# A Brief Introduction to Robust Bilevel Optimization

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Abstract. Bilevel optimization is a powerful tool for modeling hierarchical decision making processes. However, the resulting problems are challenging to solve—both in theory and practice. Fortunately, there have been significant algorithmic advances in the field so that we can solve much larger and also more complicated problems today compared to what was possible to solve two decades ago. This results in more and more challenging bilevel problems that researchers try to solve today. In this article, we give a brief introduction to one of these more challenging classes of bilevel problems: bilevel optimization under uncertainty using robust optimization techniques. To this end, we briefly state different versions of uncertain bilevel problems that result from different levels of cooperation of the follower as well as on when the uncertainty is revealed. We highlight these concepts using an academic example and discuss recent results from the literature concerning complexity as well as solution approaches. Finally, we discuss that the sources of uncertainty in bilevel optimization are much richer than in single-level optimization and, to this end, introduce the concept of decision uncertainty.

## 1. Introduction

Bilevel optimization has its roots in economics and dates back to the seminal works by von Stackelberg (1934, 1952). It has been introduced in the field of mathematical optimization much later in the publications by Bracken and McGill (1973) as well as Candler and Norton (1977). We use bilevel optimization to model hierarchical decision making processes, typically with two players, which we refer to as the leader and the follower. Despite its intrinsic hardness (Hansen et al. 1992; Jeroslow 1985), several innovative works pushed the boundaries of computational bilevel optimization so that we can tackle some relevant practical applications today; see, e.g., Kleinert et al. (2021) for a recent survey on computational bilevel optimization as well as the annotated bibliography by Dempe (2020).

The main goal of this article is to give a brief introduction to some basic concepts of bilevel optimization problems under uncertainty. The field is still in its infancy but, nevertheless, due to its relevance in many practical applications, it is developing very fast. In classic, i.e., single-level, optimization, there are two major approaches to address uncertainty: stochastic optimization (Birge and Louveaux 2011; Kall and Wallace 1994) and robust optimization (Ben-Tal, El Ghaoui, et al. 2009; Ben-Tal and Nemirovski 1998; Bertsimas, Brown, et al. 2011; Soyster 1973). The same two paths have been followed as well in bilevel optimization starting from the 1990s on. However, the sources of uncertainty are much richer in bilevel optimization compared to single-level optimization. To make this more concrete, let us consider the linear optimization problem $\min_x\{c^\top x \colon Ax \geq b\}$. It can "only" be subject to uncertainty due to uncertainties in the problem's data $c$, $A$, and $b$. Throughout this article, we will refer to this setting as *data uncertainty*. Moreover, a bilevel

optimization problem may also be subject to an additional source of uncertainty, which is due to its nature that it combines two different decision makers in one model. Hence, there can be further uncertainty involved either if the leader is not sure about the reaction of the follower or if the follower is not certain about the observed leader's decision. We will denote this additional type of uncertainty as *decision uncertainty*. Obviously, decision uncertainty does not play any role in single-level optimization since only one decision maker is involved.

In this introductory article, we will solely focus on data uncertainty that is tackled using concepts from robust optimization. For more details regarding stochastic bilevel optimization, decision uncertainty, etc. we refer to our recent survey (Beck, Ljubić, et al. 2022a).

## 2. Problem Statement

We start by considering the deterministic bilevel problem (we explain the quotation marks below)

$$\text{"}\min_{x \in X}\text{"} \quad F(x, y) \tag{1a}$$

$$\text{s.t.} \quad G(x, y) \geq 0, \tag{1b}$$

$$y \in S(x), \tag{1c}$$

where $S(x)$ denotes the set of optimal solutions of the $x$-parameterized problem

