Lec 15: Determinants of transformed matrices

In this lecture we will discover how determinants of A and B are related, where B is obtained from A by an elementary row transformation. We consider square matrices only. First of all, let's recall the definition of the determinant of a matrix. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant of A is

$$\det(A) = \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{nj_n} \tag{1}$$

where the summation is over all permutations $j_1 j_2 \ldots j_n$. We take the sign $+$ before the term $a_{1j_1} a_{2j_2} \cdots a_{nj_n}$ if the permutation $j_1 j_2 \ldots j_n$ is even, and $-$ if the permutation is odd.

For example, if $n = 2$, formula (1) becomes

$$\det(A) = \sum (\pm) a_{1j_1} a_{2j_2}.$$

Since there are only two permutations on two elements, namely 12 and 21, we have two terms in our formula, the first one with $j_1 = 1, j_2 = 2$ (corresponding to the permutation 12) and the other one with $j_1 = 2, j_2 = 1$ (permutation 21). In other words,

$$\det(A) = \pm a_{11} a_{22} \pm a_{12} a_{21}$$

Now the sign before $a_{11} a_{22}$ must be $+$ since the permutation 12 has 0 inversions, and the sign of $a_{12} a_{21}$ is $-$, because the permutation 21 has 1 inversion. Thus we arrive at a familiar formula (as it was expected) for the determinant of 2×2 matrix:

$$\det(A) = a_{11} a_{22} - a_{12} a_{21}.$$

In case $n = 3$ formula (1) has a form

$$\det(A) = \sum (\pm) a_{1j_1} a_{2j_2} a_{3j_3}. \tag{2}$$

It contains 6 terms as there are $3! = 6$ permutations on 3 elements. Namely, options for $j_1 j_2 j_3$ are $123, 213, 132, 321, 231, 312$. The number of inversions is respectively $0, 1, 1, 3, 2, 2$. Thus the expanded form of (2) is:

$$\det(A) = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32},$$

or, after rearranging the terms, $\det(A) =$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}. \tag{3}$$

Again, we recognize the formula for the determinant of a 3×3 matrix. In formula (3) we introduced another notation for $\det(A)$, that is a matrix bounded by vertical lines. So, in this notation, $\det([a_{ij}]) = |a_{ij}|$.

Now let's apply ERTs to a matrix and see what happens to its determinant. Instead of giving general proofs, we will look at illuminating examples. Take an arbitrary 3×3 matrix, interchange its first two rows and compute its determinant:

$$\det(A_{r_1 \leftrightarrow r_2}) = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$a_{21}a_{12}a_{33} + a_{22}a_{13}a_{31} + a_{23}a_{11}a_{32} - a_{23}a_{12}a_{31} - a_{21}a_{13}a_{32} - a_{22}a_{11}a_{33}.$$

Comparing this with formula (3), we see that $\det(A_{r_1 \leftrightarrow r_2}) = -\det(A)$. Now multiply the second row of A by a scalar r and find det:

$$\det(A_{r \cdot r_2 \to r_2}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ra_{21} & ra_{22} & ra_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$a_{11}(ra_{22})a_{33} + a_{12}(ra_{23})a_{31} + a_{13}(ra_{21})a_{32} - a_{13}(ra_{22})a_{31} - a_{11}(ra_{23})a_{32} - a_{12}(ra_{21})a_{33}.$$

This is r times $\det(A)$. Now add r times first row to the last one:

$$\det(A_{r \cdot r_1 + r_3 \to r_3}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ra_{11} + a_{31} & ra_{12} + a_{32} & ra_{13} + a_{33} \end{vmatrix} =$$

$$a_{11}a_{22}(ra_{13} + a_{33}) + a_{12}a_{23}(ra_{11} + a_{31}) + a_{13}a_{21}(ra_{12} + a_{32})$$
$$-a_{13}a_{22}(ra_{11} + a_{31}) - a_{11}a_{23}(ra_{12} + a_{32}) - a_{12}a_{21}(ra_{13} + a_{33}) =$$
$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} = \det(A).$$

[All terms with r vanish.] So, ERTs of the third type preserve the determinant.

