On the Estimation of Outstanding Claims

Walther Neuhaus

Gabler & Partners AS
P.O. Box 1818 Vika
N-0123 Oslo, Norway.
Tel. (47) 24130757
Fax. (47) 24130701
Email walther.neuhaus@gablerpartners.no

Abstract

This paper presents a discrete time model for the estimation of outstanding claims that comprises delay in two dimensions: reporting delay and valuation delay. This model allows a strict distinction between the cost of reported claims and the cost of unreported claims.

Keywords

Outstanding claims, Loss reserving, Credibility, IBNR, RBNS.

1. Introduction

Estimation of outstanding claims is an essential part of actuarial work in general insurance. Due to the nature of general insurance contracts and the claim settlement process, almost any actuarial task must address the question: have outstanding claims been taken into account?

Most of the common actuarial methods for estimating the cost of outstanding claims involve extrapolation of a two-dimensional development triangle. The row or vertical dimension of the development triangle is normally the accident year or underwriting year, and the column or horizontal dimension is the delay between the accident year (underwriting year) and successive valuation dates. The developing quantity, which is subject to modelling and prediction, is usually one of the following: the number of reported claims, the accumulated claim payments, or the amount of reported incurred claims. For a survey of traditional actuarial methods for loss reserving, see Taylor (2000).

While the two-dimensional models may be effective tools to predict the outstanding cost of claims per accident year, they do not allow the actuary to make a strict distinction between the outstanding cost of claims that are reported but not settled (RBNS), and claims that are incurred but not reported (IBNR). The reason for this failing is that claim development between two valuation dates comprises two separate types of development: changes in the assessment of reported incurred claims, and reports of new claims that are received by the insurer.

An explicit distinction between reported and unreported claims is made by Arjas (1989), who provides a structural framework for claim reserving but no operational models. Arjas' framework forms the basis of papers by Haastrup & Arjas (1996) and Norberg (1999a, 1999b), who also provide sketches of operational models. Implementing those models may still be a formidable task, as they are formulated in continuous time.

This paper presents a discrete time model that comprises delay in two dimensions: delay between the accident year and the reporting year (hereafter called the reporting delay), and delay between the reporting year and the valuation year (hereafter called the valuation delay). This model allows a strict distinction between the cost of reported claims and the cost of unreported claims.

The main features of model follow Norberg (1986, 1999a, 1999b). The occurrence of accidents is modelled by a mixed Poisson process, with a possibility for modelling serial correlation between accident frequencies in consecutive years. The reporting delay is assumed to be governed by a fixed pattern of delay probabilities. The severity of individual claims is assumed to be independent of the claim number process, and the model allows for a stochastic dependence between the reporting delay and the claim severity. The process of partial payments and reassessments is made conditional on the severity of claims reported.

The proposed predictors of outstanding claim cost are of the credibility-weighted form, which includes as limiting cases, the Chain ladder method and the Bornhuetter-Ferguson method. It is not the purpose of this paper to introduce new credibility models, but to show how the existing ones can be exploited in an integrated model.

To make the model operational one needs to quantify, subjectively or by estimation, several sets of fixed parameters. This paper addresses the problem of estimating those parameters only cursorily. My main concern is to argue that estimation of outstanding claims should be conducted using three, rather than only two time dimensions.

A three-dimensional model was first proposed and analysed by Ørsted (1999), who developed Kalman-filter techniques to update its estimates. The model played a role in the recognition of, and subsequent recovery from the Norwegian Workers' Compensation debacle of the mid-1990s. At that time the ultimate cost of Workers' Compensation insurance claims still was very uncertain. Being able to separate claims IBNR from claims RBNS and to show convincingly that the cost of claims RBNS was likely to escalate far beyond what most people expected, and with it the cost of claims IBNR, was crucial to gaining acceptance for the dire actuarial predictions.

2. A model of claim development – the observables

Conforming with standard actuarial terminology, the discrete time periods will be called "years" throughout this paper. In practice it is entirely possible and usually advisable to build the model with shorter time periods (quarters or months). The initial investment in doing so is more than compensated by the facility with which one can calculate updated estimates at shorter time intervals and using a consistent set of assumptions. Now let us get on with "years".

We denote accident years by *j*. For an accident year *j*, we denote the amount of risk exposed by p_j. The number of claims reported with delay *d* is denoted by N_{jd}. The individual severities of those claims we denote by $\{Y_{jd}^{(k)} : k = 1, \cdots, N_{jd}\}$ and their sum as Y_{jd}. For a given claim its ultimate severity $Y_{jd}^{(k)}$ is made up of a series of partial payments $U_{jdt}^{(k)}$ that occur at delay *t* after the reporting date:

$$(2.1) \quad Y_{jd}^{(k)} = \sum_{t=0}^{\infty} U_{jdt}^{(k)} .$$

In addition to partial payments we may observe outstanding case estimates. Denote by $V_{jdt}^{(k)}$ the change in the outstanding case estimate at delay *t* after the reporting date. Finally, let $W_{jdt}^{(k)} = U_{jdt}^{(k)} + V_{jdt}^{(k)}$ denote the change in the reported incurred claim cost. Note that

$$(2.2) \quad Y_{jd}^{(k)} = \sum_{t=0}^{\infty} U_{jdt}^{(k)} = \sum_{t=0}^{\infty} W_{jdt}^{(k)} ,$$

which states the obvious fact that the total change in the outstanding case estimate from the time when the claim is reported to the time when it is settled, is zero.

