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A New Approach to Continuous Riesz Bases

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Abstract

This paper deals with continuous frames and continuous Riesz bases. We introduce continuous Riesz bases and give some equivalent conditions for a continuous frame to be a continuous Riesz basis. It is certainly possible for a continuous frame to have only one dual. Such a continuous frame is called a Riesz-type frame [13]. We show that a continuous frame is Riesz-type if and only if it is a continuous Riesz basis. Finally we find a measure with respect to which, a continuous wavelet frame is a continuous Riesz basis.

Keywords: Continuous frame; Continuous orthonormal basis; Continuous Riesz basis

Introduction

The study of frames and Riesz bases has been an active area of functional and harmonic analysis on the one hand, as well as physics, engineering, computer science, signal and image processing etc., on the other hand. Due to the wide variety of applications, there has been a great influx of researchers into the subject in different approaches, cf. [4, 5, 6, 7, 11, 22]. The notion of continuous frames was introduced by Kaiser in [17] (which was called generalized frames) and was developed by several authors [2, 3, 13, 16, 20] in different aspects. The strong motivation to study continuous frames is that the windowed Fourier transform and the continuous wavelet transform are both their special cases. The reader is referred to [1, 9, 12, 14, 15] for a detailed account of windowed Fourier transform and wavelet transform. In [13] the authors have studied some properties of continuous frames. Also they have introduced and characterized a kind of continuous frame which possesses only one dual, called Riesz-type frame. In this paper we study continuous Riesz bases and continuous orthonormal bases as an extension of the discrete setting. We stress that the concepts and results contained in this paper have led to various important applications in frame and wavelet theory. Such a unified approach is useful, since it helps us obtain some equivalent conditions for a continuous frame to be a continuous Riesz basis. It is also beneficial to study continuous wavelet transform as an outstanding example of a continuous frame which is not in general a continuous Riesz basis.

This paper is organized as follows. In this section we recall the basic definitions, fix the notation and obtain a few results which are needed in the forthcoming sections. In Section 2, we introduce the notion of

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continuous Riesz basis with the aid of which, we give several equivalent conditions for a continuous frame to be a continuous Riesz basis. We then show that continuous Riesz bases are equivalent to Riesz type frames as defined in [13]. We shall further introduce and discuss continuous exact frames and prove that it is necessary for a continuous Riesz basis to be a continuous exact frame. In the last section, we define a continuous orthonormal basis and study its properties. As an application, we focus on continuous wavelet transform as a well known example of a continuous frame (continuous wavelet frame), which is not in general a continuous Riesz basis. Finally, we present a measure with respect to which, a continuous wavelet frame is a continuous Riesz basis.

Throughout this paper we assume that (Ω,μ) is a measure space and $\mathcal{H}$ is a Hilbert space. A mapping $F:\Omega\to\mathcal{H}$ is called *a continuous frame* with respect to (Ω,μ) if F is weakly measurable, i.e. $\omega\mapsto\langle f,F(\omega)\rangle$ is a measurable function on Ω and there exist two constants $A,B>0$ such that

$$A\|f\|\le\left(\int_\Omega|\langle f,F(\omega)\rangle|^2\,d\mu(\omega)\right)^{1/2}\le B\|f\|,\qquad(1)$$

for every $f\in\mathcal{H}$ [13].

The optimal constants A and B are called lower and upper frame bounds, respectively. A continuous frame F is said to be *tight* if we can take $A=B$. If $A=B=1$ it is called *a continuous Parseval frame*. The mapping F is called *Bessel* if the right inequality in (1) holds.

Suppose that F is a Bessel map with bound B and $\phi\in L^2(\Omega)$. Then $\int_\Omega\phi(\omega)F(\omega)d\mu(\omega)$ defines an element of $\mathcal{H}$. In fact,

$$T:L^2(\Omega)\to\mathcal{H};T\phi=\int_\Omega\phi(\omega)F(\omega)d\mu(\omega).\qquad(2)$$

is a bounded linear operator, called *the synthesis operator*. It is surjective and bounded if and only if F is a continuous frame. The continuous frame operator is defined to be $S:=TT^*$ and it is invertible as well as positive [13].

