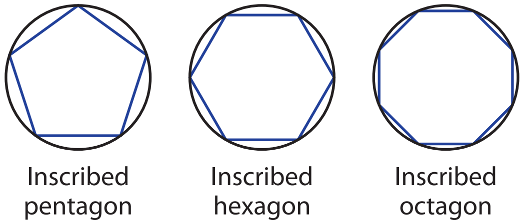
# 16.1 Justifying Circumference and Area of a Circle

**Essential Question:** How can you justify and use the formulas for the circumference and area of a circle?

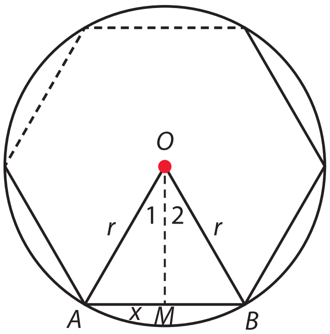
Resource Locker

## Explore Justifying the Circumference Formula

To find the circumference of a given circle, consider a regular polygon that is inscribed in the circle. As you increase the number of sides of the polygon, the perimeter of the polygon gets closer to the circumference of the circle.




Let circle $O$ be a circle with center $O$ and radius $r$. Inscribe a regular $n$–gon in circle $O$ and draw radii from $O$ to the vertices of the $n$-gon.




Let $\overline{AB}$ be one side of the $n$-gon. Draw $\overline{OM}$, the segment from $O$ to the midpoint of $\overline{AB}$.

(A) Then $\triangle AOM \cong \triangle BOM$ by ? .

(B) So, $\angle 1 \cong \angle 2$ by ? .

(C) There are $n$ triangles, all congruent to $\triangle AOB$, that surround point $O$ and fill the $n$-gon.

Therefore, $m\angle AOB =$ ? and $m\angle 1 =$ ? .

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D) Since $\angle OMA \cong \angle OMB$ by CPCTC, and $\angle OMA$ and $\angle OMB$ form a linear pair, these angles are supplementary and must have measures of 90°. So $\triangle AOM$ and $\triangle BOM$ are right triangles.

In $\triangle AOM$, $\sin\angle 1 = \dfrac{\text{length of opposite leg}}{\text{length of hypotenuse}} = \dfrac{x}{r}$.

So, $x = r \sin\angle 1$ and substituting the expression for m$\angle 1$ from above gives

$x = r \sin$ [ ? ].

E) Now express the perimeter of the $n$-gon in terms of $r$.

The length of $\overline{AB}$ is $2x$, because [ ? ].

This means the perimeter of the $n$-gon in terms of $x$ is [ ? ].

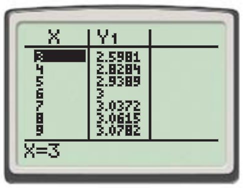
Substitute the expression for $x$ found in Step D.

The perimeter of the $n$-gon in terms of $r$ is [ ? ].

F) Your expression for the perimeter of the $n$–gon should include the factor $n \sin\left(\frac{180°}{n}\right)$. What happens to this factor as $n$ gets larger?

Use your calculator to do the following.

- Enter the expression $x \sin\left(\frac{180}{x}\right)$ as $Y_1$.
- Go to the Table Setup menu and enter the values shown.
- View a table for the function.
- Use arrow keys to scroll down.


What happens to the value of $x \sin\left(\frac{180°}{x}\right)$ as $x$ gets larger?

G) Look at the expression you wrote for the perimeter of the $n$–gon. What happens to the value of this expression, as $n$ gets larger?

**Reflect**

**1.** When $n$ is very large, does the perimeter of the $n$-gon ever equal the circumference of the circle? Why or why not?

**2.** How does the above argument justify the formula $C = 2\pi r$?

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## Explain 1 Applying the Circumference Formula

**Example 1** **Find the circumference indicated.**

(A) A Ferris wheel has a diameter of 40 feet. What is its circumference? Use 3.14 for $\pi$.

Diameter $= 2r$

$40 = 2r$

$20 = r$

Use the formula $C = 2\pi r$ to find the circumference.

$C = 2\pi r$

$C = 2\pi(20)$

$C = 2(3.14)(20)$

$C \approx 125.6$

The circumference is about 125.6 feet.

(B) A pottery wheel has a diameter of 2 feet. What is its circumference? Use 3.14 for $\pi$.

The diameter is 2 feet, so the radius in inches is $r =$ 12.

$C = 2\pi r$

$C =$ 2 · 3.14 · 12

$C \approx$ 75.36 in.

The circumference is about 75.36 inches.

**Reflect**

**3.** **Discussion** Suppose you double the radius of a circle. How does the circumference of this larger circle compare with the circumference of the smaller circle? Explain.

