

# 1. TUESDAY, 5 MARCH

In this lecture, we gave an alternate method for checking whether a function is Riemann integrable. We will give an application of this at the end of the lecture, and prove some useful facts along the way.

Recall that for a bounded function  $f : [a, b] \rightarrow \mathbb{R}$ ,  $U \int_a^b f(x)dx$ , the *upper integral* of  $f$  on  $[a, b]$ , is the infimum of  $U(f, P)$  over all partitions  $P$  of  $[a, b]$ . The *lower integral*, denoted  $L \int_a^b f(x)dx$ , is the supremum of  $L(f, P)$  as  $P$  ranges over all partitions of  $[a, b]$ .

Recall that for any partitions  $P, R$  of  $[a, b]$  such that  $P \subset R$ ,  $L(f, P) \leq L(f, R)$  and  $U(f, R) \leq U(f, P)$ . Moreover, for any  $P$ ,  $L(f, P) \leq U(f, P)$ . From this it follows that  $L \int_a^b f(x)dx \leq U \int_a^b f(x)dx$  for any bounded function. Indeed, if

$$\mathcal{L} = \{L(f, P) : P \text{ is a partition of } [a, b]\}$$

and

$$\mathcal{U} = \{U(f, P) : P \text{ is a partition of } [a, b]\},$$

then  $L \int_a^b f(x)dx = \sup \mathcal{L}$  and  $U \int_a^b f(x)dx = \inf \mathcal{U}$ . In order to show that  $L \int_a^b f(x)dx \leq U \int_a^b f(x)dx$ , we need to show that any member of  $\mathcal{L}$  is less than or equal to any member of  $\mathcal{U}$ . To that end, let  $P, Q$  be partitions of  $[a, b]$ . Then since  $R = P \cup Q$  is a partition containing  $P$  and  $Q$ ,

$$L(f, Q) \leq L(f, R) \leq U(f, R) \leq U(f, P).$$

Recall that if the function  $f$  is positive, the integral (which we will define in this lecture) should be the area under the curve. In this case, the upper Riemann sums (where the rectangles may have their tops going above the curve) may include too much area, and the lower Riemann sums may include too little. If a function is well-behaved (which, at this time, means Riemann integrable) the upper sums and lower sums should both approximate the area very well for some partitions where there is not too much area above or below. We expect this happens when the rectangles get very narrow, and indeed we saw something along these lines in the last lecture. That is, we think that the smallest values we get from upper sums (which is basically  $U \int_a^b f(x)dx$ ) should be approaching the area from above, and the largest values we get from lower sums (which is basically  $L \int_a^b f(x)dx$ ) should be approaching the area from below. Thus the upper and lower integral should be equal. It turns out this is necessary and sufficient.

**Theorem 1.1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Then  $f$  is Riemann integrable if and only if  $U \int_a^b f(x)dx = L \int_a^b f(x)dx$ .*

*Proof.* Suppose that  $f$  is Riemann integrable. Fix  $\varepsilon > 0$  and let  $\delta > 0$  be such that if  $P$  is any partition of  $[a, b]$  with  $\|P\| < \delta$ ,  $U(f, P) - L(f, P) < \varepsilon$ . Note that since  $U \int_a^b f(x)dx$  is the infimum over expressions of the form  $U(f, P)$  as  $P$  ranges over **all** partitions, if  $P$  is any partition of  $[a, b]$ ,  $\int_a^b f(x)dx \leq U(f, P)$ . Similarly,  $L \int_a^b f(x)dx \geq L(f, P)$  for any  $P$ .

Fix any partition  $P$  such that  $\|P\| < \delta$ . Then  $U(f, P) - L(f, P) < \varepsilon$ , which means  $U(f, P) < \varepsilon + L(f, P)$ , whence

$$U \int_a^b f(x) dx \leq U(f, P) < \varepsilon + L(f, P) \leq \varepsilon + \int_a^b f(x) dx.$$

Since  $\varepsilon > 0$  was arbitrary, it follows that  $U \int_a^b f(x) dx \leq L \int_a^b f(x) dx$ . Since we already showed  $L \int_a^b f(x) dx \leq U \int_a^b f(x) dx$ , we see that  $U \int_a^b f(x) dx = L \int_a^b f(x) dx$ .

