Partial Differential Equations,

An Introduction to Theory and Applications

by

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Solutions Manual

by

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Chapter 1 Solutions

1. For $u(x,t) = f(x-3t)$, $u_t = -3f'(x-3t), u_x = f'(x-3t)$. Thus, $u_t + 3u_x = 0$.

2. For $u(x,t) = e^{mt}\sin(nx)$, $u_t - ku_{xx} = mu + kn^2u = 0$ when $m = -kn^2$.

3. For $u(x,t) = \sin(mt)\sin(nx)$, $u_{tt} = -m^2u$; $u_{xx} = -n^2u$. Thus, $u_{tt} = c^2u_{xx}$ implies $m^2 = c^2n^2$.

4. For $u(x,t) = a(t)e^{2x} + b(t)e^x + c(t)$, we find $u_t = u_{xx}$ when $a(t) = e^{4t}$, $b(t) = e^t$, $c(t) =$ constant.

5. $k(u) = mu^{m-1}$.

6. $u(x,t) = (1 + (x-4t)^2)^{-1}$.

7. $$u(x,t) = \begin{cases} 0 & \text{if} \quad x \geq 4t, \\ (t - x/4)e^{-(t-x/4)} & \text{if} \quad x \leq 4t \end{cases}$$

   There is no solution if the PDE is changed to $u_t - 4u_x = 0$ because then the general solution would have the form $u(x,t) = f(x+4t)$. The initial condition $u(x,0) = 0, x > 0$ gives $f(x) = 0, x > 0$, which is incompatible with the boundary condition (along the t axis, $x = 0$), since $u(0,t) = f(4t) = 0, t > 0$.

8. (a) Verification follows from the hints.
   (b) (i) $\phi(0) = \psi(0)$; (ii) $\psi'(0) = -c\phi'(0)$.

9. By the chain rule, if $u = u(x,t)$, $u = f(x - ut)$, then $u_x = f'(x-ut)(1 - u_xt)$, so that $u_x = f'/(1+tf')$. Similarly, since $u_t = f'(x-ut)(-u_tt - u)$, we have $u_t = -f'f/(1+tf')$. Thus, $u_t + uu_x = -f'f/(1+tf') + ff'/(1+tf') = 0$.

10. Let $u_0(y) = 1 - y^2$ if $-1 \leq y \leq 1$, and $u_0(y) = 0$ otherwise.
    (i) Use the implicit solution (4.12) to find an explicit formula for $u = u(x,t)$, with $-1 < x < 1, 0 < t < \frac{1}{2}$.
    (ii) Verify that $u(1,t) = 0$, $0 < t < \frac{1}{2}$.
    (iii) Differentiate your formula to find $u_x(1^-,t)$, and deduce that $u_x(1^-,t) \to -\infty$ as $t \to \frac{1}{2}^-$.
    Note: $u_x(x,t)$ is discontinuous at $x = \pm 1$; the notation $u_x(1^-,t)$ means the one-sided limit: $u_x(1^-,t) = \lim_{x<1, x\to 1} u_x(x,t)$. Similarly, $t \to \frac{1}{2}^-$ means $t \to \frac{1}{2}$, with $t < \frac{1}{2}$.

Chapter 2 Solutions

1. (a) Hyperbolic.

   (b) Parabolic if $\alpha = 1/4$; Hyperbolic if $\alpha < 1/4$; Elliptic if $\alpha > 1/4$.

   (c) Parabolic.

   (d) Parabolic on the curve $y = x^3/4 - 1$ and on the y axis $x = 0$. These curves divide the x, y plane into 4 regions. If $x > 0$ and $y < x^3/4 - 1$, or $x < 0$ and $y > x^3/4 - 1$, then the equation is hyperbolic. In the other two regions, the equation is elliptic.

2. Hint: write everything out carefully using subscripts for components of vectors and matrices. Since you are given the result, you can work both forwards from what is given and backwards from the end result to verify that they give the same expressions.

3. This exercise is a long-winded calculation using Taylor series, combining (2.6)-(2.9) and the definition of G in terms of a, b, c, f. Best to use Maple or Mathematica, but the calculation of u_2, and then u_3, is achievable on paper.

4. Try differentiating a few times to see a pattern in the derivatives. Use the fact that the exponential decays to zero faster than any power of x.

5. (a) $\sigma(\xi) = \pm i\xi^2$. The beam equation is dispersive and not dissipative because λ is imaginary, and is nonlinear in ξ. For the wave equation, λ is also imaginary, but it is linear in ξ, so that waves with different spatial frequencies ξ all travel with the same speed. The beam equation traveling waves travel with speed $s(\xi) = \xi$ that depends on ξ. The beam equation does not dissipate energy $E(t) = \frac{1}{2}\int\{(u_t)^2 + (u_{xx})^2\}dx$: $E'(t) = 0$ (integrating by parts twice, with boundary conditions that render boundary terms equal to zero).

   (b) $\sigma(\xi) = -ic\xi/(1-\beta\xi^2)$. The speed of traveling waves is $s(\xi) = c/(1-\beta\xi^2)$. This dependence on ξ differs from the quadratic dependence of the KdV equation traveling waves in that $s(\xi)$ is bounded if $\beta < 0$, and has a singularity at $\xi = 1/\sqrt{\beta}$ if $\beta > 0$.

6. (a) $v(u)$ should be a decreasing function, so that more dense traffic moves more slowly.

   (b) $v_{max} = v(0), v(u_{max}) = 0$.

   (c) $Q'(u) = uv' + v = 0$ has at least one solution by the Intermediate Value Theorem, since $Q'(0) = v_{max} > 0$ whereas $Q'(u_{max}) = u_{max}v'(u_{max}) < 0$. At least one of these solutions is a maximum since $Q(u) \geq 0$, and $Q(0) = Q(u_{max}) = 0$.

   (d) Yes, you can choose a quartic function $Q(u) = u(u_{max} - u)q(u)$, in which $q(u) > 0$ is quadratic, $v(u) = (u_{max} - u)q(u)$ is decreasing and $Q'(u) = 0$ has three solutions between $u = 0$ and $u = 1$.