Now assume that the last calendar year and the current valuation date is *J*. At that time we will have recorded the reported number of claims $\{N_{jd} : j = 1, \cdots, J, \, d = 0, \cdots, J - j\}$, while $\{N_{jd} : j = 1, \cdots, J, \, d > J - j + 1\}$ will still be unreported. The only partial payments that we have had the chance to observe are those for which $j + d + t \le J$. The accumulated payments to the end of year *J* are

$$(2.3) \quad U_{jd, \le J-(j+d)}^{(k)} = \sum_{t=0}^{J-(j+d)} U_{jdt}^{(k)} ,$$

with corresponding formulas for the current outstanding case estimate and current reported incurred claim cost. The outstanding payments in respect of claims RBNS are

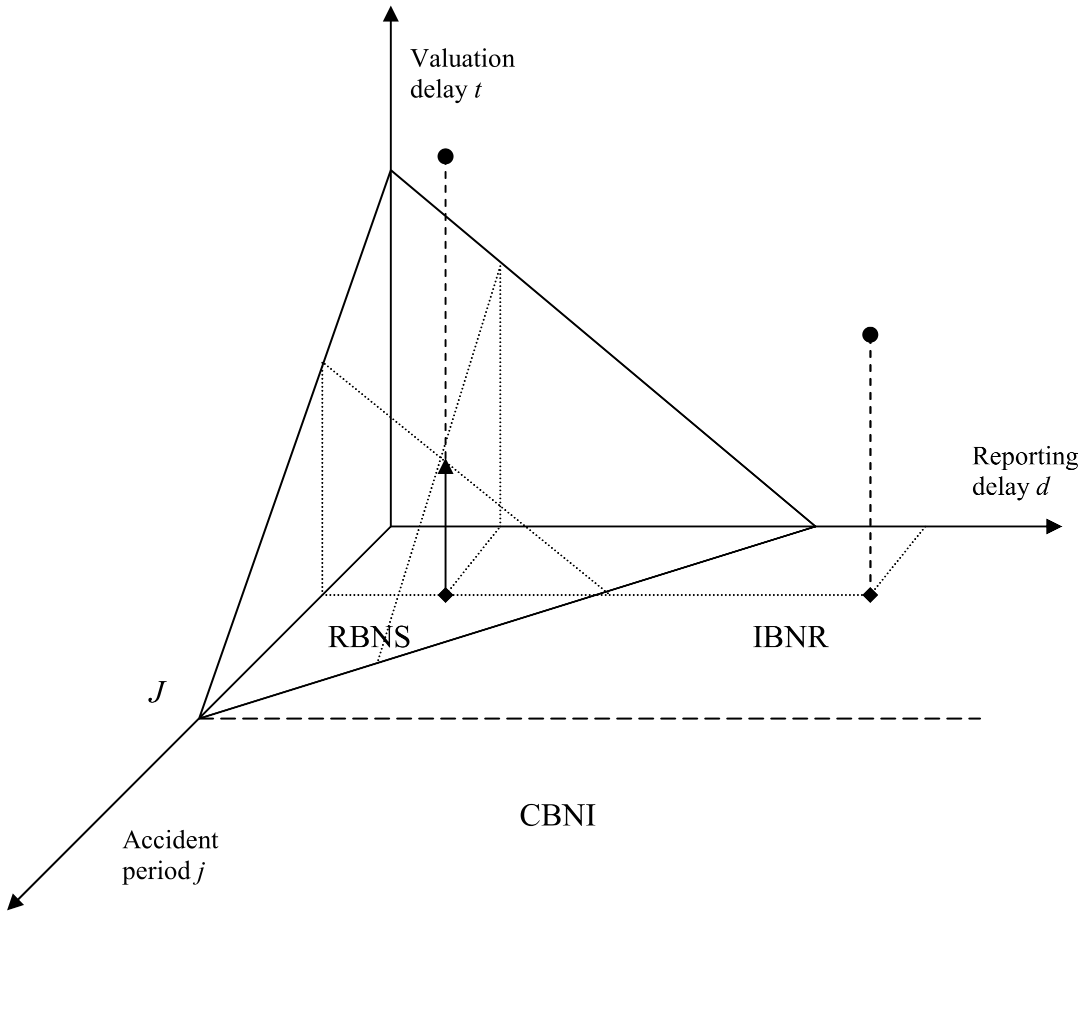
$$(2.4) \quad \mathrm{RBNS}_J = \sum_{j=1}^{J} \sum_{d=0}^{J-j} \sum_{t=J-(j+d)+1}^{\infty} U_{jdt} ,$$

and the future cost of claims IBNR is

$$(2.5) \quad \mathrm{IBNR}_J = \sum_{j=1}^{J} \sum_{d=J-j+1}^{\infty} \sum_{t=0}^{\infty} U_{jdt} .$$

The development tetrahedron (below) illustrates the three dimensions of claim development. Claims that are RBNS at time J have been reported inside the horizontal triangle given by $j + d \leq J$, as indicated by a diamond. The observed development of a reported claim is indicated by a solid vertical line lying inside the tetrahedron which is delimited by $j + d + t \leq J$, and its future development is indicated by the dotted extension of that line. The development of a claim ends at settlement, indicated by a bullet. A claim that is IBNR starts its observed development outside the horizontal triangle and its development lifeline is dotted all the way to settlement, or course. The current status of reported claims can be "read off" on the simplex given by $j + d + t = J$.

