Chapter 3

Matrix Algebra

We first learn matrices can be used as a short–handed way of representing blocks of data. We then demonstrate some possible ways of mathematically manipulating matrices, including adding, subtracting and multiplying them.

3.1 Matrix Addition and Applications

Exercise 3.1 (Matrix Addition and Applications)

1. *Stuffed animals.* Beginning inventory of different types of stuffed animals at two different store locations given in matrix, B, below:

$$B = \begin{array}{c} \\ \text{LA} \\ \text{Seattle} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Pandas} & \text{St. Bernards} & \text{Birds} \end{array} \\ \left| \begin{array}{ccc} 500 & 800 & 1300 \\ 400 & 400 & 700 \end{array} \right| \end{array}$$

or,

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 500 & 800 & 1300 \\ 400 & 400 & 700 \end{bmatrix}$$

(a) *Entry* in ith row, jth column is a_{ij}; for example, $a_{23} = 700$.
Also, $a_{22} =$ **400** / **500** / **700** / **800**.

(b) *Dimension* of matrix B is number of rows, m, by number of columns, n:
$m \times n = \mathbf{3 \times 2}$ / $\mathbf{2 \times 2}$ / $\mathbf{3 \times 3}$ / $\mathbf{2 \times 3}$.

(c) Matrix *square* if $m = n$, so matrix B **is** / **is not** square.

(d) Matrix B has **1** / **2** / **3** *row* vectors, each dimension $\mathbf{1{\times}3}$ / $\mathbf{3{\times}1}$ / $\mathbf{1{\times}2}$.

(e) Matrix B has **1** / **2** / **3** *column* vectors, each dimension $\mathbf{2{\times}1}$ / $\mathbf{1{\times}2}$/ $\mathbf{3{\times}1}$.

(f) *Transpose* of matrix B with dimension $m \times n$ is B^T with dimension $n \times m$. Rows become columns; columns become rows. Transpose of matrix B is

$$B^T = \begin{bmatrix} 500 & 400 \\ 800 & \underline{\hspace{1.5cm}} \\ 1300 & \underline{\hspace{1.5cm}} \end{bmatrix}$$

2. *Another example.* Consider matrix

$$A = \begin{bmatrix} -5 & 8 & 1.3 & 4.5 \\ 40 & 30 & 70 & 3.4 \\ 3 & -2 & -1 & 0 \end{bmatrix}$$

(a) Dimension of matrix A is **3 × 2** / **3 × 3** / **3 × 4** / **3 × 5**.

(b) a_{23} = (circle one) **−2** / **40** / **70**

(c) Matrix A **is** / **is not** square.

(d) Matrix A has **1** / **2** / **3** row vectors, each dimension **1×2** / **1×3** / **1×4**.

(e) Matrix A has **2** / **3** / **4** column vectors, dimension **1×1** / **2×1** / **3×1**.

(f) Transpose

$$A^T = \begin{bmatrix} \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \\ \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \\ \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \\ \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \end{bmatrix}$$

3. *Operations with matrices.* Let

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 6 & 4 \end{bmatrix}, C = \begin{bmatrix} 6 & 4 & 5 \\ 2 & 1 & 4 \end{bmatrix}, \text{ and } \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(a) Then

$$4A = 4\begin{bmatrix} 6 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 24 & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \\ \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \\ \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \end{bmatrix}$$

(b) Then

$$B^T = \begin{bmatrix} 2 & 4 & 6 \\ \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \end{bmatrix}$$

(c) Then

$$\left(B^T\right)^T = \begin{bmatrix} 2 & \underline{\hspace{1.5cm}} \\ 4 & \underline{\hspace{1.5cm}} \\ 6 & \underline{\hspace{1.5cm}} \end{bmatrix}$$

(d) Then

$$B + \mathbf{0} = \begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 6 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \\ \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \\ \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \end{bmatrix}$$

(e) Then

$$C + B^T = \begin{bmatrix} 6 & 4 & 5 \\ 2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 3 & -1 & 4 \end{bmatrix} = \begin{bmatrix} \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \\ \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \end{bmatrix}$$

(f) Then

$$B^T + C = \begin{bmatrix} 2 & 4 & 6 \\ 3 & -1 & 4 \end{bmatrix} + \begin{bmatrix} 6 & 4 & 5 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \\ \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \end{bmatrix}$$

Matrix addition is *commutative*; $C + B^T = B^T + C$ is an example of this.