Definition 1.1. *We denote by* $L^2(\Omega,\mu,\mathcal{H})$ *the set of all mappings* $F:\Omega\to\mathcal{H}$ *such that for all* $f\in\mathcal{H}$, *the functions* $w\mapsto\langle f,F(w)\rangle$ *defined almost everywhere on* Ω, *belong to* $L^2(\Omega)$. *We abbreviate* $L^2(\Omega,\mu,\mathcal{H})$ *to* $L^2(\Omega,\mathcal{H})$ *and we consider two elements of* $L^2(\Omega,\mathcal{H})$ *the same, when they are equal almost everywhere. So for* $F,G\in L^2(\Omega,\mathcal{H})$ *the equality* $F=G$ *means that for every* $f\in\mathcal{H}$, $\langle f,F(\omega)\rangle=\langle f,G(\omega)\rangle$ *for almost all* $\omega\in\Omega$.

A Bessel mapping $F:\Omega\to\mathcal{H}$ is called μ*-complete* if

$$cspan\{F(\omega)\}_{\omega\in\Omega}:=\{\int_\Omega\phi(\omega)F(\omega)d\mu(\omega);\phi\in L^2(\Omega)\}$$

is dense in $\mathcal{H}$.

It is easily seen that a mapping $F:\Omega\to\mathcal{H}$ is Bessel if and only if for all $f\in\mathcal{H}$, $\int_\Omega|\langle f,F(\omega)\rangle|^2\,d\omega<\infty$. It is worthwhile to mention that if $F:\Omega\to\mathcal{H}$ is μ-complete, then $\{F(\omega)\}_{\omega\in\Omega}$ is a complete subset of $\mathcal{H}$. The converse is also true when $0<\mu(\{\omega\})<+\infty$ for all $\omega\in\Omega$, since $span\{F(w)\}_{w\in\Omega}\subseteq cspan\{F(w)\}_{w\in\Omega}$.

The following proposition establishes an equivalent condition for a Bessel mapping F to be μ-complete.

Proposition 1.2. *Let* $F\in L^2(\Omega,\mathcal{H})$ *be a Bessel mpping. The following are equivalent*:

(i) F *is* μ*-complete.*

(ii) *If* $f\in\mathcal{H}$ *so that* $\langle f,F(\omega)\rangle=0$ *for almost all* $\omega\in\Omega$, *then* $f=0$.

Proof. $(i)\Rightarrow(ii)$. Assume that $f\in\mathcal{H}$ and $\langle f,F(\omega)\rangle=0$, for almost all $\omega\in\Omega$. Then

$$\langle f,\int_\Omega\phi(\omega)F(\omega)d\mu(\omega)\rangle=\int_\Omega\phi(\omega)\langle f,F(\omega)\rangle d\mu(\omega)=0,$$

for all $\phi\in L^2(\Omega)$. Therefore, $f\perp cspan\{F(\omega)\}_{\omega\in\Omega}$ which implies that $f=0$.

$(ii)\Rightarrow(i)$. Let $f\in\mathcal{H}$ such that $\langle f,\int_\Omega\phi(\omega)F(\omega)d\mu(\omega)\rangle=0$ for all $\phi\in L^2(\Omega)$. Taking $\phi=\langle f,F(.)\rangle\in L^2(\Omega)$ we get

$$0=\langle\int_\Omega\phi(\omega)F(\omega)d\mu(\omega),f\rangle$$
$$=\int_\Omega|\langle f,F(\omega)\rangle|^2d\mu(\omega),$$

which shows that $\langle f,F(\omega)\rangle=0$ for almost all $\omega\in\Omega$. Hence $f=0$. □

Continuous Riesz Bases

In this section we define a continuous Riesz basis in a Hilbert space $\mathcal{H}$ and obtain some equivalent conditions for a continuous frame to be a continuous Riesz basis.

