MATH 117 Areas of Triangles

We now will use the right-triangle trig formulas to find the areas of right triangles, equilateral triangles, and isosceles triangles. We then will use the isosceles triangles to find the area of regular n-sided polygons. Finally, we will use Heron's Formula to find the areas of other scalene triangles.

Right-Triangle Formulas

$$x^2 + y^2 = z^2 \qquad z = \sqrt{x^2 + y^2} \qquad x = \sqrt{z^2 - y^2} \qquad y = \sqrt{z^2 - x^2}$$

$$\cos\theta = \frac{\text{Adj}}{\text{Hyp}} = \frac{x}{z} \qquad \sin\theta = \frac{\text{Opp}}{\text{Hyp}} = \frac{y}{z} \qquad \tan\theta = \frac{\text{Opp}}{\text{Adj}} = \frac{y}{x}$$

$$x = z\cos\theta \quad \text{and} \quad y = z\sin\theta$$

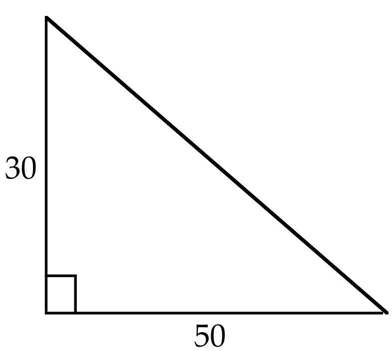
Right-Triangle Area

Given a right triangle, we can find the area using

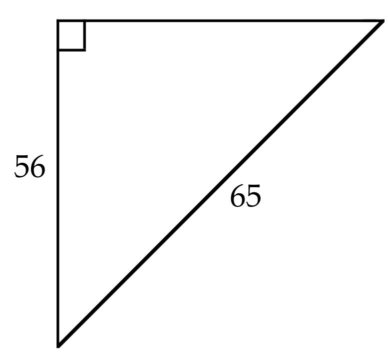
$$\text{Rt. Triangle Area} = \frac{1}{2} \times base \times height$$

The base and height are the adjacent and opposite sides of the two acute angles, so we also can say Area $= \frac{1}{2} \times opp \times adj$ or Area $= \frac{1}{2} x\,y$.

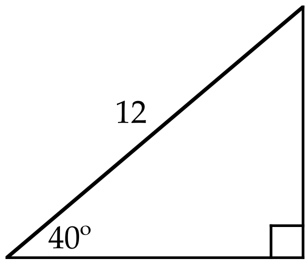
Example 1. Find the areas of the following right triangles:



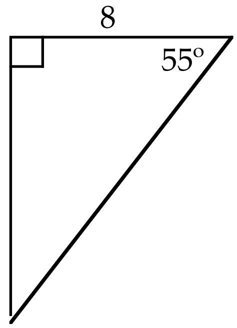
(i)



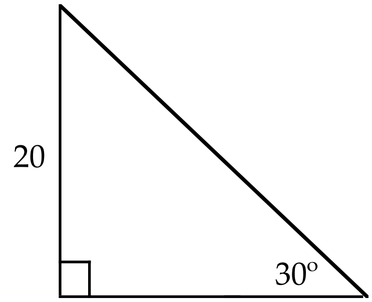
(ii)



(iii)



(iv)



(v)

Solutions. (i) We have the base and height, so the area is $\frac{1}{2}(30)(50) = \textbf{750 sq. units}$.

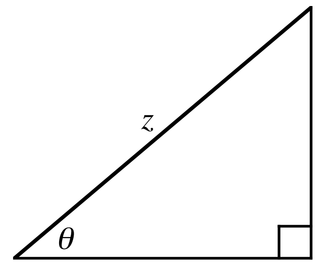
(ii) The last side is $x = \sqrt{65^2 - 56^2} = 33$, and the area is $\frac{1}{2}(33)(56) = \textbf{924 sq. units}$.

