Poly-Time Reductions

Design and Analysis of Algorithms
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Poly-Time Reductions

There are 3 types of problems:

- easy (polynomial time)
- hard (provably super polynomial time)
- we don't know

A convenient way to classify problems in the "grey zone" is polynomial time reduction

A problem Y is poly-time reducible to a problem X if there is an algorithm that solves any instance of Y making polynomially many elementary operations and polynomially many calls to a black-box solving X

Denoted $Y \le X$

Poly-Time Reductions (cntd)

Lemma

Suppose $Y \leq X$. If X can be solved in polynomial time, then Y can be solved in polynomial time

Lemma

Suppose $Y \leq X$. If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

Independent Set

A set of vertices is said to be independent, if no two of them are connected with an edge

The Independent Set Problem

Instance:

A graph G and a number k

Objective:

Does G contain an independent set of size k?

Optimization vs. decision version

Vertex Cover

A set of vertices is said to be a vertex cover if every edge of the graph has at least one end in it

The Vertex Cover Problem

Instance:

A graph G and a number k

Objective:

Does G contain a vertex cover of size k?

Independent Set vs. Vertex Cover

Lemma

Let G = (V,E) be a graph. Then S is an independent set if and only if its complement V – S is a vertex cover.

Proof

Obvious

Theorem

Independent Set ≤ Vertex Cover

and

Vertex Cover ≤ Independent Set

Satisfiability

Boolean variable is a variable that takes two values 0 and 1

Literal is a Boolean variable or its negation x or $\bar{x}$

Clause is a disjunction of literals. The clause has length k if it contains k literals

CNF is a conjunction of clauses

A k-CNF is a CNF in which every clause has length at most k

Let X be a set of Boolean variables

A truth assignment is a function $\nu: X \to \{0,1\}$

The assignment ν satisfies a clause C if it causes C to evaluate to 1

The assignment ν satisfies a CNF if it satisfies every clause in it.

Satisfiability (cntd)

The Satisfiability Problem

Instance:

A CNF Φ

Objective:

Is Φ satisfiable? That is, does there exist an assignment to variables of Φ that satisfies Φ?

The k-Satisfiability Problem

Instance:

A k-CNF Φ

Objective:

Is Φ satisfiable? That is, does there exist an assignment to variables of Φ that satisfies Φ?

3-SAT vs. Independent Set

Theorem

3-SAT ≤ Independent Set

Proof

We can view the Satisfiability problem as follows:

- pick one literal from each clause,
- select an assignment that satisfies all selected literals
  (such an assignment will satisfy all clauses, and so the CNF)
- make sure that there are no conflicts, that is, you do not pick x from one clause and $\bar{x}$ from another

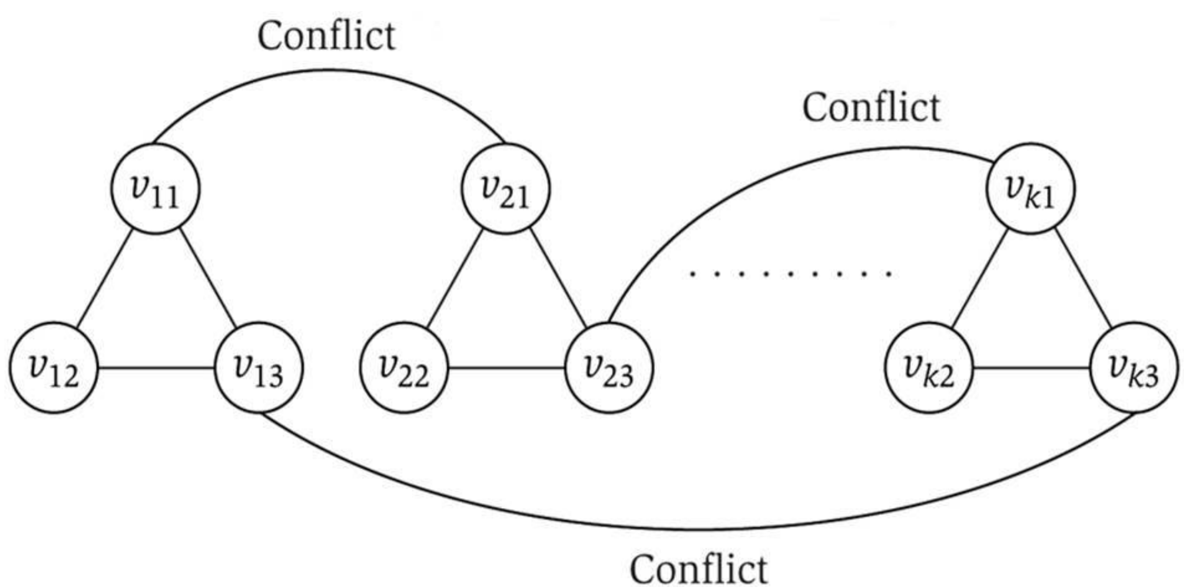
3-SAT vs. Independent Set (cntd)

