Complex Numbers and the Double Angle Formulas

Alec Leatham

In this article, we shall shall connect two seemingly distinct domains of mathematics resulting in very several very useful results. First we shall introduce some definitions of concepts that you are most likely already familiar, but perhaps never formally defined. The field of complex numbers is denoted by: C, $z \in \mathrm{C} \Rightarrow z = a + bi$ where a and b are real numbers. We can graph these numbers on a complex plane formed by the composition of the real number line and an orthogonal imaginary line intersecting at the origin. We shall define the *argument* of a complex number, z (denoted $\arg(z)$), as the angle that the line from the origin to z forms with the real number line. For example, $\arg(2 + 2i)$=45 degrees (don't just take my word for this! Get out some paper and confirm it!) to make sure you understand this concept, quickly compute the arguments of the following complex numbers (and plot them on the complex plane): $1+\sqrt{3}i$, $\sqrt{3} - i$, $7i$, $-\pi$, $5 + 5i$. What happens to the argument of of a complex number when we multiply it by i? Plot out the numbers from the previous exercise on the complex plane and observe what happens to their argument when they are multiplied by i. Now square $1 + i$ and observe what happens to the argument, both geometrically and algebraically, do you have any conjectures yet? If not, stop reading and plot out a few more complex numbers and observe the relationship between the arguments of the numbers and that of their product. After thorough investigation, we discover that when we multiply two complex numbers together, their arguments add. This property comes in handy later, so do not forget it. For now, notice that we can assert some very useful statements concerning trigonometric functions and the arguments of complex numbers, for example it is clear that $\tan(\arg(a + bi))$=$\frac{b}{a}$ (Is it clear? Verify this assertion before moving on.) Following down the same vein, find out a simple statement for $\sin(\arg(a + bi))$ and $\cos(\arg(a + bi))$. (Seriously, do it.) Now we have all the tools necessary to attack the main problem: deriving the double angle formula for the trigonometric functions. Using all that we have learned, lets derive the double angle formula for tan(x). Let $z = a+bi$, then we know that $\tan(\arg(z))$=$\frac{b}{a}$, now here's the clever part, we will let b=$\tan(\arg(z))$ and a=1. This gives us z =1+$\tan(\arg(z))i$. (Read this over a few times until you understand what is going on, this is the most important part of the proof.) Thus, z^2 = (1+$\tan(\arg(z))i)^2$=1+2$\tan(\arg(z))i$-$\tan(\arg(z))^2$=1-$\tan(\arg(z))^2$+2$\tan(\arg(z))i$. Thus, substituting z^2 for z in the original formula, we obtain: $\tan(\arg(z^2))$=$\frac{2tan(arg(z))}{1-tan(arg(z))}$. From our previous work, we know that arg(z^2)=2arg(z) because the argument of the product of two complex numbers is equal to the sum of the arguments of the numbers. If we let arg(z)=θ and substitute it back into our equation, we obtain the double angle formula for tan(θ): tan(2θ)=$\frac{2tan(2\theta)}{1-tan(\theta)}$. (Look over the proof several times and make sure that you understand it before proceeding.) Using what you have learned, find double angle formulas for sin(θ) and cos(θ). (They are a lot prettier, I swear.)