**Your Turn**

**4.** The circumference of a tree is 20 feet. What is its diameter? Round to the nearest tenth of a foot. Use 3.14 for $\pi$.

**5.** The circumference of a circular fountain is 32 feet. What is its diameter? Round to the nearest tenth of a foot. Use 3.14 for $\pi$.

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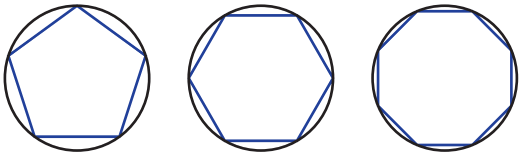
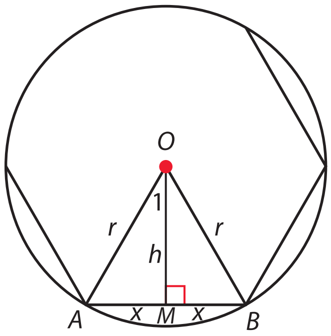
## Explain 2 Justifying the Area Formula

To find the area of a given circle, consider a regular polygon that is inscribed in the circle. As you increase the number of sides of the polygon, the area of the polygon gets closer to the area of the circle.



Inscribed pentagon | Inscribed hexagon | Inscribed octagon

Let circle $O$ be a circle with center $O$ and radius $r$. Inscribe a regular $n$–gon in circle $O$ and draw radii from $O$ to the vertices of the $n$-gon.




Let $\overline{AB}$ be one side of the $n$–gon. Draw $\overline{OM}$, the segment from $O$ to the midpoint of $\overline{AB}$.

We know that $\overline{OM}$ is perpendicular to $\overline{AB}$ because triangle $AOM$ is congruent to triangle $BOM$.

Let the length of $\overline{OM}$ be $h$.

**Example 2** **Justify the formula for the area of a circle.**

**(A)** There are $n$ triangles, all congruent to $\triangle AOB$, that surround point $O$ and fill the $n$-gon.

Therefore, the measure of $\angle AOB$ is $\frac{360°}{n}$, and the measure of $\angle 1$ is $\frac{180°}{n}$.

We know that $x = r\sin\left(\frac{180°}{n}\right)$. Write a similar expression for $h$: $h = r\cos\left(\frac{180°}{n}\right)$

**(B)** The area of $\triangle AOB$ is $\frac{1}{2}(2x)(h) = xh = r\sin\left(\frac{180°}{n}\right)h$.

Substitute your value for $h$ to get area $\triangle AOB = r^2\sin\left(\frac{180°}{n}\right)\cos\left(\frac{180°}{n}\right)$.

**(C)** There are $n$ of these triangles, so the area of the $n$-gon is $nr^2\sin\left(\frac{180°}{n}\right)\cos\left(\frac{180°}{n}\right)$.

**(D)** Your expression for the area of the $n$–gon includes the factor $n\,\sin\left(\frac{180°}{n}\right)\cos\left(\frac{180°}{n}\right)$. What happens to this expression as $n$ gets larger?

Use your graphing calculator to do the following.

- Enter the expression $x\,\sin\left(\frac{180°}{x}\right)\cos\left(\frac{180°}{x}\right)$ as $Y_1$.
- View a table for the function.
- Use arrow keys to scroll down.

What happens to the value of $x\,\sin\left(\frac{180°}{x}\right)\cos\left(\frac{180°}{x}\right)$ as $x$ gets larger? The value gets closer to $\pi$.

**(E)** Look at the expression you wrote for the area of the $n$–gon. What happens to the value of this expression as $n$ gets larger? The expression gets closer to $\pi r^2$.

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**Reflect**

6. When $n$ is very large, does the area of the $n$-gon ever equal the area of the circle? Why or why not?

7. How does the above argument justify the formula $A = \pi r^2$?

## Explain 3 Applying the Area Formula

**Example 3** **Find the area indicated.**

(A) A rectangular piece of cloth is 3 ft by 6 ft. What is the area of the largest circle that can be cut from the cloth? Round the nearest square inch.

The diameter of the largest circle is 3 feet, or 36 inches. The radius of the circle is 18 inches.

$A = \pi r^2$

$A = \pi (18)^2$

$A = 324\pi$

$A \approx 1{,}017.9 \text{ in}^2$

So, the area is about 1,018 square inches.

(B) A slice of a circular pizza measures 9 inches in length. What is the area of the entire pizza? Use 3.14 for $\pi$.

The 9-in. side of the pizza is also the length of the radius of the circle. So, $r = \boxed{9}$.

$A = \pi r^2$

$A = \pi \boxed{9}^2$

$A = \boxed{81}\pi \approx \boxed{254.34}$

To the nearest square inch, the area of the pizza is 255 $\text{in}^2$.

