Supplement: The Long Exact Homology Sequence and Applications

S1. Chain Complexes

In the supplement, we will develop some of the building blocks for algebraic topology. As we go along, we will make brief comments [in brackets] indicating the connection between the algebraic machinery and the topological setting, but for best results here, please consult a text or attend lectures on algebraic topology.

S1.1 Definitions and Comments

A *chain complex* (or simply a *complex*) C_* is a family of R-modules C_n, $n \in \mathbb{Z}$, along with R-homomorphisms $d_n\colon C_n \to C_{n-1}$ called *differentials*, satisfying $d_n d_{n+1} = 0$ for all n. A chain complex with only finitely many C_n's is allowed; it can always be extended with the aid of zero modules and zero maps. [In topology, C_n is the abelian group of n-*chains*, that is, all formal linear combinations with integer coefficients of n-simplices in a topological space X. The map d_n is the *boundary operator*, which assigns to an n-simplex an $n-1$-chain that represents the oriented boundary of the simplex.]

The kernel of d_n is written $Z_n(C_*)$ or just Z_n; elements of Z_n are called *cycles* in dimension n. The image of d_{n+1} is written $B_n(C_*)$ or just B_n; elements of B_n are called *boundaries* in dimension n. Since the composition of two successive differentials is 0, it follows that $B_n \subseteq Z_n$. The quotient Z_n/B_n is written $H_n(C_*)$ or just H_n; it is called the n^{th} *homology module* (or *homology group* if the underlying ring R is $\mathbb{Z}$).

[The key idea of algebraic topology is the association of an algebraic object, the collection of homology groups $H_n(X)$, to a topological space X. If two spaces X and Y are homeomorphic, in fact if they merely have the same homotopy type, then $H_n(X)$ and $H_n(Y)$ are isomorphic for all n. Thus the homology groups can be used to distinguish between topological spaces; if the homology groups differ, the spaces cannot be homeomorphic.]

Note that any exact sequence is a complex, since the composition of successive maps is 0.

S1.2 Definition

A *chain map* $f\colon C_* \to D_*$ from a chain complex C_* to a chain complex D_* is a collection of module homomorphisms $f_n\colon C_n \to D_n$, such that for all n, the following diagram is commutative.

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "d_n"'] & D_n \arrow[d, "d_n"'] \\ C_{n-1} \arrow[r, "f_{n-1}"'] & D_{n-1} \end{tikzcd}$$

We use the same symbol d_n to refer to the differentials in C_* and D_*.

[If $f\colon X \to Y$ is a continuous map of topological spaces and σ is a singular n-simplex in X, then $f_\#(\sigma) = f \circ \sigma$ is a singular n-simplex in Y, and $f_\#$ extends to a homomorphism of n-chains. If we assemble the $f_\#$'s for $n = 0, 1, \ldots$, the result is a chain map.]

S1.3 Proposition

A chain map f takes cycles to cycles and boundaries to boundaries. Consequently, the map $z_n + B_n(C_*) \to f_n(z_n) + B_n(D_*)$ is a well-defined homomorphism from $H_n(C_*)$ to $H_n(D_*)$. It is denoted by $H_n(f)$.

Proof. If $z \in Z_n(C_*)$, then since f is a chain map, $d_n f_n(z) = f_{n-1} d_n(z) = f_{n-1}(0) = 0$. Therefore $f_n(z) \in Z_n(D_*)$. If $b \in B_n(C_*)$, then $d_{n+1} c = b$ for some $c \in C_{n+1}$. Then $f_n(b) = f_n(d_{n+1} c) = d_{n+1} f_{n+1} c$, so $f_n(b) \in B_n(D_*)$. ♣

S1.4 The Homology Functors

We can create a category whose objects are chain complexes and whose morphisms are chain maps. The composition gf of two chain maps $f\colon C_* \to D_*$ and $g\colon D_* \to E_*$ is the collection of homomorphisms $g_n f_n$, $n \in \mathbb{Z}$. For any n, we associate with the chain complex C_* its n^{th} homology module $H_n(C_*)$, and we associate with the chain map $f\colon C_* \to D_*$ the map $H_n(f)\colon H_n(C_*) \to H_n(D_*)$ defined in (S1.3). Since $H_n(gf) = H_n(g)H_n(f)$ and $H_n(1_{C_*})$ is the identity on $H_n(C_*)$, H_n is a functor, called the n^{th} *homology functor*.

S1.5 Chain Homotopy

Let f and g be chain maps from C_* to D_*. We say that f and g are *chain homotopic* and write $f \simeq g$ if there exist homomorphisms $h_n\colon C_n \to D_{n+1}$ such that $f_n - g_n = d_{n+1} h_n + h_{n-1} d_n$; see the diagram below.

$$\begin{tikzcd} & C_n \arrow[r, "d_n"] \arrow[dl, "h_n"'] \arrow[d, "f_n - g_n"] & C_{n-1} \arrow[dl, bend left, "h_{n-1}"] \\ D_{n+1} \arrow[r, "d_{n+1}"'] & D_n & \end{tikzcd}$$

[If f and g are homotopic maps from a topological space X to a topological space Y, then the maps $f_\#$ and $g_\#$ (see the discussion in (S1.2)) are chain homotopic,]

S1.6 Proposition

If f and g are chain homotopic, then $H_n(f) = H_n(g)$.