$$\min_{y \in Y} \quad f(x, y) \tag{2a}$$

$$\text{s.t.} \quad g(x, y) \geq 0. \tag{2b}$$

Problem (1) is referred to as the upper-level (or the leader's) problem and Problem (2) is the so-called lower-level (or the follower's) problem. Moreover, we refer to $x \in X$ and $y \in Y$ as the leader's and the follower's variables, respectively. The sets $X \subseteq \mathbb{R}^{n_x}$ and $Y \subseteq \mathbb{R}^{n_y}$ can be used to include possible integrality constraints. The objective functions are given by $F, f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$ and the constraint functions by $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^m$ as well as $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^\ell$. In the case that the lower-level problem does not have a unique solution, the bilevel problem (1) and (2) is ill-posed. This ambiguity is expressed by the quotation marks in (1a). To overcome this issue, it is common to pursue either an optimistic or a pessimistic approach to bilevel optimization; see, e.g., Dempe (2002). In the optimistic setting, the leader chooses the follower's response among multiple optimal solutions of the lower-level problem such that it favors the leader's objective function value. Hence, the leader also minimizes her[1] objective in the $y$ variables, i.e., we consider the problem

$$\min_{x \in \bar{X}} \min_{y \in S(x)} \quad F(x, y) \tag{3}$$

with $\bar{X} := \{x \in X : G(x) \geq 0\}$ and $G : \mathbb{R}^{n_x} \to \mathbb{R}^m$. Here and in what follows, we focus on the setting without coupling constraints, i.e., without upper-level constraints that depend on the variables $y$. In the pessimistic setting, the leader anticipates that, among multiple optimal solutions of the follower, the worst possible response w.r.t. the upper-level objective function will be chosen by the follower. Thus, one studies the problem

$$\min_{x \in \bar{X}} \max_{y \in S(x)} \quad F(x, y).$$

In this article, we focus on bilevel problems of the above form which are additionally affected by data uncertainty.

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[1]Throughout this article, we use "her" for the leader and "his" for the follower.

2.1. **Data Uncertainty.** Data uncertainty arises when some of the players only have access to inaccurate or incomplete data. In robust optimization, it is assumed that these uncertainties take values in a given, and usually compact, uncertainty set $\mathcal{U}$. The uncertainty sets are typically modeled using boxes, polyhedra, ellipsoids, or cones; see, e.g., Ben-Tal, El Ghaoui, et al. (2009), Ben-Tal, Goryashko, et al. (2004), Ben-Tal and Nemirovski (1998), Bertsimas, Brown, et al. (2011), and Soyster (1973). In the context of single-level robust optimization, there are two possibilities to hedge against data uncertainty.

First, assuming that the coefficients of the objective function are uncertain, one searches for a solution that is optimal for the worst-case realization of the uncertain parameters. The problem can be modeled as

$$\min_{x \in \bar{X}} \max_{u \in \mathcal{U}} \quad F(x, u), \tag{4}$$

where the objective function $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ is continuous and the sets $\mathcal{U} \subseteq \mathbb{R}^{n_u}$ and $\bar{X}$ are defined as above.

Second, in the case that the uncertainty affects the coefficients of the constraints, one is interested in a solution that is feasible for all possible realizations of the uncertainty. This problem can be stated as

$$\min_{x \in X} \quad F(x) \quad \text{s.t.} \quad G(x, u) \geq 0 \quad \text{for all } u \in \mathcal{U}, \tag{5}$$

where both the objective function $F : \mathbb{R}^{n_x} \to \mathbb{R}$ and the constraint function $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^m$ are continuous. Problem (5) can be reformulated as

$$\min_{x \in X} \quad F(x) \quad \text{s.t.} \quad \min \{ G(x, u) \colon u \in \mathcal{U} \} \geq 0. \tag{6}$$

In particular, Problem (4) can be restated as an instance of Problem (6) using an epigraph reformulation, i.e.,

$$\min_{x \in \bar{X}, t \in \mathbb{R}} \quad t \quad \text{s.t.} \quad t \geq \max \{ F(x, u) \colon u \in \mathcal{U} \} .$$

Note that for the two settings discussed so far, a single decision maker has to take a here-and-now decision before the uncertainty is revealed. In bilevel optimization, however, there are two different timings that are possible—one in which the uncertainty realizes after and one in which the uncertainty realizes before the follower takes his decision.