Next, suppose that  $U \int_a^b f(x) dx = L \int_a^b f(x) dx$ . Fix  $\varepsilon > 0$ . By a result from last lecture, there exists  $\delta_1 > 0$  such that if  $\|P\| < \delta_1$ ,  $U(f, P) \leq \varepsilon/2 + U \int_a^b f(x) dx$ . Similarly, there exists  $\delta_2 > 0$  such that if  $\|P\| < \delta_2$ ,  $L(f, P) + \varepsilon/2 > L \int_a^b f(x) dx$ . Then for any partition  $P$  with  $\|P\| < \min\{\delta_1, \delta_2\}$ ,

$$U(f, P) - L(f, P) < (\varepsilon/2) + U \int_a^b f(x) dx - (L \int_a^b f(x) dx - \varepsilon/2) = \varepsilon.$$

Here we used that  $U \int_a^b f(x) = L \int_a^b f(x) dx$ . This shows that  $f$  is Riemann integrable.  $\square$

Now we may define the integral for Riemann integrable functions. If  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable, we let

$$\int_a^b f(x) dx = U \int_a^b f(x) dx = L \int_a^b f(x) dx.$$

**1.1. Applications.** If we want to show that the upper and lower integrals are equal, we only need to show the difference  $U \int_a^b f(x) dx - L \int_a^b f(x) dx$  is less than every positive number. Above, we implicitly used the following fact: If  $P$  is any partition of  $[a, b]$ ,  $U \int_a^b f(x) dx - L \int_a^b f(x) dx < U(f, P) - L(f, P)$ . Thus if we want to show that the difference between the upper and lower integrals smaller than every positive number, all we need to do is fix  $\varepsilon > 0$  and find *one* partition  $P$  of  $[a, b]$  such that  $U(f, P) - L(f, P) < \varepsilon$ . This is a convenient way to check whether a function is integrable.

**Proposition 1.2.** *Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and  $a < c < b$ . Then  $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$ . More generally, if  $a = x_0 < x_1 < \dots < x_n = b$ , then*

$$\int_a^b f(x) dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x) dx.$$

*Proof.* We know the three integrals exist, since  $f$  is continuous. We will show that

$$(1) \quad \int_a^c f(x) dx + \int_c^b f(x) dx \leq \int_a^b f(x) dx$$

and

$$(2) \quad \int_a^b f(x) dx \leq \int_a^c f(x) dx + \int_c^b f(x) dx.$$

These inequalities together will yield the desired equality.

Fix  $\varepsilon > 0$ . Fix  $\delta > 0$  such that for any partition  $P$  of  $[a, b]$  with  $\|P\| < \delta$ ,  $U(f, P) < \varepsilon + \int_a^b f(x)dx = \varepsilon + \int_a^c f(x)dx$ . Let  $Q$  be any partition of  $[a, c]$  with  $\|Q\| < \delta$  and let  $R$  be any partition of  $[c, b]$  with  $\|R\| < \delta$ . Note that  $P = Q \cup R$  is a partition of  $[a, b]$  with  $\|P\| \leq \max\{\|Q\|, \|R\|\} < \delta$ . Then

$$\int_a^c f(x)dx + \int_c^b f(x)dx \leq U(f, Q) + U(f, R) = U(f, P) < \varepsilon + \int_a^b f(x)dx.$$

Since  $\varepsilon > 0$  was arbitrary, this gives (1).

Next, fix  $\varepsilon > 0$ . Fix  $\delta_1 > 0$  such that for any partition  $P$  of  $[a, c]$  with  $\|P\| < \delta_1$ ,  $U(f, P) < \varepsilon/3 + \int_a^c f(x)dx$ . Fix  $\delta_2 > 0$  such that for any partition  $P$  of  $[c, b]$  with  $\|P\| < \delta_2$ ,  $U(f, P) < \varepsilon/3 + \int_c^b f(x)dx$ . Fix a positive number  $M$  such that  $|f(x)| \leq M$  for all  $x \in [a, b]$ . Fix  $\delta_0 > 0$  such that  $2M\delta_0 < \varepsilon/3$ . Let  $\delta = \min\{\delta_0, \delta_1, \delta_2\}$ . Recall that in the last notes we proved that if  $P$  is a partition of  $[a, b]$  and  $P'$  is a partition which is  $P$  and only one more point, then  $U(f, P) \leq U(f, P') + 2M\|P\|$ . Let  $P$  be any partition of  $[a, b]$  with  $\|P\| < \delta$ . Let  $P' = P \cup \{c\}$ . Let  $Q$  denote the points in  $P'$  which are less than or equal to  $c$ , and let  $R$  denote the points in  $P'$  which are greater than or equal to  $c$ . Note that  $Q$  is a partition of  $[a, c]$  with  $\|Q\| < \delta \leq \delta_1$ , so that  $U(f, Q) < \varepsilon/3 + \int_a^c f(x)dx$ . Similarly,  $R$  is a partition of  $[c, b]$  with  $\|R\| < \delta \leq \delta_2$ , so that  $U(f, R) < \varepsilon/3 + \int_c^b f(x)dx$ . By our remarks above,

$$U(f, P) \leq U(f, P') + 2M\|P\| < U(f, P') + 2M\delta_0 < U(f, P') + \varepsilon/3.$$

Then

$$\begin{aligned} \int_a^b f(x)dx &\leq U(f, P) \leq U(f, P') + \varepsilon/3 = U(f, Q) + U(f, R) + \varepsilon/3 \\ &< (\varepsilon/3) + \int_a^c f(x)dx + (\varepsilon/3) + \int_c^b f(x)dx + \varepsilon/3 = \int_a^c f(x)dx + \int_c^b f(x)dx + \varepsilon. \end{aligned}$$

The second statement follows by repeatedly using the first. □

Note that the above proof does not need the assumption of continuity, only the assumption that the three integrals exist.