Proof. Let $z \in Z_n(C_*)$. Then

$$f_n(z) - g_n(z) = (d_{n+1}h_n + h_{n-1}d_n)z \in B_n(D_*)$$

since $d_n z = 0$. Thus $f_n(z) + B_n(D_*) = g_n(z) + B_n(D_*)$, in other words, $H_n(f) = H_n(g)$. ♣

S2. The Snake Lemma

We isolate the main ingredient of the long exact homology sequence. After an elaborate diagram chase, a homomorphism between two modules is constructed. The domain and codomain of the homomorphism are far apart in the diagram, and the arrow joining them tends to wiggle like a serpent. First, a result about kernels and cokernels of module homomorphisms.

S2.1 Lemma

Assume that the diagram below is commutative.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "d"'] & B \arrow[d, "e"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

(i) f induces a homomorphism on kernels, that is, $f(\ker d) \subseteq \ker e$.

(ii) g induces a homomorphism on cokernels, that is, the map $y + \operatorname{im} d \to g(y) + \operatorname{im} e$, $y \in C$, is a well-defined homomorphism from coker d to coker e.

(iii) If f is injective, so is the map induced by f, and if g is surjective, so is the map induced by g.

Proof. (i) If $x \in A$ and $d(x) = 0$, then $ef(x) = gd(x) = g0 = 0$.

(ii) If $y \in \operatorname{im} d$, then $y = dx$ for some $x \in A$. Thus $gy = gdx = efx \in \operatorname{im} e$. Since g is a homomorphism, the induced map is also.

(iii) The first statement holds because the map induced by f is simply a restriction. The second statement follows from the form of the map induced by g. ♣

Now refer to our *snake diagram*, Figure S2.1. Initially, we are given only the second and third rows (ABE0 and 0CDF), along with the maps d, e and h. Commutativity of the squares ABDC and BEFD is assumed, along with exactness of the rows. The diagram is now enlarged as follows. Take A' =ker d, and let the map from A' to A be inclusion. Take C' = coker d, and let the map from C to C' be canonical. Augment columns 2 and 3 in a similar fashion. Let $A' \to B'$ be the map induced by f on kernels, and let $C' \to D'$ be the map induced by g on cokernels. Similarly, add $B' \to E'$ and $D' \to F'$. The enlarged diagram is commutative by (S2.1), and it has exact columns by construction.

$$\begin{tikzcd}
 & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & E' \arrow[d] \arrow[dddll, dashed, out=150, in=180, looseness=2] & \\
 & A \arrow[r, "f"] \arrow[d, "d"] & B \arrow[r, "s"] \arrow[d, "e"] & E \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & C \arrow[r, "g"] \arrow[d] & D \arrow[r, "t"] \arrow[d] & F \arrow[d] & \\
 & C' \arrow[r] & D' \arrow[r] & F' &
\end{tikzcd}$$

Figure S2.1

S2.2 Lemma

The first and fourth rows of the enlarged snake diagram are exact.

Proof. This is an instructive diagram chase, showing many standard patterns. Induced maps will be denoted by an overbar, and we first prove exactness at B'. If $x \in A' =$ ker d and $y = \overline{f}x = fx$, then $sy = sfx = 0$, so $y \in$ker $\overline{s}$. On the other hand, if $y \in B' \subseteq B$ and $\overline{s}y = sy = 0$, then $y = fx$ for some $x \in A$. Thus $0 = ey = efx = gdx$, and since g is injective, $dx = 0$. Therefore $y = fx$ with $x \in A'$, and $y \in$im $\overline{f}$.

Now to prove exactness at D', let $x \in C$. Then $\overline{t}(gx + \operatorname{im} e) = tgx + \operatorname{im} h = 0$ by exactness of the third row, so im $\overline{g}$ $\subseteq$ker $\overline{t}$. Conversely, if $y \in D$ and $\overline{t}(y + \operatorname{im} e) = ty + \operatorname{im} h = 0$, then $ty = hz$ for some $z \in E$. Since s is surjective, $z = sx$ for some $x \in B$. Now

$$ty = hz = hsx = tex$$

so $y - ex \in \ker t = \operatorname{im} g$, say $y - ex = gw, w \in C$. Therefore

$$y + \operatorname{im} e = \overline{g}(w + \operatorname{im} d)$$

and $y + \operatorname{im} e \in \operatorname{im} \overline{g}$. ♣

S2.3 Remark

Sometimes an even bigger snake diagram is given, with column 1 assumed to be an exact sequence

$$0 \longrightarrow A' \longrightarrow A \xrightarrow{d} C \longrightarrow C' \longrightarrow 0$$

and similarly for columns 2 and 3. This is nothing new, because by replacing modules by isomorphic copies we can assume that A' is the kernel of d, C' is the cokernel of d, $A' \to A$ is inclusion, and $C \to C'$ is canonical.

S2.4 The Connecting Homomorphism

We will now connect E' to C' in the snake diagram while preserving exactness. The idea is to zig-zag through the diagram along the path $E'EBDCC'$.