Chapter 3 Solutions

1. $u(x,y) = \frac{2}{5}(1 - e^{5(x-y)})$.

2. $u(x,t) = \sin(x - \arctan t)$.

3. $u(x,t) = \frac{1}{6}e^{(2x+t)} + (x-t)e^{-3t} - \frac{1}{6}e^{(2x-5t)}$.

7. $u(x,y,t) = \left(x - \frac{1}{2}St^2 - yt^2 + y + St\right) / \left(1 - St^2 + 2St\right)$. This solution does not depend on the domain $0 < x,\ 0 < y < 1$ and does not satisfy the inlet boundary condition at $x = 0$.

8. (a) $u(x,t) = \left(1 - t + (x - \frac{1}{2}t^2)^2\right)^{-1}$.
(b) As $t \to 1^-$, $u(x,t) \to (x - \frac{1}{2})^{-2}$, which is singular at $x = \frac{1}{2}$. The solution is finite for $0 \le t < 1$.

9. $Q(0) = 0$. There is no flux at zero density because there are no cars.
$Q(\alpha) = 0$. At maximum density, the traffic is stationary, so there is no flux.

10. The problem should refer to Example 5, Chapter 2. Let $v(\rho) = a\rho^2 + b\rho + v_m$, where v_m is the maximum speed (at zero density). Then $2a\rho + b < 0$, and there are two additional parameters, the maximum density ρ_m and either a or b, related through $v(\rho_m) = 0 : b = -a\rho_m - v_m/\rho_m$. The flux is easily made non-concave, i.e., convex over an interval. For example, choosing $b = 0, a = -v_m/\rho_m$ gives $Q''(\rho) > 0$ in the interval $0 \le \rho < \rho_m/\sqrt{3}$.

11. Let $F(u,x,t) = u - u_0(x - ut)$. Then F is smooth, $F(u_0(x_0), x_0, 0) = 0$, and $F_u(u_0(x_0), x_0, 0) = 1$. Thus, the implicit function theorem applies in a neighborhood of $(u,x,t) = (u_0(x_0), x_0, 0)$, to give $u = u(x,t)$ satisfying $F(u(x,t),x,t) = 0$.

12. $$u(x,t) = \frac{1}{2t^2}\left\{2xt - \sqrt{4xt+1}\right\}.$$

14. Consider the characteristic $x = u_0(x_0)t + x_0$, on which $u(x,t) = u_0(x_0)$ is constant. For fixed x, t with $t > 0$, let $g(y) = u_0(y)t + y - x$. Then $g'(y) = u_0'(y) + 1 > 0$. Moreover, $g(y) \to \pm\infty$ as $y \to \pm\infty$. Consequently, the equation $g(y) = 0$ has a unique smooth solution $y = y(x,t)$. Then $u(x,t) = u_0(y(x,t))$.

15. $u = u_0(x - f'(u)t)$.

Chapter 4 Solutions

1. $\int_1^3 \psi(x)\,dx = 0$.

2. (a) $e_t = u_{tt}u_t + u_{xt}u_x = (u_t u_x)_x = p_x$ and similarly for e_x.
   (b) $e_{tt} = p_{xt} = e_{xx}$ and $p_{tt} = e_{xt} = p_{xx}$.

3. (a) $\partial_{tt}(u(x-y,t)) = u_{tt}(x-y,t) = u_{xx}(x-y,t) = \partial_{xx}(u(x-y,t))$.
   (b) $(u_x)_{tt} = (u_{tt})_x = (u_{xx})_x$.
   (c) $\partial_{tt}(u(ax,at)) = a^2 u_{tt}(ax,at)$. and $\partial_{xx}(u(ax,at)) = a^2 u_{xx}(ax,at)$.

4. (a) Hint: Use the general solution (4.9).
   (b) $u(x+2h,t+k) + u(x-2h,t-k) = u(x-2k,t-h) + u(x+2k,t+h)$.

5. The wave is travels left for $t < 1/2$, reflects at $t = 1$ and travels right for $t > 3/2$.

6. (a) If $x > t$, use D'Alembert's solution. If $x < t$, take even extensions of ϕ and ψ.
   and obtain $u(x,t) = \frac{1}{2}(\phi(x+t) + \phi(t-x)) + \frac{1}{2}\int_{t-x}^{x+t} \psi(s)\,ds + \int_0^{t-x} \psi(s)\,ds$.
   (b) $u = 0$ in the first quadrant if $x + t < 1$ or $x > t + 2$.
   (c) This follows from (a).
   (d) u is continuous if ϕ is continuous and $u \in C^1$ if ψ is continuous and $\phi'(0) = 0$.

7. $F(-\xi) = G(\xi) - \int_0^\xi h(s)\,ds$ for $\xi > 0$ with G from D'Alembert's solution, so for $x < t$ we have
   $u = G(t-x) + G(x+t) - \int_0^{t-x} h(s)\,ds$. Easy to check $u_x(0,t) = h(t)$.
   The solution is continuous if ψ, ϕ, h are. If $h(0) = \phi'(0)$, then u_x, u_t are continuous.

8. $E'(t) = -\int_{-\infty}^{\infty} \mu u_t^2\,dx$.

9. (a) $E(t) = \int_0^\infty \frac{c}{2}u_t^2 + \frac{1}{2}u_x^2\,dx$ is conserved if $\psi(0) = 0$ or $u_x(0,t) = 0$.
   (b) $E'(t) = u_x u_t|_{x=0} = \psi(0)h(0)$.

10. Write $u(x,t) = \frac{1}{2c}\int_0^t \tilde{u}(x,t,s)\,ds$ where $\tilde{u}(x,t,s) = \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy$. Compute
    $u_{tt} = \partial_t \tilde{u}(x,t,t) + \int_0^t \tilde{u}_{tt}(x,t,s)\,ds$ and $\partial_{xx}u = \int_0^t \partial_{xx}g(x,t,s)\,ds$ and use that $\tilde{u}_{tt} + c^2\tilde{u}_{xx} = 0$.