Figure 1. The development tetrahedron



The following abbreviation will be used in the rest of this paper: a variable with a subscript omitted denotes the sum of the underlying variables across all values of the subscript that has been omitted. A variable with a subscript replaced by an inequality (e.g., $U_{jd,\le J-(j+d)}$) denotes the sum of the underlying variables that satisfy the inequality. A variable with a subscript replaced by $\le$ is usually the sum of the underlying variables that lie inside the tetrahedron, while a variable with a subscript replaced by > is the sum of the underlying variables that lie outside the tetrahedron.

3. A model of claim development – stochastic assumptions

Having defined the necessary notation for the observed quantities, let us now sketch out a stochastic model of their behaviour and interactions. More specific assumptions will be proposed in later sections.

We assume that conditional on unknown claim frequencies $\{\Theta_j : j = 1,\cdots,J\}$, the claim numbers N_{jd} are independent random variables, each with a Poisson distribution,

(3.1) $N_{jd} \mid \Theta_j = \theta_j \sim \text{Poisson}(p_j \theta_j \pi_d)$,

with fixed, non-negative delay probabilities $\{\pi_d : d = 0,1,\cdots\}$ that add to one. The evolution of claim frequencies will be governed by some or other stochastic process. Denote the mean of the vector $\mathbf{\Theta}_J = (\Theta_1,\cdots,\Theta_J)'$ by $\boldsymbol{\tau}_J$ its covariance matrix by $\mathbf{\Lambda}_J$.

The severities of individual claims reported in year *j*+*d* in respect of accidents incurred in year *j* we denote by $\{Y_{jd}^{(k)} : k = 1,\cdots,N_{jd}\}$. We assume that $Y_{jd}^{(k)}$ are independent random variables with a distribution G_d that may depend on the reporting delay *d*. We also assume that the severities are independent of the claim counts. Denote the mean and variance of $Y_{jd}^{(k)}$ by ξ_d and σ_d^2, and let $\rho_d = \sigma_d^2 + \xi_d^2$ denote the non-central second order moment.

Until such time as all claims are finally and irrevocably settled, the aggregate severity Y_{jd} of the claims $\{Y_{jd}^{(k)} : k = 1,\cdots,N_{jd}\}$ will be an unknown and must be estimated. Denote the unknown average severity by $\Xi_{jd} = Y_{jd} / N_{jd}$ (zero if $N_{jd} = 0$). Conditionally on the number of claims, the average severity Ξ_{jd} has mean ξ_d and variance σ_d^2 / N_{jd}.

To model the development of partial payments $\{U_{jdt} : t = 0,1,\cdots\}$ conditionally on the number of claims and the unknown average severity, an obvious candidate is the Dirichlet distribution. Thus we will assume that

$$(3.2)\quad (U_{jd0}, U_{jd1}, \cdots) \mid N_{jd}, \Xi_{jd} \quad \sim \quad \text{Dirichlet}(\alpha_0, \alpha_1, \cdots) \times N_{jd}\Xi_{jd},$$

with non-negative fixed parameters $\alpha_0, \alpha_1, \cdots$.

The Dirichlet distribution not suited to model the conditional development of reported incurred claims $\{W_{jdt} : t = 0,1,\cdots\}$, because its increments are strictly non-negative, while the increments of reported incurred claims may be negative. Therefore we will propose a model for the development of reported incurred claims, where the $\{W_{jdt} : t = 0,1,\cdots\}$ are conditionally independent, given the number of claims and the unknown average severity, and where W_{jdt} is a compound Poisson random variable with a frequency parameter that is proportional to $N_{jd}\Xi_{jd}$ and a jump size distribution H_t that allows negative jumps:

$$(3.3)\quad W_{jdt} \mid N_{jd}, \Xi_{jd} \quad \sim \quad \text{Compound Poisson } (N_{jd}\Xi_{jd}, H_t)$$

The assumptions that have been sketched above will be utilised in the sections that follow. One more assumption must be mentioned, being that for every reported claim its development (consisting of its partial payments, outstanding case estimates and ultimate severity) is stochastically independent of everything else, i.e. claim numbers, underlying claim frequencies, and the development of all other claims. This assumption is a consequence of the marked Poisson process assumption of Norberg (1999a, 1999b). It allows us to predict the amount of claims IBNR by predicting their number, and to predict separately the development of each cohort of claims RBNS that have been reported at time $j+d$ in respect of accident year j. If you need a stringent formulation, see Norberg's papers.

4. Estimation of the number of claims IBNR

4.1 General formulation

We start with a general formulation, using the notation defined in the two previous sections. Define the diagonal matrix

$$(4.1)\quad \mathbf{V}_J = \begin{bmatrix} p_1\pi_{\le J-1} & 0 & \cdots & 0 \\ 0 & p_2\pi_{\le J-2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & p_J\pi_{\le 0} \end{bmatrix}$$

At any time J, the vector of reported claim counts $\mathbf{N}_J = (N_{1,\le J-1}, \cdots, N_{J,\le 0})'$ is linearly regressed on the vector $\mathbf{\Theta}_J = (\Theta_1, \cdots, \Theta_J)'$ of claim frequencies through the equation

(4.2) $\quad \mathrm{E}(\mathbf{N}_J \mid \mathbf{\Theta}_J) = \mathbf{V}_J \cdot \mathbf{\Theta}_J ,$

and has a covariance matrix given by

(4.3) $\quad \mathrm{Cov}(\mathbf{N}_J \mid \mathbf{\Theta}_J) = \mathbf{V}_J \cdot \mathrm{diag}(\mathbf{\Theta}_J) .$