Let $z \in E' \subseteq E$; Since s is surjective, there exists $y \in B$ such that $z = sy$. Then $tey = hsy = hz = 0$ since $E' = \ker h$. Thus $ey \in \ker t = \operatorname{im} g$, so $ey = gx$ for some $x \in C$. We define the *connecting homomorphism* $\partial \colon E' \to C'$ by $\partial z = x + \operatorname{im} d$. Symbolically,

$$\partial = [g^{-1} \circ e \circ s^{-1}]$$

where the brackets indicate that ∂z is the coset of x in $C' = C/\operatorname{im} d$.

We must show that ∂ is well-defined. Suppose that y' is another element of B with $sy' = z$. Then $y - y' \in \ker s = \operatorname{im} f$, so $y - y' = fu$ for some $u \in A$. Thus $e(y - y') = efu = gdu$. Now we know from the above computation that $ey = gx$ for some $x \in C$, and similarly $ey' = gx'$ for some $x' \in C$. Therefore $g(x - x') = gdu$, so $x - x' - du \in \ker g$. Since g is injective, $x - x' = du$, so $x + \operatorname{im} d = x' + \operatorname{im} d$. Thus ∂z is independent of the choice of the representatives y and x. Since every map in the diagram is a homomorphism, so is ∂.

S2.5 Snake Lemma

The sequence

$$A' \xrightarrow{\overline{f}} B' \xrightarrow{\overline{s}} E' \xrightarrow{\partial} C' \xrightarrow{\overline{g}} D' \xrightarrow{\overline{t}} F'$$

is exact.

Proof. In view of (S2.2), we need only show exactness at E' and C'. If $z = sy$, $y \in B' = \ker e$, then $ey = 0$, so $\partial z = 0$ by definition of ∂. Thus $\operatorname{im} \overline{s} \subseteq \ker \partial$. Conversely, assume $\partial z = 0$, and let x and y be as in the definition of ∂. Then $x = du$ for some $u \in A$, hence $gx = gdu = efu$. But $gx = ey$ by definition of ∂, so $y - fu \in \ker e = B'$. Since $z = sy$ by definition of ∂, we have

$$z = s(y - fu + fu) = s(y - fu) \in \operatorname{im}\, \overline{s}.$$

To show exactness at C', consider an element ∂z in the image of ∂. Then $\partial z = x + \operatorname{im} d$, so $\overline{g}\partial z = gx + \operatorname{im} e$. But $gx = ey$ by definition of ∂, so $\overline{g}\partial z = 0$ and $\partial z \in \ker \overline{g}$. Conversely, suppose $x \in C$ and $\overline{g}(x + \operatorname{im} d) = gx + \operatorname{im} e = 0$. Then $gx = ey$ for some $y \in B$. If $z = sy$, then $hsy = tey = tgx = 0$ by exactness of the third row. Thus $z \in E'$ and (by definition of ∂) we have $\partial z = x + \operatorname{im} d$. Consequently, $x + \operatorname{im} d \in \operatorname{im} \partial$. ♣

S3. The Long Exact Homology Sequence

S3.1 Definition

We say that

$$0 \longrightarrow C_* \xrightarrow{f} D_* \xrightarrow{g} E_* \longrightarrow 0$$

where f and g are chain maps, is a *short exact sequence of chain complexes* if for each n, the corresponding sequence formed by the component maps $f_n \colon C_n \to D_n$ and $g_n \colon D_n \to E_n$, is short exact. We will construct *connecting homomorphisms* $\partial_n \colon H_n(E_*) \to H_{n-1}(C_*)$ such that the sequence

$$\cdots \xrightarrow{g} H_{n+1}(E_*) \xrightarrow{\partial} H_n(C_*) \xrightarrow{f} H_n(D_*) \xrightarrow{g} H_n(E_*) \xrightarrow{\partial} H_{n-1}(C_*) \xrightarrow{f} \cdots$$

is exact. [We have taken some liberties with the notation. In the second diagram, f stands for the map induced by f_n on homology, namely, $H_n(f)$; similarly for g.] The second diagram is the *long exact homology sequence*, and the result may be summarized as follows.

S3.2 Theorem

A short exact sequence of chain complexes induces a long exact sequence of homology modules.

Proof. This is a double application of the snake lemma. The main ingredient is the following snake diagram.

$$\begin{tikzcd}
 & C_n/B_n(C_*) \arrow[r] \arrow[d, "d"'] & D_n/B_n(D_*) \arrow[r] \arrow[d, "d"'] & E_n/B_n(E_*) \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & Z_{n-1}(C_*) \arrow[r] & Z_{n-1}(D_*) \arrow[r] & Z_{n-1}(E_*) &
\end{tikzcd}$$

The horizontal maps are derived from the chain maps f and g, and the vertical maps are given by $d(x_n + B_n) = dx_n$. The kernel of a vertical map is $\{x_n + B_n \colon x_n \in Z_n\} = H_n$, and the cokernel is $Z_{n-1}/B_{n-1} = H_{n-1}$. The diagram is commutative by the definition of a chain map. But in order to apply the snake lemma, we must verify that the rows are exact, and this involves another application of the snake lemma. The appropriate diagram is

$$\begin{tikzcd}
0 \arrow[r] & C_n \arrow[r] \arrow[d, "d"'] & D_n \arrow[r] \arrow[d, "d"'] & E_n \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r] & D_{n-1} \arrow[r] & E_{n-1} \arrow[r] & 0
\end{tikzcd}$$

where the horizontal maps are again derived from f and g. The exactness of the rows of the first diagram follows from (S2.2) and part (iii) of (S2.1), shifting indices from n to $n \pm 1$ as needed. ♣

S3.3 The connecting homomorphism explicitly

If $z \in H_n(E_*)$, then $z = z_n + B_n(E_*)$ for some $z_n \in Z_n(E_*)$. We apply (S2.4) to compute ∂z. We have $z_n + B_n(E_*) = g_n(y_n + B_n(D_*))$ for some $y_n \in D_n$. Then $dy_n \in Z_{n-1}(D_*)$ and $dy_n = f_{n-1}(x_{n-1})$ for some $x_{n-1} \in Z_{n-1}(C_*)$. Finally, $\partial z = x_{n-1} + B_{n-1}(C_*)$.

