Let $\vec{F}(x,y,z) = (xyz, 0, -x^2y)$, and compute curl $\vec{F}$ and div $\vec{F}$. Is it possible that $\vec{F} = \operatorname{grad} f$ for some scalar-valued function $f : \mathbb{R}^3 \to \mathbb{R}$? Explain.

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 0 & -x^2y \end{bmatrix} = (-x^2+0)\vec{i} - (-2xy - xy)\vec{j} + (0 - xz)\vec{k}$$

$$= (-x^2, 3xy, -xz)$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = yz + 0 + 0 = yz.$$

We know that curl(grad f) $= \vec{0}$, so $\vec{F}$ can't be the gradient of any f.

2. Consider the path

$$\vec{x}(t) = (2\sin t, 5t, 2\cos t).$$

(a) Find the length of the path from $t = -10$ to $t = 10$.

arc length $= \int_{-10}^{10} \|\vec{v}(t)\|\, dt$

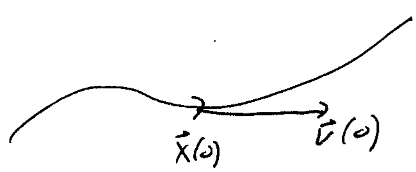
$\vec{v}(t) = \vec{x}'(t) = (2\cos t, 5, -2\sin t)$

$\|\vec{v}(t)\| = \sqrt{4\cos^2 t + 25 + 4\sin^2 t}$

$= \sqrt{29}$

So arc length $= \int_{-10}^{10} \sqrt{29}\, dt = 20\sqrt{29}$

(b) Find a parametric equation for the tangent line to the curve at the point $\vec{x}(0)$.



line is $\ell(t) = \vec{x}(0) + \vec{v}(0)t$.

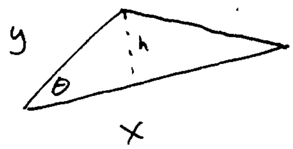
$\vec{x}(0) = (0, 0, 2)$

$\vec{v}(0) = (2, 5, 0)$

So $\ell(t) = (0, 0, 2) + t(2, 5, 0)$

or $\ell(t) = (2t, 5t, 2)$

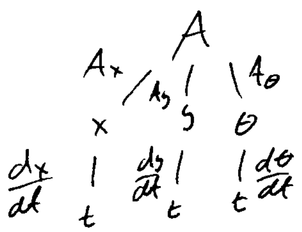
3 ~~4~~. The length x of a side of a triangle is increasing at a rate of 3 in/s, the length y of another side is decreasing at a rate of 2 in/s, and the contained angle θ is increasing at a rate of 0.05 radian/s. How fast is the area of the triangle changing when $x = 40$ in, $y = 50$ in, and $\theta = \pi/4$? (Recall that $\pi/4$ radians is 45°.)



Area $A = \frac{1}{2}(\text{base})(\text{height})$

$= \frac{1}{2} x h$

$= \frac{1}{2} x y \sin\theta$



So $\dfrac{dA}{dt} = \dfrac{\partial A}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial A}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial A}{\partial \theta}\dfrac{d\theta}{dt}$

$= \frac{1}{2} y \sin\theta \frac{dx}{dt} + \frac{1}{2} x \sin\theta \frac{dy}{dt} + \frac{1}{2} xy\cos\theta \frac{d\theta}{dt}$

Plug in the values for x, y, θ, x', y' & θ' & set

$x = 40,\ y = 50,\ \theta = \pi/4,\ x' = 3,\ y' = -2,\ \theta' = \frac{1}{20}$

$$\frac{dA}{dt} = \frac{1}{2}\cdot 50 \cdot \frac{1}{\sqrt{2}} 3 + \frac{1}{2}\cdot 40 \cdot \frac{1}{\sqrt{2}}\cdot(-2) + \frac{1}{2}\cdot 40\cdot 50 \cdot \frac{1}{\sqrt{2}}\cdot\frac{1}{20}$$

$$= \frac{1}{2}\left(\frac{70}{\sqrt{2}} + \frac{100}{\sqrt{2}}\right)$$

$$= \frac{85}{\sqrt{2}} \ \text{in}^2/\text{s}$$

4. Let $f(x,y) = e^{x^2y}$. Find an approximate value for $f(0.95, 0.10)$.

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Take $(a,b) = (1,0)$.

Then $f(1,0) = e^0 = 1$

$f_x(x,y) = 2xy\, e^{x^2y}$, so $f_x(1,0) = 0$.

$f_y(x,y) = x^2 e^{x^2y}$, so $f_y(1,0) = 1$.

So: $f(0.95, 0.10) \approx 1 + 0\cdot(-.05) + 1\cdot(0.1)$

$= 1 + 0 + .1$

$= 1.1$

5. The function $f(x,y)$ gives the temperature (in °C) at the point (x,y), where x and y are in centimeters. A bug leaves the point $(2,1)$ at 3 cm/min so that it cools off as fast as possible. In which direction does the bug head? At what rate does it cool off, in °C/min?

Move in the direction of minus the gradient of f at that point, $-\text{grad } f(2,1)$.

Since its speed is 3, it cools off at a rate $-3 \cdot \|\text{grad } f(2,1)\|$ °C/min

EXTRA CREDIT Which is the best, grad, curl, or div? Explain.