On the Connection between Minimal Fractal Manifold and Quantum Spin

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Abstract

As natural outcome of the Renormalization Group program, the *minimal fractal manifold* (MFM) describes a space-time continuum having arbitrarily small deviations from four-dimensions $(\varepsilon = 4 - D, \varepsilon << 1)$. Recent studies point out that MFM offers unexplored solutions to the challenges raised by the Standard Model for particle physics. Here we show that the inner connection between MFM and local conformal field theory (CFT) makes *quantum spin* a topological property of the MFM.

Key words: Minimal Fractal Manifold, Local Conformal Field Theory, Spin Operator, Renormalization Group.

1. Introduction

In his seminal paper of 1939, Wigner has shown that the concept of *quantum spin* follows naturally from the unitary representation of the Poincaré group [1-3]. The two invariant Casimir operators of the Poincaré group, $P_\mu P^\mu = m^2$ and $W^\mu W_\mu = -ms(s+1)$ supply the rest mass m and the spin s of the particle, respectively. Here P^μ is the generator of translations and W^μ the Pauli-Lubanski operator defined as

$$W^\mu = \varepsilon^{\mu\nu\rho\sigma} P_\nu J_{\rho\sigma} \tag{1}$$

in which $\varepsilon^{\mu\nu\rho\sigma}$ stands for the four-dimensional Levi-Civita index and $J^{\mu\nu}$ are the generators of the Lorentz group. The second Casimir invariant implies that the square of the spin three-vector of a massive particle ($\mathbf{S}$) relates to the Pauli-Lubanski operator via

$$\mathbf{S}\cdot\mathbf{S}=\frac{1}{m^2}W^{\mu}W_{\mu} \tag{2}$$

Our brief analysis reveals that quantum spin may be understood outside the traditional framework of representation theory, specifically as emerging attribute of the so-called *minimal fractal manifold* (MFM). As inherent outcome of the Renormalization Group, MFM describes a space-time continuum endowed with arbitrarily small deviations from four-dimensions ($\varepsilon = 4 - D,\ \varepsilon \ll 1$). As recently shown, the MFM is a source of unexplored solutions to the challenges raised by the Standard Model for particle physics [4]. Expanding on these ideas, here we suggest that the inner connection between MFM and local conformal field theory (CFT) makes *quantum spin* a topological property of the MFM.

2. Quantum spin and the MFM

Consider a flat four-dimensional space-time with constant metric having the standard signature $\eta_{\mu\nu} = diag(-1,...,+1)$. A differentiable map $x' = \zeta(x)$ is called a *conformal transformation* if the metric tensor changes as [5]

$$\eta_{\mu\nu} \to \overline{\eta_{\mu\nu}} = \eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Omega^2(x)\eta_{\mu\nu} \tag{3}$$

in which $\Omega^2(x)$ represents the scale factor and Einstein's summation convention is implied. The scale factor is *strictly equal to unity* on flat space-times ($\Omega^2(x)=1$), a condition matching the translations and rotations group of Lorentz transformations. In general, if the underlying space-time background deviates from flatness and is characterized by a metric $g_{\mu\nu}(x) \neq \eta_{\mu\nu}$, the condition for local conformal transformation (3) reads

$$g_{\mu\nu}(x) \to \overline{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \tag{4}$$

where $\Omega^2(x) \neq 1$. A *nearly conformal transformation* (NCT) is defined by a scale factor departing slightly and continuously from unity, that is,

$$\Omega(x) = 1 + \varepsilon(x) \approx \exp[\varepsilon(x)] \;, \;\; \varepsilon(x) << 1 \tag{5}$$

Consider next infinitesimal coordinate transformations which, up to a first order in a small parameter $\upsilon(x) << 1$, can be presented as

$$x'^{\rho} = x^{\rho} + \upsilon^{\rho}(x) + O(\upsilon^2) \tag{6}$$

Demanding that (6) represents a local conformal transformation amounts to [5]

$$\partial_{\mu}\upsilon_{\nu} + \partial_{\nu}\upsilon_{\mu} = \frac{2}{D}(\partial \cdot \upsilon)\,\eta_{\mu\nu} \tag{7}$$

The scale factor corresponding to (6) is given by

$$\Omega^2(x) = 1 + \frac{2(\partial \cdot \upsilon)}{D} + O(\upsilon^2) \tag{8}$$