(iii) The two lateral sides are given by $x = 12\cos 40^\circ$ and $y = 12\sin 40^\circ$. So the area is given by

$$\text{Area} = \frac{1}{2} \times 12\cos 40^\circ \times 12\sin 40^\circ = \frac{12^2 \cos 40^\circ \sin 40^\circ}{2} \approx \textbf{35.453 sq. units}.$$

(iv) To find the height y, we use $\tan(55^\circ) = \frac{y}{8}$ and $y = 8\tan(55^\circ)$. So the area is given by $\frac{1}{2} \times 8 \times 8\tan(55^\circ) \approx \textbf{45.7 sq. units}$.

(v) To find the base x, we use $\tan(30^\circ) = \frac{20}{x}$ and $x = \frac{20}{\tan(30^\circ)}$. So the area is given by $\frac{1}{2} \times 20 \times \frac{20}{\tan(30^\circ)} \approx \textbf{346.41 sq. units}$.



Another general form of right-triangle area can be given when we have the hypotenuse z and one angle θ. First, recall that $\sin(2\theta) = 2\sin\theta\cos\theta$ so that $\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$. Because $x = z\cos\theta$ and $y = z\sin\theta$, we obtain Area = $\frac{1}{2}x\,y$ = $\frac{1}{2}\times z\cos\theta\times z\sin\theta = \frac{z^2}{2}\cos\theta\sin\theta = \frac{z^2}{2}\left(\frac{1}{2}\sin(2\theta)\right) = \frac{z^2}{4}\sin(2\theta)$. The forms to use are

$$\text{Rt. Triangle Area} = \frac{z^2}{2}\cos\theta\sin\theta \quad \text{or} \quad \text{Rt. Triangle Area} = \frac{z^2}{4}\sin(2\theta)$$

In Example (iii) above, we have Area = $\frac{12^2}{4}\sin(2\times 40^\circ) \approx$ **35.453 sq. units**.

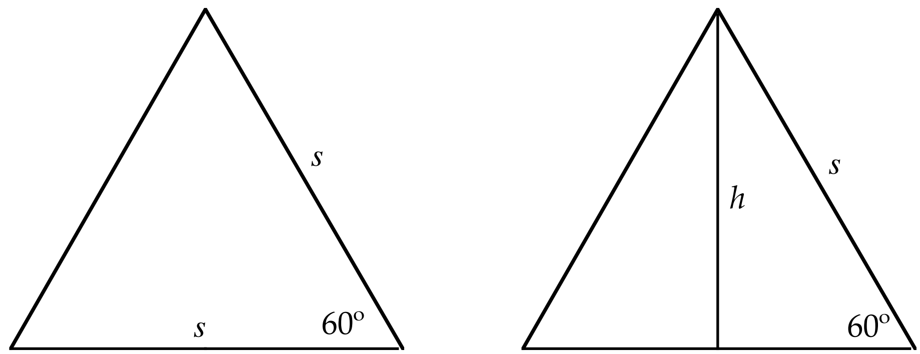
Examples (iv) and (v) also demonstrate other formulas that can be used. If we have an angle θ in a right triangle with x being adjacent and y being opposite, then

$$\text{Rt. Triangle Area} = \frac{x^2}{2}\tan\theta \quad \text{and} \quad \text{Rt. Triangle Area} = \frac{y^2}{2\tan\theta}$$

In most cases though, it is easiest to use Area = $\frac{1}{2}\times base\times height$, and simply find the base and height using right-triangle trig.

Equilateral Triangle Area

Given an equilateral triangle with three sides of length s and three 60° angles, we can still find the area using $\frac{1}{2}\times base\times height$. We note that $\sin 60^\circ = \frac{h}{s}$ which gives $h = s\times\sin 60^\circ = \frac{\sqrt{3}}{2}\times s$.



The base of the entire triangle is s and the height is $\frac{\sqrt{3}}{2}s$, so the area of an equilateral triangle is $(1/2)\times s \times \frac{\sqrt{3}}{2}s = \frac{\sqrt{3}\,s^2}{4}$.