(g) Then

$$2(C+B^T) = 2\left\{\begin{bmatrix} 6 & 4 & 5 \\ 2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 3 & -1 & 4 \end{bmatrix}\right\} = \begin{bmatrix} \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \\ \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \end{bmatrix}$$

(h) Then

$$C+B^T+3C = 4C+B^T = 4\begin{bmatrix} 6 & 4 & 5 \\ 2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 3 & -1 & 4 \end{bmatrix} = \begin{bmatrix} \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \\ \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \end{bmatrix}$$

(i) Then

$$C - B^T = \begin{bmatrix} 6 & 4 & 5 \\ 2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 3 & -1 & 4 \end{bmatrix} = \begin{bmatrix} \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \\ \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} & \underline{\hspace{1.5cm}} \end{bmatrix}$$

4. *Inventory matrices: stuffed animals again.* Matrix B is beginning inventory, P is purchases (by retailer), S is sales (from customers) and E is ending inventory of different types of stuffed animals produced at two different store locations,

$$B = \begin{array}{cc} & \begin{array}{ccc} \text{Pandas} & \text{St. Bernards} & \text{Birds} \end{array} \\ \begin{array}{l} \text{LA} \\ \text{Seattle} \end{array} & \begin{vmatrix} 500 & 800 & 1300 \\ 400 & 400 & 700 \end{vmatrix} \end{array}$$

$$P = \begin{array}{cc} & \begin{array}{ccc} \text{Pandas} & \text{St. Bernards} & \text{Birds} \end{array} \\ \begin{array}{l} \text{LA} \\ \text{Seattle} \end{array} & \begin{vmatrix} 30 & 300 & 40 \\ 50 & 400 & 10 \end{vmatrix} \end{array}$$

$$S = \begin{array}{r} \\ \text{LA} \\ \text{Seattle} \end{array} \begin{array}{ccc} \text{Pandas} & \text{St. Bernards} & \text{Birds} \\ \left| \begin{array}{c} 400 \\ 350 \end{array} \right. & \begin{array}{c} 1000 \\ 700 \end{array} & \left. \begin{array}{c} 1200 \\ 600 \end{array} \right| \end{array}$$

2nd MATRIX [A] EDIT ENTER 2 ENTER 3 ENTER, data: 500 ENTER 800 ENTER ... 700 ENTER.

2nd MATRIX EDIT [B] ENTER 2 ENTER 3 ENTER, data: 30 ENTER 300 ENTER ... 10 ENTER.

2nd MATRIX EDIT [C] ENTER 2 ENTER 3 ENTER, data: 400 ENTER 1000 ENTER ... 600 ENTER.

(a) Ending inventory is

$$E = B+P-S = \begin{bmatrix} 500 & 800 & 1300 \\ 400 & 400 & 700 \end{bmatrix} + \begin{bmatrix} 30 & 300 & 40 \\ 50 & 400 & 10 \end{bmatrix} - \begin{bmatrix} 400 & 1000 & 1200 \\ 350 & 700 & 600 \end{bmatrix} =$$

$$\begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}$$

$[A] + [B] - [C]$ STO $[D]$

(b) Transpose

$$E^T = (B + P - S)^T = \left\{ \begin{bmatrix} 130 & 100 & 140 \\ 100 & 100 & 110 \end{bmatrix} \right\}^T = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$

2nd MATRIX $[D]$ 2nd MATRIX MATH ^{T} ENTER

(c) **True / False**

$$(B + P) - S = B + (P - S)$$

This is an example of *associative* rule of matrix addition/subtraction.

Compare $([A] + [B]) - [C]$ with $[A] + ([B] - [C])$

(d) If

$$\begin{bmatrix} 500 & 800 & 1300 \\ 400 & 400 & 700 \end{bmatrix} = \begin{bmatrix} 450 + x & 800 & 1300 \\ 400 & 400 & 700 \end{bmatrix}$$

$500 = 450 + x$ then $x =$ **-25 / -50 / 50**

(e) If

$$\begin{bmatrix} 500 + 3x & 800 & 1300 \\ 400 & 400 & 700 \end{bmatrix} = \begin{bmatrix} 450 + x & 800 & 1300 \\ 400 & 400 & 700 \end{bmatrix}$$