Using the apparatus of linear greatest accuracy credibility theory, we know that the best linear estimator of $\mathbf{\Theta}_J$ based on the vector of observations $\mathbf{N}_J$ is

(4.4) $\quad \overline{\mathbf{\Theta}}_J = \mathbf{Z}_J \hat{\mathbf{\Theta}}_J + (\mathbf{I} - \mathbf{Z}_J)\boldsymbol{\tau}_J ,$

i.e., it is a credibility-weighted average of the "chain-ladder estimates"

(4.5) $\quad \hat{\mathbf{\Theta}}_J = \mathbf{V}_J^{-1}\mathbf{N}_J = \left(\dfrac{N_{1,\le J-1}}{p_1 \pi_{\le J-1}}, \cdots, \dfrac{N_{J,\le 0}}{p_J \pi_{\le 0}} \right)' ,$

and the prior mean $\boldsymbol{\tau}_J$, where the credibility matrix is

(4.6) $\quad \mathbf{Z}_J = \mathbf{\Lambda}_J \left(\mathbf{\Lambda}_J + \mathrm{diag}(\boldsymbol{\tau}_J) \cdot \mathbf{V}_J^{-1} \right)^{-1} .$

It is relatively easy to verify that the mean squared error matrix of the estimator $\overline{\mathbf{\Theta}}_J$ is

(4.7) $\quad \mathbf{Q}(\mathbf{Z}_J) = \mathrm{E}(\overline{\mathbf{\Theta}}_J - \mathbf{\Theta}_J)(\overline{\mathbf{\Theta}}_J - \mathbf{\Theta}_J)' = \mathbf{Z}_J \mathbf{V}_J^{-1} \mathrm{diag}(\boldsymbol{\tau}_J) \mathbf{Z}_J' + (\mathbf{I} - \mathbf{Z}_J) \mathbf{\Lambda}_J (\mathbf{I} - \mathbf{Z}_J)' .$

The credibility predictor of the number of claims IBNR in respect of accidents incurred in year j, is

(4.8) $\quad \overline{N}_{j,>J-j} = p_j \overline{\Theta}_j \pi_{>J-j} ,$

and its mean squared error is

(4.9) $\quad \mathrm{E}(\overline{N}_{j,>J-j} - N_{j,>J-j})^2 = (p_j \pi_{>J-j})^2 [\mathbf{Q}(\mathbf{Z}_J)]_{jj} + p_j \pi_{>J-j} \tau_j .$

The credibility predictor of the total number of claims IBNR is

(4.10) $\quad \overline{N}_{>} = \sum_{j=1}^{J} p_j \overline{\Theta}_j \pi_{>J-j} ,$

with mean squared error

(4.11) $\quad \mathrm{E}(\overline{N}_{>} - N_{>})^2 = \sum_{j=1}^{J} \sum_{j'=1}^{J} (p_j \pi_{>J-j}) [\mathbf{Q}(\mathbf{Z}_J)]_{jj'} (p_{j'} \pi_{>J-j'}) + \sum_{j=1}^{J} p_j \pi_{>J-j} \tau_j .$

We now turn to estimating the cost of claims IBNR. In the conditional distribution given $\mathbf{\Theta}_J = (\theta_1, \cdots, \theta_J)'$, the amounts $\{Y_{j,>J-j} : j = 1, \cdots, J\}$ of claims IBNR are independent random variables, and $Y_{j,>J-j}$ has a compound Poisson distribution with frequency parameter $p_j\theta_j\pi_{>J-j}$ and a mixed severity distribution (the tail severity distribution)

(4.12) $\overline{G}_{>J-j} = \pi_{>J-j}^{-1} \sum_{d=J-j+1}^{\infty} \pi_d G_d$.

Slightly abusing notation, we let the inequality subscript in conjunction with a bar denote a π − weighted average. The non-central first and second order moments of the tail severity distribution are then

(4.13) $\overline{\xi}_{>J-j} = \pi_{>J-j}^{-1} \sum_{d=J-j+1}^{\infty} \pi_d \xi_d$ and

(4.14) $\overline{\rho}_{>J-j} = \pi_{>J-j}^{-1} \sum_{d=J-j+1}^{\infty} \pi_d \rho_d$.

The credibility predictor of the amount of claims IBNR in respect of accidents incurred in year *j*, is

(4.15) $\overline{Y}_{j,>J-j} = p_j \overline{\Theta}_j \pi_{>J-j} \overline{\xi}_{>J-j}$,

and its mean squared error is

(4.16) $\mathrm{E}\left(\overline{Y}_{j,>J-j} - Y_{j,>J-j}\right)^2 = \left(p_j \pi_{>J-j} \overline{\xi}_{>J-j}\right)^2 \left[\mathbf{Q}(\mathbf{Z}_J)\right]_{jj} + p_j \tau_j \pi_{>J-j} \overline{\rho}_{>J-j}$.