2.1.1. *Here-and-Now Follower.* In this case, both the leader and the follower have to make their decisions before the uncertainty is revealed, i.e., one considers the timing

$$\text{leader } x \quad \curvearrowright \quad \text{follower } y = y(x) \quad \curvearrowright \quad \text{uncertainty } u. \tag{7}$$

This means that the leader anticipates an optimal response of the follower who hedges against data uncertainty. Hence, the lower-level problem is an $x$-parameterized problem in which we can embed any of the concepts known for single-level optimization under uncertainty. For instance, if only the lower-level objective function is uncertain and the follower is assumed to behave in an optimistic way, we are solving Problem (3) with

$$S(x) := \arg\min_{y' \in Y} \left\{ \max_{u \in \mathcal{U}} \ f(x, u, y') \colon g(x, y') \geq 0 \right\} .$$

2.1.2. *Wait-and-See Follower.* In this setting, the leader first takes a here-and-now decision, i.e., without knowing the realization of uncertainty. Then, the uncertainty is revealed and, finally, the follower decides in a wait-and-see fashion, taking the leader's decision as well as the realization of the uncertainty into account. Hence, one considers the timing

$$\text{leader } x \quad \curvearrowright \quad \text{uncertainty } u \quad \curvearrowright \quad \text{follower } y = y(x, u). \tag{8}$$

This means that the leader does not have full knowledge about the lower-level problem. Thus, she wants to hedge against the worst-case reaction of the follower. Here, "worst-case" may not only imply the robustness of the leader w.r.t. lower-level data uncertainty but also her conservatism regarding the cooperation of the follower. For instance, to protect against the worst-case realization of the uncertainties w.r.t. the leader's objective function, we consider the problem

$$\text{“} \min_{x \in \bar{X}} \max_{u \in \mathcal{U}} \text{”} \quad F(x, y) \quad \text{s.t.} \quad y \in S(x, u), \tag{9}$$

where $S(x, u)$ is the set of optimal solutions of the $(x, u)$-parameterized problem

$$\min_{y \in Y} \quad f(x, u, y) \quad \text{s.t.} \quad g(x, u, y) \geq 0.$$

The quotation marks in (9) express the ill-posedness of the bilevel problem in the case that the set $S(x, u)$ is not a singleton. Hence, one also needs to distinguish between the optimistic and the pessimistic case in the robust setting. Indeed, both situations can be motivated by practical applications. For instance, the pessimistic robust bilevel problem appears when the leader wants to hedge against the worst-case both w.r.t. lower-level data uncertainty as well as w.r.t. the potentially unknown level of cooperation of the follower. On the other hand, there may also be situations in which the follower still hedges against his uncertainties in a robust way but, in the case of ambiguous optimal solutions, acts in an optimistic way. This might be the case in energy markets with sufficiently regulated agents, where a strong level of regulation might lead to an optimistic robust bilevel problem.

## 3. An Academic Example

Let us consider the linear bilevel problem taken from Beck, Ljubić, et al. (2022a) that is given by

$$\text{“} \min_{x \in \mathbb{R}} \text{”} \quad F(x, y) = x + y \tag{10a}$$

$$\text{s.t.} \quad x - y \geq -1, \tag{10b}$$

$$3x + y \geq 3, \tag{10c}$$

$$y \in S(x), \tag{10d}$$

where $S(x)$ denotes the set of optimal solutions of the $x$-parameterized lower level