Definition 2.1. *Let* (Ω,μ) *be a measure space. A mapping* $F\in L^2(\Omega,\mathrm{H})$ *is called a continuous Riesz basis for* H *with respect to* (Ω,μ), *if* $\{F(\omega)\}_{\omega\in\Omega}$ *is* μ*-complete and there are two positive numbers* A *and* B *such that*

$$A\left(\int_{\Omega_1}|\phi(\omega)|^2\,d\mu(\omega)\right)^{1/2} \leq \left\|\int_{\Omega_1}\phi(\omega)F(\omega)d\mu(\omega)\right\| \leq B\left(\int_{\Omega_1}|\phi(\omega)|^2\,d\mu(\omega)\right)^{1/2}, \tag{3}$$

for every $\phi\in L^2(\Omega)$ *and measurable subset* Ω_1 *of* Ω *with* $\mu(\Omega_1)<+\infty$. *The integral is taken in the weak sense and the constants* A *and* B *are called Riesz basis bounds.*

Note that in the discrete case, where $\Omega=\mathbb{N}$ and μ is the counting measure on $\mathbb{N}$, a continuous Riesz basis is actually a Riesz basis. To obtain an equivalent condition for a continuous Riesz basis first we need to extend the notion of ω-independence to the continuous case.

Definition 2.2. *A Bessel mapping* $F\in L^2(\Omega,\mathrm{H})$ *is said to be* L^2*-independent if* $\int_\Omega\phi(\omega)F(\omega)d\mu(\omega)=0$ *for* $\phi\in L^2(\Omega,\mu)$, *implies that* $\phi=0$ *almost everywhere.*

We can now give a characterization of a continuous Riesz basis, expressed in terms of L^2-independence.

Theorem 2.3. *Let* H *be a Hilbert space,* (Ω,μ) *a measure space. A continuous frame* $F\in L^2(\Omega,\mathrm{H})$ *is a continuous Riesz basis for* H *if and only if* F *is* μ*-complete and* L^2*-independent.*

Proof. Suppose that F is a μ-completeand L^2-independent continuous frame with bounds A, B. For $\phi\in L^2(\Omega)$ and a measurable subset Ω_1 of Ω with finite measure, put $f=\int_{\Omega_1}\phi(\omega)F(\omega)d\mu(\omega)$. We have

$$\int_\Omega\phi(\omega)\chi_{\Omega_1}(\omega)F(\omega)d\mu(\omega) = \int_\Omega\langle f,S^{-1}F(\omega)\rangle F(\omega)d\mu(\omega),$$

where S isthe continuous frame operator of F. Therefore,

$$\phi(\omega)\chi_{\Omega_1}(\omega)=\langle f,S^{-1}F(\omega)\rangle \quad (\mu\text{–almost all } \omega\in\Omega).$$

Moreover,

$$B^{-1}\|f\|\leq\langle S^{-1}f,f\rangle^{1/2}\leq A^{-1}\|f\|$$

and

$$\langle S^{-1}f,f\rangle=\langle f,S^{-1}\int_{\Omega_1}\phi(\omega)F(\omega)d\mu(\omega)\rangle = \int_{\Omega_1}\overline{\phi(\omega)}\langle f,S^{-1}F(\omega)\rangle d\mu(\omega) = \int_{\Omega_1}|\phi(\omega)|^2 d\mu(\omega).$$

Thus, F is a continuous Riesz basis for H with bounds A,B.

For the converse, assume that F is a continuous Riesz basis. Obviously, F is an L^2-independent continuous frame. Using (3), for $\phi\in L^2(\Omega)$ and $f\in\mathrm{H}$, we get

$$\left|\int_\Omega\phi(\omega)\langle F(\omega),f\rangle d\mu(\omega)\right| = \left|\left\langle\int_\Omega\phi(\omega)F(\omega)d\mu(\omega),f\right\rangle\right| \leq \left\|\int_\Omega\phi(\omega)F(\omega)d\mu(\omega)\right\| \leq B\|f\|\|\phi\|_2.$$

It turns out that $\omega\mapsto\langle f,F(\omega)\rangle\in L^2(\Omega)$ satisfies

$$\left(\int_\Omega|\langle f,F(\omega)\rangle|^2 d\mu(\omega)\right)^{1/2}\leq B\|f\|.$$

By the above argument, the synthesis operator T defined as in (2), is a bounded linear operator. T is also one-to-one and onto because of the L^2-independence and μ-completeness of F. Hence T is invertible and for all $f\in\mathrm{H}$ we can write

$$\|f\|^4=|\langle T^{-1}f,T^*f\rangle|^2 \leq \|T^{-1}\|^2\|f\|^2\|T^*f\|.$$

Therefore, using $\|T^{-1}\|\leq A^{-1}$, we get

$$A\|f\|\leq\left(\int_\Omega|\langle f,F(\omega)\rangle|^2\,d\mu(\omega)\right)^{\frac{1}{2}}. \qquad \square$$

Here we define the dual of a continuous frame.