S3.4 Naturality

Suppose that we have a commutative diagram of short exact sequences of chain complexes, as shown below.

$$
\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \arrow[r] & 0
\end{tikzcd}
$$

Then there is a corresponding commutative diagram of long exact sequences:

$$
\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_n(C_*) \arrow[r] \arrow[d] & H_n(D_*) \arrow[r] \arrow[d] & H_n(E_*) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(C_*) \arrow[r] & \cdots \\
\cdots \arrow[r, "\partial"] & H_n(C'_*) \arrow[r] & H_n(D'_*) \arrow[r] & H_n(E'_*) \arrow[r, "\partial"] & H_{n-1}(C'_*) \arrow[r] & \cdots
\end{tikzcd}
$$

Proof. The homology functor, indeed any functor, preserves commutative diagrams, so the two squares on the left commute. For the third square, an informal argument may help to illuminate the idea. Trace through the explicit construction of ∂ in (S3.3), and let f be the vertical chain map in the commutative diagram of short exact sequences. The first step in the process is

$$z_n + B_n(E_*) \to y_n + B_n(D_*).$$

By commutativity,

$$fz_n + B_n(E'_*) \to fy_n + B_n(D'_*).$$

Continuing in this fashion, we find that if $\partial z = x_{n-1} + B_{n-1}(C_*)$, then

$$\partial(fz) = fx_{n-1} + B_{n-1}(C'_*) = f(x_{n-1} + B_{n-1}(C_*)) = f(\partial z). \quad ♣$$

A formal proof can be found in "An Introduction to Algebraic Topology" by J. Rotman, page 95.

S4. Projective and Injective Resolutions

The functors Tor and Ext are developed with the aid of projective and injective resolutions of a module, and we will now examine these constructions.

S4.1 Definitions and Comments

A *left resolution* of a module M is an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

A left resolution is a *projective resolution* if every P_i is projective, a *free resolution* if every P_i is free. By the first isomorphism theorem, M is isomorphic to the cokernel of the map $P_1 \to P_0$, so in a sense no information is lost if M is removed. A *deleted projective resolution* is of the form

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] \arrow[d] & 0 \\
& & & M &
\end{tikzcd}$$

and the deleted version turns out to be more convenient in computations. Notice that in a deleted projective resolution, exactness at P_0 no longer holds because the map $P_1 \to P_0$ need not be surjective. Resolutions with only finitely many P_n's are allowed, provided that the module on the extreme left is 0. The sequence can then be extended via zero modules and zero maps.

Dually, a *right resolution* of M is an exact sequence

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow E_2 \cdots ;$$

we have an *injective resolution* if every E_i is injective, . A *deleted injective resolution* has the form

$$\begin{tikzcd}
0 \arrow[r] & E_0 \arrow[r] & E_1 \arrow[r] & E_2 \arrow[r] & \cdots \\
& M \arrow[u] & & &
\end{tikzcd}$$

Exactness at E_0 no longer holds because the map $E_0 \to E_1$ need not be injective.

We will use the notation $P_* \to M$ for a projective resolution, and $M \to E_*$ for an injective resolution.

S4.2 Proposition

Every module M has a free (hence projective) resolution.

Proof. By (4.3.6), M is a homomorphic image of a free module F_0. Let K_0 be the kernel of the map from F_0 onto M. In turn, there is a homomorphism with kernel K_1 from a free module F_1 onto K_0, and we have the following diagram:

$$0 \longrightarrow K_1 \longrightarrow F_1 \longrightarrow K_0 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

Composing the maps $F_1 \to K_0$ and $K_0 \to F_0$, we get

$$0 \longrightarrow K_1 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

which is exact. But now we can find a free module F_2 and a homomorphism with kernel K_2 mapping F_2 onto K_1. The above process can be iterated to produce the desired free resolution. ♣

Specifying a module by generators and relations (see (4.6.6) for abelian groups) involves finding an appropriate F_0 and K_0, as in the first step of the above iterative process. Thus a projective resolution may be regarded as a generalization of a specification by generators and relations.

Injective resolutions can be handled by dualizing the proof of (S4.2).

S4.3 Proposition

Every module M has an injective resolution.