$$\min_{y \in \mathbb{R}} \quad f(x, y) = -0.1y \tag{11a}$$

$$\text{s.t.} \quad -2x + y \geq -7, \tag{11b}$$

$$-3x - 2y \geq -14, \tag{11c}$$

$$0 \leq y \leq 2.5. \tag{11d}$$

The problem is depicted in Figure 1 (left). The upper- and lower-level constraints are represented with dashed and solid lines, respectively. The optimal solution $(x^*, y^*) = (1.5, 2.5)$ is the same for both the optimistic and the pessimistic setting and it is illustrated by the thick dot. Suppose now that the lower-level objective function is uncertain. To this end, we consider $\tilde{f}(x, u, y) = (-0.1 + u)y$ and assume that $u$ only takes values in the uncertainty set $\mathcal{U} = \{u \in \mathbb{R} \colon |u| \leq 0.5\}$. In what follows,
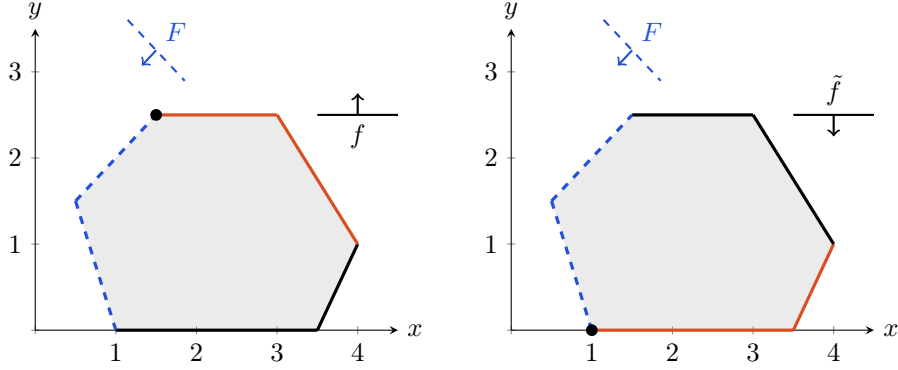
FIGURE 1. Both figures show the upper-level constraints (dashed blue lines), the lower-level constraints (solid black and orange lines), the shared constraint set (gray area), and the bilevel feasible set (solid orange lines) of the bilevel problem (10) and (11). The deterministic variant of the problem is depicted on the left and the variant with a here-and-now follower is given on the right.

we distinguish between a follower taking a here-and-now or a wait-and-see decision to illustrate how the considered timing may affect the solution of the problem.

3.1. **Here-and-Now Follower.** We first consider the timing in (7). The robustified lower-level problem is thus given by

$$\min_{y \in \mathbb{R}} \max_{u \in \mathcal{U}} \quad \tilde{f}(x, u, y) = (-0.1 + u)y \quad \text{s.t.} \quad (11b)-(11d).$$

Using classic techniques from robust optimization, we obtain a modified gradient of the lower-level objective function, which is shown in Figure 1 (right). The optimal solution $(x^*, y^*) = (1, 0)$ of this problem is represented by the thick dot. In particular, there is a unique lower-level response for every feasible $x$, which is why we do not need to distinguish between the optimistic and the pessimistic case.

3.2. **Wait-and-See Follower.** We now consider the timing in (8), i.e., the overall robustified bilevel problem reads

$$\text{``}\min_{x \in \mathbb{R}} \max_{u \in \mathcal{U}}\text{''} \quad F(x, y) \quad \text{s.t.} \quad (10b)-(10c), \ y \in S(x, u),$$

where $S(x, u)$ is the set of optimal solutions of the $(x, u)$-parameterized lower level

$$\min_{y \in \mathbb{R}} \quad \tilde{f}(x, u, y) = (-0.1 + u)y \quad \text{s.t.} \quad (11b)-(11d).$$

To solve this problem, we need to distinguish the following three cases.

(i) $-0.5 \leq u < 0.1$: This case corresponds to the setting that is depicted in Figure 1 (left). The optimal follower's reaction is thus given by

$$y(x, u) = \begin{cases} 2.5, & x \leq 3, \\ -1.5x + 7, & 3 \leq x \leq 4. \end{cases} \tag{12}$$

Note, however, that the bilevel problem is infeasible for $x < 1.5$. In particular, this means that the robust optimal leader's decision $x^* = 1$ for the case with a here-and-now follower is no longer bilevel feasible if the follower decides in a wait-and-see fashion.

(ii) $u = 0.1$: Any feasible decision of the follower, i.e., any $y \in \mathbb{R}$ that satisfies (11b)–(11d), is optimal for the $x$-parameterized lower level. Hence, the distinction between an optimistic and a pessimistic follower is necessary. In the optimistic setting, the follower would react with

$$y(x, u) = \begin{cases} 0, & x \leq 3.5, \\ 2x - 7, & 3.5 \leq x \leq 4. \end{cases} \quad (13)$$

This corresponds to the setting that is depicted in Figure 1 (right). A pessimistic follower, however, would select (12). Note that the bilevel problem with an optimistic follower turns out to be infeasible for $x < 1$ and, again, the problem is infeasible for $x < 1.5$ if a pessimistic follower is considered.

