Linear Programming

Design and Analysis of Algorithms
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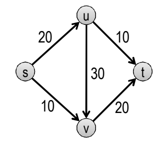
Maximal Flow

Let f_{xy} be a real variable meaning the amount of flow through edge xy



Maximize $Z = f_{su} + f_{sv}$
Subject to capacity constraints

$0 \le f_{su}, f_{vt} \le 20$
$0 \le f_{ut}, f_{sv} \le 10$
$0 \le f_{uv} \le 30$

conservation constraints

$f_{su} = f_{ut} + f_{uv}$
$f_{sv} + f_{uv} = f_{vt}$

Linear Programming

Linear Programming

Instance

Objective function $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

Constraints:

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le b_1$
$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le b_2$
$\vdots$
$a_{m1}x1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m$

Objective

Find values of the variables that satisfy all the constraints and maximize the objective function

Example

PRODUCT	Labor (hr/unit)	Clay (lb/unit)	Revenue ($/unit)
Bowl	1	4	40
Mug	2	3	50

There are 40 hours of labor and 120 pounds of clay available each day

Decision variables
x_1 = number of bowls to produce
x_2 = number of mugs to produce

Example

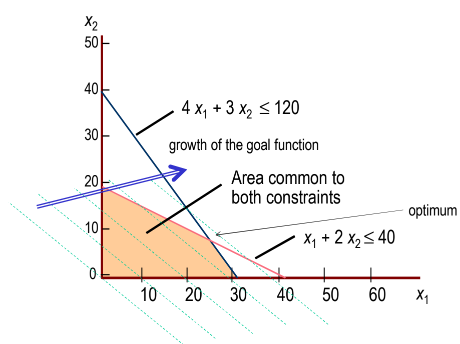
Maximize $Z = \$40\, x_1 + 50\, x_2$

Subject to

$x_1 + 2x_2 \le 40$ hr (labor constraint)
$4x_1 + 3x_2 \le 120$ lb (clay constraint)
$x_1, x_2 \ge 0$

Solution is $x_1 = 24$ bowls $x_2 = 8$ mugs
Revenue = $1,360

Example



Fundamental Theorem

Convex set

A set (or region) is convex if, for any two points (say, x_1 and x_2) in that set, the line segment joining these points lies entirely within the set.

A point is by definition convex.

Theorem (Extreme point or Simplex filter theorem)

If the maximum or minimum value of a linear function defined over a polygonal convex region exists, then it is to be found at the boundary of the region.

Simplex Method

A finite number of extreme points implies a finite number of solutions

Hence, search is reduced to a finite set of points

However, a finite set can still be too large for practical purposes

Simplex method provides an efficient systematic search guaranteed to converge in a finite number of steps.

Simplex Method

1. Convert to slack form
2. Begin the search at an extreme point (i.e., a basic feasible solution).
3. Determine if the movement to an adjacent extreme can improve on the optimization of the objective function. If not, the current solution is optimal. If, however, improvement is possible, then proceed to the next step.
4. Move to the adjacent extreme point which offers (or, perhaps, *appears* to offer) the most improvement in the objective function.
5. Continue steps 3 and 4 until the optimal solution is found or it can be shown that the problem is either unbounded or infeasible.

Simplex Method: Slack Form

maximize: $3x_1 + x_2 + 2x_3$

subject to:

$$x_1 + x_2 + 3x_3 \le 30$$
$$2x_1 + 2x_2 + 5x_3 \le 24$$
$$4x_1 + x_2 + 2x_3 \le 36$$
$$x_1, x_2, x_3 \ge 0$$

The corresponding slack form is

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Simplex Method: Basic Solution

We have

$$z = 3x_1 + x_2 + 2x_3$$
$$x_4 = 30 - x_1 - x_2 - 3x_3$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Variables on the right are called basic

Variables on the left --- nonbasic

Set basic variables to 0 and compute the nonbasic variables:

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (0,0,0,30,24,36)$$
$$z = 0$$

Simplex Method: Pivot

Increase one of the basic variables that appears in the objective function with nonnegative coefficient

x_1 can be increased to 9, that makes the last constraint tight

Solve that constraint for x_1 and substitute the solution into other constraints and the objective function

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

x_1 is entering variable

x_6 is leaving variable

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4}$$
$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$
$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4}$$
$$x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2}$$

Simplex Method: Pivot

Repeat the procedure

x_3 can be increased to 3/2, that makes the last constraint tight

$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$

$$z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16}$$
$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16}$$
$$x_3 = \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8}$$
$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16}$$

Simplex Method: Pivot

Repeat the procedure again

x_2 can be increased to 4, that makes the second constraint tight

$$x_2 = 4 - \frac{8x_3}{8} - \frac{2x_5}{3} + \frac{x_6}{3}$$

$$z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3}$$
$$x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3}$$
$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3}$$
$$x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2}$$

Simplex Method: Results

Lemma

The Pivot procedure creates a linear program equivalent to the original one

Lemma

Assuming that a slack form is such that the basic solution is feasible, Simplex either reports that a linear program is unbounded, or it terminates with a feasible solution that delivers a maximum to the objective function