$500 + 3x = 450 + x$ then $2x = -50$ or $x =$ **-25 / -50 / 50**

5. *Communication diagrams and corresponding matrices.*

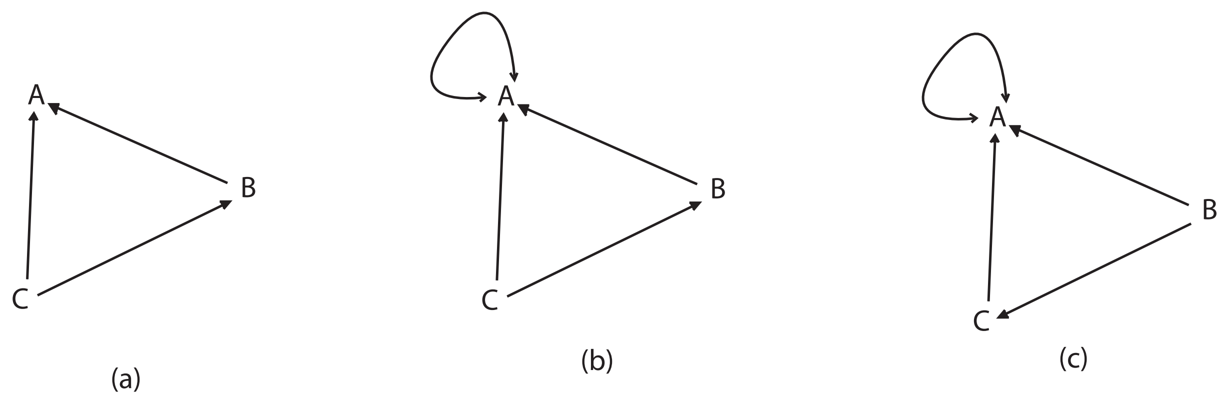

Figure 3.1 (Communication diagrams)

(a) Consider matrices

$$V = \begin{array}{cc} & \begin{array}{ccc} \mathrm{A} & \mathrm{B} & \mathrm{C} \end{array} \\ \begin{array}{c} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \end{array} & \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \end{array}, \quad W = \begin{array}{cc} & \begin{array}{ccc} \mathrm{A} & \mathrm{B} & \mathrm{C} \end{array} \\ \begin{array}{c} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \end{array} & \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} \end{array}, \quad X = \begin{array}{cc} & \begin{array}{ccc} \mathrm{A} & \mathrm{B} & \mathrm{C} \end{array} \\ \begin{array}{c} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \end{array} & \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} \end{array}$$

Match diagrams with matrices.

diagrams	(a)	(b)	(c)
matrices			

3.2 Matrix Multiplication and Applications

Exercise 3.2 (Matrix Multiplication and Applications)

1. *Identity matrix.* An example of an *identity* matrix is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And so

$$I_2 = \begin{bmatrix} \underline{\hspace{2cm}} & \underline{\hspace{2cm}} \\ \underline{\hspace{2cm}} & \underline{\hspace{2cm}} \end{bmatrix}$$

Identity matrix **is always / is not always** *square.*

2. *Another example.* Let

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 6 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 4 & 5 \\ 2 & 1 & 4 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Then

$$A \times I = \begin{bmatrix} 6 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}$$

Multiplying A by *identity* matrix I is like multiplying by "1": A is returned.

Use previous matrices, then 2nd MATRIX [A] ENTER $\times$ 2nd MATRIX MATH Identity(3) ENTER.

(b) Then

$$A^2 = A \times A = \begin{bmatrix} 6 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix} \times \begin{bmatrix} 6 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}$$

Use previous matrices, then 2nd MATRIX [A] ENTER x^2 ENTER.

(c) Then

$$A^3 = A \times A \times A = \begin{bmatrix} 6 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix} \times \begin{bmatrix} 6 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix} \times \begin{bmatrix} 6 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}$$

2nd MATRIX [A] ENTER $\wedge$3 ENTER.

(d) Then

$$BC = B \times C = \begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 6 & 4 \end{bmatrix} \times \begin{bmatrix} 6 & 4 & 5 \\ 2 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}$$

which has dimension $(3 \times \boxed{2}) \times (\boxed{2} \times 3) = 3 \times 3$.

Use previous matrices, then 2nd MATRIX [B] ENTER $\times$ 2nd MATRIX [C] ENTER.

(e) Then

$$CB = C \times B = \begin{bmatrix} 6 & 4 & 5 \\ 2 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 6 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$

which has dimension $(2 \times \boxed{3}) \times (\boxed{3} \times 2) = 2 \times 2$.
Notice $BC \neq CB$; matrix multiplication is *not* commutative.
2nd MATRIX [C] ENTER × 2nd MATRIX [B] ENTER.