(iii) $0.1 < u \leq 0.5$: The optimal follower's reaction is given by (13). Again, the overall bilevel problem turns out to be infeasible for $x < 1$.

To determine an optimal solution of the bilevel problem (10) and (11) with a wait-and-see follower, we thus consider the worst-case realization of each of the previous three cases w.r.t. the leader's decision $x$. Hence, we need to solve

$$\min_x \quad \hat{F}(x) \quad \text{s.t.} \quad 1.5 \leq x \leq 4 \quad (14)$$

with the piecewise-linear function

$$\hat{F}(x) = \begin{cases} x + 2.5, & 1.5 \leq x \leq 3, \\ -0.5x + 7, & 3 \leq x \leq 4. \end{cases}$$

In particular, the solution $x^* = 1.5$ of Problem (14) is an optimal decision of the leader in both the optimistic and the pessimistic setting. After observing the realization of the uncertainty, the corresponding response of the follower is then given by

$$y_o^*(x^*, u) = \begin{cases} 2.5, & -0.5 \leq u < 0.1, \\ 0, & 0.1 \leq u \leq 0.5 \end{cases}$$

in the optimistic setting, whereas, for the pessimistic case, we have

$$y_p^*(x^*, u) = \begin{cases} 2.5, & -0.5 \leq u \leq 0.1, \\ 0, & 0.1 < u \leq 0.5. \end{cases}$$

Note that, if $u \in [-0.5, 0.1)$ realizes, at the point $x^* = 1.5$, the deterministic solution $(x^*, y(x^*))$ and the robust bilevel solutions $(x^*, y(x^*, u))$ coincide. However, the optimal follower's response $y(x^*, u)$ in the robust setting may change significantly for $u \geq 0.1$.

## 4. Selected Results from the Literature

The field of robust bilevel optimization is still in its infancy. For a detailed discussion of existing modeling and solution approaches, we refer to our recent survey (Beck, Ljubić, et al. 2022a). In deterministic bilevel optimization, a standard solution approach is to reformulate the problem as a classic, i.e., single-level, problem. This can be done, e.g., by replacing the lower level with its Karush–Kuhn–Tucker (KKT) conditions (Fortuny-Amat and McCarl 1981). The same holds true for robust bilevel problems whenever the robust counterpart of the lower-level problem can be reformulated as a deterministic problem for which the KKT conditions are necessary and sufficient. However, these reformulation techniques cannot be applied anymore if discrete variables are introduced in the lower level. Due to their intrinsic hardness, approaches for discrete robust bilevel problems have not been investigated a lot up to now. In single-level optimization, the knapsack problem is one of the

most thoroughly studied discrete optimization problem due to its relevance both in theory and practice; see, e.g., Pisinger and Toth (1998). Bilevel knapsack problems naturally extend their single-level counterparts such as to capture hierarchical and, in particular, competitive settings (Caprara et al. 2013; Della Croce and Scatamacchia 2020; DeNegre 2011; Fischetti, Ljubić, et al. 2019; Fischetti, Monaci, et al. 2018). Moreover, the bilevel knapsack interdiction problem is commonly used as a benchmark for testing bilevel optimization solvers; see, e.g., DeNegre and Ralphs (2009) and Tang et al. (2016). It is thus not surprising that bilevel knapsack problems are also among the first discrete bilevel problems studied under uncertainty—both in terms of complexity questions and solution approaches. The remainder of this section is thus dedicated to a brief overview of recent results from the literature for robust bilevel knapsack problems.

4.1. **Complexity Results for Robust Continuous Bilevel Knapsack Problems with a Wait-and-See Follower.** We start by considering the robust continuous bilevel knapsack problem with an uncertain lower-level objective, i.e., we consider the problem

$$\max_{x \in [x^-, x^+]} \min_{c \in \mathcal{U}, y \in \mathbb{R}^n} \quad d^\top y \tag{15a}$$

$$\text{s.t.} \quad y \in \arg\max_{y'} \left\{ c^\top y' : a^\top y' \le x, \, 0 \le y' \le 1 \right\} \tag{15b}$$