Our observations in case of matrices of order $n = 3$ are valid for any n. Moreover, they are the same for column transformations. Namely,

1. switching two rows (or two columns) changes the sign of det;

2. multiplying a row (or a column) by a number multiplies det by the same number;

3. adding a multiple of one row to another one (or a column to another one) does not change det.

Or, in the other direction,

$$\det(A) = -\det(A_{r_i \leftrightarrow r_j});$$

$$\det(A) = \frac{1}{r}\det(A_{r \cdot r_i \to r_i});$$

$$\det(A) = \det(A_{r \cdot r_i + r_j \to r_j}).$$

Before giving examples, point out one important

Theorem. *1. If a matrix A has a zero row or a zero column, then* $\det(A) = 0$.

2. *If A is upper-triangular or lower-triangular, then* $\det(A) = a_{11}a_{22}\cdots a_{nn}$ *(product of all diagonal entries).*

This follows directly from formula (1). A term $\pm a_{1j_1}a_{2j_2}\cdots a_{nj_n}$ contains one element from each row and one element from each column. Hence if A has a zero row or a zero column, all terms in the sum (1) are 0, and $\det(A) = 0$. Consider now the case of upper-triangular A. Find all nonzero terms in formula (1): $\pm a_{1j_1}a_{2j_2}\cdots a_{nj_n} \neq 0$ implies $a_{nj_n} \neq 0$. Then $j_n = n$, because all other entries of the bottom row of A are 0. Next, $a_{(n-1)j_{n-1}}$ must be nonzero. Then $j_{n-1} = n-1$ or n. But $j_n = n$ already. Hence $j_{n-1} = n-1$. Proceeding in this way, we get $j_{n-2} = n-2, \ldots, j_1 = 1$. Thus the only nonzero term is $\pm a_{11}a_{22}\cdots a_{nn}$. The sign must be $+$ for the permutation $12\ldots n$ has no inversions. We conclude $\det(A) = a_{11}a_{22}\cdots a_{nn}$. A similar argument works for lower-triangular A.

Now let A be an $n \times n$ matrix. By ERTs we can produce an upper-triangular matrix B from A. Knowing $\det(B)$ we can find $\det(A)$.

Example.

$$\begin{vmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{vmatrix} = (\text{switch rows 1 and 3}) = -\begin{vmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{vmatrix} = (\text{zero out below } (1,1) \text{ entry}) =$$

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{vmatrix} = (\text{zero out below } (2,2) \text{ entry}) = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

By words 'zero out below $(1,1)$ entry' we mean subtracting multiples of the first row from other rows so that all entries below $(1,1)$ entry are 0. Similarly for $(2,2)$ entry in the next step. The latter determinant is 0 because of the first or the second statement of the theorem above.

This example illustrates that straightforward computing of det by formula (3) is not always the fastest way: verifying $9\cdot5\cdot1+8\cdot4\cdot3+7\cdot2\cdot6-7\cdot5\cdot3-9\cdot2\cdot4-1\cdot8\cdot6 = 0$ takes more time then reducing to an upper-triangular matrix above.

Example.

$$\begin{vmatrix} 0 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & 1 & 3 \end{vmatrix} = (\text{multiply row 2 by } \frac{1}{2}) = 2\begin{vmatrix} 0 & 2 & 3 \\ 1 & 2 & 3 \\ 5 & 1 & 3 \end{vmatrix} = (\text{switch rows 1 and 2}) =$$

$$-2\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 5 & 1 & 3 \end{vmatrix} = (\text{zero out below } (1,1) \text{ entry}) = -2\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & -9 & -12 \end{vmatrix} =$$

$$(\text{zero out below } (2,2) \text{ entry}) = -2\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{3}{2} \end{vmatrix} = -2(1\cdot2\cdot\frac{3}{2}) = -6.$$

In this example the straightforward computation by formula (3) would be, probably, easier (do it!).