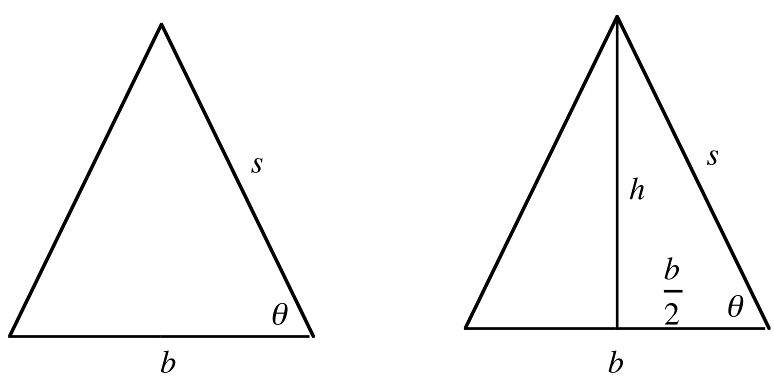
$$\text{Equilateral Triangle Area} = \frac{\sqrt{3}\,s^2}{4}$$

***Example* 2**. Find the area of an equilateral triangle with sides of length 20 inches.

Solution. The area is $\frac{\sqrt{3}\times 20^2}{4} \approx$ **173.2 square inches**.

Isosceles Triangle Area

Given an isosceles triangle, we can find the area using $\frac{1}{2}\times base \times height$ provided we know the base angle θ and either the base length b or vertical side s. Generally, we are given only one of b or s.



If we have s, then $\frac{b}{2} = s \times \cos\theta$ and $h = s \times \sin\theta$, which gives

$$\text{Isosceles Triangle Area} = \frac{b}{2} \times h = s^2 \cos\theta \ \sin\theta$$

If we have b, then $\tan\theta = h / (b/2)$ which gives $h = \frac{b}{2}\tan\theta$. The area is then

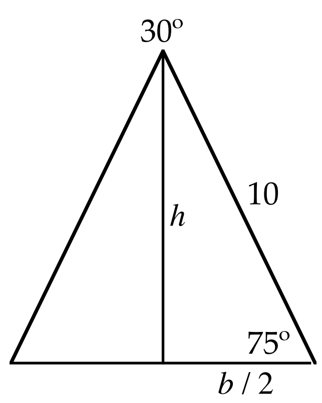
$$\text{Isosceles Triangle Area} = \frac{b}{2} \times h = \frac{b^2}{4}\tan\theta$$

Example 3. Find the areas of the following isosceles triangles:

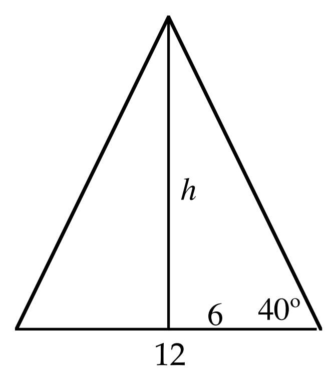
(i) Vertical sides of length 10 inches and a vertical angle of 30°
(ii) A base of 12 feet and base angles of 40°

Solution. (i) If the vertical angle is 30°, then each base angle is $\theta = \frac{180-30}{2} = 75^\circ$. So the height is $h = 10 \times \sin 75^\circ$ and half the base is $b/2 = 10 \times \cos 75^\circ$. So the area is

$$\frac{b}{2} \times h = 10^2 \cos 75^\circ \ \sin 75^\circ = \mathbf{25\ in^2}.$$



(i)



(ii)