Definition 2.4. *Let $F:\Omega\to\mathcal{H}$ be a Bessel mapping. A Bessel mapping $F_1:\Omega\to\mathcal{H}$ is called a dual for F if*

$$f=\int_\Omega\langle f,F_1(\omega)\rangle F(\omega)d\mu(\omega)\quad (f\in\mathcal{H}). \tag{4}$$

One may easily see that for every continuous frame $F:\Omega\to\mathcal{H}$ with frame operator S,

$$\begin{aligned} f&=SS^{-1}f\\ &=\int_\Omega\langle f,S^{-1}F(\omega)\rangle F(\omega)d\mu(\omega),\end{aligned}$$

where $f\in\mathcal{H}$. Thus, F has at least one dual frame, namely $S^{-1}F$. It is certainly possible for a continuous frame to have only one dual. This type of frames, which is called *Riesz-type frame*, has been introduced by J. P. Gabardo and D. Han in [13]. Riesz-type frames are actually frames for which the adjoint of the synthesis operator is onto. In fact:

Proposition 2.5. *[13] Let F be a (Ω,μ) frame. Then F is a Riesz-type frame if and only if $Range(T^*)=L^2(\Omega)$.*

Using Proposition 2.5, we show that continuous Riesz bases and Riesz-type continuous frames are the same.

Theorem 2.6. *Let $F:\Omega\to\mathcal{H}$ be a continuous frame. Then F is a continuous Riesz basis if and only if F is a Riesz-type continuous frame.*

Proof. Let $F:\Omega\to\mathcal{H}$ be a continuous Riesz basis with duals F_1 and F_2. Then for all $f\in\mathcal{H}$ we have

$$\int_\Omega\langle f,F_1(\omega)-F_2(\omega)\rangle F(\omega)d\mu(\omega)=0,$$

which shows that

$$\langle f,F_1(\omega)\rangle=\langle f,F_2(\omega)\rangle\quad(\mu-\text{almost all }\ \omega\in\Omega).$$

Hence, $F_1=F_2$ in $L^2(\Omega,\mathcal{H})$.

For the converse, let F be a Riesz-type continuous frame. If $\phi\in L^2(\Omega)$ such that $\int_\Omega\phi(\omega)F(\omega)d\mu(\omega)=0$, then obviously by Proposition 2.5

$$\phi\in\ker(T)=(Range(T^*))^\perp=\{0\}.$$

Hence $\phi=0$ almost everywhere. Now Theorem 2.3 implies that F is a continuous Riesz basis. □

We will now present a further equivalent condition for a mapping F to be a continuous Riesz basis.

Proposition 2.7. *A mapping $F\in L^2(\Omega,\mathcal{H})$ is a continuous Riesz basis for $\mathcal{H}$ if and only if F is μ-complete and the mapping*

$$\begin{cases} U:L^2(\Omega,\mathcal{H})\to L^2(\Omega,\mathcal{H})\\ \phi\mapsto\int_\Omega\phi(\gamma)\langle F(\gamma),F(.)\rangle d\mu(\gamma)\end{cases}$$

defines a bounded invertible operator.

Proof. Assume that F is a continuous Riesz basis with bounds A and B. Then the synthesis operator T satisfies $\|T\|\le B$. Putting $U=T^*T:L^2(\Omega,\mathcal{H})\to L^2(\Omega,\mathcal{H})$ we have $\|U\|\le B^2$. U is also injective. Indeed, the following calculations:

$$\begin{aligned}0&=U\phi(\omega)\\ &=T^*(T\phi(\omega))\\ &=\langle T\phi,F(\omega)\rangle\\ &=\int_\Omega\phi(\gamma)\langle F(\gamma),F(\omega)\rangle d\mu(\gamma),\end{aligned}$$

$$\text{for almost all }\ \omega\in\Omega$$

together with L^2-independence of F implies that $\phi=0$. Moreover, U is onto by Theorems 2.6. The converse holds trivially.

In the sequel we study continuous Riesz bases with regard to the exactness property. First we define a continuous exact frame and present two auxiliary results.