The credibility predictor of the total amount of claims IBNR in respect of all accident years is

(4.17) $\overline{Y}_{>} = \sum_{j=1}^{J} p_j \overline{\Theta}_j \pi_{>J-j} \overline{\xi}_{>J-j}$,

with mean squared error

(4.18) $\mathrm{E}(\overline{Y}_{>} - Y_{>})^2 = \sum_{j=1}^{J}\sum_{j'=1}^{J} (p_j \pi_{>J-j} \overline{\xi}_{>J-j}) \left[\mathbf{Q}(\mathbf{Z}_J)\right]_{jj'} (p_{j'} \pi_{>J-j'} \overline{\xi}_{>J-j'}) + \sum_{j=1}^{J} p_j \pi_{>J-j} \tau_j \overline{\rho}_{>J-j}$.

Having written up general formulas for the credibility predictors and their mean squared error, we will now propose a handful of models for the process $\{\mathbf{\Theta}_J\}_{J=1}^{\infty}$ that can be used to determine the mean $\boldsymbol{\tau}_J$ and the covariance matrix $\mathbf{\Lambda}_J$. The purpose in this paper is not to study these models in any detail, only to show how they fit into the general framework.

4.2 Bühlmann-Straub model

The Bühlmann-Straub model makes the assumption that the single-year accident frequencies $\{\Theta_j\}_{j=1}^{J}$ are independent and identically distributed random variables with a known mean τ

and a known variance λ. In that case one finds that $\boldsymbol{\tau}_J = \tau \cdot \mathbf{1}^{J\times 1}$ and $\boldsymbol{\Lambda}_J = \lambda \cdot \mathbf{I}^{J\times J}$ and easily derives the optimal credibility matrix $\mathbf{Z}_J = \mathrm{diag}\left(\frac{\lambda p_j \pi_{\le J-j}}{\lambda p_j \pi_{\le J-j} + \tau}\right)$ and its mean squared error matrix $\mathbf{Q}(\mathbf{Z}_J) = \tau \cdot \mathbf{Z}_J \mathbf{V}_J^{-1} \mathbf{Z}_J^{'} + \lambda \cdot (\mathbf{I} - \mathbf{Z}_J)(\mathbf{I} - \mathbf{Z}_J)'$. Note that since the optimal credibility matrix is diagonal, each accident year's claim frequency is estimated on the basis of that accident year's claim numbers alone.

4.3 Hierarchical model

One can replace the known mean τ of the Bühlmann-Straub model with an unknown random variable T that has mean τ_0 and variance λ_0 and assume that conditionally on $T = \tau$ the Θ_j are i.i.d. random variables with mean τ and a known variance λ. In that case one finds that $\boldsymbol{\tau}_J = \tau_0 \cdot \mathbf{1}$ and $\boldsymbol{\Lambda}_J = \lambda_0 \cdot \mathbf{11}' + \lambda \cdot \mathbf{I}$. The optimal credibility estimator may be written up explicitly, but in my opinion one may just as well stick to the matrix formulas (4.4)-(4.6).

4.4 Random walk model

The Bühlmann-Straub model stipulates that the claim frequencies are statistically constant, in the sense that each accident year´s claim frequency a priori has the same expected value. It also stipulates that the claim frequencies $\Theta_1, \cdots, \Theta_J$ are independent; therefore in estimating the claim frequency of a specific accident year, nothing can be gained by including data from other accident years. The hierarchical model allows for transfer of information between accident years, but it still retains the underlying assumption that claim frequencies are statistically constant.

In real-life situations, claim frequencies are neither constant nor independent, but rather behave like a correlated time series. A simple assumption that reflects that observation would be that the claim frequencies follow a random walk, $\Theta_j = \Theta_{j-1} + \varepsilon_j$, where $\varepsilon_1, \cdots, \varepsilon_J$ are independent and identically distributed error terms with mean zero and variance λ. Assume also, pro forma, that there exists an initial random variable Θ_0 that has mean $\tau_0 = \mathrm{E}(\Theta_0)$ and variance $\lambda_0 = \mathrm{Var}(\Theta_0)$. Then it is easy to verify that the random vector

$\mathbf{\Theta}_J = (\Theta_1, \cdots, \Theta_J)'$ has a mean vector $\boldsymbol{\tau}_J = \tau_0 \cdot \mathbf{1}$ and a covariance matrix $\mathbf{\Lambda} = (\lambda_{jj'})$, with elements $\lambda_{jj'} = \lambda_0 + \min(j, j')\lambda$. These can be inserted into (4.4)-(4.7).

One could argue that strictly positive claim frequencies cannot be modelled as a random walk, i.e. a martingale, that will converge almost surely when bounded. In my opinion, the error that one commits in making the random walk assumption, is of the same nature as the error one commits by modelling recruits' height by a normal distribution - i.e., negligible for practical purposes.

One can develop more sophisticated models for the time series of claim frequencies. For example, if the basic time period is shorter than a year, it may be necessary to model seasonal variation. This can be done, at the expense of having to specify a larger number of model parameters.

4.5 Kalman filter

Several authors have proposed the Kalman filter as a tool in the estimation of outstanding claims. In my opinion, the Kalman filter is an elegant tool, but not particularly well suited in the estimation of outstanding claims. I will put forward some arguments for my view.