The idea is to encode a CNF as a graph, and satisfying assignments as independent sets

Let a 3-SAT instance contains variables $x_1,\dots,x_n$ and clauses $C_1,\dots,C_k$

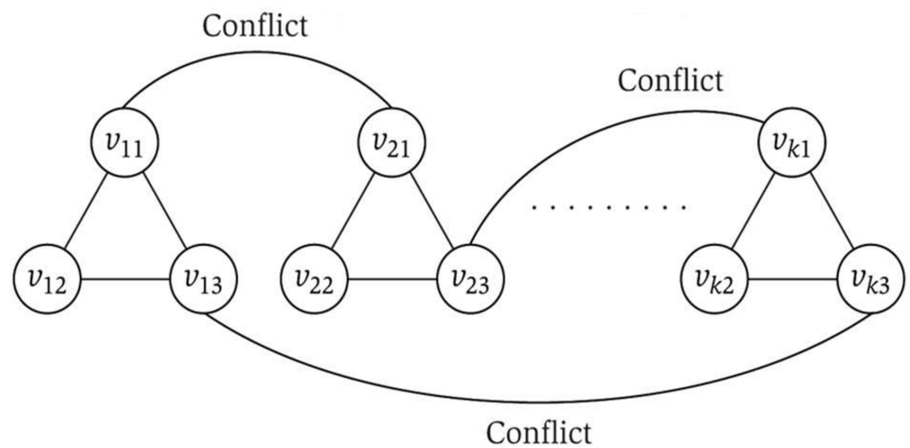
Graph G = (V,E) consists of 3k vertices grouped into k triangles:

v_{i1}, v_{i2}, v_{i3} so that v_{ij} corresponds (or labeled) with j-th literal of C_i



3-SAT vs. Independent Set (cntd)

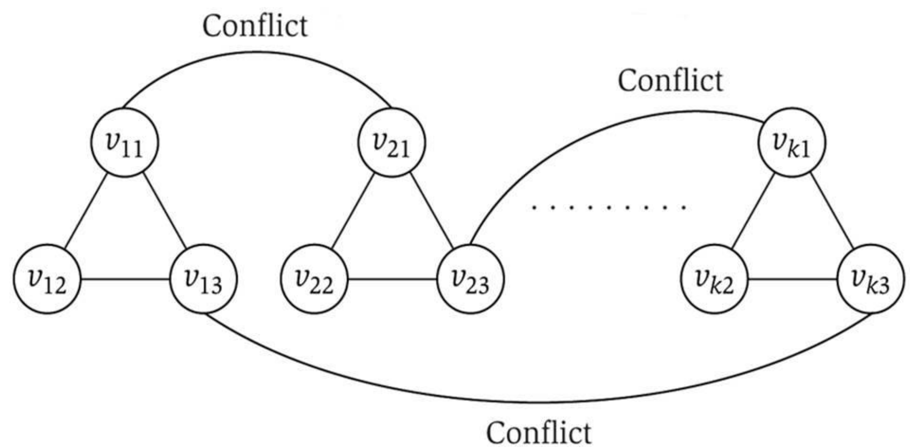
For each pair of vertices whose labels are in conflict we add an edge



We show that the CNF has a satisfying assignment if and only if G has an independent set of size k

At most one vertex in each triangle can be in an independent set, so the size of a set cannot be more than k

3-SAT vs. Independent Set (cntd)



If there is a satisfying assignment,
there is a satisfied literal in each clause (triangle).

Pick such a literal and include it into an independent set

As there are no conflicts, it is really an independent set

If there is an independent set S of size k, every triangle contains a vertex from S

Choose an assignment so that all literals – labels of vertices from S – are satisfied.

It is possible, as they are not involved in any conflict

And it is a satisfying assignment