

## Homework #2

- 2.1 (a) What is the optimal solution of this problem?

Let us consider that  $x_1$ ,  $x_2$  and  $x_3$  are slack variables and write an equivalent LP formulation of the given problem:

$$\begin{aligned} \max & 10 \cdot x_4 + 32 \\ \text{s.t} & \\ & 2 \cdot x_4 \leq 8 \\ & 3 \cdot x_4 \leq 6 \\ & 8 \cdot x_4 \leq 24 \\ & x_4 \geq 0 \end{aligned} \tag{1}$$

From this formulation it follows directly that the optimal value of  $x_4$  is:

$$x_4^* = \min\left\{\frac{8}{2}, \frac{6}{3}, \frac{24}{8}\right\} = 2$$

Then from the definition of  $x_1, x_2, x_3$  as slack variables we have that the corresponding optimal solutions are:

$$\begin{aligned} x_1^* &= 8 - 2 \cdot 2 = 4 \\ x_2^* &= 6 - 3 \cdot 2 = 0 \\ x_3^* &= 24 - 8 \cdot 2 = 8 \end{aligned}$$

And finally the optimal value is:

$$z^* = 32 + 10 \cdot 2 = 52$$

- (b) Change the coefficient of  $x_4$  in the z-equation to -3. What is the optimal solution now?  
Following the same line of ideas than in item 0a we write an equivalent LP formulation:

$$\begin{aligned} \max & -3 \cdot x_4 + 32 \\ \text{s.t} & \\ & 2 \cdot x_4 \leq 8 \\ & 3 \cdot x_4 \leq 6 \\ & 8 \cdot x_4 \leq 24 \\ & x_4 \geq 0 \end{aligned} \tag{2}$$

Then we would like to make  $x_4 \geq 0$  as small as possible keeping feasibility. Given that  $x_4 = 0$  is a feasible solution and is the lower bound of  $x_4$  establish by the non negativity constraint. Then we can conclude that:

$$\begin{aligned} x_1^* &= 8 - 2 \cdot 0 = 8 \\ x_2^* &= 6 - 3 \cdot 0 = 6 \\ x_3^* &= 24 - 8 \cdot 0 = 24 \\ x_4^* &= 0 \end{aligned}$$

And finally the optimal value is:

$$z^* = 32 + 10 \cdot 0 = 32$$

- (c) Change the signs on all  $x_4$  coefficients to be negative. What is the optimal solution now? Changing the signs on all  $x_4$  coefficients to be negative including the coefficient of  $x_4$  in the objective function. It follows from the same analysis that we have done in item 0b that optimal solution is:

$$\begin{aligned} x_1^* &= 8 - 2 \cdot 0 = 8 \\ x_2^* &= 6 - 3 \cdot 0 = 6 \\ x_3^* &= 24 - 8 \cdot 0 = 24 \\ x_4^* &= 0 \end{aligned}$$

In the case in which we just change the coefficients of the constraints (keep 10 as the coefficient of the variable  $x_4$  in the objective function). The problem is unbounded, in other words even though the problem is feasible there is no optimal solution, we can make  $x_4$  arbitrarily large and still be feasible then we can make  $z = x_4 + 32$  as large as we want.

- 2.2 (a) Find an initial basic feasible solution, specify values of the decision variables, and tell which are basic.

Given that the problem is already given in standard form directly we can obtain matrix  $A$  and vector  $b$ :

$$A = \begin{bmatrix} 0 & 5 & 50 & 1 & 1 & 0 \\ 1 & -15 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 10 \\ 2 \\ 6 \end{bmatrix}$$

By inspection we can conclude that  $A_1, A_4, A_6$  are linearly independent so the corresponding variables  $x_1, x_4, x_6$  constitute a basis with corresponding basic solution:

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 0 \\ x_3 &= \\ x_4 &= 10 \\ x_5 &= 0 \\ x_6 &= 6 \end{aligned}$$

Given the basic variables and using the fact that all the nonbasic variables have value zero, we can replace the values of the nonbasic variables (zeros) in the constraints and obtain a unique solution for the values of the basic variables.

- (b) Transform the system of equations to the canonical form for carrying out the simplex routine.

From the last constraint we have:

$$x_6 = 6 - x_2 - x_3 - x_5$$

Replacing this expression of  $x_6$  in the objective function we have:

$$z = 10 \cdot x_2 + 2 \cdot x_3 - x_5 - 6$$

Then we can rewrite the system in canonical form as:

$$\begin{array}{ll} \max z = 10 \cdot x_2 + 2 \cdot x_3 - x_5 - 6 \\ \text{s.t} \\ 5 \cdot x_2 + 50 \cdot x_3 + x_4 + x_5 = 10 \\ x_1 - 15 \cdot x_2 + 2 \cdot x_3 = 2 \\ x_2 + x_3 + x_5 + x_6 = 6 \end{array}$$

- (c) Is your initial basic feasible solution optimal? Why?

No, because  $x = [32, 2, 0, 0, 0, 4]$  is a feasible solution that have a higher value than my initial basic feasible solution. My initial BFS have objective value  $-6$  and this solution have objective value:  $10 \cdot 2 - 6 = 14$ .  $x$  is feasible because it satisfy all constraints (to verify replace the values of  $\vec{x}$  in constraints).