It goes beyond the scope of this paper to introduce the Kalman filter for readers who are not familiar with it. Let it suffice to say that the Kalman filter updating formula is of the form (using the same model and notation as before)

(4.19) $$\overline{\mathbf{\Theta}}_{J|J} = \mathbf{K}_J \begin{pmatrix} N_{1,J-1} / p_1\pi_{J-1} \\ \vdots \\ N_{J,0} / p_J\pi_0 \end{pmatrix} + (\mathbf{I} - \mathbf{K}_J)\overline{\mathbf{\Theta}}_{J|J-1}.$$

Here, $\overline{\mathbf{\Theta}}_{J|J}$ denotes the credibility estimator of $\mathbf{\Theta}_J = (\Theta_1, \cdots, \Theta_J)'$ at valuation date *J*. The vector $\overline{\mathbf{\Theta}}_{J|J-1} = \left(\overline{\mathbf{\Theta}}'_{J-1|J-1}, \overline{\Theta}_{J|J-1}\right)'$ consists of the credibility estimator of $\mathbf{\Theta}_{J-1} = (\Theta_1, \cdots, \Theta_{J-1})'$ at valuation date J -1, and a credibility predictor of Θ_J based on what was known at time J - 1. The credibility predictor of Θ_J depends of course on the dynamics of the underlying process model; in the random walk model it is $\overline{\Theta}_{J|J-1} = \overline{\Theta}_{J-1|J-1}$. The Kalman gain matrix $\mathbf{K}_J$ can be calculated recursively by formulas that are similar to the formula for the credibility estimator (4.6), which involves the inversion of a $J \times J$ matrix. The vector of observations in big brackets consists of the incremental claim counts - i.e. new claims reported in period J -

scaled by the appropriate exposures. To sum up, the Kalman filter is a device to update the estimate of $\mathbf{\Theta}_J$ in the light of new information as it emerges. Why don't I like it then?

In normal time-series applications, with a long time series and observation vectors of fixed dimension, the Kalman filter is an algorithm that allows one to calculate the latest state estimates without having to invert large matrices. In estimation of outstanding claims, however, the dimension of the matrix to be inverted is always be equal to the length of the time series. Therefore the Kalman filter does not reduce computational effort compared with (4.4)-(4.7).

Secondly, the Kalman filter is intended for automatic updating of estimates as new data becomes available. I have yet to see a line of insurance where the estimation of outstanding claims can be left to the automatic pilot for any length of time. Any adjustment in the parameters necessitates a whole new run of the filter through all time points j=1,…,J, which can be more easily accomplished by a straight application of (4.4)-(4.7).

After these critical comments about the Kalman filter, I must add that dynamic linear modelling, of which the random walk model is the very simplest example, fits perfectly into the framework of estimating the number of claims IBNR.

5. Estimation of the amount of claims RBNS

Let us now turn to the problem of estimating the ultimate cost of a cohort of claims that has been reported in calendar year j+d and was incurred in accident year j, where of course $j+d \leq J$. We know with certainty the number of claims that have been reported (N_{jd}) and any activity that has already been recorded on the claims. We pretend to know the ultimate cost of claims that are closed but, let’s face it, they could be reopened. In fact, the ultimate claim cost of those claims will never be known with full certainty.

In this section two models will be proposed to estimate the ultimate cost. One model is based on payments and the other model is based on reported incurred claims, i.e. payments plus case estimates. It would be nice to have formulated a model that utilises payment information and case estimate information simultaneously, but I have not found any elegant and tractable model yet. Anyone who has cared to read so far is hereby invited to join the search party.

5.1 Estimation of claims RBNS by payment data

For the cohort of claims that has been reported in calendar year *j*+*d* and was incurred in accident year *j*, we denote the payments at delay *t* after the reporting year by U_{jdt}. The unknown ultimate claim cost we denote by Y_{jd} and the unknown average severity by Ξ_{jd}.

To model the development of partial payments $\{U_{jdt} : t = 0,1,\cdots\}$ conditionally on the number of claims and the unknown average severity, an obvious candidate is the Dirichlet distribution. Thus let us make the assumption that

(5.1) $$\left(U_{jd0}, U_{jd1}, \cdots\right) \mid N_{jd}, \Xi_{jd} \sim \text{Dirichlet}(\alpha_0, \alpha_1, \cdots) \times N_{jd} \Xi_{jd},$$

with non-negative fixed parameters $\alpha_0, \alpha_1, \cdots$ summing to $\alpha > 0$. Let $\upsilon_t = \alpha_t / \alpha$. The conditional moments of the partial payments are then

(5.2) $$\mathrm{E}\left(U_{jdt} \mid N_{jd}, \Xi_{jd}\right) = \upsilon_t N_{jd} \Xi_{jd} \text{ and}$$

(5.3) $$\mathrm{Cov}\left(U_{jdt}, U_{jdt'} \mid N_{jd}, \Xi_{jd}\right) = \left(\frac{\delta_{tt'}\upsilon_t - \upsilon_t \upsilon_{t'}}{\alpha + 1}\right)\left(N_{jd}\Xi_{jd}\right)^2.$$

Conditional on only N_{jd} and before any payments have been recorded, the average severity Ξ_{jd} has a "prior mean" of ξ_d and a variance of σ_d^2 / N_{jd}. We now use the apparatus of linear greatest accuracy credibility theory to find the best linear predictor of Ξ_{jd} in the conditional model. It is

(5.4) $$\overline{\Xi}_{jd} = z_{jd} \hat{\Xi}_{jd} + (1 - z_{jd})\xi_d,$$

with "chain ladder estimate"

(5.5) $$\hat{\Xi}_{jd} = \frac{U_{jdt}}{N_{jd} \cdot \upsilon_{\le J-(j+d)}},$$

and a credibility factor of

(5.6) $$z_{jd} = \frac{\sigma_d^2(\alpha+1)\upsilon_{\le J-(j+d)}}{\sigma_d^2(\alpha+1)\upsilon_{\le J-(j+d)} + \left(\sigma_d^2 + N_{jd}\xi_d^2\right)\upsilon_{>J-(j+d)}}.$$

