)$ for fixed σ and large t.

THEOREM A.11. *If*

$$\sigma_1 \leqq \sigma \leqq \sigma_2 ,$$

then for some constant K, and for $|t| > 1$,

$$|\Gamma(\sigma + it)| = K|t|^{\sigma - 1/2}e^{-\pi|t|/2}\left(1 + O\left(\frac{1}{|t|}\right)\right), \qquad (25)$$

the constant implied by O depending only on σ_1 and σ_2.

PROOF. From (21) of Theorem A.9,

$$\log \Gamma(\sigma + it) = \left(\sigma + it - \frac{1}{2}\right)\log(\sigma + it) - (\sigma + it) + a + O\left(\frac{1}{|t|}\right), \qquad (26)$$

but

$$\log(\sigma + it) = \log(\sigma^2 + t^2)^{1/2} + i \arctan\frac{t}{\sigma} ;$$

hence

$$\mathscr{R}\left(\left(\sigma + it - \frac{1}{2}\right)\log(\sigma + it)\right) = \left(\sigma - \frac{1}{2}\right)\log(\sigma^2 + t^2)^{1/2} - t \arctan\frac{t}{\sigma} . \tag{27}$$

On the other hand,

$$\log(\sigma^2 + t^2) - \log t^2 = \log\left(1 + \left(\frac{\sigma}{t}\right)^2\right) = O\left(\frac{\sigma}{t}\right)^2 = O\left(\frac{1}{t^2}\right),$$

that is,

$$\log(\sigma^2 + t^2)^{\frac{1}{2}} = \log|t| + O\left(\frac{1}{t^2}\right). \tag{28}$$

Moreover, because

$$\arctan\frac{t}{\sigma} + \arctan\frac{\sigma}{t} = \begin{cases} \pi/2 & \text{if } t > 0 , \\ -\pi/2 & \text{if } t < 0 , \end{cases}$$

it follows that

$$\arctan\frac{t}{\sigma} = \pm\frac{\pi}{2} - \arctan\frac{\sigma}{t} = \pm\frac{\pi}{2} - \frac{\sigma}{t} + O\left(\frac{1}{t^2}\right)$$

on expanding the arc tan in a power series. This, together with (27) and (28) gives us

$$\mathscr{R}\left\{\left(\sigma + it - \frac{1}{2}\right)\log(\sigma + it)\right\} = \left(\sigma - \frac{1}{2}\right)\log|t| - \frac{\pi}{2}|t| + \sigma + O\left(\frac{1}{|t|}\right). \tag{29}$$

Therefore from (29),

$$\log|\Gamma(\sigma + it)| = \left(\sigma - \frac{1}{2}\right)\log|t| - \frac{\pi}{2}|t| + a + O\left(\frac{1}{|t|}\right),$$

or

$$\begin{aligned} |\Gamma(\sigma + it)| &= K|t|^{\sigma - 1/2}e^{-\pi|t|/2}\, e^{O(1/|t|)} \\ &= K|t|^{\sigma-1/2}e^{-\pi|t|/2}\left(1 + O\left(\frac{1}{|t|}\right)\right). \end{aligned}$$

Actually, it can be shown that $K = \sqrt{2\pi}$ but we never need this fact.
Finally, concerning the gamma function, we prove

Theorem A.12. *The residue of $\Gamma(s)$ at the pole $s = -k$ is $(-1)^k/k!$.*

Proof. The residue at $s = -k$ is

$$\lim_{s \to -k} (s + k)\Gamma(s) ,$$

which, by Theorem A.3, is

$$\lim_{s\to -k} (s+k) \lim_{n\to\infty} \frac{n!n^s}{s(s+1)\cdots(s+n)}$$

$$= \lim_{s\to -k} \lim_{n\to\infty} \frac{n!n^s(s+k)}{s(s+1)\cdots(s+k-1)(s+k)(s+k+1)\cdots(s+n)}$$

$$= \lim_{n\to\infty} \lim_{s\to -k} \frac{n!n^s}{s(s+1)\cdots(s+k-1)(s+k+1)\cdots(s+n)}$$

$$= \lim_{n\to\infty} \frac{n!n^{-k}}{(-k)(-k+1)\cdots(-1)(1)(2)\cdots(n-k)}$$

$$= \lim_{n\to\infty} \frac{(-1)^k}{k!} \frac{n!}{n^k\cdot(n-k)!}$$

$$= \frac{(-1)^k}{k!} \lim_{n\to\infty} \frac{n(n-1)\cdots(n-k+1)}{n^k} = \frac{(-1)^k}{k!} .$$