(f) Then

$$CB = \begin{bmatrix} 6 & 4 & 5 \\ 2 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 6 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} \begin{bmatrix} 6 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} & \begin{bmatrix} 6 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} & \begin{bmatrix} 2 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \end{bmatrix}$$
$$= \begin{bmatrix} 6(2)+4(4)+5(6) & 6(3)+4(-1)+5(4) \\ 2(2)+1(4)+4(6) & 2(3)+1(-1)+4(4) \end{bmatrix} = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$

Each *row* of first matrix is multiplied by each *column* of second matrix. Number of columns of first matrix must equal number of rows of second matrix: $(m \times n) \times (n \times q) = m \times q$.

(g) Then

$$BCA = \begin{bmatrix} 2 & 3 \\ 4 & -1 \\ 6 & 4 \end{bmatrix} \times \begin{bmatrix} 6 & 4 & 5 \\ 2 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 6 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}$$

2nd MATRIX [B] ENTER × 2nd MATRIX [C] × 2nd MATRIX [A] ENTER.

(h) Impossible to calculate CBA because

i. mismatch in first and second matrices: $(2 \times 2) \times (3 \times 2) \times (3 \times 3)$

ii. mismatch in second and third matrices: $(2 \times 3) \times (3 \times 2) \times (3 \times 3)$

3. *Cholesterol and Diets.* Three patients, at beginning of first, second, third and fourth months, had following cholesterol levels,

$$A = \begin{array}{l|cccc|} & \text{1st} & \text{2nd} & \text{3rd} & \text{4th month} \\ \text{Mahoney} & 220 & 215 & 210 & 205 \\ \text{Sky} & 220 & 210 & 200 & 195 \\ \text{Pavlov} & 210 & 205 & 195 & 190 \end{array}$$

Furthermore, cholesterol levels for four months under three different diets are related by following matrix of factor gain per cholesterol values,

$$B = \begin{array}{l|ccc|} \text{diet} \rightarrow & \text{mild} & \text{moderate} & \text{heavy} \\ \text{1st month} & 2 & 1.7 & 1.9 \\ \text{2nd month} & 2.2 & 2.3 & 2.4 \\ \text{3rd month} & 2 & 2.7 & 1.6 \\ \text{4th month} & 1.8 & 1.3 & 2.4 \end{array}$$

(a) Amount of cholesterol gained by three patients in four months under four diets is

$$A \times B = \begin{bmatrix} 220 & 215 & 210 & 205 \\ 220 & 210 & 200 & 195 \\ 210 & 205 & 195 & 190 \end{bmatrix} \times \begin{bmatrix} 2 & 1.7 & 1.9 \\ 2.2 & 2.3 & 2.4 \\ 2 & 2.7 & 1.6 \\ 1.8 & 1.3 & 2.4 \end{bmatrix}$$

$$= \begin{bmatrix} 220(2)+215(2.2)+210(2)+205(1.8) & 220(1.7)+215(2.3)+210(2.7)+205(1.3) & \underline{\hspace{2cm}} \\ 220(2)+210(2.2)+200(2)+195(1.8) & \underline{\hspace{2cm}} & 220(1.9)+210(2.4)+200(1.6)+195(2.4) \\ \underline{\hspace{2cm}} & 210(1.7)+205(2.3)+195(2.7)+190(1.3) & 210(1.9)+205(2.4)+195(1.6)+190(2.4) \end{bmatrix}$$

(b) In other words,

$$A \times B = \begin{array}{l|ccc|} \text{exercise level} \rightarrow & \underline{\hspace{2cm}} & \text{moderate} & \text{heavy} \\ \text{Mahoney} & \underline{\hspace{2cm}} & 1702 & 1762 \\ \underline{\hspace{2cm}} & 1653 & 1650.5 & 1710 \\ \underline{\hspace{2cm}} & 1603 & \underline{\hspace{2cm}} & 1659 \end{array}$$

(c) Matrix A has dimension $\mathbf{4 \times 3}$ / $\mathbf{3 \times 4}$ / $\mathbf{4 \times 4}$ / $\mathbf{3 \times 3}$,
Matrix B has dimension $\mathbf{4 \times 3}$ / $\mathbf{3 \times 4}$ / $\mathbf{4 \times 4}$ / $\mathbf{3 \times 3}$.
Matrix $A \times B$ has dimension $\mathbf{4 \times 3}$ / $\mathbf{3 \times 4}$ / $\mathbf{4 \times 4}$ / $\mathbf{3 \times 3}$.