Proof. By (10.7.4), M can be embedded in an injective module E_0. Let C_0 be the cokernel of $M \to E_0$, and map E_0 canonically onto C_0. Embed C_0 in an injective module E_1, and let C_1 be the cokernel of the embedding map. We have the following diagram:

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow C_0 \longrightarrow E_1 \longrightarrow C_1 \longrightarrow 0$$

Composing $E_0 \to C_0$ and $C_0 \to E_1$, we have

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow C_1 \longrightarrow 0$$

which is exact. Iterate to produce the desired injective resolution. ♣

S5. Derived Functors

S5.1 Left Derived Functors

Suppose that F is a right exact functor from modules to modules. (In general, the domain and codomain of F can be abelian categories, but the example we have in mind is $M \otimes_R -$.) Given a short exact sequence $0 \to A \to B \to C \to 0$, we form deleted projective resolutions $P_{A*} \to A$, $P_{B*} \to B$, $P_{C*} \to C$. It is shown in texts on homological algebra that it is possible to define chain maps to produce a short exact sequence of complexes as shown below.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & P_{A_*} \arrow[r] \arrow[u] & P_{B_*} \arrow[r] \arrow[u] & P_{C_*} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The functor F will preserve exactness in the diagram, except at the top row, where we only have $FA \to FB \to FC \to 0$ exact. But remember that we are using deleted resolutions, so that the first row is suppressed. The *left derived functors* of F are defined by taking the homology of the complex $F(P)$, that is,

$$(L_nF)(A) = H_n[F(P_{A*})].$$

The word "left" is used because the L_nF are computed using left resolutions. It can be shown that up to natural equivalence, the derived functors are independent of the particular projective resolutions chosen. By (S3.2), we have the following long exact sequence:

$$\cdots \xrightarrow{\partial} (L_nF)(A) \longrightarrow (L_nF)(B) \longrightarrow (L_nF)(C) \xrightarrow{\partial} (L_{n-1}F)(A) \longrightarrow \cdots$$

S5.2 Right Derived Functors

Suppose now that F is a left exact functor from modules to modules, e.g., $\operatorname{Hom}_R(M, -)$. We can dualize the discussion in (S5.1) by reversing the vertical arrows in the commutative diagram of complexes, and replacing projective resolutions such as P_{A*} by injective resolutions E_{A*}. The *right derived functors* of F are defined by taking the homology of $F(E)$. In other words,

$$(R^nF)(A) = H^n[F(E_{A*})]$$

where the superscript n indicates that we are using right resolutions and the indices are increasing as we move away from the starting point. By (S3.2), we have the following long exact sequence:

$$\cdots \xrightarrow{\partial} (R^nF)(A) \longrightarrow (R^nF)(B) \longrightarrow (R^nF)(C) \xrightarrow{\partial} (R^{n+1}F)(A) \longrightarrow \cdots$$

S5.3 Lemma

$(L_0F)(A) \cong F(A) \cong (R^0F)(A)$

Proof. This is a good illustration of the advantage of deleted resolutions. If $P_* \to A$, we have the following diagram:

$$\begin{tikzcd} F(P_1) \arrow[r] & F(P_0) \arrow[r] \arrow[d] & 0 \\ & F(A) \arrow[d] & \\ & 0 & \end{tikzcd}$$

The kernel of $F(P_0) \to 0$ is $F(P_0)$, so the 0^{th} homology module $(L_0F)(A)$ is $F(P_0)$ mod the image of $F(P_1) \to F(P_0)$ [=the kernel of $F(P_0) \to F(A)$.] By the first isomorphism theorem and the right exactness of F, $(L_0F)(A) \cong F(A)$. To establish the other isomorphism, we switch to injective resolutions and reverse arrows:

$$\begin{tikzcd} 0 \arrow[r] & F(E_0) \arrow[r] & F(E_1) \\ & F(A) \arrow[u] & \\ & 0 \arrow[u] & \end{tikzcd}$$

The kernel of $F(E_0) \to F(E_1)$ is isomorphic to $F(A)$ by left exactness of F, and the image of $0 \to F(E_0)$ is 0. Thus $(R^0F)(A) \cong F(A)$. ♣

S5.4 Lemma

If A is projective, then $(L_nF)(A) = 0$ for every $n > 0$; if A is injective, then $(R^nF)(A) = 0$ for every $n > 0$.

Proof. If A is projective [resp. injective], then $0 \to A \to A \to 0$ is a projective [resp. injective] resolution of A. Switching to a deleted resolution, we have $0 \to A \to 0$ in each case, and the result follows. ♣

S5.5 Definitions and Comments

If F is the right exact functor $M \otimes_R -$, the left derived functor L_nF is called $\operatorname{Tor}_n^R(M, -)$. If F is the left exact functor $\operatorname{Hom}_R(M, -)$, the right derived functor R^nF is called $\operatorname{Ext}_R^n(M, -)$. It can be shown that the Ext functors can also be computed using projective resolutions and the contravariant hom functor. Specifically,

$$\operatorname{Ext}_R^n(M, N) = [R^n \operatorname{Hom}_R(-, N)](M).$$

A switch from injective to projective resolutions is a simplification, because projective resolutions are easier to find in practice.

The next three results sharpen Lemma S5.4. [The ring R is assumed fixed, and we write $\otimes_R$ simply as $\otimes$. Similarly, we drop the R in Tor^R and Ext_R. When discussing Tor, we assume R commutative.]

S5.6 Proposition

If M is an R-module, the following conditions are equivalent.

(i) M is flat;

(ii) $\operatorname{Tor}_n(M, N) = 0$ for all $n \geq 1$ and all modules N;

(iii) $\operatorname{Tor}_1(M, N) = 0$ for all modules N.

