Chapter 3: Dynamics

3.1 Forces and equations of motion.

Up to today, we have discussed the ocean basins, observational methods, observed ocean circulation and water masses. One important question we will now ask is: why does the ocean circulation possess the observed features, say subtropical gyres (STG), subpolar gyres (SPG), and strong western boundary current? In order to provide a theoretical explanation for the observed ocean circulation, we will first understand the dynamics that govern the fluid motion.

Although most of you have probably seen the equations of motion for the atmosphere, or from other GFD classes, it is useful to demonstrate them for the ocean, even though they are essentially the same as those for the atmosphere. It is always important for us to know where the equations we use are from, and what approximations were made to obtain them.

In fluid dynamics, we typically confine our attention to the movement of small fluid elements. To derive the equations of motion for a fluid, we consider the balance of forces acting on such fluid elements.

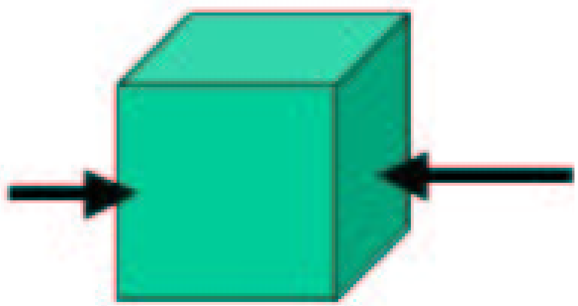

Figure 1: Schematic diagram showing the forces acting on a fluid element in the ocean.

Forces. The forces that may act on a rectangular sea water element in Fig 1 are:

- (i) Pressure gradient force (PGF);
- (ii) Gravity, $\mathbf{g}$;
- (iii) Friction, wind forcing, buoyancy forcing, etc.

The fluid element feels the sum of all the forces, and its motion is given by Newton's second law of motion.

$$\mathbf{F} = m\mathbf{a}, \tag{1}$$

where $\mathbf{F}$ is the net force acting on a body of mass m, and $\mathbf{a}$ is the resulting acceleration of the body. Bold-faced variables represent vectors.

By applying Newton's second law to a fluid element measured relative to axes that are fixed in space, relative to the stars (not relative to the earth yet), we obtain the equation of motion.

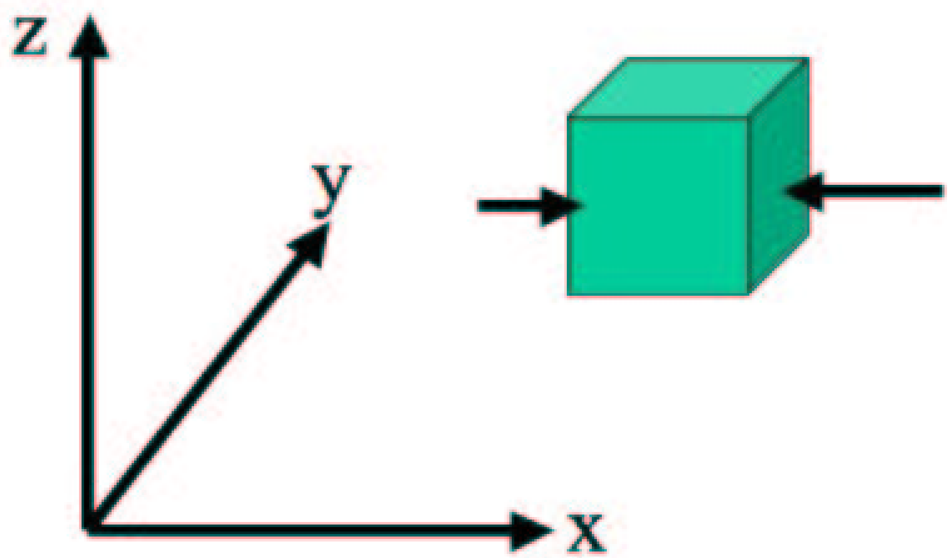

Figure 2: Schematic diagram showing a fluid element in a coordinate fixed in space, which does not rotate with the earth.