4. *More Properties.*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ 8 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 \\ 4 & 1 \end{bmatrix}$$

(a) Since

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 5 \\ 8 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 0 \end{bmatrix}$$

and

$$AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 0 \end{bmatrix}$$

i. $AB = AC$ but sometimes $B \neq C$

ii. $AB = AC$ and $B = C$ always

(b) Since

$$AD = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

i. $AD = \mathbf{0}$ but sometimes $A \neq \mathbf{0}$ and $D \neq \mathbf{0}$

ii. $AD = \mathbf{0}$ but $A = \mathbf{0}$ and $D = \mathbf{0}$ always

5. *Systems of equations and matrices.*

(a) Consider three matrices A, X and B,

$$A = \begin{bmatrix} 220 & 215 & 210 & 205 \\ 220 & 210 & 200 & 195 \\ 210 & 205 & 195 & 190 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 20 \\ -1 \\ 30 \end{bmatrix}$$

Then linear system of equations of $AX = B$ is

$$\begin{array}{ccccccccc} 220x & + & 215y & + & \underline{\qquad}z & + & 205w & = & 20 \\ \underline{\qquad}x & + & \underline{\qquad}y & + & \underline{\qquad}z & + & 195w & = & \underline{\qquad} \\ \underline{\qquad}x & + & \underline{\qquad}y & + & 195z & + & \underline{\qquad}w & = & \underline{\qquad} \end{array}$$

(b) If

$$\begin{array}{ccccccc} x & + & 2y & + & z & = & 3 \\ 7x & + & 10y & + & 9z & = & -98 \\ x & + & 3y & + & 5z & = & 6 \end{array}$$

then $AX = B$ where

$$A = \begin{bmatrix} 1 & \underline{\qquad} & 1 \\ 7 & 10 & \underline{\qquad} \\ \underline{\qquad} & 3 & 5 \end{bmatrix}$$

and

$$X = \begin{bmatrix} \text{x} \\ \underline{\qquad} \\ \underline{\qquad} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \underline{\qquad} \\ -98 \\ \underline{\qquad} \end{bmatrix}$$

3.3 The Inversion of a Matrix

In same way $\frac{1}{5}$ is related to 5 ($\frac{1}{5} \times 5 = 1$), matrix A^{-1} is related to matrix A ($A^{-1} \times A = I$, where I is identity matrix).

Exercise 3.3 (Inversion of a Matrix)

1. *A First Example.* Let

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 6 & 4 & 5 \\ 2 & 1 & 4 \end{bmatrix}$$

(a) Inverse of A

$$A^{-1} = \begin{bmatrix} 6 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 5 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}$$

2nd MATRIX EDIT [A] ENTER 3 ENTER 3 ENTER, type in data, 2nd QUIT,
2nd MATRIX [A] ENTER x^{-1} ENTER, then to get fractional form, MATH ENTER.

(b) Inverse of B

$$B^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$

2nd MATRIX EDIT [B] ENTER 2 ENTER 2 ENTER, type in data, 2nd QUIT,
2nd MATRIX [B] ENTER x^{-1} ENTER, then to get fractional form, MATH ENTER.

(c) **True / False.** $B^{-1} = B^T$

(d) Then

$$B^{-1}B = B \times B^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}^{-1} \times \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{14} & \frac{3}{14} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix} = I$$

2nd MATRIX [B] ENTER x^{-1} ENTER × 2nd MATRIX [B] ENTER.

(e) **True / False.** $B^{-1}B = BB^{-1} = I$
2nd MATRIX [B] ENTER × 2nd MATRIX [B] ENTER x^{-1} ENTER.

(f) **True / False.** C^{-1} cannot be calculated because matrix C is *not* square.

2. *More Properties of Inverse.*

(a) If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 5 & 4 \end{bmatrix}$$

then inverse

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(3)(4)-(7)(5)}\begin{bmatrix} 4 & -7 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ \end{bmatrix}$$

Check: 2nd MATRIX [A] 2 ENTER 2, type in data, then 2nd MATRIX [A] ENTER x^{-1} ENTER, then, for fractional form, MATH ENTER.

Inverse for larger dimension matrices solved using Gauss-Jordan.