11. (a) Note that the integrals in (4.23) depend only on $|x|$.
    (b) If $ct - \epsilon \le |x| \le ct + \epsilon$ then $S(x,ct) \cap \{|x| < \epsilon\} = \emptyset$ so $u(x,t) = 0$.

Chapter 5 Solutions

1. (a) Follow the hint.
   (b) Bound the integrand by something easy to integrate.

2. Note that for $\mathbf{x} \in \mathbb{R}^n$, $\nabla\phi = \frac{\mathbf{x}}{r}\phi_r$ and $\nabla \cdot (\mathbf{x}/r) = (n-1)/r$.

3. Observe that $\Phi(x,t)$ is even in x.

4. $u(x,t) = \dfrac{1}{4}\mathrm{Erf}\left(\dfrac{1+x}{\sqrt{4kt}}\right) - \dfrac{1}{4}\mathrm{Erf}\left(\dfrac{1-x}{\sqrt{4kt}}\right)$.

5. $u(x,t) = e^{(p^2kt+px)}$.

6. $u(x,t) = \dfrac{1}{2}e^{kt-x}\left(1 - \mathrm{Erf}\left(\dfrac{2kt-x}{\sqrt{4kt}}\right)\right)$.

7. $\frac{\partial}{\partial t}(\int \frac{1}{2}u^2\,dx) = \int u\partial_x(ku_x)\,dx = -\int ku_x^2\,dx$ if $u \in L^2$.

8. a) Commute x and t derivatives on u.
   b) By uniqueness, $v(x,t) = 0$ for all $t > 0$, so $u(x,t) = x^2f(t) + xg(t) + h(t)$. Substituting into the initial condition and the heat equation leads to $u(x,t) = x^2 + 2kt$.

9. (a) We find that $v_t = kv_{xx}$ with $v(x,0) = g(x)$.
   (b) du acts as a source term.
   (c) Set $v = u\exp(\int_0^t d(s)\,ds)$.

10. Set $v(x,t) = u(x+ct,t)$.

11. Define the energy $E(t) = \frac{1}{2}\int_\Omega u_t^2\,dx + \frac{c^2}{2}\int_\Omega |\nabla u|^2\,dx$. We get

$$E'(t) = c^2 \int_{\partial\Omega} u_t(\nabla u \cdot n)\,dS$$

where n is the outward unit normal; this is zero if $u =$ const. or $\nabla u \cdot n = 0$ on $\partial\Omega$.

Chapter 6 Solutions

1. $u(x,t) = e^{-4t} \sin x - 3e^{-100t} \sin 5x$.

2. $u(x,t) = (2\cos 3\pi t + \frac{2}{3\pi} \sin 3\pi t) \sin \pi x + 7 \cos 9\pi t \sin 3\pi x$.

3. $\lambda = 0$ is an eigenvalue if and only if $a_0 + La_0 a_L + a_L = 0$ with eigenfunction $u(x) = a_0 x + 1$.

4. a) Set $\lambda = -k^2$. Then k satisfies $\tanh kL = k|a_0 + a_L|/(k^2 + a_0 a_L) := f(k)$. This is maximized at $k = \sqrt{a_0 a_L}$ with $f(k^*) \geq 1$, so it must intersect $\tanh kL$ at some value $k > 0$.

   b) If $f'(0) < L$ then f crosses $\tanh kL$ twice to produce two eigenvalues (which is also sufficient). This is equivalent to $a_0 + a_L + La_0 a_L > 0$.

5. a) Formalize the graphical arguments in the section.

   b) $\beta_n = \tan^{-1}(f(\beta_n)) + (n-1)\pi/L$ but $f(\beta_n) \to 0$ as $n \to \infty$ so $\theta_n := \beta_n - (n-1)\pi/L \to 0$.

   c) Note that $f(Cn + \theta_n) = (a_0 + a_L)/(Cn) + O(1/n^2)$ and $\tan(1/n) = 1/n + O(1/n^2)$, which gives that $\theta_n = (a_0 + a_L)/\pi n + O(1/n^2)$.

6. There are two eigenvalues if $a_0 + a_L + La_0 a_L > 0$, one if it equals zero and none otherwise. Neumann boundary conditions correspond to $a_0 = a_L = 0$ and Dirichlet to $a_0 = a_L = \pm\infty$.

7. Set $\alpha_i = \sqrt{r_i/p_i}$ and $\lambda = k^2$. Continuity of u and u' at $x = m$ gives two equations, along with two more for the boundary conditions. Solving separately for $u = u_1(x), x < m$, $u = u_2(x), x > m$ we find $u_1(x) = A_1 \sin(k\alpha_1 x)$, $u_2(x) = A_2 \sin(k\alpha_2(L-x))$. Let $A = A_2/A_1$. Then $u_1(m) = u_2(m)$ gives $A = \sin(k\alpha_1 m)/\sin(k\alpha_2(L-m))$. Similarly, continuity of u' at $x = m$ gives $k\alpha_1 \cos(k\alpha_1 m) = -Ak\alpha_2 \cos(k\alpha_2(L-m))$. Thus, the equation that determines values of k, and hence $\lambda = k^2$, is
   $$\frac{\tan(k\alpha_1 m)}{\tan(k\alpha_2(L-m))} = -\frac{\alpha_1}{\alpha_2}.$$

8. a) Look for values of $\mu > 0$ such that $\cos \mu L = -1/\cosh \mu L$, by graphing the functions of μL on each side of this equation. Note that $\cos \mu L$ has zeros at $\mu L = n\pi + \pi/2$ and has minima at $\mu L = \pi + 2n\pi$; solutions exist between minima and adjacent zeros.

   b) Pretty clearly from the graph, $L\mu_n \sim (n - 1/2)\pi$ as $n \to \infty$. More precisely, define θ_n by $L\mu_n = (n-1/2)\pi + \theta_n$. Then, expanding $\cos(L\mu_n)$ and using $-1/\cosh(\mu_n L) \sim -2e^{-\mu_n L}$ we are led to $\theta_n \sim 2(-1)^{n+1} e^{\pi/2} e^{-\pi n}$.

