Math 1B: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. Introduce yourself to your new friends, and write all of your names at the top of the chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: some are very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, are from the mathematical folklore, or are independently marked.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Trigonometric Integrals

The most important rule to know for trigonometric integrals is the Pythagorean identity:

$$\boxed{\cos^2 x + \sin^2 x = 1}$$

This lets us translate between (squares of) cosines and (squares of) sines. This is helpful for finding u-substitutions, since $\cos' = -\sin$ and $\sin' = \cos$. For example, to integrate $\cos^7(x)$, we can break off a cos and write the rest in terms of sin, and then substitute: $\int \cos^7 x\, dx = \int (\cos^2 x)^3 \cos x\, dx = \int (1-\sin^2 x)^3 \cos x\, dx = \int (1-u^2)^3 (du) = \int (-u^6 + 3u^4 - 3u^2 + 1) du = u^7/7 - 3u^5/5 + u^3 - u + C = \sin^7 x/7 - 3\sin^5 x/5 + \sin^3 x - \sin x + C$. This trick turns the integral of $\cos^n x \sin^m x$ into the integral of a polynomial provided that n and m are non-negative and at least one of n and m are odd. (When they can be negative, but still at least one is odd, we get rational functions, which we will learn how to integrate in section 7.4.)

By dividing the Pythagorean identity by $\sin^2$ or $\cos^2$, we get two more versions of the rule: $\boxed{1 + \tan^2 x = \sec^2 x}$ and $\boxed{1 + \cot^2 x = \csc^2 x}$. Since $\tan' = \sec^2$ and $\sec' = \tan \sec$, we can integrate $\tan^n \sec^m$ if m is even or n is odd. (It's similar for cot and csc.)

Sometimes, though, these aren't enough. Then it's important the remember the double-angle formulas:

$$\boxed{\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \sin x \, \cos x = \frac{1}{2} \sin 2x}$$

The product-to-sum formulas are also occasionally helpful:

$$\boxed{\begin{aligned} 2\sin A \sin B &= \cos(A-B) - \cos(A+B) \\ 2\sin A \cos B &= \sin(A-B) + \sin(A+B) \\ 2\cos A \cos B &= \cos(A-B) + \cos(A+B) \end{aligned}}$$

1. § Evaluate the integrals:

(a)	$\displaystyle\int \sin^6 x \cos^3 x\,dx$	(b)	$\displaystyle\int_0^{\pi/2} \cos^5 x\,dx$	(c)	$\displaystyle\int \frac{\sin^3\sqrt{x}}{\sqrt{x}}\,dx$
(d)	$\displaystyle\int x\cos^2 x\,dx$	(e)	$\displaystyle\int \cos\theta\cos^5(\sin\theta)\,d\theta$	(f)	$\displaystyle\int \cot^5\theta\sin^4\theta\,d\theta$
(g)	$\displaystyle\int \tan^3(2x)\sec^5(2x)\,dx$	(h)	$\displaystyle\int \tan^6(ay)\,dy$	(i)	$\displaystyle\int \frac{\sin\phi}{\cos^3\phi}\,d\phi$
(j)	$\displaystyle\int \csc^4 x\cot^6 x\,dx$	(k)	$\displaystyle\int \frac{\cos x+\sin x}{\sin 2x}\,dx$	(l)	$\displaystyle\int \frac{dx}{\cos x - 1}$
(m)	$\displaystyle\int_{x=0}^{3\pi/2} (\sin x+\cos x)^3\,dx$	(n)	$\displaystyle\int \cos 3x\sin 2x\,dx$	(o)	$\displaystyle\int \cos(2x+1)\cos(4x-2)\sin(x)\,dx$

2. § Evaluate $\int \sin x\cos x\,dx$ in four different ways: (a) by substituting $u=\cos x$; (b) by substituting $u=\sin x$; (c) by using the double angle formula for $\sin 2x$; (d) by integrating by parts. Explain the different appearances of the answers.

3. Let a be a number such that $0 < a < \pi/2$. Compute the volume obtained by rotating the region bounded by the curves

$$y=\tan x,\;\; y=0,\;\; x=a$$

about the x-axis. Your answer should be a function of a.

4. Find the average value of $\sin^2 x$: $\frac{1}{2\pi}\int_0^{2\pi}\sin^2 x\,dx$. Find the average values of $\sin^4 x$ and $\sin^2 x\cos^2 x$.

5. § Let m and n be positive integers. Prove that:

   (a) $$\int_{-\pi}^{\pi} \sin mx\cos nx\,dx = 0$$

   (b) $$\int_{-\pi}^{\pi} \sin mx\sin nx\,dx = \begin{cases} 0 & \text{if } m\neq n \\ \pi & \text{if } m=n \end{cases}$$

   (c) $$\int_{-\pi}^{\pi} \cos mx\cos nx\,dx = \begin{cases} 0 & \text{if } m\neq n \\ \pi & \text{if } m=n \end{cases}$$

6. (a) Use integration by parts to find a reduction formula for $\int_{x=0}^{\pi/2}\cos^n x\,dx$.

   (b) Let $n=2k+1$ be odd. Make a substitution to turn $\int_{x=0}^{\pi/2}\cos^n x\,dx$ into a polynomial integral. For any particular value of k you could expand this out and integrate. Instead, find a reduction formula for this integral.

   (c) When $n=2k+1$ is odd, solve the reduction formulas from parts (a) and (b) to find the value of $\int_{x=0}^{\pi/2}\cos^n x\,dx$. Hint: what is $\int_{x=0}^{\pi/2}\cos x\,dx$?

   (d) When $n=2k$ is even, the method in part (b) doesn't work directly, and using double-angle formulas would be extremely messy. Solve the reduction formula from part (a) to evaluate the integral. Hint: what is $\int_{x=0}^{\pi/2}dx$?