with $x^-$, $x^+ \in \mathbb{R}$, $x^- \le x^+$, $a$, $c$, $d \in \mathbb{R}^n_{\ge 0}$, and an uncertainty set $\mathcal{U} \subseteq \mathbb{R}^n$. In this setting, the leader first decides on the knapsack's capacity $x$. Then, the uncertainties regarding the lower-level objective function coefficients realize. Finally, the follower solves a knapsack problem according to the realization of his own profits, which may differ from those of the leader. Hence, the follower decides in a wait-and-see fashion, i.e., the timing in (8) is considered. The leader's aim is to choose the capacity of the knapsack in such a way that her own profit of the items packed by the follower is maximized. Whenever the follower's choice of items is not unique, the pessimistic approach is considered. The deterministic variant of Problem (15) can be solved in polynomial time, which makes it a good starting point to address the question of how uncertainties may affect the hardness of the underlying bilevel problem.

Driven by this question, Buchheim and Henke (2020, 2022) show that the complexity of Problem (15) strongly depends on the considered type of the uncertainty set. For discrete uncertainty sets as well as for interval uncertainty under the independence assumption, i.e., for the case in which the follower's objective function coefficients independently take values in given intervals, Problem (15) remains solvable in polynomial time. However, the problem becomes NP-hard if the uncertainty set is the Cartesian product of discrete sets. In particular, this shows that replacing the uncertainty set by its convex hull may significantly change the problem, which is very much in contrast to the situation in single-level robust optimization. NP-hardness is also shown for the variants of the problem with polytopal uncertainty sets and uncertainty sets that are defined by a $p$-norm with $p \in [1, \infty)$. In particular, for all NP-hard variants of the problem, even the evaluation of the leader's objective function is NP-hard.

As a generalization of the aforementioned works, Buchheim, Henke, and Hommelsheim (2021) are concerned with complexity questions for robust bilevel combinatorial problems of the form

$$\text{``}\max_{x \in X} \min_{c \in \mathcal{U}}\text{''} \quad d^\top y \tag{16a}$$

$$\text{s.t.} \quad y \in \arg\max_{y' \in \mathbb{R}^{n_y}} \left\{ c^\top y' : By' \le Ax + b \right\}. \tag{16b}$$

with $X \subseteq \{0,1\}^{n_x}$, $A \in \mathbb{R}^{m \times n_x}$, $B \in \mathbb{R}^{m \times n_y}$, $c, d \in \mathbb{R}^{n_y}$, and $b \in \mathbb{R}^m$. Again, it is assumed that the lower-level objective function coefficients are uncertain, that the uncertainties take values in a given uncertainty set $\mathcal{U} \subseteq \mathbb{R}^{n_y}$, and that the follower decides in a wait-and-see fashion. As before, the quotation marks in (16a) express the ambiguity in the case that the lower level does not have a unique solution. The deterministic variant of Problem (16) is known to be NP-easy.[2] However, it is shown that interval uncertainty renders Problem (16) significantly harder than the consideration of discrete uncertainty sets. More precisely, the robust counterpart can be $\Sigma_2^P$-hard[3] for interval uncertainty under the independence assumption, whereas it can be NP-hard for uncertainty sets $\mathcal{U}$ with $|\mathcal{U}| = 2$ and strongly NP-hard for general discrete uncertainty sets. In particular, it is shown that replacing the discrete uncertainty set by its convex hull may increase the complexity of the problem at hand, which is in line with the results in Buchheim and Henke (2020, 2022).

4.2. **Solution Approaches for the Bilevel Knapsack Interdiction Problem with a Here-and-Now Follower.** Beck, Ljubić, et al. (2022b) study discrete linear min-max problems with uncertainties regarding the lower-level objective function coefficients. In contrast to the aforementioned works, which all follow the notion of strict robustness, the authors consider a $\Gamma$-robust approach (Bertsimas and Sim 2003, 2004). The problem under consideration thus reads

$$\min_x \quad c^\top x + d^\top y \tag{17a}$$

$$\text{s.t.} \quad Ax \geq a, \, x \in X \subseteq \mathbb{Z}^{n_x}, \tag{17b}$$

$$y \in \underset{y' \in Y(x)}{\arg\max} \left\{ d^\top y' - \max_{\{S \subseteq [n_y]: \, |S| \leq \Gamma\}} \sum_{i \in S} \Delta d_i y_i' \right\}, \tag{17c}$$