- (i) Consider first the pressure forces acting on the fluid element as illustrated in Fig 2.

  The pressure force on the element in the x-direction will be $+p\delta y\delta z\mathbf{i}$ on the left, and $-(p+\delta p)\delta y\delta z\mathbf{i}$ on the right.

  The net pressure force in the x-direction is:

  $\mathbf{i}(p\delta y\delta z - (p+\delta p)\delta y\delta z) = -\delta p\delta y\delta z\mathbf{i} = -\mathbf{i}(\frac{\partial p}{\partial x})\delta x\delta y\delta z,$

  Similarly, the net force in the y-direction is:

  $-\mathbf{j}(\frac{\partial p}{\partial y})\delta x\delta y\delta z,$

  and in the z-direction:

  $-\mathbf{k}(\frac{\partial p}{\partial z})\delta x\delta y\delta z.$

  Therefore, the total pressure force acting on the fluid element is:

  $-\mathbf{\nabla} p\delta x\delta y\delta z,$

  where $\mathbf{\nabla} = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$

- (ii) The gravitational forcing acting onto fluid element is given by:

  $-\mathbf{k}\rho g\delta x\delta y\delta z = \rho\mathbf{g_f}\delta x\delta y\delta z.$

  $\mathbf{g_f}$ is acceleration of gravity relative to a fixed frame, say stars.

- (iii) Let us denote the effects of friction and other forces as $\mathbf{F}^*$.

The Newton's law yields, for a unit mass:
$\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{d\mathbf{V}'}{dt}$,

$$\frac{d\mathbf{V}'}{dt} = \frac{-\nabla p\delta x\delta y\delta z + \rho\mathbf{g_f}\delta x\delta y\delta z + \mathbf{F}^*}{\rho\delta x\delta y\delta z} = -\frac{1}{\rho}\nabla p + \mathbf{g_f} + \mathbf{F}, \tag{2}$$

where $\mathbf{F} = \mathbf{F}^*/(\rho\delta x\delta y\delta z)$ is force per unit mass, $\mathbf{V}'$ is velocity relative to a fixed frame.

This is the equation of motion for a fluid element relative to a fixed Cartesian coordinate system in space.

Now the earth is rotating, so it is more convenient for us as observers on earth to express the equations of motion in terms of a coordinate system that is rotating with the earth. (Motion relative to the earth.)

General GFD theory. For a coordinate system that is rotating with the earth, we must apply the following transformation:
$(\frac{d\mathbf{V}}{dt})_e = (\frac{d\mathbf{V}'}{dt})_f - 2\mathbf{\Omega}\times\mathbf{V} - \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{R})$,
where $\mathbf{\Omega}$ =angular velocity of the earth. $\mathbf{R}$ = vector distance of fluid parcel from center of earth, $\mathbf{V}$ = velocity of fluid parcel relative to the earth.

The above relation is derived as follows.
$(\frac{d\mathbf{R}}{dt})_f = (\frac{d\mathbf{R}}{dt})_r + \mathbf{\Omega}\times\mathbf{R}$. That is:
$\mathbf{V}' = \mathbf{V} + \mathbf{\Omega}\times\mathbf{R}$.
So, $(\frac{d\mathbf{V}'}{dt})_f = \frac{d}{dt}(\mathbf{V}+\mathbf{\Omega}\times\mathbf{R})_r + \mathbf{\Omega}\times(\mathbf{V}+\mathbf{\Omega}\times\mathbf{R})_r = \frac{d\mathbf{V}}{dt} + 2\mathbf{\Omega}\times\mathbf{V} + \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{R}$.
Therefore,
$(\frac{d\mathbf{V}'}{dt})_f = (\frac{d\mathbf{V}}{dt})_e + 2\mathbf{\Omega}\times\mathbf{V} + \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{R}$. In the above, subscript "f" denotes fixed frame, and subscript "r" denotes rotating frame.