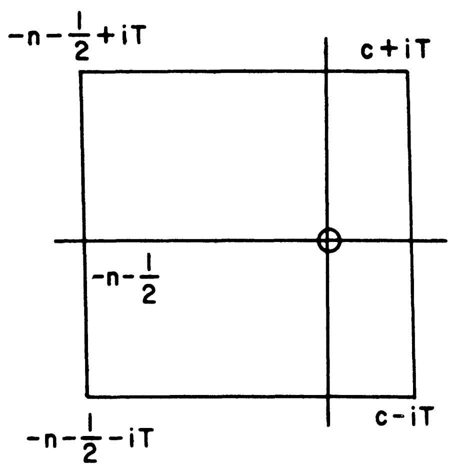
We are now in a position to prove Mellin's formula which was stated without proof and used in §6, Chapter II.

THEOREM A.13. *If $c > 0$, then*

$$e^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}\,ds . \tag{30}$$

PROOF. The formula is, so to speak, an inversion of formula (12) of Theorem A.6. The proof uses contour integration. The right-hand side is

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \Gamma(s)x^{-s}\,ds . \tag{31}$$



We consider the contour shown in the diagram. Then

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \Gamma(s)x^{-s}\,ds &= \frac{1}{2\pi i}\int_{c-iT}^{-n-1/2-iT} \Gamma(s)x^{-s}ds + \frac{1}{2\pi i}\int_{-n-1/2-iT}^{-n-1/2+iT} \Gamma(s)x^{-s}ds \\ &+ \frac{1}{2\pi i}\int_{-n-1/2+iT}^{c+iT} \Gamma(s)x^{-s}ds + \text{sum of the residues} .\end{aligned} \tag{32}$$

The integrand has simple poles at $s = 0, -1, \cdots, -n$, and the residue at $s = -k$ is $(-1)^k x^k/k!$. We call the integrals in (32) I_1, I_2, I_3, respectively. Then

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\Gamma(s)x^{-s}ds = I_1 + I_2 + I_3 + \sum_{k=0}^{n}\frac{(-1)^k}{k!}x^k . \tag{33}$$

It remains to show that I_1, I_2, I_3 converge to 0 as $n, T \to \infty$. We consider first I_3; I_1 is treated in the same way:

$$\begin{aligned} I_3 &= \frac{1}{2\pi i}\int_{-n-1/2+iT}^{c+iT}\Gamma(s)x^{-s}ds = \frac{1}{2\pi i}\int_{-n-1/2}^{c}\Gamma(\sigma+iT)x^{-\sigma-iT}d\sigma \\ &= O\left(\int_{-n-1/2}^{c} e^{-\pi|T|/2}\,|T|^{\sigma-1/2}x^{-\sigma}\,d\sigma\right), \end{aligned} \tag{34}$$

by Theorem A.11. The integral in (34), however, is

$$O\left\{\frac{e^{-\pi|T|/2}}{T^{1/2}}\left(\frac{(|T|x^{-1})^c}{\log|T|x^{-1}} - \frac{(|T|x^{-1})^{-n-1/2}}{\log|T|x^{-1}}\right)\right\} = o(1) \qquad \text{as } T\to\infty . \tag{35}$$

We have therefore shown that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)x^{-s}ds = \sum_{k=0}^{n}\frac{(-1)^k}{k}x^k + \int_{-n-1/2-i\infty}^{-n-1/2+i\infty}\Gamma(s)x^{-s}ds . \tag{36}$$