The conditional mean squared error of the predictor (5.4) is

(5.7) $$q_d(z_{jd} \mid N_{jd}) = \mathrm{E}\left((\overline{\Xi}_{jd} - \Xi_{jd})^2 \mid N_{jd}\right) = N_{jd}^{-1}\left(z_{jd}^2 \frac{\left(\sigma_d^2 + N_{jd}\xi_d^2\right)\upsilon_{>J-(j+d)}}{(\alpha+1)\upsilon_{\le J-(j+d)}} \;+\; (1 - z_{jd})^2 \sigma_d^2 \right).$$

The best linear predictor of the outstanding payments is

(5.8) $$\overline{U}_{jd,>J-(j+d)} = N_{jd}\overline{\Xi}_{jd} - U_{jd,\le J-(j+d)},$$

with conditional mean squared error

(5.9) $\mathrm{E}\left(\left(\overline{U}_{jd,>J-(j+d)} - U_{jd,>J-(j+d)}\right)^2 \mid N_{jd}\right) = N_{jd}^2 \cdot q_d(z_{jd} \mid N_{jd})$.

Due to the independence between the different cohorts, the mean squared error of the overall amount of outstanding payments for reported claims is additive.

The assumption of the payment pattern being the same for claims at all notification delays, is not necessarily realistic. To see why this need not be the case, contrast claims notified in the accident year (d=0) with claims notified in the subsequent year (d=1). If accidents are spread evenly over the accident year, claim notifications in the accident year will be skewed towards the end of the year because of the notification delay. On the other hand, unless the reporting pattern is very flat-tailed, claim notifications in the subsequent year will occur mostly at the start of the year before they start tailing off. Thus on average, claims that are reported in the accident year will have less time for the first batch of payments (t=0) to be processed, than claims reported in the subsequent year. Therefore one should expect that υ_0 is smaller for d=0 than for d=1. The formulas above extend readily to a model with payment patterns that depend on d, i.e. $\{\upsilon_{dt} : d, t = 0,1,\cdots\}$. However, this comes at the expense of having to set more parameters.

5.2 *Estimation of claims RBNS by reported incurred claims*

For the cohort of claims that has been reported in calendar year $j+d$ and was incurred in accident year j, we denote the change in the reported incurred claim amount at delay t after the reporting year by W_{jdt}. As in the previous section we denote the unknown ultimate claim cost by Y_{jd} and the unknown average severity by Ξ_{jd}.

To model the development of $\{W_{jdt} : t = 0,1,\cdots\}$ conditionally on the number of claims and the unknown average severity, one needs a distribution that allows negative as well as positive increments. That requirement excludes the Dirichlet model.

Consider the following model: given the number of reported claims N_{jd} and the average claim amount Ξ_{jd}, we assume that the W_{jdt} at different delays t are conditionally independent and that

(5.10) $W_{jdt} \sim \text{Compound Poisson}\,(N_{jd}\Xi_{jd}, H_t)$.

The assumption (5.10) implies that the expected number of claim reassessments at delay t (a claim reassessment being a partial payments and/or a change to outstanding case estimate) is proportional to the unknown overall claim amount $Y_{jd} = N_{jd}\Xi_{jd}$, and that the individual reassessments have a size distribution H_t. Let us briefly discuss this assumption.

To assume that the expected number of claim reassessments is proportional to the *number* of claims reported, is quite reasonable. To assume that it is actually proportional not to the number of claims but to the amount of claims, stretches the imagination a bit more. That could be wrong, but it could also be approximately right. I will postulate here that it is approximately right, because this assumption makes for nice mathematics. I am of course not saying that the expected number of claim reassessments is equal to the aggregate claim amount (expressed in some currency or other); it is only the proportionality that counts. The distribution function H_t will have a high point mass at zero, so that the number of actual claim reassessments will be much smaller. One could generate the same compound Poisson distribution using a different model formulation with an explicit proportionality factor in the claim frequency parameter and a distribution function H_t that is strictly non-zero.

Also take note that we are not constraining the aggregate claim development to equal the aggregate severity, i.e. we are not demanding that $\sum_{t=0}^{\infty} W_{jdt} = N_{jd}\Xi_{jd}$, as we did in the payment model. Thus the aggregate severity takes on the role of the expected level of ultimate payments, given the (abstract) severities of claims reported, rather than the definitive level of ultimate payments. Thinking about it, it strikes me as quite a realistic assumption that with given severities, there is residual randomness in the compensation paid to the claimants. So let us get on with the model.

Denote the first and second order moments of the distribution H_t by

(5.11) $\omega_t = \int_{-\infty}^{\infty} u \; H_t(du)$ and

(5.12) $\eta_t = \int_{-\infty}^{\infty} u^2 \; H_t(du)$.

Then we can easily establish the following conditional moments:

(5.13) $\mathrm{E}(W_{jdt} \mid N_{jd}, \Xi_{jd}) = N_{jd}\Xi_{jd}\omega_t$, and

(5.14) $\mathrm{Var}(W_{jdt} \mid N_{jd}, \Xi_{jd}) = N_{jd}\Xi_{jd}\eta_t$.

We are assuming that $\sum_{t=0}^{\infty}\omega_t = 1$ and $\sum_{t=0}^{\infty}\eta_t < \infty$, but not all ω_t need to be non-negative.