9. (a) Use (6.5) to obtain $b_n = 4/\pi n$ for n odd and zero otherwise.
   (b) Evaluating the series at $x = \pi/4$ gives $1 + 1/3 - 1/5 - 1/7 + 1/9 + \cdots = \frac{\pi}{4}\sqrt{2}$.

Chapter 7 Solutions

1. Integrate by parts to obtain $\mathcal{L}^*v = (av)'' - (bv)' + cv$.

2. The proof is as in Theorem 7.1, but using the weighted norm $||u||_r^2 = \int r(x)|u(x)|^2\,dx$, ie. $\lambda||u||_r^2 = (\lambda ru, u) = (\mathcal{L}u, u) = \cdots = \overline{\lambda}||u||_r^2$ with (u, v) the unweighted L^2 inner product.

3. Hint: If f is continuous except for a jump at x_0 from a to b then $f - (b-a)H(x - x_0)$ is continuous.

4. Let $\{v_n\}$ be the orthonormal basis for $L^2((-\pi, \pi))$. Approximate f in L^2 by $h \in C^1([-\pi, \pi])$ and use Bessel's inequality to bound $||S_n(f) - S_n(h)||_{L^2} \leq ||f - h||_{L^2}$. We have $S_n(h) \to h$ uniformly (hence in L^2) by Theorem 7.5.
By the triangle inequality, $||f - S_n(f)||_{L^2} \leq 2||f - h||_{L^2} + ||h - S_n(h)||_{L^2}$.

5. Consider the difference quotient $D^h f = h^{-1}(f(x+h) - f(x))$. We have the estimate
$$||D^h(f * g) - f' * g||_{L^1} \leq ||D^h f - f'||_{L^1} ||g||_{L^1}.$$
Now $D^h f \to f'$ in L^1 by dominated convergence since $D^h f \to f'$ pointwise and
$$||D^h f||_{L^1} = \int h^{-1} \left| \int_0^h f'(x+s)\,ds \right| dx \leq ||f'||_{L^1}$$
using Fubini's theorem. Thus $D^h(f * g) \to f' * g$ in L^1 so $(f * g)' = f' * g$.

6. (a) By the inversion formula, $f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ixn} \hat{f}(x)\,dx$, which are just the Fourier series coefficients for f.

   (b) Take the inverse transform of $\hat{f}$ from (a) and use that the inverse transform of the function equal to $e^{-in\omega}$ for $|x| < \pi$ and zero otherwise is $\frac{1}{\pi}\sin(\pi(x-n))/(x-n)$.

7. (a) Compute $a_{2n} = -\frac{4}{\pi(4n^2-1)}$ for $n \geq 1$ and $a_0 = 4/\pi$ with $a_{2n+1} = 0$. Evaluate at $x = \pi$ and $x = \pi/2$ to obtain $\sum_{n=1}^{\infty} 1/(4n^2 - 1) = 1/2$ and $\sum_{n=1}^{\infty} (-1)^n/(4n^2 - 1) = (2 - \pi)/4$.

   (b) Take $f(x) = x^2$, which has Fourier series $x^2 = \frac{\pi^2}{3} + 4\sum_{n \geq 1} \frac{(-1)^n}{n^2} \cos nx$.

Chapter 8 Solutions

1. Let $M = \max_{\partial\Omega} u(x)$, let $\epsilon > 0$ and $v_\epsilon = u(x) + \epsilon|x|^2$. If v_ϵ has a maximum at $x_0 \in \Omega$ then $0 \geq \Delta v_\epsilon = 2n\epsilon$, a contradiction. Hence $|v_\epsilon| \leq M + O(\epsilon)$. This implies that $|u(x)| \leq M + O(\epsilon)$ so $|u(x)| \leq M$.

2. If u is constant in $\overline{U}$ then clearly the weak maximum principle holds. Otherwise, by the strong maximum principle, $u(x) < \max_{y\in\partial U} u(y)$, which is a stronger condition.

3. (a) If u and v are solutions then $w = u - v$ solves $\Delta w = 0$ with $\partial w/\partial\nu + \alpha w = 0$ on ∂U. Then $\int |\nabla w|^2 = -\int w\Delta w + \int w\frac{\partial w}{\partial \nu}\, dS = -\int \alpha w^2\, dS$ which implies that $w = 0$.

   (b) If $\alpha = 0$ then (a) implies $\nabla w = 0$, i.e. $u - v = \text{const}$.

   (c) Consider $u'' = 0$ in $(0, 1)$ with boundary conditions $u' - 2u = 0$ at $x = 1$ and $-u' - 2u = 0$ at $x = 0$. Then $u = b(1 - 2x)$ is a solution for any $b \in \mathbb{R}$.

4. Compute the geometric series to obtain $(1 - \frac{r}{a}z)^{-1} + (1 - \frac{r}{a}\overline{z})^{-1} - 1$ where $z = e^{i(\theta-\phi)}$. This is then equal to $\frac{1-(r/a)^2 z\overline{z}}{1-2r/a(z+\overline{z})+(r/a)^2 z\overline{z}}$ which gives Poisson's formula.

5. Evaluate the integral $V_r = \int_{B(0,r)} 1\, dx$ in polar coordinates.

6. If $\Delta u \neq 0$ with $u \in C^2(U)$ then $\Delta u > 0$ (or $\Delta u < 0$) in some ball $B(x, r) \subset U$. This contradictions the calculation in (8.2).

7. See Theorem 8.4 where a short proof is given.

Chapter 9 Solutions

1. (a) Integrate the ODE.

   (b) If u, v are solutions then $w = u - v$ satisfies $w'' = 0$ and $w'(0) = w'(1) = 0$, so $w = C$.