It remains to show that the integral on the right converges to 0 as $n\to\infty$. Indeed

$$I = \int_{-n-1/2-i\infty}^{-n-1/2+i\infty}\Gamma(s)x^{-s}ds = \int_{-\infty}^{\infty}\Gamma\left(-n-\frac{1}{2}+it\right)x^{n+1/2-it}dt . \tag{37}$$

Using the functional equation for $\Gamma(s)$ in the integrand on the right, we find

$$\Gamma\left(-n-\frac{1}{2}+it\right) = \frac{\Gamma(\frac{1}{2}+it)}{(-n-\frac{1}{2}+it)\cdots(-\frac{1}{2}+it)} = O\left(\frac{|\Gamma(\frac{1}{2}+it)|}{(n+1)!}\right). \tag{38}$$

Then using (25), we get from (37) and (38),

$$\begin{aligned} I &= O\left(\int_{-1}^{1}\frac{|\Gamma(\frac{1}{2}+it)|}{(n+1)!}x^{n+1/2}dt\right) + O\left(\int_{1}^{\infty}\frac{e^{-\pi t/2}x^{n+1/2}}{(n+1)!}dt\right) \\ &= O\left(\frac{x^{n+1/2}}{(n+1)!}\right)\int_{-1}^{1}\left|\Gamma\left(\frac{1}{2}+it\right)\right|dt + O\left(\frac{x^{n+1/2}}{(n+1)!}\right). \end{aligned} \tag{39}$$

The constants implied by the O are independent of n and t. On the other hand

$$\int_{-1}^{1}\left|\Gamma\left(\frac{1}{2}+it\right)\right|dt = O(1) . \tag{40}$$

Letting $n\to\infty$, the assertion of the theorem follows from (39) and (40).

APPENDIX B

The functional equations of the functions $\zeta(s)$ and $L(s,\chi)$. In Theorem 3.5, of Chapter I, we showed that $\zeta(s)$ is analytic for $\sigma > 0$ except for a simple pole at $s = 1$. We shall show here that $\zeta(s)$ is a meromorphic function whose only singularity is at $s = 1$ and moreover that it satisfies a relatively simple functional equation.

In addition, the same ideas applied to $L(s,\chi)$ show that $L(s,\chi)$ for $\chi \neq \chi_1$, is entire and satisfies a similar type of functional equation.

The proof for the zeta function stems from Riemann. The starting point is the gamma function. Since

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t}t^{s/2-1}dt\,, \qquad \sigma > 0\,, \tag{1}$$

we replace t by $\pi n^2 u$ and find directly that

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)n^{-s} = \int_0^\infty e^{-\pi n^2 u}u^{s/2-1}du \tag{2}$$

therefore

$$\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty \sum_{n=1}^\infty e^{-\pi n^2 u}u^{s/2-1}du\,, \tag{3}$$

the interchange of integration and summation being clearly justified. Riemann's object in (3) is to introduce the function

$$\omega(u) = \sum_{n=1}^\infty e^{-\pi n^2 u}\,,$$

which is closely allied to the function

$$\theta(u) = \sum_{r=-\infty}^\infty e^{-\pi r^2 u}\,, \tag{4}$$

which is an elliptic function satisfying the simple functional equation

$$\theta(u) = \frac{1}{\sqrt{u}}\,\theta\left(\frac{1}{u}\right). \tag{5}$$

The integral in (3) is well behaved for $\sigma > 0$ but for $\sigma \leqq 0$, trouble occurs in the neighborhood of the lower end point. The object of (5) is to improve matters. Before proceeding therefore, we study in more detail the function $\theta(u)$ defined in (4) or rather a slight generalization of it.

We consider the function

$$\Psi(\tau, \alpha) = \sum_{n=-\infty}^\infty e^{-\pi(n+\alpha)^2\tau}\,, \tag{6}$$

for real α and $\tau > 0$. The series converges absolutely. It is our first object to prove the following

THEOREM B.1.

$$\Psi(\tau, \alpha) = \frac{1}{\sqrt{\tau}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/\tau - 2\pi i n\alpha} . \tag{7}$$

The formula will then hold by analytic continuation for all τ such that $\mathscr{R}(\tau) > 0$.