Proof. (i) implies (ii): Let $P_* \to N$ be a projective resolution of N. Since $M \otimes -$ is an exact functor (see (10.8.1)), the sequence

$$\cdots \to M \otimes P_1 \to M \otimes P_0 \to M \otimes N \to 0$$

is exact. Switching to a deleted resolution, we have exactness up to $M \otimes P_1$ but not at $M \otimes P_0$. Since the homology modules derived from an exact sequence are 0, the result follows.

(ii) implies (iii): Take $n = 1$.

(iii) implies (i): If $0 \to A \to B \to C \to 0$ is a short exact sequence, then by (S5.1), we have the following long exact sequence:

$$\cdots \operatorname{Tor}_1(M, C) \to \operatorname{Tor}_0(M, A) \to \operatorname{Tor}_0(M, B) \to \operatorname{Tor}_0(M, C) \to 0.$$

By hypothesis, $\operatorname{Tor}_1(M, C) = 0$, so by (S5.3),

$$0 \to M \otimes A \to M \otimes B \to M \otimes C \to 0$$

is exact, and therefore M is flat. ♣

S5.7 Proposition

If M is an R-module, the following conditions are equivalent.

(i) M is projective;

(ii) $\operatorname{Ext}^n(M, N) = 0$ for all $n \geq 1$ and all modules N;

(iii) $\operatorname{Ext}^1(M, N) = 0$ for all modules N.

Proof. (i) implies (ii): By (S5.4) and (S5.5), $\operatorname{Ext}^n(M, N) = [\operatorname{Ext}^n(-, N)](M) = 0$ for $n \geq 1$.

(ii) implies (iii): Take $n = 1$.

(iii) implies (i): Let $0 \to A \to B \to M \to 0$ be a short exact sequence. If N is any module, then using projective resolutions and the contravariant hom functor to construct Ext, as in (S5.5), we get the following long exact sequence:

$$0 \to \operatorname{Ext}^0(M, N) \to \operatorname{Ext}^0(B, N) \to \operatorname{Ext}^0(A, N) \to \operatorname{Ext}^1(M, N) \to \cdots$$

By (iii) and (S5.3),

$$0 \to \operatorname{Hom}(M, N) \to \operatorname{Hom}(B, N) \to \operatorname{Hom}(A, N) \to 0$$

is exact. Take $N = A$ and let g be the map from A to B. Then the map g^* from $\operatorname{Hom}(B, A)$ to $\operatorname{Hom}(A, A)$ is surjective. But $1_A \in \operatorname{Hom}(A, A)$, so there is a homomorphism $f \colon B \to A$ such that $g^*(f) = fg = 1_A$. Therefore the sequence $0 \to A \to B \to M \to 0$ splits, so by (10.5.3), M is projective. ♣

S5.8 Corollary

If N is an R-module, the following conditions are equivalent.

(a) N is injective;

(b) $\text{Ext}^n(M, N) = 0$ for all $n \geq 1$ and all modules M;

(c) $\text{Ext}^1(M, N) = 0$ for all modules M.

Proof. Simply saying "duality" may be unconvincing, so let's give some details. For (a) implies (b), we have $[\text{Ext}^n(M, -)](N) = 0$. For (c) implies (a), note that the exact sequence $0 \to N \to A \to B \to 0$ induces the exact sequence

$$0 \to \text{Ext}^0(M, N) \to \text{Ext}^0(M, A) \to \text{Ext}^0(M, B) \to 0.$$

Replace $\text{Ext}^0(M, N)$ by $\text{Hom}(M, N)$, and similarly for the other terms. Then take $M = B$ and proceed exactly as in (S5.7). ♣

S6. Some Properties of Ext and Tor

We will compute $\text{Ext}^n_R(A, B)$ and $\text{Tor}^R_n(A, B)$ in several interesting cases.

S6.1 Example

We will calculate $\text{Ext}^n_{\mathbb{Z}}(\mathbb{Z}_m, B)$ for an arbitrary abelian group B. To ease the notational burden slightly, we will omit the subscript $\mathbb{Z}$ in Ext and Hom, and use $=$ (most of the time) when we really mean $\cong$. We have the following projective resolution of $\mathbb{Z}_m$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}_m \longrightarrow 0$$

where the m over the arrow indicates multiplication by m. Switching to a deleted resolution and applying the contravariant hom functor, we get

$$\begin{tikzcd} 0 \arrow[r] & \text{Hom}(\mathbb{Z}, B) \arrow[r, "m"] & \text{Hom}(\mathbb{Z}, B) \arrow[r] & 0 \\ & \text{Hom}(\mathbb{Z}_m, B) \arrow[u] & & \end{tikzcd}$$

But by (9.4.1), we have

$$\text{Hom}_R(R, B) \cong B \tag{1}$$

and the above diagram becomes

$$0 \longrightarrow B \xrightarrow{m} B \longrightarrow 0 \tag{2}$$

By (S5.3), $\text{Ext}^0(\mathbb{Z}_m, B) = \text{Hom}(\mathbb{Z}_m, B)$. Now a homomorphism f from $\mathbb{Z}_m$ to B is determined by $f(1)$, and $f(m) = mf(1) = 0$. If $B(m)$ is the set of all elements of B

that are annihilated by m, that is, $B(m) = \{x \in B \colon mx = 0\}$, then the map of $B(m)$ to $\operatorname{Hom}(\mathbb{Z}_m, B)$ given by $x \to f$ where $f(1) = x$, is an isomorphism. Thus

$$\operatorname{Ext}^0(\mathbb{Z}_m, B) = B(m).$$

It follows from (2) that

$$\operatorname{Ext}^n(\mathbb{Z}_m, B) = 0,\ n \geq 2$$

and

$$\operatorname{Ext}^1(\mathbb{Z}_m, B) = \ker(B \to 0)/\operatorname{im}(B \to B) = B/mB.$$

The computation for $n \geq 2$ is a special case of a more general result.