Definition 2.8. *A continuous frame $F:\Omega\to\mathcal{H}$ is called exact if for every measurable subset $\Omega_1\subseteq\Omega$ with $0<\mu(\Omega_1)<\infty$, the mapping $F:\Omega\setminus\Omega_1\to\mathcal{H}$ is not a continuous frame for $\mathcal{H}$.*

The proof of the following lemma which we omit, is similar to the discrete case [8, Proposition 5.3.6].

Lemma 2.9. *Let $F:\Omega\to\mathcal{H}$ be a continuous frame and $f\in\mathcal{H}$. If f has a representation $f=\int_\Omega\phi(\omega)F(\omega)d\mu(\omega)$ for some $\phi\in L^2(\Omega)$, then*

$$\begin{aligned}\|\phi\|^2&=\int_\Omega|\langle f,S^{-1}F(\omega)\rangle|^2\,d\mu(\omega)\\ &\quad+\int_\Omega|\phi(\omega)-\langle f,S^{-1}F(\omega)\rangle|^2\,d\mu(\omega)\end{aligned}$$

where S is the continuous frame operator of F.

Proposition 2.10. *Let $F:\Omega\to H$ be a continuous frame with the continuous frame operator S. Also, let Ω_1 be a measurable subset of Ω such that $0<\mu(\Omega_1)<\infty$ and $f=\int_{\Omega_1}F(\omega)d\mu(\omega)$. If $\chi_{\Omega_1}=\langle f,S^{-1}F(.)\rangle$, then $F:\Omega\setminus\Omega_1\to H$ is not a continuous frame for H.*

Proof. By the frame decomposition,

$$\int_{\Omega_1}F(\omega)d\mu(\omega)=f=\int_{\Omega}\langle f,S^{-1}F(\omega)\rangle F(\omega)d\mu(\omega).$$

So Lemma 2.9 yields,

$$\mu(\Omega_1)=\int_{\Omega}|\langle f,S^{-1}F(\omega)\rangle|^2\,d\mu(\omega)$$
$$+\int_{\Omega}|\chi_{\Omega_1}(\omega)-\langle f,S^{-1}F(\omega)\rangle|^2\,d\mu(\omega).$$

Now if $\chi_{\Omega_1}=\langle f,S^{-1}F(.)\rangle$, then $\int_{\Omega\setminus\Omega_1}|\langle f,S^{-1}F(\omega)\rangle|^2\,d\mu(\omega)=0$. Hence by Proposition 1.2 $F:\Omega\setminus\Omega_1\to H$ is not μ-complete. □

We now give a necessary condition for a continuous frame to be a continuous Riesz basis via exactness.

Proposition 2.11. *A continuous Riesz basis in a Hilbert space is a continuous exact frame.*

Proof. Let $F:\Omega\to H$ be a continuous Riesz basis and $\Omega_1\subseteq\Omega$ be a measurable subset with positive measure. Then $\int_{\Omega_1}F(\omega)d\mu(\omega)\neq 0$. By the completeness of F, there exists $\phi_0\in L^2(\Omega\setminus\Omega_1)$ such that

$$\int_{\Omega_1}F(\omega)d\mu(\omega)=\int_{\Omega\setminus\Omega_1}\phi_0(\omega)F(\omega)d\mu(\omega).$$

Now L^2-independence implies that $\phi_0=0$. Therefore, $\int_{\Omega_1}F(\omega)d\mu(\omega)=0$, which is a contradiction. □

We conjecture whether the converse of Proposition 2.11 is true or not.

Results and Discusion

In this section, we present some examples to illustrate how our results can be applied. Continuous wavelet and Gabor frames, which play an important role in many applications are continuous frames (cf. [1, 12, 14, 17]). Our aim is to find a particular measure with respect to which, this continuous frame is also a continuous Riesz basis. To this end, we define an orthonormal basis. We begin with the following lemma whose proof is obvious.

Lemma 3.1. *For a mapping $F\in L^2(\Omega,H)$ the following statements are equivalent:*

(i) For all $f\in H$, $f=\int_{\Omega}\langle f,F(\omega)\rangle F(\omega)d\mu(\omega)$.

(ii) For all $f\in H$, $\|f\|^2=\int_{\Omega}|\langle f,F(\omega)\rangle|^2 d\mu(\omega)$.

(iii) For all $f_1,f_2\in H$, $\langle f_1,f_2\rangle=\int_{\Omega}\langle f_1,F(\omega)\rangle\langle F(\omega),f_2\rangle d\mu(\omega)$.