   (c) $u'(z) = \int_0^z f(y)\,dy$ so (up to a constant) $u(x) = \int_0^x \int_0^z f(y)\,dy\,dz = \int N(x,y)f(y)\,dy$ where $N(x,y) = x - y$ if $x \geq y$ and $N(x,y) = 0$ otherwise.

2. From (9.6) compute $\phi^y(x) = G(x,y) + \frac{1}{2}|x - y| = \frac{1}{2}(x + y) - xy$, which is linear in x.

3. If $v = e^{-cx}u$ then $v' = e^{cx}\delta = \delta$ so $v = H(x) + C$ for some constant C. The solutions satisfying $|u| \to 0$ as $|x| \to \infty$ are $u = e^{-cx}H(x)$ for $c > 0$ and $u = -e^{-cx}H(-x)$ for $c < 0$.

4. $(g * f)(x) = \int_x^{x+1} g(y)\,dy$ which is piecewise quadratic for $-2 < x < 1$ and zero otherwise.

5. Linearity is immediate. If $\phi_n \to \phi$ in $\mathcal{D}$ then $(f_i, \phi_n) \to (f_i, \phi)$ in each case since $\phi_n \to \phi$ uniformly and the functions ϕ_n have shared compact support.

6. If $\phi \in C_c^\infty$ then $\int \eta^\epsilon(y)\phi(x - y)\,dy = \int \eta(y)\phi(x - \epsilon y)\,dy \to \phi(x)$ as $\epsilon \to 0$ since $\int \eta\,dy = 1$, $\phi(x - \epsilon y) \to \phi(x)$ uniformly and the integral for each ϵ is over a fixed compact set.

7. (a) $(\partial u/\partial x, \phi) = -(u, \partial\phi/\partial x)$ for $\phi \in \mathcal{D}$.

   (b) We have $(\partial u/\partial x, \phi) = -(H(y), \partial\phi/\partial x) = -\int_0^\infty \int \frac{\partial \phi}{\partial x}\,dx\,dy = 0$ for any $\phi \in \mathcal{D}$.

8. (a) Let $u(x) = \frac{1}{2k}e^{-k|x|}$. Then $-u'' + k^2 u = \delta$ in $\mathcal{D}'$. For $\phi \in \mathcal{D}$ and $\epsilon > 0$,

$$\int_{|x|\geq\epsilon} -u\phi'' + k^2 u\phi\,dx = (u\phi' - u'\phi)|_{-\epsilon}^{\epsilon} = \frac{1}{2k}e^{-k\epsilon}(\phi'(\epsilon) - \phi'(-\epsilon)) + e^{-k\epsilon}\phi(\epsilon)$$

   which tends to $\phi(0)$ as $\epsilon \to 0$.

   (b) Solve (9.14) to obtain $\phi^y(x) = \frac{1}{2k}e^{-k|y|}e^{-kx}$ and $G(x,y) = \frac{1}{2k}(e^{-k|x-y|} - e^{-k|y|-kx})$.

9. The fundamental solution is $u = \frac{1}{4\pi|x|}e^{-k|x|}$. Using the change of variable $u = v/r$ we get $-v''/r + k^2 v/r = f$ which suggests $v = C_k e^{-kr}$. The integral in the source condition can then be evaluated explicitly to obtain $C_k = 1/4\pi$.

10. $u \in L^2(B)$ if and only if $\alpha < 1/2$, which is the condition that ensures $|x_2|^{-\alpha} \in L^2(B)$.

11. (a) $((gf)', \phi) = -(f, g\phi') = -(f, (g\phi)' - g'\phi) = (gf', \phi) + (g'f, \phi)$.

    (b) Apply (a) with $f = \delta$ and note that $g\delta = g(0)\delta$.

12. Compute $u_\epsilon(x) = \int_{B(x,\epsilon)} \eta_\epsilon(x - y)u(y)\,dy = \int_0^\epsilon r^{n-1}\eta_\epsilon(r)\int_{\partial B(x,r)} u(y)\,dS\,dr$ and use the mean value property and the fact that $\int_{B(x,\epsilon)} \eta_\epsilon(y)\,dy = 1$.

Chapter 10 Solutions

1. (a) Use the triangle inequality for $v = (u - v) + v$ and $u = (v - u) + u$.

   (b) Take e.g. u, v of opposite signs.

2. $f(a, b) = a^p/p + b^q/q - ab$ is smooth and has $\nabla f = (a^{p-1} - a, b^{p-1} - b)$ which has critical points when a and b are 0 or 1. The Hessian is negative definite if $p, q > 1$ so the global minimum is at $f(0, 0) = 0$.

3. (a) Apply Hölder using exponents r and $s = (1/p + 1/q)^{-1}$ to $\int (uv)w\,dx$ and then Hölder using p/s and q/s to $\int (uv)^s\,dx$.

   (b) Write the integral as $\int u^{q\lambda} u^{q(1-\lambda)}\,dx$ and use Hölder with exponents $p/(q\lambda)$ and $r/(q(1-\lambda))$.

4. Show that $\int_U |\partial u/\partial x_j|^p = C \int_0^1 r^{-p(\beta+1)+n-1}\,dr$ so $u \in W^{1,p}$ if $\beta < n/p - 1$.

5. Observe that $u^{(\alpha)} = |x|^{1+\alpha}(\sin|x|/|x|)^\alpha \sim |x|^{1+\alpha}$ and $\partial_{x_j} u^{(\alpha)} \sim (x_j/|x|)|x|^\alpha$ near $x = 0$. It follows that $u^{(\alpha)} \in H^1(B)$ if and only if $\alpha > -1$.

Chapter 11 Solutions

1. If $u \in H_0^1(U)$ then $\int |u|^2 \le C \int |\nabla u|^2 = 0$ by Poincare's inequality, a contradiction.

2. If $Av = \lambda v$ then $\lambda |v|^2 = v^T A v \ge \theta |v|^2$ so $\lambda \ge \theta$.

3. We have $\int_0^1 u'^2\,dx \ge C \int_0^1 u^2\,dx$ for $u \in H_0^1((0,1))$ where $C = \pi^2$ is the smallest eigenvalue of $-d^2/dx^2$. Thus Theorem 11.8 applies with $\gamma = -c - \pi^2$.