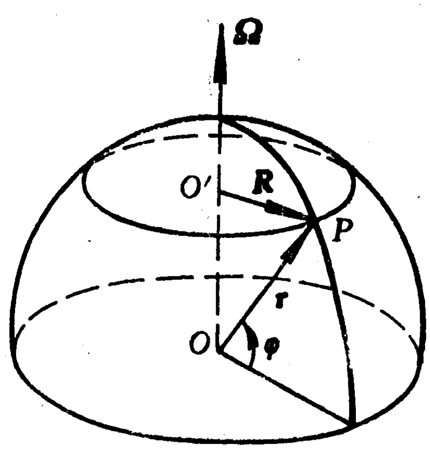

Figure 3: The fixed and rotational frame of the Earth's coordinate system

Substituting for $(\frac{d\mathbf{V}'}{dt})_f$ in equation (2), we have:

$$(\frac{d\mathbf{V}}{dt})_e = -\frac{1}{\rho}\nabla p - 2\mathbf{\Omega}\times\mathbf{V} + \mathbf{g_f} - \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{R}) + \mathbf{F}. \tag{3}$$

Each term in the above equation from left to right is:

Acceleration of unite mass (left hand side of the equation), PGF, Coriolis force, Gravity, centripetal force, friction and other forces. $\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{R})$ is only 0.3% of $\mathbf{g_f}$, so it has small effect and is combined with $\mathbf{g_f}$ to give $\mathbf{g}$.

Consider a local, rotating, Cartesian coordinate system on the surface of the earth with origin at latitude ϕ, we can then write the components of the equations of motion in a rotating frame in which $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ and $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\Omega sin\phi v - 2\Omega cos\phi w + F_x, \tag{4a}$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - 2\Omega sin\phi u + F_y, \tag{4b}$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + 2\Omega cos\phi u - g + F_z. \tag{4c}$$

In general, as we will show, the Coriolis terms involving $cos\phi$ are small and can be neglected. The total derivative (Lagrangian form) $\frac{d}{dt}$ can be expressed in Eulerian form: $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$. The left hand side of the above equations can be written in Eulerian form.

$$\frac{\partial u}{dt} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\Omega sin\phi v - 2\Omega cos\phi w + F_x, \tag{5a}$$

$$\frac{\partial v}{dt} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - 2\Omega sin\phi u + F_y, \tag{5b}$$

$$\frac{\partial w}{dt} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + 2\Omega cos\phi u - g + F_z. \tag{5c}$$

We also denote $f = 2\Omega sin\phi$=Coriolis parameter.

3.2 Frictional effects.

From GFD class ATOC 5400. If molecular viscosity were the only form of dissipation, the straining rates and shear rates acting on the flow are given by:

$\nu\frac{\partial u}{\partial x}$; $\nu\frac{\partial v}{\partial y}$; $\nu\frac{\partial w}{\partial z}$,

where ν is kinematic viscosity, and is in the order of $10^{-6}m^2/s$. So the rate of change of momentum per unit mass is given by:

$\frac{\partial}{\partial x}(\nu\frac{\partial u}{\partial x})$, $\frac{\partial}{\partial y}(\nu\frac{\partial v}{\partial y})$, and $\frac{\partial}{\partial z}(\nu\frac{\partial w}{\partial z})$.

In physical oceanography we are interested in the scales of motion that are much larger than molecular scales (greater than 1–10km). The nonlinear terms in the momentum equations can amplify small disturbances through the processes of barotropic and baroclinic instabilities, producing eddies. These eddies dissipate momentum, heat, salinity, and energy.