PROOF. The left-hand side of (7) is

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2\tau - 2\pi n\alpha\tau - \pi\alpha^2\tau} .$$

We are therefore required to prove that

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2\tau - 2n\pi\alpha\tau} = \frac{e^{\pi\alpha^2\tau}}{\sqrt{\tau}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/\tau - 2\pi i n\alpha} . \tag{8}$$

Our natural recourse is Cauchy's theorem and the calculus of residues. In fact if z is the complex variable $x + iy$, then the function

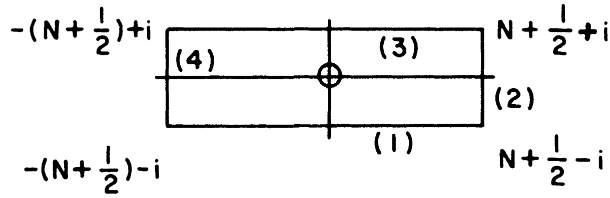
$$f(z) = \frac{e^{-\pi z^2\tau - 2\pi\alpha z\tau}}{e^{2\pi i z} - 1} \tag{9}$$

has simple poles at $z = 0, \pm 1, \pm 2, \cdots$ with residue

$$\frac{1}{2\pi i} e^{-\pi r^2\tau - 2\pi r\alpha\tau} \tag{10}$$

at the simple pole $z = r$.

We consider the rectangle $\mathscr{C}$ in the z plane with vertices at $N + \frac{1}{2} \pm i$,



$-(N + \frac{1}{2}) \pm i$ where N is a positive integer. We label the segments of the path (1), (2), (3), (4). By Cauchy's theorem, we get from (9) and (10),

$$\int_{\mathscr{C}} \frac{e^{-\pi\tau z^2 - 2\pi\alpha\tau z}}{e^{2\pi i z} - 1} dz = \sum_{n=-N}^{N} e^{-\pi n^2\tau - 2\pi n\alpha\tau} . \tag{11}$$

The integrals along the vertical sides (2) and (4) are $o(1)$ as $N \to \infty$. Along (2), $z = N + \frac{1}{2} + iy$, and a simple calculation shows that for some constant c

$$\begin{aligned} \int_{(2)} \frac{e^{-\pi\tau z^2 - 2\pi\alpha\tau z}}{e^{2\pi i z} - 1} dz &= O\left(\int_{-1}^{1} \frac{e^{-cN}}{||e^{-2\pi y} \cdot e^{2\pi i (N+1/2)}| - 1|} dy\right) \\ &= O\left(\frac{e^{-cN}}{e^{-2\pi y} + 1}\right) = o(1) . \end{aligned} \tag{12}$$

A similar argument holds for the integral along (4). Thus letting $N \to \infty$, we conclude from (11) and (12) that

$$e^{\pi\alpha^2\tau}\Psi(\tau, \alpha) = \int_{-\infty-i}^{\infty-i} f(z)dz - \int_{-\infty+i}^{\infty+i} f(z)dz \,, \tag{13}$$

the integrals being absolutely convergent as a simple calculation will show. Since along the path in the first integral of (13)

$$|e^{2\pi i z}| = e^{2\pi} > 1,$$

it follows that

$$\sum_{n=-1}^{-\infty} e^{2\pi i n z} = \frac{1}{e^{2\pi i z} - 1} \,,$$

the series converging uniformly and therefore

$$\int_{-\infty-i}^{\infty-i} f(z)dz = \sum_{n=-1}^{-\infty} \int_{-\infty-i}^{\infty-i} e^{-\pi\tau z^2 - 2\pi z(\alpha\tau - ni)} dz \,. \tag{14}$$

A similar argument shows that

$$\int_{-\infty+i}^{\infty+i} f(z)dz = \sum_{n=0}^{\infty} \int_{-\infty+i}^{\infty+i} e^{-\pi\tau z^2 - 2\pi z(\alpha\tau - ni)} dz \,. \tag{15}$$