(b) If

$$A = \begin{bmatrix} 2 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.25 & 0 & 0.5 \\ 0.55 & -0.2 & -0.5 \\ -0.35 & 0.4 & 0.5 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} \_\_\_\_ & \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ & \_\_\_\_ \end{bmatrix} = I$$

so (choose *two*!)

i. B is inverse of A
ii. A is inverse of B
iii. A is transpose of B
iv. B is transpose of A

(c) Not all square matrices have inverses.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix}$$

does not have an inverse because

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(2)(6)-(-4)(-3)}\begin{bmatrix} 6 & -(-4) \\ -(-3) & 2 \end{bmatrix}$$

where $D = ad - bc = (2)(6) - (-4)(-3) = 0$, so $\frac{1}{D}$ **does** / **does not** exist.

3. *Solving equations using inverse.* Given linear system of equations

$$AX = B$$

where

$$A = \begin{bmatrix} 2 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$$

2nd MATRIX [A] 3 ENTER 3 ENTER, enter numbers; and 2nd MATRIX [B] 3 ENTER 1, enter numbers

intersection given by

$$X = A^{-1}B$$

or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{5}{20} & 0 & \frac{1}{2} \\ \frac{11}{20} & \underline{\qquad} & -\frac{1}{2} \\ \underline{\qquad} & \frac{4}{10} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} \underline{\qquad} \\ 0 \\ \underline{\qquad} \end{bmatrix}$$

2nd MATRIX [A] x^{-1} × 2nd MATRIX [B] ENTER

in other words, $(x, y, z) = (1, 0, 2)$.

4. *Calculating inverse using Gauss–Jordan.* Inverse of matrix

$$A = \begin{bmatrix} 2 & 4 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

given by first setting up following augmented matrix,

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 2 & 1 & 0 & 0 \\ -2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

2nd MATRIX [A] 3 ENTER 6 ENTER, then type in data

$$\xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & \underline{\qquad} & \frac{1}{2} & 0 & \underline{\qquad} \\ -2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

2nd MATRIX MATH *row ($\frac{1}{2}$, [A], 1) STO→ [B] then MATH ENTER

$$\xrightarrow{R_2+2R_1 \to R_2,\ R_3-3R_1 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & \underline{\qquad} & 5 & \underline{\qquad} & 1 & 0 \\ 0 & -4 & \underline{\qquad} & -\frac{3}{2} & 0 & 1 \end{array}\right]$$

2nd MATRIX MATH *row+(2, [B], 1, 2) STO→ [C], 2nd MATRIX MATH *row+(-3, [C], 1, 3) STO→ [D]

$$\xrightarrow{\frac{1}{5}R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \underline{\qquad} & \frac{1}{5} & 0 \\ 0 & -4 & -2 & -\frac{3}{2} & 0 & 1 \end{array}\right]$$

2nd MATRIX MATH *row ($\frac{1}{5}$, [D], 2) STO→ [E] then MATH ENTER

$$\xrightarrow{R_1-2R_2 \to R_1,\ R_3+4R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & \underline{\qquad} & -\frac{2}{5} & 0 \\ 0 & 1 & 1 & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & \underline{\qquad} & 2 & -\frac{7}{10} & \frac{4}{5} & 1 \end{array}\right]$$

2nd MATRIX MATH *row+(-2, [E], 2, 1) STO→ [F], 2nd MATRIX MATH *row+(4, [F], 2, 3) STO→ [G]

$$\xrightarrow{\frac{1}{2}R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & \frac{1}{10} & -\frac{2}{5} & 0 \\ 0 & 1 & 1 & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \underline{\qquad} & -\frac{4}{10} & \frac{1}{2} \end{array}\right]$$

2nd MATRIX MATH *row ($\frac{1}{2}$, [G], 3) STO→ [H] then MATH ENTER

$$\xrightarrow{R_1+R_3 \to R_1,\ R_2-R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \underline{\qquad} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{11}{20} & -\frac{1}{5} & \underline{\qquad} \\ 0 & 0 & 1 & -\frac{7}{20} & \frac{2}{5} & \frac{1}{2} \end{array}\right]$$

2nd MATRIX MATH *row+(1, [H], 3, 1) STO→ [I], 2nd MATRIX MATH *row+(-1, [I], 3, 2), MATH ENTER

where inverse of matrix A is given on right hand side of line.

3.4 More Applications of Inverses

We look at three applications of inverses: cryptography, economics and manufacturing.