**Proposition 1.3.** *Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is a bounded function and  $a < c < b$  are such that  $f$  is continuous on  $[a, b] \setminus \{c\}$ . Then  $f$  is integrable on  $[a, b]$ .*

*Proof.* Fix  $M > 0$  such that  $|f(x)| \leq M$  for all  $x \in [a, b]$ . Fix  $\varepsilon > 0$ . We need to find one partition  $P$  of  $[a, b]$  such that  $U(f, P) - L(f, P) < \varepsilon$ . But if  $P$  is any partition, say  $P = \{x_0, \dots, x_n\}$ , then

$$(3) \quad U(f, P) - L(f, P) = \sum_{i=1}^n (x_i - x_{i-1}) \left[ \sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right].$$

Recall that we think of this as being the sums of areas of rectangles (this is not true if the function is negative, and in this case we can interpret this as signed area), but the picture of widths and heights is still a good one to keep in mind.

Our strategy will be as follows. The  $x_0, \dots, x_n$  points give us the left and right endpoints of our rectangles. If  $x_i - x_{i-1}$  is small and  $c$  is not in this interval, then hopefully the function (being continuous on this interval) will not vary too much. Therefore the height from this term of the sum in (3) will be small (which is good, since we want (3) to be small). But what about when the interval  $[x_{i-1}, x_i]$  *does* contain  $c$ ? We cannot make the height  $\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x)$  may not be small (if there is a jump in the function, for instance). But the height will still be bounded by  $2M$ , since the supremum is at most  $M$  and the infimum is at least  $-M$ . And there are at most two rectangles containing  $c$ , and they will have small width. That is how we will keep the contribution from these terms in (3) from giving us something big.

Fix points  $x, y \in [a, b]$  such that  $x < c < y$  and  $y - x < \varepsilon/4M$ . Note that  $A = [a, x] \cup [y, b]$  is compact, and (since it does not contain  $c$ ),  $f$  is continuous on this set. Therefore  $f$  is uniformly continuous on the set  $A$ . This means there exists  $\delta > 0$  such that for any  $s, t \in A$  with  $|s - t| < \delta$ ,  $|f(s) - f(t)| < \varepsilon/2(b - a)$ .

Let  $P$  be any partition of  $[a, b]$  with  $\|P\| < \delta$  and, by replacing  $P$  with  $P \cup \{x, y\}$ , we may assume  $P$  contains  $x, y$ . Write  $P = \{x_0, \dots, x_n\}$ . Since  $x, y \in P$ , there exist  $j < k$  such that  $x = x_j$  and  $y = x_k$ . Then we break the rectangles in the partition into three subsets: Those rectangles between  $a$  and  $x$  (the rectangles between points  $x_0 < \dots < x_j = x$ ), those rectangles between  $x$  and  $y$  (with endpoints between  $x_j < \dots < x_k$ ), and those rectangles between  $y$  and  $c$  (with endpoints  $x_k < \dots < x_n$ ). In the first and last kinds of rectangles, the difference between the supremum and the infimum of the function value will be less than

$\frac{\varepsilon}{2(b-a)}$ , since the interval  $[x_{i-1}, x_i]$  has length less than  $\delta$  and is contained in the set  $A$ . Then

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^n (x_i - x_{i-1}) \left[ \sup_{z \in [x_{i-1}, x_i]} f(z) - \inf_{z \in [x_{i-1}, x_i]} f(z) \right] \\
&= \sum_{i=1}^j (x_i - x_{i-1}) \left[ \sup_{z \in [x_{i-1}, x_i]} f(z) - \inf_{z \in [x_{i-1}, x_i]} f(z) \right] \\
&\quad + \sum_{i=j+1}^k (x_i - x_{i-1}) \left[ \sup_{z \in [x_{i-1}, x_i]} f(z) - \inf_{z \in [x_{i-1}, x_i]} f(z) \right] \\
&\quad + \sum_{i=k+1}^n (x_i - x_{i-1}) \left[ \sup_{z \in [x_{i-1}, x_i]} f(z) - \inf_{z \in [x_{i-1}, x_i]} f(z) \right] \\
&< \frac{\varepsilon}{2(b-a)} \sum_{i=1}^j (x_i - x_{i-1}) \\
&\quad + 2M \sum_{i=j+1}^k (x_i - x_{i-1}) \\
&\quad + \frac{\varepsilon}{2(b-a)} \sum_{i=k+1}^n (x_i - x_{i-1}) \\
&= \frac{\varepsilon}{2(b-a)} (x - a) + 2M(y - x) + \frac{\varepsilon}{2(b-y)} \\
&= \frac{\varepsilon}{2} \cdot \frac{b-y+x-a}{b-a} + 2M \cdot \frac{\varepsilon}{4M} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Here we have used the fact that  $\frac{b-y+x-a}{b-a} < 1$ , since  $b-y$  is the length of  $[y, b]$ ,  $x-a$  is the length of  $[a, x]$ , and  $b-a$  is the length of  $[a, b]$ , (which is greater than the sum of the other two lengths, since they are lengths of disjoint subintervals of  $[a, b]$ ).