On the other hand, completing the square in z, we get

$$e^{\pi\tau(\alpha - ni/\tau)^2} \int_{-\infty\pm i}^{\infty\pm i} e^{-\pi\tau(z+\alpha-ni/\tau)^2} dz = e^{\pi\tau(\alpha-ni/\tau)^2} \int_L e^{-\pi\tau u^2} du \,, \tag{16}$$

where the path L is along a line parallel to the real axis with imaginary part μ (say). Applying Cauchy's theorem again to the rectangle with vertices $\pm W$, $\pm W + i\mu$ (with real W) we find as $W \to \infty$

$$\int_L e^{-\pi\tau u^2} du = \int_{-\infty}^{\infty} e^{-\pi\tau u^2} du \,. \tag{17}$$

Thus since

$$\int_{-\infty}^{\infty} e^{-\pi\tau u^2} du = \frac{1}{\sqrt{(\pi\tau)}} \cdot \sqrt{\pi} = \frac{1}{\sqrt{\tau}} \,, \tag{18}$$

we get from (13), (14), (15), (16), (17), (18)

$$\begin{aligned} e^{\pi\alpha^2\tau}\Psi(\tau, \alpha) &= \frac{1}{\sqrt{\tau}} \sum_{n=-\infty}^{\infty} e^{\pi\tau(\alpha - ni/\tau)^2} \\ &= \frac{1}{\sqrt{\tau}} e^{\pi\tau\alpha^2} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/\tau - 2\pi i n\alpha} \,, \end{aligned}$$

and this is what we set out to prove.

Several corollaries follow readily.

Theorem B.2.

$$\Psi(\tau, \alpha) = \frac{1}{\sqrt{\tau}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/\tau} \cos 2\pi n\alpha \,. \tag{19}$$

PROOF. The proof is an immediate consequence of (7) and the fact that

$$e^{-\pi n^2/\tau}\sin 2\pi n\alpha\ ,$$

is an odd function in n, for then the series

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2/\tau}\sin 2\pi n\alpha$$

must vanish.

If we specialize α, we get at once

THEOREM B.3. *If*

$$\Psi(\tau, 0) = \theta(\tau)\ ,$$

then

$$\theta(\tau) = \frac{1}{\sqrt{\tau}}\,\theta\left(\frac{1}{\tau}\right). \tag{20}$$

If we differentiate both sides of (19) with respect to α, we get

THEOREM B.4.

$$\sum_{n=-\infty}^{\infty}(n+\alpha)e^{-\pi\tau(n+\alpha)^2} = \frac{1}{\tau\sqrt{\tau}}\sum_{n=-\infty}^{\infty} ne^{-n^2\pi/\tau}\sin(2\pi n\alpha)\ . \tag{21}$$

PROOF. The proof follows from the uniform convergence of both sides in α.

For dealing with the L functions, we shall require series similar to the above but involving characters. Let χ be a primitive character modulo k. Two cases arise in a natural way. In the first place $\chi(-1) = \pm 1$; we therefore consider

Case (i). Suppose that

$$\chi(-1) = 1\ . \tag{22}$$

We define

$$\psi(\tau, \chi) = 2\sum_{n=1}^{\infty}\chi(n)e^{-\pi n^2\tau/k} \tag{23}$$

and shall show that $\psi(\tau, \chi)$ satisfies a functional equation.

In fact since

$$\chi(n)e^{-\pi n^2\tau/k}$$

is, by (22), an even function of n and since $\chi(0) = 0$, we get

$$\psi(\tau, \chi) = \sum_{n=-\infty}^{\infty}\chi(n)e^{-\pi n^2\tau/k}\ . \tag{24}$$

We break the summation in (24) into residue classes modulo k by putting $n = mk + r$, and we get

$$\psi(\tau, \chi) = \sum_{r=0}^{k} \chi(r) \sum_{m=-\infty}^{\infty} e^{-\pi(m+r/k)^2 k\tau} = \sum_{r=0}^{k} \chi(r) \Psi\left(k\tau, \frac{r}{k}\right), \tag{25}$$

where $\Psi(k\tau, r/k)$ is defined by (6).