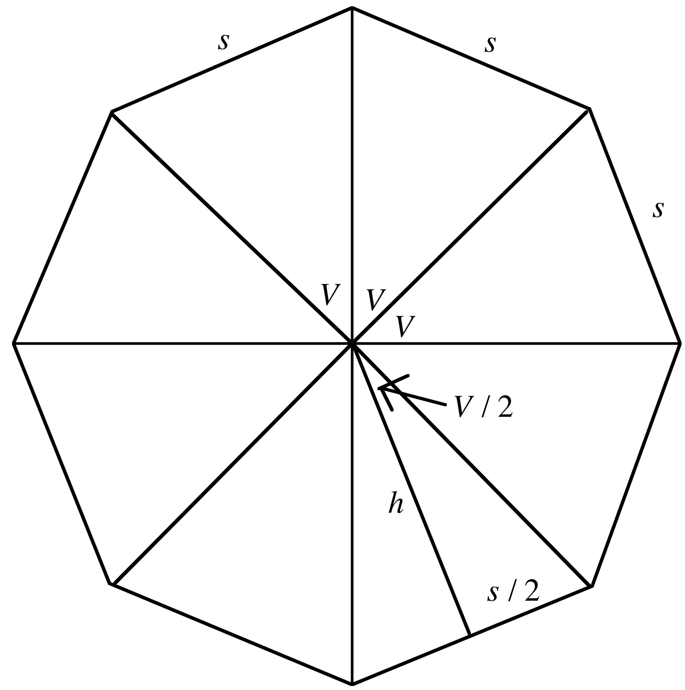
(ii) With $b = 12$ and $\theta = 40^\circ$, then $h/6 = \tan 40^\circ$ or $h = 6\tan 40^\circ$. So the area is

$$\frac{b}{2} \times h = 6 \times 6\tan 40^\circ \left(= \frac{12^2}{4}\tan 40^\circ \right) = \mathbf{30.2\ ft^2}.$$

Regular n-Sided Polygons

A regular n-sided polygon makes n congruent isosceles triangles where the base of a triangle equals one side s of the polygon. If there are n sides, then the vertical angle of each interior triangle is $V = \dfrac{360^\circ}{n}$.

In order to find the area of the polygon, we first must find the area of each interior isosceles triangle. But now we will do so in terms of the *vertical* angle V. We note that $\tan\left(\dfrac{V}{2}\right) = \dfrac{s/2}{h}$, so that $h = \dfrac{s/2}{\tan\left(\dfrac{V}{2}\right)}$.



The area of one interior isosceles triangle is then $\dfrac{s}{2} \times h = \dfrac{s^2}{4\tan\left(\dfrac{V}{2}\right)}$. Using $V = \dfrac{360^\circ}{n}$ and $\dfrac{V}{2} = \dfrac{180^\circ}{n}$, we obtain the area of a regular n-sided polygon:

$$\text{Regular } n\text{-gon area} = \frac{n\,s^2}{4\tan\left(\dfrac{180^\circ}{n}\right)}$$

Example 4. Find the area of a regular octagon ($n = 8$) with sides of length 10 inches.

Solution. Each vertical angle is $V = 360^\circ/8 = 45^\circ$. Bisecting an interior triangle, we have $\tan(22.5^\circ) = \dfrac{5}{h}$; so $h = \dfrac{5}{\tan(22.5^\circ)}$. The overall area is then $8\left(\dfrac{1}{2} \times base \times ht\right)$ = $8\left(\dfrac{1}{2} \times 10 \times \dfrac{5}{\tan(22.5^\circ)}\right)$, or $\dfrac{8 \times 10^2}{4\tan\left(\dfrac{180^\circ}{8}\right)}$, which gives about **482.8427 square inches**.

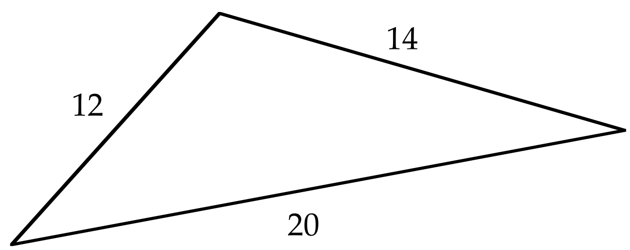
Scalene Triangles and Heron's Formula

Suppose a triangle has sides of length a, b, and c. Then *Heron's Formula* gives the area as

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{where } s = \frac{1}{2}(a+b+c)$$

Example 5. Find the area of the following triangle:



Solution. Let $s = \frac{1}{2}(12+14+20) = 23$. Then the area is

$$\sqrt{23(23-20)(23-14)(23-12)} = \sqrt{23 \times 3 \times 9 \times 11} = \sqrt{6831} \approx \mathbf{82.65\ sq.\ units}.$$