**Reflect**

8. Suppose the slice of pizza represents $\frac{1}{6}$ of the whole pizza. Does this affect your answer to Example 3B? What additional information can you determine with this fact?

**Your Turn**

9. A circular swimming pool has a diameter of 18 feet. To the nearest square foot, what is the smallest amount of material needed to cover the surface of the pool? Use 3.14 for $\pi$.

## Elaborate

10. If the radius of a circle is doubled, is the area doubled? Explain.

11. **Essential Question Check-In** How do you justify and use the formula for the circumference of a circle?

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## Evaluate: Homework and Practice

Personal Math Trainer
- Online Homework
- Hints and Help
- Extra Practice

**1.** Which inscribed figure has a perimeter closer to the circumference of a circle, a regular polygon with 20 sides or a regular polygon with 40 sides? Explain.

**Find the circumference of each circle with the given radius or diameter. Round to the nearest tenth. Use 3.14 for $\pi$.**

**2.** $r = 9$ cm

**3.** $r = 24$ in.

**4.** $d = 14.2$ mm

**5.** A basketball rim has a radius of 9 inches. Find the circumference of the rim. Round to the nearest tenth. Use 3.14 for $\pi$.

**6.** The diameter of a circular swimming pool is 12 feet. Find its circumference. Use 3.14 for $\pi$.

**7.** The diameter of the U.S. Capitol Building's dome is 96 feet at its widest point. Find its circumference. Use 3.14 for $\pi$.

**Find the area of each circle with the given radius or diameter. Use 3.14 for $\pi$.**

**8.** $r = 7$ yd

**9.** $d = 5$ m

**10.** $d = 16$ ft

**11.** A drum has a diameter of 10 inches. Find the area of the top of the drum. Use 3.14 for $\pi$.

**12.** The circumference of a quarter is about 76 mm. What is the area? Round to the nearest tenth.

**Algebra** **Find the area of the circle with the given circumference $C$. Use 3.14 for $\pi$.**

**13.** $C = 31.4$ ft

**14.** $C = 21.98$ ft

**15.** $C = 69.08$ ft

**16.** A Ferris wheel has a diameter of 56 ft. How far will a rider travel during a 4-minute ride if the wheel rotates once every 20 seconds?
Use $\frac{22}{7}$ for $\pi$.

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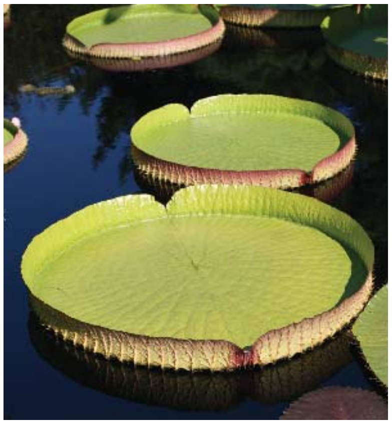
**17.** A giant water lily pad is shaped like a circle with a diameter of up to 5 feet. Find the circumference and area of the pad. Round to the nearest tenth.



**18.** A pizza parlor offers pizzas with diameters of 8 in., 10 in., and 12 in. Find the area of each size pizza. Round to the nearest tenth. If the pizzas cost \$9, \$12, and \$18 respectively, which is the better buy?

**19.** **Critical Thinking** Which do you think would seat more people, a 4 ft by 6 ft rectangular table or a circular table with a diameter of 6 ft? How many people would you sit at each table? Explain your reasoning.

**20.** You can estimate a tree's age in years by using the formula $a = \frac{r}{w}$, where $r$ is the tree's radius without bark and $w$ is the average thickness of the tree's rings. The circumference of a white oak tree is 100 inches. The bark is 0.5 in. thick, and the average thickness of a ring is 0.2 in. Estimate the tree's age and the area enclosed by the outer circumference of the widest ring.

**21.** **Multi-Step** A circular track for a model train has a diameter of 8.5 feet. The train moves around the track at a constant speed of 0.7 ft/s.

**a.** To the nearest foot, how far does the train travel when it goes completely around the track 10 times?

**b.** To the nearest minute, how long does it take the train to go completely around the track 10 times?

**22.** The Parthenon is a Greek temple dating to about 445 BCE. The temple features 46 Doric columns, which are roughly cylindrical. The circumference of each column at the base is about 5.65 meters. What is the approximate diameter of each column? Round to the nearest tenth.

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**23.** **Explain the Error** A circle has a circumference of $2\pi$ in. Which calculation of the area is incorrect? Explain.