where $\Gamma \in [n_y] := \{1, \ldots, n_y\}$ and $Y(x) \subseteq \mathbb{Z}_+^{n_y}$ denotes the lower-level feasible set. Here, the timing in (7) is considered, i.e., both the leader and the follower decide before the uncertainty realizes. The authors present two approaches to reformulate Problem (17). The first approach is based on an extended formulation, whereas the second one exploits the fact that Problem (17) can be interpreted as a single-leader multi-follower problem with independent followers. Based on these reformulations, the authors propose generic branch-and-cut frameworks to solve the problem. Moreover, it is shown that the same techniques can also be used for the case in which uncertainties only arise in a single packing-type constraint on the lower level. To assess the applicability of the proposed branch-and-cut methods, the authors focus on the $\Gamma$-robust knapsack interdiction problem (Caprara et al. 2016). In this setting, both players share a common set of items and the leader has the ability to influence the follower's decision by prohibiting the usage of certain items by the follower. The authors derive problem-tailored cuts and perform a computational study on 200 robustified knapsack interdiction instances with up to 55 items, i.e., with up to 55 variables on both the upper and the lower level.

## 5. A First Glimpse at Decision Uncertainty

Although being subject to data uncertainty, both decision makers in the bilevel problem are assumed to take perfectly rational decisions in the sense that they can perfectly anticipate or observe the other's decision and that they can solve their problem to global optimality. In decision making theory, however, it is well known

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[2]A decision problem is NP-easy if it can be polynomially reduced to an NP-complete decision problem (Buchheim, Henke, and Hommelsheim 2021).

[3]This class contains those problems that can be solved in nondeterministic polynomial time, provided that there exists an oracle that solves problems that are in NP in constant time.

that these assumptions regarding perfect information and rationality are rarely satisfied in a real-world context. Luckily, bilevel optimization under uncertainty allows to relax these assumptions in multiple ways. Throughout this article, we assumed that the major source of uncertainty stems from unknown or noisy input data. However, bilevel optimization involves (at least) two decision makers and, hence, other uncertainties in the decision making process are also possible. Another possible one is *decision uncertainty* in which, e.g., the leader is not sure about the reaction of the follower (for instance if the follower does not necessarily choose an optimal solution) or in which the follower is not sure about the observed leader's decision. We are not going into the details here but want to give a few pointers to the relevant literature that covers such aspects. If the leader is uncertain about her anticipation of the follower's optimal reaction and, thus, may want to hedge against sub-optimal follower reactions, the resulting setup can be modeled using so-called near-optimal robust bilevel models; see, e.g., Besançon et al. (2019). As an extreme case of the former aspect it may be the case that the upper-level player knows that the follower will play against her. This is the setting of a pessimistic bilevel optimization problem, which is also rather naturally connected to the field of robust optimization; see, e.g., Wiesemann et al. (2013). However, if the level of cooperation or confrontation of the follower is not known, this leads to intermediate cases in between of the optimistic and the pessimistic case; see, e.g., Aboussoror and Loridan (1995) and Mallozzi and Morgan (1996). Moreover, in many situations it is not possible for the follower to perfectly observe the optimal decision of the leader and the follower thus may want to hedge against all possible leader decisions in some uncertainty set around the observation. Such settings are tackled in, e.g., Bagwell (1995), Beck and Schmidt (2021), and van Damme and Hurkens (1997). Finally, even if all data and the rational reaction of the follower is known and even if the leader can, in principle, fully anticipate the (globally) optimal reaction of the follower, it might still be the case that limited intellectual or computational resources render it impossible for the follower to take a globally optimal decision. In such situations, a follower might resort to heuristic approaches and the leader may be uncertain w.r.t. which heuristic is used. For a good primer in this context, we refer to the recent paper by Zare et al. (2020).

The above list is by far not comprehensive. A much more detailed discussion of these and other aspects can be found in our recent survey (Beck, Ljubić, et al. 2022a). However, it is hopefully clear now how much more diverse the sources of uncertainty can be in bilevel optimization as compared to single-level optimization. Hence, we expect a lot of research in this area in future years.

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