4. $\begin{pmatrix} x & -y - 1/2 \\ -y - 1/2 & 2 \end{pmatrix}$ is positive definite if and only if $x > \frac{1}{2}(1 + y/2)^2$, which defines the region on which L is uniformly elliptic.

5. $\begin{pmatrix} x & 1/2 \\ 1/2 & 1 \end{pmatrix}$ is positive definite when $x > 1/4$ with smallest eigenvalue $x - 1/4$ so $c = 1/4$.

6. From the triangle inequality, $(u,u) + 2(u,v) + (v,v) = ||u+v||^2 \le ||u||^2 + ||v||^2 + 2||u||\,||v||$.

Chapter 12 Solutions

1. (a) $u' = \frac{1}{2}(u - u_-)(u - u_+)$.

   (b) Solutions travel from the right to the left so we need $u_- > u_+$.

   (c) The solution satisfies $(u - u_+)/(u - u_-) = Ce^{\beta\xi}$ where $\beta = (u_- - u_+)/2$. After some rearrangement, we get $u = u_- - \beta(1 + \tanh(\frac{1}{2}\beta\xi))$.

   (d) The wave speed is $s = 1/2$.

2. Equation (12.5): $u_t + uu_x + \gamma u_{xxx} = 0$.

   (a) Under the appropriate decay conditions, $\frac{\partial}{\partial t}\int u^2\,dx = -2\int \frac{\partial}{\partial x}(\frac{1}{3}u^3 + u_x^2) = 0$ and $\int u_t = \int -uu_x - \gamma u_{xxx} = -\int \frac{\partial}{\partial x}(u^2) = 0$.

   (b) The identity $(u^2u_{xx})_x = u^2u_{xxx} + 2uu_xu_{xx}$ gives

$$\frac{\partial}{\partial t}(\int_{-\infty}^{\infty} \frac{1}{2}u_x^2 - \eta u^3\,dx) = \int_{-\infty}^{\infty} -uu_xu_{xx} - 3\eta\gamma u^2u_{xxx}\,dx = \int(1/2 - 3\eta)u^2u_{xxx}\,dx$$

   so $\eta = 1/6\gamma$ produces the desired invariant.

3. Take scales $t = \alpha u_{max}\hat{t}$, $x = \sqrt{dT}\hat{x}$ and $u = u_{max}\hat{u}$.

5. Solve $v' = kv(v-1)$ to obtain $v(x - st)$ with $s = (2a-1)/\sqrt{2}$ as in Example 1. The traveling wave is then $u(x,t) = (1 - Ce^{k(x-st)})^{-1}$.

6. (a) Using (b), the ODE is $\beta v'' = -\alpha v' + \frac{1}{2}(v - v_+)(v - v_-)$.

   (b) The boundary conditions are consistent if and only if $\frac{1}{2}v_+^2 - sv_+ = \frac{1}{2}v_-^2 - sv_- = C$ which gives $s = \frac{1}{2}(v_+ + v_-)$. The right equilibrium point is a saddle point, while the other is a stable node or spiral so we need $v_- > v_+$. For sufficiency, show that one branch of the unstable manifold of $(v_-, 0)$ must cross the v-axis.

   (c) The solution is monotonic if and only if $(v^+, 0)$ is a nodal sink (rather than a spiral), which holds if and only if $\alpha^2 > 2(v^- - v^+)\beta$. Write the ODE as a system and show that eigenvalues of the equilibria at $v = v_\pm, v' = 0$ are real and distinct if and only if the condition on α, β holds.

7. After integrating once, the ODE for $u(\xi)$ is $u'/(b - su + f(u)) = 1$. Integrating again gives the desired equation.

Chapter 13 Solutions

1. The PDE is $xu_x + tu_t = 0$ with characteristics $(x(s), t(s)) = (As^2, Bs^2)$, i.e. $x/t =$ const. u is constant on characteristics, so u is a function of x/t.

2. Hint: Take locally linear approximations of u near $(x_0, t_0) \in C$ on either side of the curve $x = \gamma(t)$ and evaluate at a point on the curve. Show that they agree only when $\gamma'(t) = f'(u)$ by differentiating $u(\gamma(t)+, t) = u(\gamma(t)-, t)$ and using the pde $u_t + f'(u)u_x = 0$.

3. We consider $f(u)$ only for $0 \le u \le 1$, since u is a volume fraction. Set $A = k_0u^2 + k_w(1-u)^2$. A direct computation (using MAPLE for example) shows that $A^2f'(u) = 2k_0k_wu(1-u)$ and $f''(u)A^3 = 2k_wk_0((k_w + k_0)(2u^3 - 3u^2) + k_w)$. Thus, $f(u)$ is positive for $0 < u \le 1$, and $f"(0) > 0, f"(1) < 0$. Thus, there is at least one zero in the interval $0 < u < 1$. (You can also use Rolle's Theorem.) Since the cubic in $f"(u)A^3$ is readily seen to be a decreasing function in this interval, the zero of $f"$ is unique.

4. For v fixed, $f(u) = \frac{1}{2}(u-v)f'(u) + \frac{1}{2}(u-v)f'(v) - f(v)$ which, as an ODE for $f(u)$, has solution $f(u) = A(u-v)^2 + 2f'(v)u - f(v) - vf'(v)$. In particular, f is quadratic. This characterizes when a scalar conservation law has shock speed equal to the average of the characteristic speeds.

5. $u(x,t) = x/t$ for $0 < x/t < \sqrt{2}t^{-1/2}$ and $u = 0$ otherwise. The shock height decays as $t^{-1/2}$ so $u(x,t) \to 0$ uniformly with $\max|u| \le \sqrt{2}t^{-1/2}$.