□

## 2. THURSDAY, 7 MARCH

**2.1. The fundamental theorem of the calculus.** The main result of this lecture was our favorite theorem from integral calculus:

**Theorem 2.1.** *Let  $a < b$  and let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function.*

- (i) *If  $F : [a, b] \rightarrow \mathbb{R}$  is given by  $F(x) = \int_a^x f(t)dt$ , then  $F$  is differentiable and  $F' = f$ .*
- (ii) *If  $G : [a, b] \rightarrow \mathbb{R}$  is any differentiable function such that  $G' = f$ ,  $\int_a^b f(x)dx = G(b) - G(a)$ .*

In order for the function  $F$  to be defined at  $a$ , we agree that  $\int_a^a f(x)dx = 0$ . We also agree that if  $c < d$ ,  $\int_d^c f(t)dt = -\int_c^d f(t)dt$ . Note that with these conventions, the formula

$$\int_x^z f(t)dt = \int_x^y f(t)dt + \int_y^z f(t)dt$$

holds for **any**  $x, y, z \in [a, b]$  when  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Note that we only proved this formula before in the case that  $x < y < z$ .

*Proof of Theorem 2.1.* (i) We must show that for any  $x \in [a, b]$ ,

$$\lim_{y \rightarrow x} \frac{F(y) - F(x)}{h} = f(x),$$

with the appropriate one-sided limit if  $x = a$  or  $x = b$ . This limit means that for each  $x \in [a, b]$  and  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any  $y \in [a, b]$  with  $0 < |x - y| < \delta$ ,

$$\left| \frac{F(y) - F(x)}{y - x} - f(x) \right| < \varepsilon.$$

To that end, fix  $x \in [a, b]$  and  $\varepsilon > 0$ . Fix  $\delta > 0$  such that for any  $t \in [a, b]$  with  $|t - x| < \delta$ ,  $|f(t) - f(x)| < \varepsilon/2$ . We may do this since  $f$  is continuous at  $x$ . Fix  $y \in [a, b]$  with  $0 < |y - x| < \delta$ . First consider the case that  $y > x$  (which we omit if  $x = b$ ). Then  $y = x + h$  for some number  $h \in (0, \delta)$ . Note that for any  $x \leq t \leq x + h$ ,  $|x - t| < h$ , so that  $|f(t) - f(x)| < \varepsilon/2$  for all such  $t$ . Note also that, since  $f(x)$  (the value of the function at the fixed point  $x$ ) is a single number,  $\int_x^{x+h} f(x)dt = hf(x)$ . Also,

$$F(y) = F(x + h) = \int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt.$$

Then

$$\begin{aligned} \left| \frac{F(y) - F(x)}{y - x} - f(x) \right| &= \left| h^{-1} \left( \int_a^x f(t)dt + \int_x^{x+h} f(t)dt - \int_x^{x+h} f(t)dt \right) - h^{-1} \int_x^{x+h} f(x)dx \right| \\ &= h^{-1} \left| \int_x^{x+h} (f(t) - f(x))dt \right| \leq h^{-1} \left( \frac{\varepsilon}{2} \cdot h \right) = \varepsilon/2 < \varepsilon. \end{aligned}$$

Here we have used the fact that if a function  $g$  has absolute value less than or equal to some number  $c$  on an interval, then the integral of  $g$  on that interval is less than or equal to  $c$  times the length of the interval.

The case that  $y < x$  is similar, except this time  $h$  is negative.

(ii) Suppose  $G' = f$ . Let  $F(x) = \int_a^x f(t)dt$ . Then  $(G - F)' = f - f = 0$ , so  $G - F$  is constant by the mean value theorem. Moreover,

$$(G - F)(a) = G(a),$$

since  $F(a) = \int_a^a f(t)dx = 0$ . Then since  $G - F$  is constant, it is constantly  $G(a)$ , whence

$$G(b) - G(a) = F(b) = \int_a^b f(t)dt,$$

as claimed.