S6.2 Proposition

$\operatorname{Ext}^n_{\mathbb{Z}}(A, B) = 0$ for all $n \geq 2$ and all abelian groups A and B.

Proof. If B is embedded in an injective module E, we have the exact sequence

$$0 \to B \to E \to E/B \to 0.$$

This is an injective resolution of B since E/B is divisible, hence injective; see (10.6.5) and (10.6.6). Applying the functor $\operatorname{Hom}(A, -) = \operatorname{Hom}_{\mathbb{Z}}(A, -)$ and switching to a deleted resolution, we get the sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}(A,E) \arrow[r] & \operatorname{Hom}(A,E/B) \arrow[r] & 0 \\ & \operatorname{Hom}(A,B) \arrow[u] & & \end{tikzcd}$$

whose homology is 0 for all $n \geq 2$. ♣

S6.3 Lemma

$\operatorname{Ext}^0_{\mathbb{Z}}(\mathbb{Z}, B) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) = B$ and $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, B) = 0$.

Proof. The first equality follows from (S5.3) and the second from (1) of (S6.1). Since $\mathbb{Z}$ is projective, the last statement follows from (S5.7). ♣

S6.4 Example

We will compute $\operatorname{Tor}^{\mathbb{Z}}_n(\mathbb{Z}_m, B)$ for an arbitrary abelian group B. As before, we drop the superscript $\mathbb{Z}$ and write $=$ for $\cong$. We use the same projective resolution of $\mathbb{Z}_m$ as in (S6.1), and apply the functor $-\otimes B$. Since $R \otimes_R B \cong B$ by (8.7.6), we reach diagram (2) as before. Thus

$$\begin{gathered}\operatorname{Tor}_n(\mathbb{Z}_m, B) = 0,\ n \geq 2;\\ \operatorname{Tor}_1(\mathbb{Z}_m, B) = \ker(B \to B) = \{x \in B \colon mx = 0\} = B(m);\\ \operatorname{Tor}_0(\mathbb{Z}_m, B) = \mathbb{Z}_m \otimes B = B/mB.\end{gathered}$$

[To verify the last equality, use the universal mapping property of the tensor product to produce a map of $\mathbb{Z}_m \otimes B$ to B/mB such that $n \otimes x \to n(x+mB)$. The inverse of this map is $x + mB \to 1 \otimes x$.]

The result for $n \geq 2$ generalizes as in (S6.2):

S6.5 Proposition

$\operatorname{Tor}_n^{\mathbb{Z}}(A, B) = 0$ for all $n \geq 2$ and all abelian groups A and B.

Proof. B is the homomorphic image of a free module F. If K is the kernel of the homomorphism, then the exact sequence $0 \to K \to F \to B \to 0$ is a free resolution of B. [K is a submodule of a free module over a PID, hence is free.] Switching to a deleted resolution and applying the tensor functor, we get a four term sequence as in (S6.2), and the homology must be 0 for $n \geq 2$. ♣

S6.6 Lemma

$\operatorname{Tor}_1(\mathbb{Z}, B) = \operatorname{Tor}_1(A, \mathbb{Z}) = 0$; $\operatorname{Tor}_0(\mathbb{Z}, B) = \mathbb{Z} \otimes B = B$.

Proof. The first two equalities follow from (S5.6) since $\mathbb{Z}$ is flat. The other two equalities follow from (S5.3) and (8.7.6). ♣

S6.7 Finitely generated abelian groups

We will show how to compute $\operatorname{Ext}^n(A, B)$ and $\operatorname{Tor}_n(A, B)$ for arbitrary finitely generated abelian groups A and B. By (4.6.3), A and B can be expressed as a finite direct sum of cyclic groups. Now Tor commutes with direct sums:

$$\operatorname{Tor}_n(A, \oplus_{j=1}^r B_j) = \oplus_{j=1}^r \operatorname{Tor}_n(A, B_j).$$

[The point is that if P_{j*} is a projective resolution of B_j, then the direct sum of the P_{j*} is a projective resolution of $\oplus_j B_j$, by (10.5.4). Since the tensor functor is additive on direct sums, by (8.8.6(b)), the Tor functor will be additive as well. Similar results hold when the direct sum is in the first coordinate, and when Tor is replaced by Ext. (We use (10.6.3) and Problems 5 and 6 of Section 10.9, and note that the direct product and the direct sum are isomorphic when there are only finitely many factors.)]