Following the classic work of Reynolds, the eddies can exert "stresses" and "strains" on the ocean circulation in much the same way as molecular viscosity. So we express dissipation by eddies as:

$F_x = \frac{\partial}{\partial x}(A_x \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(A_y \frac{\partial u}{\partial y}) + \frac{\partial}{\partial z}(A_z \frac{\partial u}{\partial z})$,

$F_y = \frac{\partial}{\partial x}(A_x \frac{\partial v}{\partial x}) + \frac{\partial}{\partial y}(A_y \frac{\partial v}{\partial y}) + \frac{\partial}{\partial z}(A_z \frac{\partial v}{\partial z})$,

and

$F_z = \frac{\partial}{\partial x}(A_x \frac{\partial w}{\partial x}) + \frac{\partial}{\partial y}(A_y \frac{\partial w}{\partial y}) + \frac{\partial}{\partial z}(A_z \frac{\partial w}{\partial z})$,

where (A_x,A_y) are coefficients of horizontal eddy viscosity. 10–$10^5 m^2/s$.

A_z is coefficient of vertical eddy viscosity. $10^{-5} - 10^{-1} m^2/s$.

You may have seen these forms in many numerical models.

3.3 Continuity of volume–mass conservation.

In the previous class, we derived the equations of motion for zonal, meridional, and vertical momentum. So we obtained 3 equations for u, v, w but we have 5 unknowns: u, v, w, ρ, p. We need two more equations in order to find solutions to the equation set. [In order to find the solutions, the number of equations should be the same as the number of unknowns.]

Consider the flow through a fixed volume in space as indicated in Fig 4.

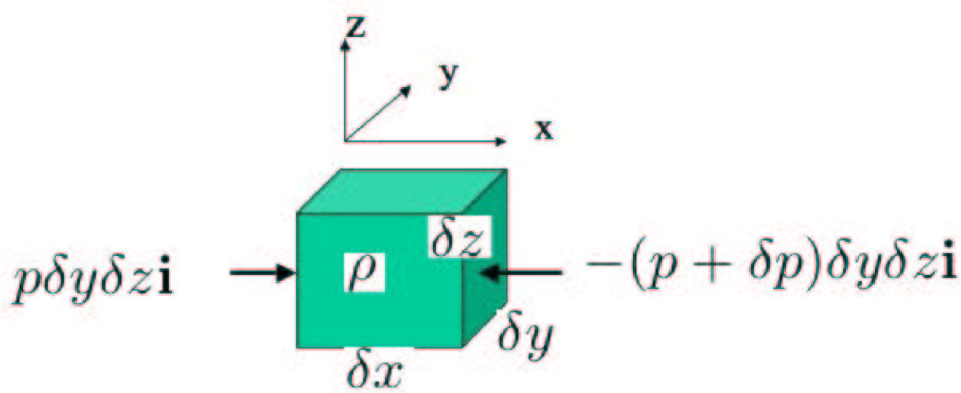

Figure 4: Flow through a fixed volume in space.

In x-direction, the rate of mass that flows into the volume is:

$\rho u \delta y \delta z$ (kg/s).

The mass flows out of the volume is:

$(\rho + \delta\rho)(u + \delta u)\delta y\delta z = (\rho + \frac{\partial\rho}{\partial x}\delta x)(u + \frac{\partial u}{\partial x}\delta x)\delta y\delta z$.

To the first order (ignore the second order term), the net mass that flows into the volume is: Mass in–Mass out:

$-[u\frac{\partial\rho}{\partial x} + \rho\frac{\partial u}{\partial x}]\delta x\delta y\delta z = -\frac{\partial(\rho u)}{\partial x}\delta x\delta y\delta z$.

Similarly, we can derive the net mass that flows out of the box in both y and z directions. So we obtain the total net mass that flows out of the volume:

$-[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}]\delta x\delta y\delta z$.

The net mass in or out must cause mass change in the parcel. That is:

$\frac{\partial\rho}{\partial t}\delta x\delta y\delta z = -[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}]\delta x\delta y\delta z$,

$\frac{\partial\rho}{\partial t}\delta x\delta y\delta z + [\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}]\delta x\delta y\delta z = 0.$
Thus,

$$\frac{\partial\rho}{\partial t} + [\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}] = 0. \tag{6}$$

This is one form of the mass continuity equation.
Rearranging the above equation and using
$\boldsymbol{\nabla}\cdot\mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$, which is called "divergence",
$\mathbf{V}\cdot\nabla\rho = u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}$, which is referred to as advection of ρ,
we have:
$\frac{\partial\rho}{\partial t} + \rho\boldsymbol{\nabla}\cdot\mathbf{V} + \mathbf{V}\cdot\nabla\rho = 0.$
Or,

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla\cdot\mathbf{V} = 0. \tag{7}$$

If the fluid is incompressible, then $\frac{1}{\rho}\frac{d\rho}{dt} = 0$, and thus

$$\nabla\cdot\mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{8}$$

Incompressibility is a useful approximation in oceanography.