If we apply (19) to $\Psi(k\tau, r/k)$, we find from (25)

$$\begin{aligned} \psi(\tau, \chi) &= \sum_{r=0}^{k} \chi(r) \frac{1}{\sqrt{(k\tau)}} \sum_{m=-\infty}^{\infty} e^{-\pi m^2/k\tau} \cos \frac{2\pi rm}{k} \\ &= \frac{1}{\sqrt{(k\tau)}} \sum_{m=-\infty}^{\infty} e^{-\pi m^2/k\tau} \sum_{r=0}^{k} \chi(r) \cos \frac{2\pi rm}{k} . \end{aligned} \tag{26}$$

On the other hand, since $\chi(-1) = 1$, and therefore $\chi(k-n) = \chi(n)$ and since $\sin(2\pi m(k-n)/k) = -\sin(2\pi mn/k)$, it follows that

$$\sum_{r=0}^{k} \chi(r) \sin \frac{2\pi mr}{k} = 0 . \tag{27}$$

Accordingly from (26) and (27),

$$\psi(\tau, \chi) = \frac{1}{\sqrt{(k\tau)}} \sum_{m=-\infty}^{\infty} e^{-\pi m^2/k\tau} \sum_{r=0}^{k} \chi(r) e^{2\pi imr/k} . \tag{28}$$

The inner sum, however, is the familiar Gaussian sum

$$G(m, \chi) = \sum_{r=0}^{k} \chi(r) e^{2\pi imr/k} .$$

By Theorem 4.12,

$$G(m, \chi) = \bar{\chi}(m) G(1, \chi) \tag{29}$$

and therefore from (27), (28) and (29),

$$\psi(\tau, \chi) = \frac{1}{\sqrt{\tau}} \frac{G(1, \chi)}{\sqrt{k}} \sum_{m=-\infty}^{\infty} \bar{\chi}(m) e^{-\pi m^2/k\tau} .$$

Consequently we get

THEOREM B.5. *If $\psi(\tau, \chi)$ is defined by (23) and $G(1, \chi)$ is a Gaussian sum, then*

$$\psi(\tau, \chi) = \frac{1}{\sqrt{\tau}} \frac{G(1, \chi)}{\sqrt{k}} \psi\left(\frac{1}{\tau}, \bar{\chi}\right) . \tag{30}$$

For simplicity, we put

$$\varepsilon(\chi) = \frac{G(1, \chi)}{\sqrt{k}} ; \tag{31}$$

then, by Theorem 4.13, Chapter V, we get

$$|\varepsilon(x)| = \frac{|G(1, \chi)|}{\sqrt{k}} = 1 . \tag{32}$$

Moreover, since

$$G(1,\bar{\chi}) = \sum_{r=0}^{k} \bar{\chi}(r)e^{2\pi ir/k} = \sum_{r=0}^{k} \bar{\chi}(r)e^{-2\pi ir/k} = \overline{G(1,\chi)},$$

because $\chi(-1) = 1$. Therefore it follows that

$$\overline{\varepsilon(\chi)} = \varepsilon(\bar{\chi}).$$

Thus by (32),

$$\varepsilon(\bar{\chi}) = \frac{1}{\varepsilon(\chi)}. \tag{33}$$

Case (ii). $\chi(-1) = -1$. In this case we modify the function $\psi(\tau,\chi)$ for later application. Let

$$\psi_1(\tau,\chi) = 2\sum_{n=1}^{\infty} n\chi(n)e^{-\pi n^2\tau/k}; \tag{34}$$

then exactly as in Case (i), we show using (21) that

$$\begin{aligned}\psi_1(\tau,\chi) &= \sum_{n=-\infty}^{\infty} n\chi(n)e^{-\pi n^2\tau/k} = \frac{1}{i\tau\sqrt{(k\tau)}} \sum_{m=-\infty}^{\infty} me^{(-\pi m^2/k)\tau} \sum_{r=0}^{k} \chi(r)e^{2\pi imr/k} \\ &= \frac{-i\,G(1,\chi)}{\tau\sqrt{\tau}\sqrt{k}}\psi_1\!\left(\frac{1}{\tau},\bar{\chi}\right) = \varepsilon_1(\chi)\frac{1}{\tau\sqrt{\tau}}\psi_1\!\left(\frac{1}{\tau},\bar{\chi}\right),\end{aligned} \tag{35}$$

where

$$\varepsilon_1(\chi) = \frac{-iG(1,\chi)}{\sqrt{k}}.$$

In this case because $\chi(-1) = -1$,

$$\overline{G(1,\chi)} = \sum_{r=0}^{k} \bar{\chi}(r)e^{-2\pi ir/k} = -\sum_{r=0}^{k} \bar{\chi}(r)e^{2\pi ir/k} = -G(1,\bar{\chi}).$$