6. The PDE $u_t + uu_x = 0$ provides the equations $c = 1$ and $ab = 2$, using the assumption that a, b, c, d are all positive. The Rankine-Hugoniot condition holds (for the shock at $x(t) = -dt^2$ with speed $s(t) = -2dt$) if $a(1 + \sqrt{c - bd}) = 4d$. Substituting $b = 2/a$ and isolating the square root, we find an equation for d/a, leading to $d = 3a/8$. Thus, $b = 2/a,\ c = 1,\ d = 3a/8, a > 0$ is the solution. Moreover, the entropy condition follows from $d > 0$, on the $t > 0$ branch of the shock.

7. (a) The Rankine-Hugoniot condition is $3s = u_+^2 + u_+u_- + u_-^2$.

   (b) The entropy condition is $3 > 1 + u_+ + u_+^2 > 3u_+^2$, which holds when $-1/2 < u_+ < 1$.

   (c) The rarefaction $u(x,t) = \hat{u}(x/t)$ satisfies $\hat{u}'(\xi) = 1/f''(\hat{u}(\xi))$ which gives $\hat{u}(\xi) = \sqrt{\xi}$. The solution is a rarefaction when $u^+ > 1$ (and a shock-rarefaction pair if $u^+ < -1/2$).

8. Along a characteristic, $v = u_x$ satisfies $v' = -f''(u)v^2$ which has solution $v = 1/(C - f''(u)t)$. Hence solutions remain smooth for all time if $f''(u) < 0$ and blow up if $f''(u) > 0$. The time of blow up is $t^* = \inf_{x\in\mathbb{R}} u_x(x,0)/f''(u(x,0))$ (it depends on the slope of the initial data).

9. (a) First note that $x_2 = -\bar{\rho}t_2/(\rho_m v_m)$ and $t_2 = t_1(1 - \bar{\rho}/\rho_m)^{-1}$. For $t > t_2$ the rarefaction and the Rankine-Hugoniot condition provide an ODE for $\gamma(t)$ which can be solved to obtain $\gamma(t) = 2v_m(1 - \bar{\rho}/\rho_m)t - v_m t_2^{1/2}(2 - \bar{\rho}/\rho_m)t^{1/2}$.

   (b) Using (a), the time $t_3 = \frac{t_1}{4}(2 - \bar{\rho}/\rho_m)^2(1 - \bar{\rho}/\rho_m)^{-2}$.

10. The density to the right of the slowed car will become zero, and cars will reach a density ρ_1 behind it. Since the velocity-density relation is $v = v_m(1-\rho/\rho_m)$, we find $\rho_1 = \rho_m(1-\frac{v_0}{2v_m}.)$ There will then be a trailing shock connecting the original density $\bar{\rho} = \rho_m(1-\frac{v_0}{v_m}.)$ to ρ_1 with speed given by the Rankine-Hugoniot condition:

$$\bar{s} = \frac{\bar{\rho}v(\bar{\rho}) - \rho_1 v(\rho_1)}{\bar{\rho}-\rho_1} = v_m(1 - \frac{(\bar{\rho}+\rho_1)}{\rho_m}).$$

11. (a) Taking $u(x,t) \to u(-x,t)$ reverses $(u^2)_x$ and the dispersive term. To undo the $(u^2)_x$ sign change we can take $u \to -u$ in the KdV-Burgers equation but not the modified equation.

    (b) The system is $u' = w$ and $w' = \gamma w - (u-u_-)(u-u_+)(u-u_0)$ with $u_0 = -(u_- + u_+)$. If $u^+ < u_0 < u^-$ then u^- and u^+ are unstable spirals and u_0 is a saddle point. Thus any heteroclinic orbit must be from $u^\pm$ to u_0. The condition $u_+ < u_0 < u_-$ guarantees that the entropy condition holds for such shocks.

    (c) The corresponding shocks here are Lax shocks, but are undercompressive in the example. Note that here the outer equilibria are unstable spirals rather than saddle nodes.

12. (a) If $u_R < -1+\sqrt{2}\gamma/3 = u_M$ then we have an undercompressive shock from u_L to u_M followed by a rarefaction $\hat{u}$ from u_M to u_R given by $\hat{u}(\xi) = (u_M^2 + \frac{1}{3}(\xi - s))^{1/2}$ for $s < \xi < 3u_R^2$.
    (b) Assume $u_R > -\sqrt{2}/3\gamma$ and γ is such that $2\sqrt{2}/3\gamma < 1 = u_L$. Then $u_R > -1/2$ so the shock $u_L \to u_R$ satisfies the entropy condition. The traveling wave exists by Theorem 13.3. When $u_M = -1+\sqrt{2}\gamma/3 < u_R < -\sqrt{2}\gamma/3$, there is a shock from u_L to u_M (by Theorem 13.3) and then a second from u_M to u_R which is a Lax shock since $u_M < u_R < 0$.

13. By the remarks after (13.39), $\Gamma(v)$ for $v_0 < 0$ is the least value of w_0 such that $q(t)$ has a positive zero. Such a zero exists when $v_0 < 0$ if and only if $q'(0) < 0$ and $\min q \le 0$, which hold precisely when $w_0 < -\sqrt{2v_0}$. Hence $\Gamma(v) = -\sqrt{2v}$.

Chapter 14 Solutions

1. The system is genuinely non-linear since $\nabla\lambda_\pm \cdot r_\pm = \pm\frac{3}{2}(gh)^{1/2}$.

2. If T is convex and $T'(1) > 0$ then $T'(\xi) \geq \frac{1}{\xi-1}T(\xi) > T(\xi)/\xi$ and the system is hyperbolic by (14.18). If T is concave then $T(\xi)/\xi > T'(\xi)$ but there is some ξ^* such that $T'(\xi) \leq 0$ for $\xi \geq \xi^*$ and hyperbolicity is lost.