Any locally defined MFM is characterized by a space-time dimension $D(x) = 4 - \varepsilon(x)$, where the onset of the fractal dimension $\varepsilon(x) << 1$ reflects a nearly-vanishing deviation from strict conformal invariance expected at the fixed points of the Renormalization Group flow [4, 8]. Conformal behavior in flat space-time matches the scale-invariant (constant) metric $\eta_{\mu\nu}$, whereby $\Omega^2(x) \to 1$ and $\varepsilon(x) \to 0$ as a result of (3) and (5). In field-theoretic language, reaching the conformal limit on the flat four dimensional space-time means that the Renormalization

Group trajectories flow into stable fixed points where they settle down to steady equilibria. One arrives at similar conclusions by following the prescription of the dimensional regularization program [4, 7-8]. All these observations enable us to draw a natural connection between the fractal dimension $\varepsilon(x)$ << 1 and the NCT, namely,

$$\boxed{D(x) = 4 - \varepsilon(x) \Leftrightarrow \Omega^2(x) = 1 + \varepsilon(x)} \tag{9}$$

Replacing (9) into (8) and ignoring the contribution of quadratic terms yields

$$2\varepsilon(x) = \partial \cdot \upsilon(x) \tag{10}$$

Furthermore, setting the fractal dimension as divergence of a locally defined "dimensional" field $\xi(x)$

$$2\varepsilon(x) = \partial_\mu \xi^\mu = \partial \cdot \xi \tag{11}$$

leads to the following condition for conformal invariance on the MFM

$$\partial \cdot (\upsilon - \xi) = 0 \tag{12}$$

A typical ansatz in CFT is to assume that the infinitesimal coordinate transformations $\upsilon_\mu(x)$ are at most quadratic in x^ν, that is,

$$\upsilon_\mu(x) = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \tag{13}$$

where $a_\mu, b_{\mu\nu}, c_{\mu\nu\rho}$ << 1 are constant coefficients with $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. The individual terms of expansion (13) describe various conformal transformations and their respective generators. In particular,

1) The constant coefficient a_μ represents an infinitesimal translation $x'^\mu = x^\mu + a^\mu$ whose generator is the momentum operator $P_\mu = -i\partial_\mu$.

2) The next term can be split into a symmetric and an anti-symmetric contribution according to

$$b_{\mu\nu} = \lambda\eta_{\mu\nu} + m_{\mu\nu} \tag{14}$$

where $m_{\mu\nu} = -m_{\nu\mu}$. The symmetric part $\lambda\eta_{\mu\nu}$ labels infinitesimal scale transformations (dilatations) of the generic form $x'^\mu = (1+\lambda)x^\mu$ and generator $D = -ix^\mu\partial_\mu$. The anti-symmetric part $m_{\mu\nu}$ describes infinitesimal rotations $x'^\mu = (\delta^\mu_\nu + m^\mu_\nu)x^\nu$ whose associated generator is the angular momentum operator $L_{\mu\nu} = i(x_\mu\partial_\nu - \partial_\mu x_\nu)$.

3) The last term at the quadratic order in x defines the so-called "special conformal transformations".

Returning to (9) to (12), a reasonable hypothesis is to assume that the dimensional field $\xi(x)$ is at most linear in x, which corresponds to a nearly-constant fractal dimension $\varepsilon(x) \approx \varepsilon$. Thus we take

$$\xi_\mu(x) = d_\mu + e_{\mu\nu}x^\nu \tag{15}$$

subject to the requirement of infinitesimal coefficients $d_\mu, e_{\mu\nu} << 1$. Retracing previous steps, we split $e_{\mu\nu}$ into a symmetric and anti-symmetric contribution

$$e_{\mu\nu} = \lambda\eta_{\mu\nu} + f_{\mu\nu} \tag{16}$$

subject to the condition $f_{\mu\nu} = -f_{\nu\mu}$. The symmetric part denotes a scale transformation similar to $x'^{\mu} = (1+\lambda)x^{\mu}$, whereas the anti-symmetric part defines an "intrinsic" rotation of the form

$$\boxed{x'^{\mu} = (\delta^{\mu}_{\nu} + f^{\mu}_{\nu})x^{\nu}} \tag{17}$$

It follows that the "rotation-like" transformation (17) stems from the fractal topology of the MFM and may be associated with the generator of *quantum spin* $S_{\mu\nu}$. A favorable consequence of this brief analysis is that, by construction, $S_{\mu\nu}$ replicates the algebra of the angular momentum operator $L_{\mu\nu}$. In closing we mention that these findings are consistent with the body of ideas developed in [6].

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