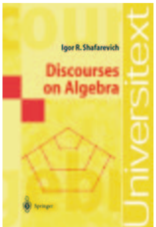
Discourses on Algebra

Rajaram Nityananda



Discourses on Algebra
Igor R Shafarevich
Narosa Publishing
Pages: 273, Price in India: ₹ 1750

To the Indian reader, the word discourse, evokes a respected figure interpreting divine wisdom to common folk in an accessible fashion. I dug a bit deeper with Google translate, and found that the original Russian title of Shafarevich's book was more like *Selected Chapters of Algebra* and that it was first published in a journal of mathematics education. Nevertheless, this book can be read as a discourse in our sense of the word, especially given Shafarevich's own stature among mathematicians and his religious, and philosophical beliefs. Apparently, he believed in the existence of mathematical truth, prior to its discovery by the human mind. This almost evokes Ramanujan's often quoted, and now filmed statement that an equation had no meaning for him unless it was a thought of God!

The article on Shafarevich's work elsewhere in this issue leaves no doubt that he is a master of algebra among other things. Yet he prefaces this book with a complaint – students beginning mathematics find geometry in the Euclidean style more attractive, and look upon algebra as an assorted bag of tricks and results, more in the style of the ancient Egyptians and Mesopotamians than the Greeks. Maybe one should not be surprised – after all, a very large fraction of the human brain is devoted to the visual system, and hence perhaps takes to geometry. But the author is clear that the problem is not with algebra itself, but needs to be addressed through proper choice and presentation of the introductory material. The stated goal then, is to develop the basics of algebra in such a way that it does not fall short of geometry in terms of general principles and logical methods.

Readers expecting 'higher' algebra are in for a surprise. There are no complex numbers and hence no fundamental theorem of algebra. Nor do we find groups, vector spaces, rings, or fields. This is partly because of the target audience – high school students (presumably Olympiad material!). But one more reason is that the material covered in this book precedes the more formal developments historically. Like many other Russian authors, he has a consciousness of history which stands out from most of their contemporaries. This historical sense is a part of the Russian culture and educational system – Kolmogorov almost became a historian! Interestingly, neither Diophantus nor the Indian mathematicians are mentioned. My guess is that the omission is deliberate. Diophantus sticks out like a sore thumb among the geometric Greeks, and works like a magician. The same holds for the Indian school. One can wonder why they chose the problems they did, and how they found their solutions. The only answer may be Feynman's derivation of the Schrödinger equation – "It came out of the mind of Schrödinger". Shafarevich has little room in this book for magic or magicians. His choice of material suggests that he would like the historical basics to come first educationally as well, but viewed in a modern light. These are the roots of the subject, and one must delve deep into them before moving to the trunk, branches, flowers, and fruits. The heroes of the story are listed at the end, and clearly, Euler is near the top of the list in terms of citation with only Jakob Bernoulli coming close.

The book is built around three themes – numbers, polynomials, and sets, which occur in that order in three cycles – rather like an elaborate musical piece. Each time, the level is raised, building on the earlier movements. The first section on numbers moves upto the unique factorisation into primes. The second delves into the primes and their distribution, and the third – now merged with polynomials and their (real!) roots – faces up to real numbers and irrationality. Transcendence comes in towards the end under infinite sets, a section with interesting historical details of correspondence between Cantor and Dedekind. Measure comes in *via* a single definition – a thin set (measure zero). By this time, enough fire-power has been assembled to prove very non-trivial results. For example, the majority of real numbers are 'normal'. This means that a given digit has equal probability of being anything from 0 to 9 (in base 10). One notices a shift from the finite paradise of integers and rationals to the (hellish?) infinities of the real world, as the book progresses. However, this is not a book on analysis. Limits are studiously avoided or mentioned in passing. His definition of derivative of a polynomial is an example of this algebraic spirit. We are all taught the 'remainder theorem' – $f(x)$ divided by $(x - a)$ leaves remainder $f(a)$. But the quotient term evaluated at a is nothing but the derivative at a,which is a polynomial in a! Division of polynomials requires no infinitesimals or limits.

In the presentation, there are hardly any 'it is easy to sees', everything is set out in a way that great teachers are famous for – one feels impatient when they dwell on apparently elementary points one by one, and then suddenly realises that they have moved into high ground at the same steady pace! This kind of writing leaves the reader with no excuse. He chooses proofs which are logical and flow from general principles, rather than the shortest or the most elegant.

One interesting device is the use of supplementary sections which cover material not needed for the main development – this is where the author can liberate himself from his own manifesto and give expression to his taste. One such section is Chebyshev's first breakthrough in the problem of the distribution of primes. In Chebyshev's time, it was a long-standing empirical observation that the fraction of primes upto N behaves as $1/(logN)$ In this one case, the author confesses that his principle of logical development is defeated. Some things just have to just fall out of the sky! In this case, the sky is the apparently trivial observation that the number of combinations of n objects taken r at a time is an integer. So the numerator in the formula must be divisible by the denominator. Apparently Euler described such a situation by saying that sometimes his pencil acquired an intelligence of its own. Shafarevich also makes the terse remark that this is the first time in the development that we – the author and readers – meet the work of a Russian mathematician.

The other supplementary section which stands out is the Euler pentagon theorem. This is an apparently mysterious observation, due to Euler, that the infinite product $((1-x)(1-x^2)(1-x^3)..$ when expanded as a power series in x, only contains powers with are 'pentagonal numbers' of the form $((3n^2-n))/2$, e.g., 1, 5, 12, etc. Further, the coefficients are just + or -1. This leads to a recursion relation for partitions, a theme which should resonate with an Indian audience, especially after the Ramanujan film. Many properties of all kinds of partitions are encountered in this part of the journey.

This review is written from the perspective of a test subject. I consider myself a geometrically inclined physicist, sharing the perception of algebra that Shafarevich begins by promising to rectify. My purpose in reading the book was to see how the other side functions. Following the thought processes of a great mathematician is an experience I would recommend to teachers and students alike. More correctly, these are just the thought processes that he would like us to start with. As the article on his research featured in this issue brings out, his own thoughts soared far higher.

My personal favourite was the section on power series which grew naturally out of polynomials. To a physicist reader who has calculus dinned into his head from an early age, the Shafarevich treatment of infinite series was a revelation. Issues of convergence and even numerical values of terms were completely bypassed in favour of a purely symbolic approach, which he traces to Newton. Fittingly, the book ends with a series which generates the Bernoulli numbers. It is denoted by $((e(x)-1))/x$, where $e(x)$ is defined as a power series. The coefficients are the familiar $1/n!$ The powers are just powers of the great unknown, x which stands for nothing but itself and the rules it obeys. If this way of looking at e^x is not pure algebra, what is? Such a book by a master should be made widely available – in its current form only libraries can afford it but perhaps a cheaper Indian edition is possible?

Rajaram Nityananda
School of Liberal Studies, Azim Premji University
PES South Campus, Electronic City, Bengaluru 560 100, India.Email:
rajaram.nityananda@gmail.com