Exercise 3.4 (More Applications of Inverses)

1. *Cryptography.* Inverses can be used to decode a coded message, where message assigns numbers to letters:

$$\begin{bmatrix} A & B & C & \ldots & Y & Z & \text{space} & \text{period} & \text{apostrophe} \\ 1 & 2 & 3 & \ldots & 25 & 26 & 30 & 40 & 60 \end{bmatrix}$$

and an *encoding* matrix is given.

(a) *Code and decode HELLO.* Using both encoding matrix

$$E = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

and word HELLO (8, 5, 12, 12, 15) with corresponding code matrix

$$C = \begin{bmatrix} 8 & 5 \\ 12 & 12 \\ 15 & 30 \end{bmatrix}$$

message is coded as

$$CE = \begin{bmatrix} 8 & 5 \\ 12 & 12 \\ 15 & 30 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \end{bmatrix}$$

Enter numbers in matrices [A] and [B], [A] × [B] STO→ [C] ENTER.

Message can then be *de*coded as

$$(CE)E^{-1} = C = \begin{bmatrix} 8 & 2 \\ 12 & 12 \\ 15 & 45 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ \end{bmatrix}$$

2nd MATRIX [C] × 2nd MATRIX [B] x^{-1} ENTER.

which, of course, is HELLO.

(b) *Decode message.* Using encoding matrix

$$E = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

decode message (38, 48, 85, 2, 15, 11, 39, 58 92), which has matrix

$$CE = \begin{bmatrix} 38 & 48 & 85 \\ 2 & 15 & 11 \\ 39 & 58 & 92 \end{bmatrix}$$

Message can then be decoded as

$$(CE)E^{-1} = C = \begin{bmatrix} 38 & 48 & 85 \\ 2 & 15 & 11 \\ 39 & 58 & 92 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \_\_\_\_ & \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ & \_\_\_\_ \end{bmatrix}$$

Type numbers into matrices [A] and [B]

2nd MATRIX [A] × 2nd MATRIX [B] x^{-1} ENTER.

which decodes as (8, 9, 30, 1, 7, 1, 9, 14 30) or HI AGAIN

2. *Economics: Leontief's model.* This model is

total production = internal consumption by sectors + consumer demand

or more exactly

$$X = TX + D$$

where X is total production, TX is internal consumption, D is consumer demand and T is *input-output matrix*, a matrix of proportions of sector inputs required to produce sector outputs. Some algebra gives

$$\begin{aligned} X &= TX + D \\ X - TX &= D \\ (I - T)X &= D \\ X &= (I - T)^{-1}D \end{aligned}$$

where I is identity matrix and $(I - T)^{-1}$ exists.

(a) *A First Example.* If input-output matrix and consumer demand are

$$T = \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.1 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 200 \\ 100 \\ 350 \end{bmatrix}$$

then production is

$$X = (I - T)^{-1} D$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.1 \\ 0.4 & 0.2 & 0.3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 200 \\ 100 \\ 350 \end{bmatrix} = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$$

Enter data in matrices [A] and [B],

then (2nd MATRIX MATH identity(3) $-$ 2nd MATRIX [A])$^{-1}$ $\times$ [B] ENTER.

(b) *An Economics Example.* If input–output matrix is

output (amount produced) →	agriculture	manufacturing	service
input (amount used in production) ↓			
agriculture	0.2	0.1	0.1
manufacturing	0.2	0.3	0.1
service	0.1	0.2	0.3

what is total production of economy, X, to meet a demand for 100 units of agriculture, 200 units of manufactured goods and 150 units of service? Since

$$T = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.2 & 0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 100 \\ 200 \\ 150 \end{bmatrix}$$

then

$$X = (I - T)^{-1} D$$

$$= \left(\begin{bmatrix} 1 & \_\_\_\_ & 0 \\ 0 & 1 & 0 \\ \_\_\_\_ & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.2 & \_\_\_\_ & 0.1 \\ 0.1 & 0.2 & 0.3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 100 \\ 200 \\ \_\_\_\_ \end{bmatrix}$$

$$= \begin{pmatrix} 1.3 & \_\_\_\_ & 0.2 \\ 0.4 & 1.6 & 0.3 \\ \_\_\_\_ & 0.5 & 1.5 \end{pmatrix} \begin{bmatrix} 100 \\ 200 \\ 150 \end{bmatrix} = \begin{bmatrix} \_\_\_\_ \\ 400 \\ 360 \end{bmatrix}$$

Enter data in matrices [A] and [B], then

(2nd MATRIX MATH identity(3) $-$ 2nd MATRIX [A])$^{-1}$ for first, then $\times$ [B] ENTER for second.