Now we have 4 equations but 5 unknowns. We need one more equation to close the equation group. Equation of state: $\rho = \rho(T, S, P)$. Then we need to write down the equations for **T and S**. We will not discuss the T and S equations here, but will discuss them later in the thermodynamics section.

3.4 Scaling.

The equations of motion are complicated and nonlinear and do not possess general analytic solutions. We'll have to solve them numerically.

For certain motions, however, only a few terms in the equations are important, and others play a secondary role. In order to understand the underlying physics that causes the motion, we wish to keep only the important terms and thus simplify the equations.

To do so, we will use observations of the real ocean to estimate the size of each term.

We will consider first, the main body of the ocean away from the surface and side boundaries, and this is usually called ocean "interior". We will see, to the lowest order, what equations the large-scale motion in the interior ocean satisfy.

Scales: Pacific Ocean width–12,000km across; Atlantic–6000km. So, here we choose horizontal length scale

$L = 1000km = 10^6 m$ as representative of the large scale ocean circulation.

Current speed in the interior ocean: $U = 0.1m/s$,

Vertical scale $H = 10^3 m.$

Scale for time T=L/U=10^7 (10 days);—(gravity waves have time scales of days–filtered out by this scaling!)

Scale for w: $\frac{\partial w}{\partial z} = -(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$ [from continuity equation for incompressible fluid).

Then $\frac{W}{H} - \frac{U}{L}$, $W - UH/L - 10^{-4}$. [In fact w scale is much smaller than this scale, since for large scale motion in the ocean interior, $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ generally compensates each other, so that their sum $(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$ is much smaller than each individual. Additionally, vertical motions in the ocean have scales much smaller than 1000m]. Since $W << U$, the interior ocean circulation is generally horizontal–quasi-two-dimensional.

Now Assume the following scales according to the observations:

$L - 10^6 m$,
$U - 0.1 m/s$,
$H - 10^3 m$,
$W - 10^{-4} m/s$,
$T - L/U - -10^7 s$,
$f = 2\Omega sin 45° - 10^{-4}$, [$f$ is Coriolis parameter]
$\rho - 10^3 kg/m^3$,

P: We can let P be undetermined first, since usually there are no direct observations for P. [Note: The P in CTD cast is used to measure depth.]

$A_x, A_y - 10^5 m^2/s$–upper bound,
$A_z - 10^{-1} m^2/s$–upper bound.

Let's consider w equation first.

$$\frac{\partial w}{dt}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z}+2\Omega cos\phi u-g+\frac{\partial}{\partial x}(A_x\frac{\partial w}{\partial x})+\frac{\partial}{\partial y}(A_y\frac{\partial w}{\partial y})+\frac{\partial}{\partial z}(A_z\frac{\partial w}{\partial z}). \quad (9)$$

In the above equation, the nonlinear, advection terms are often referred to as "inertial terms".

The scales for each term are:
$\frac{W}{T} + \frac{UW}{L} + \frac{VW}{L} + \frac{WW}{H} = -\frac{1}{\rho}\frac{P}{H} + fU - g + A_x\frac{W}{LL} + A_y\frac{W}{LL} + A_z\frac{W}{HH}$,
Substituting the numbers listed above, we have:
$10^{-11} \quad +10^{-11} \quad +10^{-11} \quad +10^{-11} \quad = \quad ?? + 10^{-5} \quad +10 \quad +10^{-11} \quad +10^{-11} \quad + 10^{-11}$.