Definition 3.2. *A continuous orthonormal basis for H with respect to (Ω,μ) is a continuous Parseval frame F for which*

$$\left\|\int_{\Omega}\phi(\omega)F(\omega)d\mu(\omega)\right\|=\|\phi\|_2\quad(\phi\in L^2(\Omega)).$$

One may easily see that if F is a continuousParseval frame for H, then H can be embedded isometrically in $L^2(\Omega,\mu)$ with the mapping $H\to L^2(\Omega); f\mapsto\langle f,F(.)\rangle$. In the next Proposition, we show that the above embedding is onto provided that F is a continuous orthonormal basis.

Proposition 3.3. *If there exists a continuous orthonormal basis for H with respect to (Ω,μ), then H is isometrically isomorphic to $L^2(\Omega,\mu)$.*

Proof. Suppose that F is a continuous orthonormal basis for H with respect to (Ω,μ). Define $\Phi:H\to L^2(\Omega)$ by $f\mapsto\Phi_f$, where

$$\Phi_f(\omega)=\langle f,F(\omega)\rangle\quad(\mu-\text{almost all }\ \omega\in\Omega).$$

Clearly, Φ is an isometry and if $\phi\in L^2(\Omega)$, then by taking $f=\int_{\Omega}\phi(\omega)F(\omega)d\mu(\omega)$ we have

$$\|\phi-\Phi_f\|_2^2=\int_{\Omega}|\phi(\omega)-\langle f,F(\omega)\rangle|^2 d\mu(\omega)$$
$$=\int_{\Omega}(|\phi(\omega)|^2-\langle f,F(\omega)\rangle\overline{\phi(\omega)}$$
$$-\overline{\langle f,F(\omega)\rangle}\phi(\omega)+|\langle f,F(\omega)\rangle|^2)d\mu(\omega)$$
$$=\|\phi\|_2^2-\|f\|^2=0,$$

which shows that $\phi=\Phi_f$.

Corollary 3.4. *If F is a continuous orthonormal basis* *for* $\mathcal{H}$ *with respect to* $(\Omega,\mathcal{H})$, *then for all* $f\in\mathcal{H}$ *there exists a unique* $\phi\in L^2(\Omega)$ *for which* $f=\int_\Omega \phi(\omega)F(\omega)d\mu(\omega)$.

In the remainder of this section we assume that G is a locally compact group equipped with a left Haar measure $d\lambda$.

Example 3.5. *Let* $(\pi,\mathcal{H})$ *be an irreducible representation on a locally compact group* G *and* $\psi\in\mathcal{H}$ *be an admissible (wavelet) vector, i.e.*

$$C_\psi := \frac{1}{\|\psi\|^2}\int_G |<\psi,\pi(g)\psi>|^2\, d\lambda(g) < +\infty.$$

Then the mapping

$$W_\psi : \mathcal{H}\to L^2(G),$$

$$(W_\psi f)(g) = C_\psi^{\frac{-1}{2}} < f,\pi(g)\psi > \quad (f\in\mathcal{H}, g\in G),$$

is called a continuous wavelet transform on G. *It is a linear isometry onto its range; i.e.*

$$C_\psi^{-1}\int_G |< f,\pi(g)\psi >|^2\, d\lambda(g)$$

$$= \|W_\psi f\|_2^2 = \|f\|^2 \qquad (f\in\mathcal{H}).$$

Namely, $\{\pi(g)\psi\}_{g\in G}$ *is a continuous tight frame with bound* C_ψ *with respect to the measure space* (G,λ). *For a more general statement of this fact and further details see [1, Chapter 8]. In this case* W_ψ *is the adjoint of synthesis operator. Combining Proposition 2.5 and Theorem 2.6 with the fact that* W_ψ *is not onto in general (cf. [1]), we conclude that* $\{\pi(g)\psi\}_{g\in G}$ *is not a continuous Riesz basis.*

In the rest, we assume that G is a locally compact abelian group with the dual group $\hat{G}$. The Fourier transform is defined by

$$\hat{f}(\xi) = \int_G f(x)\overline{\xi(x)}d\lambda(x),$$

for $f\in L^1(G)$. Thanks to the Plancherel Theorem [10], the Fourier transform from $L^1(G)\cap L^2(G)$ uniquely extends to an isometric isomorphism from $L^2(G)$ onto $L^2(\hat{G})$. For a more detailed exposition of locally compact abelian groups the reader is referred to [10].