In other words, (agriculture,manufacturing,service) $= (220, 400, 360)$ units.

3. *Manufacturing.* Very similar to Leontief's model, this model is

total output = internal consumption by assembly + external demand

or more exactly

$$X = PX + D$$

where X is total output, PX is internal consumption, D is external demand and P is *parts matrix*, a matrix of number of parts required by other parts in assembly. Some algebra gives

$$\begin{aligned} X &= PX + D \\ X - PX &= D \\ (I - P)X &= D \\ X &= (I - P)^{-1} D \end{aligned}$$

where I is the identity matrix and $(I - P)^{-1}$ exists.

(a) *A first look.*

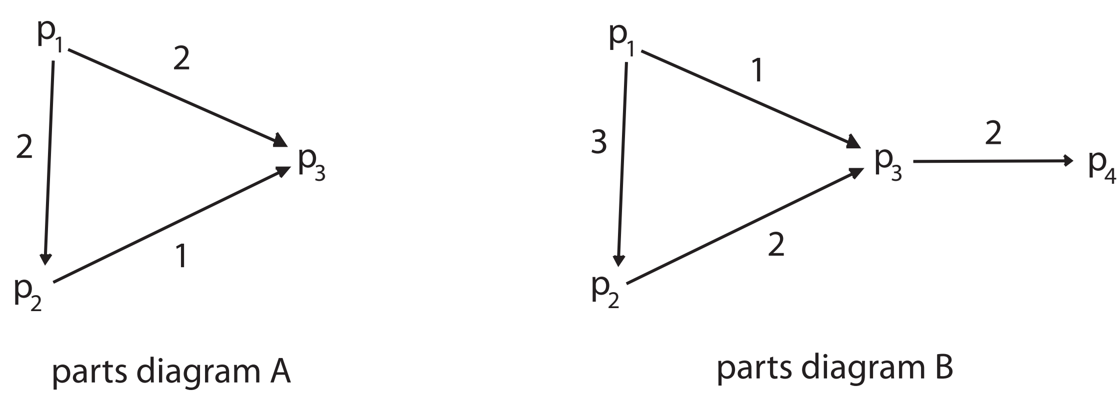

Figure 3.2 (Parts diagrams)

Parts matrices corresponding to parts diagrams are:

$$A = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \end{array} \begin{array}{c} \begin{array}{ccc} p_1 & p_2 & p_3 \end{array} \\ \left| \begin{array}{ccc} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right| \end{array}, \quad B = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \end{array} \\ \left| \begin{array}{cccc} 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right| \end{array}$$

In parts diagram A, two (2) of part p_1 required in parts p_2 and p_3 and

i. one (1) of part p_2 required in part p_3

ii. one (1) of part p_3 required in part p_2

(b) *Parts diagram and matrix A.*
If parts matrix is given by matrix A and external demand is

$$D = \begin{bmatrix} 200 \\ 100 \\ 350 \end{bmatrix}$$

then total output is

$$X = (I - P)^{-1}D =$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 200 \\ 100 \\ 350 \end{bmatrix} = \begin{bmatrix} \underline{\qquad\quad} \\ \underline{\qquad\quad} \\ \underline{\qquad\quad} \end{bmatrix}$$

Enter data in matrices [A] and [B],
then (2nd MATRIX MATH identity(3) $-$ 2nd MATRIX [A] $)^{-1}$ $\times$ [B] ENTER then MATH ENTER.

(c) *Parts diagram and matrix B.*
If parts matrix is given by matrix B and external demand is

$$D = \begin{bmatrix} 200 \\ 100 \\ 350 \\ 100 \end{bmatrix}$$

then total output is

$$X = (I - P)^{-1}D =$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 200 \\ 100 \\ 350 \\ 100 \end{bmatrix} = \begin{bmatrix} \underline{\qquad\quad} \\ \underline{\qquad\quad} \\ \underline{\qquad\quad} \\ \underline{\qquad\quad} \end{bmatrix}$$

Enter data in matrices [A] and [B],
then (2nd MATRIX MATH identity(4) $-$ 2nd MATRIX [A] $)^{-1}$ $\times$ [B] ENTER.