3. The rarefaction curves are $\hat{u}(\xi) = (\pm\frac{3}{\sqrt{8}}\xi + u_0^{3/2})^{2/3}$ and $\hat{v}(\xi) = -\xi + v_0$ for $\xi < 0$.

4. (a) DF has eigenvalues $\lambda = \pm\sqrt{v^2 - \alpha u^2}$ so the system is strictly hyperbolic iff $\alpha \geq 0$.

   (b) The conditions are $s(u_- - u_+) = u_-^2 - u_+^2 + \alpha(v_-^2 - v_+^2)$ and $-2(u_- v_- - u_+ v_+) = s(v_- - v_+)$.

   (c) From (b), we need $v_+ = 0$ or $1 + 2u_+ - 3u_+^2 + v_+^2 = 0$.

   (d) The equations are $\epsilon u' = (u-1)(u+2) - v^2$ and $\epsilon v' = v(1-2u)$. An orbit with $v = 0$ connects $(1,0)$ to $(-2,0)$ for the wave speed $s = -1$. However, $\lambda_1(U^\pm) < s < \lambda_2(U^\pm)$, which violates the entropy condition.

5. Setting $r = \sqrt{u^2+v^2}$, the system becomes $u_t + (f(r)u)_x = 0$ and $v_t + (f(r)v)_x = 0$.
   (a) $\lambda_1 = f(r)$ and $\lambda_2 = f(r) + rf'(r) = (rf(r))'$ and eigenvectors $r_1 = (v, -u)$ and $r_2 = (u, v)$.

   (b) Since $\nabla\lambda_1 \cdot r_1 = r^{-1}f'(r)(u,v)\cdot(v,-u) = 0$ the first field is linearly degenerate. If $(rf(r))'' \neq 0$ then $\nabla\lambda_2 \cdot r_2 = \frac{1}{r}(rf(r)''(u,v)\cdot(u,v) = r(rf(r))'' \neq 0$ for $r > 0$.

   (c) Rotation invariance implies that the eigenvalues depend only on r and the eigenvectors must be (u,v) and $(v,-u)$, so one field is automatically linearly degenerate.

   (d) The Rankine-Hugoniot conditions for the shock $(1,0) \to (u,v)$ give $u(u^2+v^2)-1 = s(u-1)$ and $v(u^2+v^2) = sv$ which has a solution when $u^2+v^2 = s = 1$ (a contact discontinuity) or when $v = 0$ and $u^2+u+1 = s$ (a shock). For shocks, the entropy condition requires $-1 < u < 0$. Rarefactions exist from $(1,0)$ to $(u_R, 0)$ for any $u_R > 1$.

6. (a) For $\lambda_\pm^T$ the eigenvectors (using the calculations in (14.17)) are $r_\pm^T = (-q, p, \mp q\lambda, \pm\lambda p)$. Since $\lambda_\pm^T$ is a function only of ξ it is parallel to $(p,q,0,0)$ and thus $\nabla\lambda_\pm^T \cdot r_\pm^T = 0$.

   (b) For $\lambda_\pm^T$ we have $r_\pm^L = (p, q, \pm\lambda p, \pm\lambda q)$ and $\nabla\lambda_\pm^L \cdot r_\pm^L = \frac{T''(\xi)}{\xi}(p^2+q^2) = \xi T''(\xi)$ so the field is genuinely nonlinear provided $T'' \neq 0$.

7. (a) If $\psi^+(U)$ is a Riemann invariant of a 2×2 system for λ^+ then $\nabla\psi^+$ is a left eigenvector for $\lambda^- I - DF$. It follows that $\psi_t^+ + \lambda^-\psi_x^+ = \nabla\psi^+ \cdot u_t + \lambda^-\nabla\psi\cdot u_x = \nabla\psi\cdot(-DF + \lambda^+ I)u_x = 0$.

   (b) Riemann invariants for the p-system are $\psi^\pm = \int_0^u \sqrt{\sigma'(u)}\,ds \pm v$ and it easily checked that they satisfy $\psi_t^\pm \mp \sqrt{\sigma'(u)}\psi_x^\pm = 0$.

8. The normalization for rarefaction waves is chosen so that $\nabla\lambda_k \cdot r_k = 1$, which reduces (14.39) to $\tilde{s}'(0) = 1/2$. This implies that $\tilde{s}(\xi) < s(0) = \lambda_k(U_-)$ for $\xi < 0$ small. Since $\tilde{s}'(0) < 1$, (14.40) implies that $\lambda_k(U^+) < s(\xi)$ for $\xi < 0$ small.

Chapter 15 Solutions

1. For an incompressible flow, $\nabla(u \otimes u) = (\nabla \cdot u)u + (u \cdot \nabla)u = (u \cdot \nabla)u$. If we set $E = \frac{1}{2}\rho|u|^2$ then $\partial E/\partial t + \nabla \cdot ((E + p)u) = 0$, which is the same as the energy in the compressible case (if an external force $G = \nabla V$ arises from a potential V then $E = \frac{1}{2}\rho|u|^2 + V$).

2. The equations are, with $G = (G_1, G_2, G_3)$ and subscripts denoting derivatives,

$$
\begin{aligned}
u_t + uu_x + vu_y + wu_z &= -p_x + \frac{1}{Re}(u_{xx} + u_{yy} + u_{zz}) + G_1 \\
v_t + uv_x + vv_y + wv_z &= -p_y + \frac{1}{Re}(v_{xx} + v_{yy} + v_{zz}) + G_2, \\
w_t + uw_x + vw_y + ww_z &= -p_z + \frac{1}{Re}(w_{xx} + w_{yy} + w_{zz}) + G_3, \\
u_x + v_y + w_z &= 0.
\end{aligned}
$$

3. For a steady state flow with no y-dependence we assume $v = 0$ and all derivatives with respect to t or x are zero. Incompressibility, $u_x + w_z = 0$ then implies $w = 0$, so only u is non-zero. The resulting equations are simply $u_{zz} = 0$ with $u(z = h) = U$ and $u(z = 0) = 0$ so the solution is $u = Uz/h$.