Therefore

$$\overline{\varepsilon_1(x)} = \frac{i\,\overline{G(1,\chi)}}{\sqrt{k}} = \frac{-i\,G(1,\bar{\chi})}{\sqrt{k}} = \varepsilon_1(\bar{\chi}),$$

and since as above

$$|\varepsilon_1(\chi)| = 1,$$

we get

$$\varepsilon_1(\bar{\chi}) = \frac{1}{\varepsilon_1(\chi)}. \tag{36}$$

We return to proofs of the functional equations for $\zeta(s)$ and $L(s,\chi)$.

THEOREM B.6. *If*

$$\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{37}$$

then $\xi(s)$ *is regular for all* s *except for simple poles at* $s = 0$, $s = 1$. $\xi(s)$ *satisfies the functional equation* $\xi(s) = \xi(1-s)$.

Proof. We had from (3),

$$\xi(s) = \int_0^\infty \omega(u)\, u^{s/2-1} du \,,$$

where

$$\omega(u) = \sum_{n=1}^\infty e^{-\pi n^2 u} \,.$$

We break the interval of integration

$$\xi(s) = \int_0^1 \omega(u) u^{s/2-1} du + \int_1^\infty \omega(u) u^{s/2-1} du \,. \tag{38}$$

In the first integral we replace u by $1/u$, and find from (38),

$$\xi(s) = \int_1^\infty \omega\left(\frac{1}{u}\right) u^{-s/2-1} du + \int_1^\infty \omega(u) u^{s/2-1} du \,. \tag{39}$$

Now

$$1 + 2\omega(u) = \theta(u)$$

and using Theorem B.3, we get

$$1 + 2\omega(u) = \frac{1}{\sqrt{u}}\left(1 + 2\omega\left(\frac{1}{u}\right)\right) . \tag{40}$$

Inserting $\omega(1/u)$ from (40) in the first integral of (39), we deduce, on performing the simple integrations,

$$\xi(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \omega(u)(u^{s/2} + u^{(1-s)/2}) \frac{du}{u} \,. \tag{41}$$

The integral in (41) is regular for all s and the right-hand side is clearly invariant on replacing s by $1-s$. This completes the proof.

We turn to the functional equation for $L(s, \chi)$ for χ a primitive character modulo k. The argument is much the same as the one we used for $\xi(s)$. Naturally there are added complications but we have prepared for these.

We consider again 2 cases.

Case (i). $\chi(-1) = 1$. We start from the gamma function and get

$$\left(\frac{\pi}{k}\right)^{-s/2} \Gamma\left(\frac{s}{2}\right) n^{-s} = \int_0^\infty e^{-\pi n^2 u/k} u^{s/2-1} du$$

and therefore using (23),

$$\begin{aligned} \xi(s, \chi) = \left(\frac{\pi}{k}\right)^{-s/2} \Gamma\left(\frac{s}{2}\right) L(s, \chi) &= \int_0^\infty \sum_{n=1}^\infty \chi(n) e^{-\pi n^2 u/k} u^{s/2-1} du \\ &= \frac{1}{2} \int_0^\infty \psi(u, \chi) u^{s/2-1} du \\ &= \frac{1}{2} \int_0^1 \psi(u, \chi) u^{s/2-1} du + \frac{1}{2} \int_1^\infty \psi(u, \chi) u^{s/2-1} du \,. \end{aligned} \tag{42}$$