So we can see immediately that the pressure gradient term must balance gravity term since all other terms are a few orders of smaller.

Therefore, we obtain $P - \rho g H - 10^7$.

Thus, the dominant terms are: pressure gradient force and the gravity.

$$-\frac{1}{\rho}\frac{\partial P}{\partial z} - g = 0, \quad (10)$$

Or:

$$\frac{\partial P}{\partial z} = -\rho g. \quad (11)$$

That is, to the lowest order, the vertical pressure gradient force is balanced by gravity. It is called "Hydrostatic equation".

Similarly, we can obtain the scaling for zonal and meridional momentum equations.

Consider the zonal component of the momentum, and assume the eddy coefficients are constants:

$\frac{\partial u}{dt} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\Omega sin\phi v - 2\Omega cos\phi w + A_x\frac{\partial^2 u}{\partial x^2} + A_y\frac{\partial^2 u}{\partial y^2} + A_z\frac{\partial^2 u}{\partial z^2}$.
If we ignore the mixing, that is, $A_x = 0$, $A_y = 0$, and $A_z = 0$, we call the above equation "inviscid".

$$\frac{U}{T} + \frac{U^2}{L} + \frac{U^2}{L} + \frac{WU}{H} = ?? + fU + 2\Omega cos\phi W + \frac{A_x U}{L^2} + \frac{A_y U}{L^2} + \frac{A_z U}{H^2},$$

$$10^{-8} + 10^{-8} + 10^{-8} + 10^{-8} = ?? + 10^{-5} + 10^{-8} + 10^{-8} + 10^{-8} + 10^{-8}$$

Obviously, the pressure gradient term must balance the Coriolis term, which is a few orders larger than all other terms.

To the lowest order, we have:
$-\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\Omega sin\phi v = 0$, or:

$$fv = \frac{1}{\rho}\frac{\partial P}{\partial x}. \tag{12}$$

This is geostrophic balance for meridional current v.

Similarly, we obtain the geostrophic balance for zonal current u by scaling the meridional momentum equation (assignment #2):

$$fu = -\frac{1}{\rho}\frac{\partial P}{\partial y}. \tag{13}$$

Thus, to the lowest order, the motion in the ocean interior satisfies "geostrophic balance" and "hydrostatic balance".

Therefore, if we have the CTD data, we can obtain the "geostrophic" current between two station pairs, which is a good estimation for the interior oceanic current.

3.5 The Rossby Number and Ekman Number.

(a).The Rossby Number.

Given the dominant role played by the Coriolis force in the momentum equations, it can be used as a measure of the importance for other terms. Consider the ratio of a typical nonlinear term (so-called "inertial term") and the Coriolis term in the horizontal momentum equations,

$$\frac{Inertial}{Coriolis} = \frac{U^2}{L} \times \frac{1}{fU} = \frac{U}{fL} = R_o.$$

R_o is a measure of the relative importance of the inertial effects and rotational effects on the flow, and is called the "Rossby Number". For the ocean interior,
$R_o = \frac{10^{-1}}{10^{-4} \times 10^6} = 10^{-3} << 1$. Therefore, rotational effects dominate inertial effects.

(b). The Ekman Number.

Consider now the ratio of the eddy viscosity terms and the Coriolis term, so that:

$$\frac{Frictional}{Coriolis} = \frac{A_x U}{L^2} \times \frac{1}{fU} = \frac{A_x}{fL^2} = E_x.$$

Or $E_y = \frac{A_y}{fL^2}$ and $E_z = \frac{A_z}{fH^2}$.

E_x and E_y are called the horizontal "Ekman numbers", and E_z is the "vertical Ekman number". For the ocean interior:

$E_x - \frac{10^5}{10^{-4} \times 10^{12}} - 10^{-3} << 1$,

$E_y - 10^{-3} << 1$,

$E_z - 10^{-3} << 1$. [Upper bounds used for A_x, A_y, and A_z].

In the ocean interior, rotational effects dominate frictional effects.