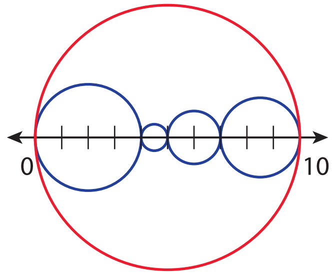
**A**

The circumference is $2\pi$ in., so the diameter is 2 in. The area is $A = \pi(2^2) = 4\pi$ in$^2$.

**B**

The circumference is $2\pi$ in., so the radius is 1 in. The area is $A = \pi(1^2) = \pi$ in$^2$.

**24.** **Write About It** The center of each circle in the figure lies on the number line. Describe the relationship between the circumference of the largest circle and the circumferences of the four smaller circles.




**25.** Find the diameter of a data storage disk with an area 113.1 cm$^2$.

**26.** Which of the following ratios can be derived from the formula for the circumference of a circle? Determine all that apply.

**A.** $\frac{C}{d}$

**B.** $\frac{C}{2r}$

**C.** $\frac{C}{\pi}$

**D.** $\frac{C}{2\pi}$

**E.** $\frac{C}{2c}$

**27.** A meteorologist measured the eyes of hurricanes to be 15 to 20 miles in diameter during one season. What is the range of areas of the land underneath the eyes of the hurricanes?

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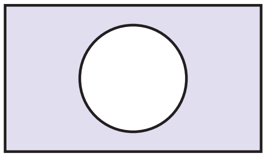
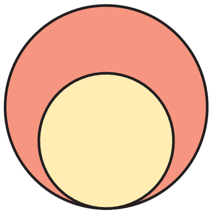
**28.** A circle with a 6 in. diameter is stamped out of a rectangular piece of metal as shown. Find the area of the remaining piece of metal. Use 3.14 for $\pi$.



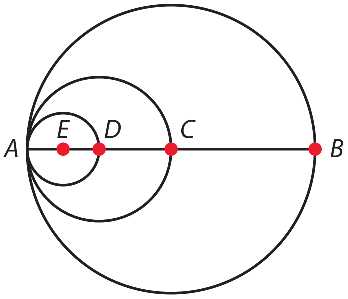

**H.O.T. Focus on Higher Order Thinking**

**29. Critique Reasoning** A standard bicycle wheel has a diameter of 26 inches. A student claims that during a one-mile bike ride the wheel makes more than 1000 complete revolutions. Do you agree or disagree? Explain. (*Hint:* 1 mile = 5280 feet)

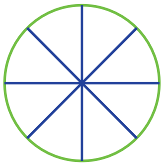
**30. Algebra** A graphic artist created a company logo from two tangent circles whose diameters are in the ratio 3:2. What percent of the total logo is the area of the region outside of the smaller circle?

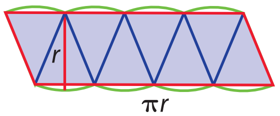


**31. Communicate Mathematical Ideas** In the figure, $\overline{AB}$ is a diameter of circle *C*, *D* is the midpoint of $\overline{AC}$, and *E* is the midpoint of $\overline{AD}$. How does the circumference of circle *E* compare to the circumference of circle *C*? Explain.




**32. Critical Thinking** Evelyn divides the circle and rearranges the pieces to make a shape that resembles a parallelogram.






She then divides the circle into more pieces and rearranges them into a shape that resembles a parallelogram. She thinks that the area of the new parallelogram will be closer to the area of a circle. Is she correct? Explain.

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# Lesson Performance Task

In the lesson, you saw that the more wedges into which you divide a circle, the more closely each wedge resembles a triangle, and the closer to the area of a circle the area of the reassembled wedges becomes. In the branch of mathematics called calculus, this process is called finding a *limit.* Even though you can't cut a circle into millions of tiny wedges to calculate the actual area of the circle, you can see what the area is going to be long before that, by spotting a pattern. You can apply this method in many ways, some of them unexpected.

Mac is a race walker. He is training for a race. He has decided that in the weeks leading up to the race, he'll work up to a level where he is walking 20 kilometers a day. His plan is to walk 10 kilometers the first day of training and increase the distance he walks each day until he reaches his goal.

**1.** Copy and complete the table for Mac's first 6 days of training.

| Day | Increase in distance walked from the day before (in kilometers) | Total distance walked that day (in kilometers) |
|---|---|---|
| 1 | 10 | 10 |
| 2 | 5 | $10 + 5 =$ ? |
| 3 | 2.5 | ? $+ 5 =$ ? |
| 4 | ? | ? + ? = ? |
| 5 | ? | ? + ? = ? |
| 6 | ? | ? + ? = ? |

**2.** Describe your results in relation to Mac's plan to reach a level where he walks 20 kilometers a day.

**3.** Suppose Mac continues his training plan. Will Mac ever reach his goal?

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