We apply (30) to the first integral of (42) after replacing u by $1/u$,

$$\xi(s,\chi) = \frac{\varepsilon(\chi)}{2}\int_1^\infty \psi(u,\bar{\chi})u^{(1-s)/2}\frac{du}{u} + \frac{1}{2}\int_1^\infty \psi(u,\chi)u^{s/2}\frac{du}{u}\,. \tag{43}$$

The integrals on the right of (43) are regular for all s and therefore so is $\xi(s,\chi)$. Moreover

$$\xi(1-s,\bar{\chi}) = \frac{\varepsilon(\bar{\chi})}{2}\int_1^\infty \psi(u,\chi)u^{s/2}\frac{du}{u} + \frac{1}{2}\int_1^\infty \psi(u,\bar{\chi})u^{(1-s)/2}\frac{du}{u}\,.$$

Using (33), however, it follows that

$$\varepsilon(\chi)\ \xi(1-s,\bar{\chi}) = \xi(s,\chi)\,.$$

Case (ii). $\chi(-1) = -1$. In this case we start from

$$\left(\frac{\pi}{k}\right)^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)n^{-s} = \int_0^\infty ne^{-\pi n^2u/k}u^{(s+1)/2-1}du\,.$$

Then it follows that

$$\xi_1(s,\chi) = \left(\frac{\pi}{k}\right)^{-(s+1)/2}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi) = \frac{1}{2}\int_0^\infty \psi_1(u,\chi)u^{(s+1)/2}\frac{du}{u}\,,$$

where $\psi_1(u,\chi)$ is defined by (34). We break the interval of integration as before and apply (35) and deduce

$$\xi_1(s,\chi) = \frac{\varepsilon_1(\chi)}{2}\int_1^\infty \psi_1(u,\bar{\chi})u^{-s/2}du + \frac{1}{2}\int_1^\infty \psi_1(u,\chi)u^{-(1-s)/2}du\,.$$

Again the right-hand side is regular in s and using (36), we deduce

$$\xi_1(s,\chi) = \varepsilon_1(\chi)\,\xi_1(1-s,\bar{\chi})\,. \tag{44}$$

We combine these two into the same

THEOREM B.7. *If χ is a primitive character modulo k which is nonprincipal,*

$$a = \begin{cases} 0 & \text{if } \chi(-1) = 1\,, \\ 1 & \text{if } \chi(-1) = -1 \end{cases}$$

and

$$\xi(s,\chi) = \left(\frac{\pi}{k}\right)^{-(s+a)/2}\Gamma\left(\frac{s+a}{2}\right)L(s,\chi)\,, \tag{45}$$

then $\xi(s,\chi)$ is an entire function of s and satisfies the functional equation

$$\xi(s,\chi) = \varepsilon(\chi)\xi(1-s,\bar{\chi})$$

where

$$\varepsilon(\chi) = \begin{cases} \dfrac{G(1,\chi)}{\sqrt{k}} & \text{if } a = 0\,, \\[2ex] \dfrac{-iG(1,\chi)}{\sqrt{k}} & \text{if } a = 1\,. \end{cases}$$

In the particular case when $\chi(n) = \chi_d(n) = (d/n)$, we have by Theorem 4.17

$$G(1, \chi) = \begin{cases} i\sqrt{|d|} & \text{if } d < 0 . \\ \sqrt{d} & \text{if } d > 0 \end{cases}$$

Therefore in either case ,

$$\varepsilon(\chi) = 1 . \tag{46}$$

COROLLARY 1. *If $a = 0$, the function $L(s, \chi)$ vanishes for $s = 0, -2, -4, \cdots$. If $a = 1$, $L(s, \chi)$ vanishes for $s = -1, -3, -5, \cdots$.*

PROOF. We showed that $L(s, \chi)$ is analytic for $\sigma \geqq \delta > 0$. The poles of $\Gamma((s + a)/2)$ must be cancelled by zeros of $L(s, \chi)$.

If χ is not primitive, we reduce the case to the primitive one by using Theorem 4.7 of Chapter V.

References

The following is a list of books on number theory which are devoted to analytic theory of numbers or which contain sections devoted to the analytic theory.

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Index of Symbols Used

SUBJECT INDEX