Thus to complete the computation, we need to know $\operatorname{Ext}(A, B)$ and $\operatorname{Tor}(A, B)$ when $A = \mathbb{Z}$ or $\mathbb{Z}_m$ and $B = \mathbb{Z}$ or $\mathbb{Z}_n$. We have already done most of the work. By (S6.2) and

(S6.5), Ext^n and Tor_n are identically 0 for $n \geq 2$. By (S6.1),

$$\text{Ext}^0(\mathbb{Z}_m, \mathbb{Z}) = \mathbb{Z}(m) = \{x \in \mathbb{Z} \colon mx = 0\} = 0;$$
$$\text{Ext}^0(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_n(m) = \{x \in \mathbb{Z}_n \colon mx = 0\} = \mathbb{Z}_d$$

where d is the greatest common divisor of m and n. [For example, $\mathbb{Z}_{12}(8) = \{0, 3, 6, 9\} \cong \mathbb{Z}_4$. The point is that the product of two integers is their greatest common divisor times their least common multiple.] By (S6.3),

$$\text{Ext}^0(\mathbb{Z}, \mathbb{Z}) = \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z};\ \text{Ext}^0(\mathbb{Z}, \mathbb{Z}_n) = \mathbb{Z}_n.$$

By (S6.1),

$$\text{Ext}^1(\mathbb{Z}_m, \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}_m;$$
$$\text{Ext}^1(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_n/m\mathbb{Z}_n = \mathbb{Z}_d$$

as above. By (S5.7),

$$\text{Ext}^1(\mathbb{Z}, \mathbb{Z}) = \text{Ext}^1(\mathbb{Z}, \mathbb{Z}_n) = 0$$

By (S6.4),

$$\text{Tor}_1(\mathbb{Z}_m, \mathbb{Z}) = \text{Tor}_1(\mathbb{Z}, \mathbb{Z}_m) = \mathbb{Z}(m) = 0;$$
$$\text{Tor}_1(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_n(m) = \mathbb{Z}_d.$$

By (8.7.6) and (S6.4),

$$\text{Tor}_0(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z};$$
$$\text{Tor}_0(\mathbb{Z}_m, \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}_m;$$
$$\text{Tor}_0(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_n/m\mathbb{Z}_n = \mathbb{Z}_d.$$

Notice that $\text{Tor}_1(A, B)$ is a torsion group for all finitely generated abelian groups A and B. This is a partial explanation of the term "Tor". The Ext functor arises in the study of group extensions.

S7. Base Change in the Tensor Product

Let M be an A-module, and suppose that we have a ring homomorphism from A to B (all rings are assumed commutative). Then $B \otimes_A M$ becomes a B module (hence an A-module) via $b(b' \otimes m) = bb' \otimes m$. This is an example of base change, as discussed in (10.8.8). We examine some frequently occurring cases. First, consider $B = A/I$, where I is an ideal of A.

S7.1 Proposition

$(A/I) \otimes_A M \cong M/IM$.

Proof. Apply the (right exact) tensor functor to the exact sequence of A-modules

$$0 \to I \to A \to A/I \to 0$$

to get the exact sequence

$$I \otimes_A M \to A \otimes_A M \to (A/I) \otimes_A M \to 0.$$

Recall from (8.7.6) that $A \otimes_A M$ is isomorphic to M via $a \otimes m \to am$. By the first isomorphism theorem, $(A/I) \otimes_A M$ is isomorphic to M mod the image of the map from $I \otimes_A M$ to M. This image is the collection of all finite sums $\sum a_i m_i$ with $a_i \in I$ and $m_i \in M$, which is IM. ♣

Now consider $B = S^{-1}A$, where S is a multiplicative subset of A.

S7.2 Proposition

$(S^{-1}A) \otimes_A M \cong S^{-1}M$.

Proof. The map from $S^{-1}A \times M$ to $S^{-1}M$ given by $(a/s, m) \to am/s$ is A-bilinear, so by the universal mapping property of the tensor product, there is a linear map $\alpha\colon S^{-1}A \otimes_A M \to S^{-1}M$ such that $\alpha((a/s) \otimes m) = am/s$. The inverse map β is given by $\beta(m/s) = (1/s) \otimes m$. To show that β is well-defined, suppose that $m/s = m'/s'$. Then for some $t \in S$ we have $ts'm = tsm'$. Thus

$$1/s \otimes m = ts'/tss' \otimes m = 1/tss' \otimes ts'm = 1/tss' \otimes tsm' = 1/s' \otimes m'.$$

Now α followed by β takes $a/s \otimes m$ to am/s and then to $1/s \otimes am = a/s \otimes m$. On the other hand, β followed by α takes m/s to $1/s \otimes m$ and then to m/s. Consequently, α and β are inverses of each other and yield the desired isomorphism of $S^{-1}A \otimes_A M$ and $S^{-1}M$. ♣

Finally, we look at $B = A[X]$.

S7.3 Proposition

$$[X] \otimes_A M \cong M[X]$$

where the elements of $M[X]$ are of the form $a_0 m_0 + a_1 X m_1 + a_2 X^2 m_2 + \cdots + a_n X^n m_n$, $a_i \in A$, $m_i \in M$, $n = 0, 1, \ldots$.

Proof. This is very similar to (S7.2). In this case, the map α from $A[X] \otimes_A M$ to $M[X]$ takes $f(X) \otimes m$ to $f(X)m$, and the map β from $M[X]$ to $A[X] \otimes_A M$ takes $X^i m$ to $X^i \otimes m$. Here, there is no need to show that β is well-defined. ♣