Example 3.6. *(i) Let* G *be a locally compact abelian group. If* G *is compact, then* $\hat{G}$ *is a continuous orthonormal basis for* $L^2(G)$ *[10].*

(ii) Let G *be a second countable locally compact abelian group. By a uniform lattice we mean a discrete and cocompact subgroup of* G *(cf. [18, 19, 21]). Fix a uniform lattice* K *in* G. *A fundamental domain for* K *is a Borel subset* S *of* G *such that every* $x\in G$ *can be uniquely written in the form* $x=sk$ *where* $s\in S$ *and* $k\in K$. *The existence of a fundamental domain for* K *is guaranteed by [19, Lemma 2]. Let* $\mathcal{H}=L^2(S)$ *and* $\Omega=\hat{G}$. *Define* $F:\Omega\to L^2(S)$ *by* $F(\omega)(s)=\omega(s)\chi_S(s)$, *for each* $\omega\in\hat{G}$. *It is obvious that* $\langle f,F(\omega)\rangle=\hat{f}(\omega)$, *for each* $f\in\mathcal{H}$ *and* $\omega\in\hat{G}$. *Hence, by the Plancherel theorem,* F *is a continuous Parseval frame. However,* T^*, *is not onto and Proposition 2.5 implies that* F *is not a Riesz-type frame. By Theorem 2.6* F *is not a continuous Riesz basis. Now we want to establish a measure with respect to which,* F *is a continuous Riesz basis. Consider the dual group* $\hat{G}$ *of* G *and the uniform lattice* $\Gamma=K^\perp$ *in* $\hat{G}$ *[18, 21]. Take* $\nu_\alpha=\sum_{n\in\Gamma}\delta_{n\alpha}$, *where* α *is a fixed element in* $\hat{G}$ *and* δ_b *is the Dirac measure at the point* $b\in G$. *Then F is also a continuous Parseval frame with respect to* ν_α. *Indeed,*

$$\left\|\int_\Omega |\langle f,F(\omega)\rangle|^2\, d\nu_\alpha(\omega)\right\|^2 = \sum_{n\in\Gamma}|\hat{f}(n\alpha)|^2$$

$$= \int_{\hat{G}} |\hat{f}(\omega)|^2\, d\nu_\alpha(\omega) = \|f\|_2^2.$$

Furthermore, F *is a continuous Riesz basis with respect to* ν_α. *To see this, by Theorem 2.3 it is enough to show* F *is* L^2*-independent. Assume that* $\int_\Omega \varphi(\omega)F(\omega)d\nu_a(\omega)=0$ *for some* $\varphi\in L^2(S)$, *then by using the compactness of S we have*

$$0 = \left\|\int_\Omega \varphi(\omega)F(\omega)d\nu_\alpha(\omega)\right\|^2$$

$$= \langle \int_\Omega \varphi(\omega)F(\omega)d\nu_\alpha(\omega), \int_\Omega \varphi(\omega)F(\omega)d\nu_\alpha(\omega)\rangle$$

$$= \langle \sum_{n\in\Gamma}\varphi(n\alpha)F(n\alpha), \sum_{m\in\Gamma}\varphi(m\alpha)F(m\alpha)\rangle$$

$$= \sum_{n\in\Gamma}\sum_{m\in\Gamma}\varphi(n\alpha)\overline{\varphi(m\alpha)}\langle n\alpha\chi_S, m\alpha\chi_S\rangle$$

$$= \sum_{n\in\Gamma}|\varphi(n\alpha)|^2 = \|\varphi\|_2^2.$$

The above calculations indicate that not only F is a continuous Riesz basis but also it is a continuous orthonormal basis. We can summarize it as follows:

Proposition 3.7. *Let G be a locally compact abelian group with a uniform lattice K and fundamental domain S. Then for any $\alpha \in \hat{G}$ the mapping $F : (\hat{G}, \nu_\alpha) \to L^2(S)$ defined by $F(\omega)(s) = \omega(s)\chi_S(s)$ is a continuous orthonormal basis for $L^2(S)$, where* $\nu_\alpha = \sum_{n \in (G/K)^\wedge} \delta_{n\alpha}$.

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