Another way to analyze this question, is checking the coefficients of the nonbasic variables in the objective function, if all of them are non positive, then we can immediately conclude that current solution is optimal. But if at least one of them is not positive we can suspect that is not optimal but is not guaranteed.

- (d) How would you select a column in which to pivot in carrying out the simplex algorithm?

To select a column in which to pivot we can pick any column among the ones corresponding to nonbasic variables with positive coefficient in the objective function.

- (e) Having chosen a pivot column, now select a row in which to pivot and describe the selection rule. How does this rule guarantee that the new basic solution is feasible? It is possible that no row meets the criterion of your rule? If this happens, what does this indicate about the original problem?

We select the row corresponding to the constraint that is first going to be violated (that is not going to hold anymore) when we are increasing the nonbasic variable that wants to enter the basis. Constraints in which the coefficient of the entering variable is negative will never be violated, then we should just check among rows for which the entering variable have a positive coefficient. Then we should pick the minimum among the coefficients  $\frac{b_i}{a_{ij}}$  In which  $i$  correspond to the row index and  $j$  the entering variable index:

$$i^* = \min_{\{i|a_{ij}>0\}} \frac{b_i}{a_{ij}}$$

If more than one row achieve such min value, pick any one of them.

This rule guarantee that the new basic solution is feasible because as we previously described we pick the minimum among those ratios to guarantee that no constraint is going to be violated moving from current basic solution to new basic solution.

Yes it is possible that no row meets the criterion, for example if all rows have non positive coefficient in variable corresponding to the entering variable, which implies that we can increase the entering variable as much as we want without violating any constraint. In this case we can conclude that the original problem is unbounded.

- (f) Without carrying out the pivot operation, compute the new basic feasible solution.

From the rule described in previous item we would like that  $x_2$  enter the basis we one of the current basic variables need to leave the basis. From previous described rule to pick the basic variable that will leave the basis we have:

$$\min\left\{\frac{10}{5}, \frac{6}{1}\right\}$$

Then let us take out of the basis the variable  $x_4$  which is the basic variable associated with the first constraint. Then we have that the new basic variables are:  $x_1, x_2, x_6$  then it follows from  $x_B = [x_1, x_2, x_6] = B^{-1} \cdot b$  that the new basic feasible solution is:

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 32 \\ x_3 &= 0 \\ x_4 &= 0 \\ x_5 &= 0 \\ x_6 &= 4 \end{aligned}$$

- (g) Perform the pivot operation indicated by (d) and (e) and check your answer to (f). Substitute your basic feasible solution in the original equations as an additional check.

Let us write the dictionary after performing the pivot operation in which  $x_2$  enters the basis and  $x_4$  leaves the basis.

$$\begin{aligned} x_2 + 10 \cdot x_3 + \frac{1}{5} \cdot x_4 + \frac{1}{5} \cdot c_5 &= 2 \\ x_1 + 152 \cdot x_3 + 3 \cdot x_4 + 3 \cdot x_5 &= 32 \\ -9 \cdot x_3 - \frac{1}{5} \cdot x_4 - \frac{4}{5} \cdot x_5 + x_6 &= 4 \\ -z - 98 \cdot x_3 - 2 \cdot x_4 - 3 \cdot x_5 &= -14 \end{aligned}$$

- (h) Is your solution optimal now? why? Yes, because the coefficient of all nonbasic variables in the row of the dictionary corresponding to the objective function are negative.

2.7 Let us define the decision variables as  $x_S$  and  $x_J$ , the number of seniors and juniors workers that are hired in a month, respectively.

$$\begin{aligned} \max & 2000 \cdot x_S + 1800 \cdot x_J \\ \text{s.t} & \\ & 1000 \cdot x_S + 800 \cdot x_J \leq 9000 \quad \text{Budget for salaries constraint} \\ & x_S + x_J \leq 10 \quad \text{Limited space for assemblers constraint} \\ & x_S, x_J \geq 0 \end{aligned}$$

2.6 Let us define the decision variables as the number of tires produced of the different types:  $x_1, x_2, x_3$ .

$$\max 6 \cdot x_1 + 4 \cdot x_2 + 8 \cdot x_3$$

s.t

$$\frac{2}{100} \cdot x_1 + \frac{2}{100} \cdot x_2 + \frac{2}{100} \cdot x_3 \leq 12 \quad \text{Molding time constraint}$$

$$\frac{3}{100} \cdot x_1 + \frac{2}{100} \cdot x_2 + \frac{1}{100} \cdot x_3 \leq 9 \quad \text{Curing time constraint}$$

$$\frac{2}{100} \cdot x_1 + \frac{1}{100} \cdot x_2 + \frac{3}{100} \cdot x_3 \leq 16 \quad \text{Assembly constraint}$$

$$x_1, x_2, x_3 \geq 0$$

2.8 Let us define the decision variables as the number of pounds of each ingredient that are in 2000 pounds production:  $x_A, x_B, x_C$ .

$$\min 4 \cdot x_A + 3 \cdot x_B + 2 \cdot x_C$$

s.t

$$x_A + x_B + x_C = 2000 \quad \text{Demand constraint}$$

$$x_A \geq 10\% \cdot (x_A + x_B + x_C) \quad \text{Government regulation constraint for ingredient A}$$

$$x_B \geq 20\% \cdot (x_A + x_B + x_C) \quad \text{Government regulation constraint for ingredient A}$$

$$x_C \leq 800$$

$$x_A, x_B, x_C \geq 0$$