Conditional on only N_{jd} and before any payments have been recorded, the average severity Ξ_{jd} has a "prior mean" of ξ_d and a variance of σ_d^2 / N_{jd} . Using the apparatus of linear greatest accuracy credibility theory, one can show that the best linear estimator of Ξ_{jd} in the conditional model, given N_{jd} , is

(5.15) $\overline{\Xi}_{jd} = z_{jd}\hat{\Xi}_{jd} + (1 - z_{jd})\xi_d$, with

(5.16) $\hat{\Xi}_{jd} = \left(\sum_{t=0}^{J-(j+d)} \frac{\omega_t^2}{\eta_t} \right)^{-1} \sum_{t=0}^{J-(j+d)} \frac{\omega_t}{\eta_t} \cdot \frac{W_{jdt}}{N_{jd}}$, and

(5.17) $z_{jd} = \sigma_d^2 \sum_{t=0}^{J-(j+d)} \frac{\omega_t^2}{\eta_t} \cdot \left(\xi_d + \sigma_d^2 \sum_{t=0}^{J-(j+d)} \frac{\omega_t^2}{\eta_t} \right)^{-1}$.

It is interesting to note that the number of claims N_{jd} does not enter into the credibility factor z_{jd} . The reason for this lies in the assumption that the "prior" variance of the unknown Ξ_{jd} is inversely proportional to N_{jd} in the conditional model. The conditional mean squared error of $\overline{\Xi}_{jd}$ is

(5.18) $r_d(z_{jd} \mid N_{jd}) = \mathrm{E}\left((\overline{\Xi}_{jd} - \Xi_{jd})^2 \mid N_{jd} \right) = N_{jd}^{-1} \left(z_{jd}^2 \left(\sum_{t=0}^{J-(j+d)} \frac{\omega_t^2}{\eta_t} \right)^{-1} \xi_d + (1 - z_{jd})^2 \sigma_d^2 \right)$

Having estimated the average severity by the credibility formula (5.15), the estimator of outstanding claim development becomes

(5.19) $\overline{W}_{jd,>J-(J+d)} = N_{jd}\overline{\Xi}_{jd}\omega_{>J-(J+d)}$,

with mean squared error

(5.20) $\mathrm{E}\left((\overline{W}_{jd,>J-(J+d)} - W_{jd,>J-(J+d)})^2 \mid N_{jd} \right) = N_{jd}\xi_d\eta_{>J-(J+d)} + \left(N_{jd}\omega_{>J-(J+d)} \right)^2 r_d(z_{jd} \mid N_{jd})$.

6. Inflation and discounting

It is easy to write down expressions for the inflated and possibly discounted value of future payments. Denote the rate of inflation by ε and the discount rate by δ. The inflated, discounted value of the estimated cost of claims IBNR is

$$\text{(6.1)} \qquad \mathrm{IBNR}_J^{(\mathrm{ID})} = \sum_{j=1}^{J} \sum_{d=J-j+1}^{\infty} \sum_{t=0}^{\infty} \left(p_j \overline{\Theta}_j \pi_d \xi_d \upsilon_t\right) \cdot \left(\frac{1+\varepsilon}{1+\delta}\right)^{(j+d+t)-J-0.5} ,$$

and the inflated, discounted value of the estimated future payments on claims RBNS is

$$\text{(6.2)} \qquad \mathrm{RBNS}_J^{(\mathrm{ID})} = \sum_{j=1}^{J} \sum_{d=0}^{J-j} \sum_{t=J-(j+d)+1}^{\infty} \left(\overline{Y}_{jd} - U_{jd,\leq J-(j+d)}\right) \cdot \left(\frac{\upsilon_t}{\upsilon_{>J-(j+d)}}\right) \cdot \left(\frac{1+\varepsilon}{1+\delta}\right)^{(j+d+t)-J-0.5} .$$

By subtracting 0.5 in the exponent we have made allowance for the assumption that claim payments will be spread evenly over the payment year. These equations can easily be extended to variable rates of inflation or interest.

7. A numerical example

The numerical example is taken from a small portfolio of liability insurances. The data came on a file with the following records:

Claim number	Accident date	Reporting date	Valuation date	Accumulated payments until the valuation date	Outstanding case estimate on the valuation date
Unique identifier of every claim	dd.mm.yyyy (starting 01.01.1988)	dd.mm.yyyy	Every year end between the reporting date and 31.12.2000		

This file contains sufficient information to fill the development tetrahedron with cumulative payment and case estimate data and claim counts. Tables 1-3 shows the traditional triangles. In order to protect information, I have converted all amounts to a non-existent currency that will be denoted N€ (Neuro). It should be obvious from the summary statistics that predicting the claim development in the portfolio is not an easy task. In what follows, I will briefly outline the estimations that have been made.

Table 1. Claim counts by reporting+valuation delay

Number of claims	Delay d+t											
Accident year	-	1	2	3	4	5	6	7	8	9	10	11	12
1 988	8	12	16	19	20	21	22	23	24	26	27	28	29
1 989	3	6	10	14	20	24	28	28	29	30	30	30	
1 990	3	13	17	24	28	29	29	32	32	32	33		
1 991	2	10	21	27	30	32	37	40	40	41			
1 992	4	17	28	39	41	43	44	45	45				
1 993	12	27	41	50	53	56	56	58					
1 994	9	28	35	42	46	50	52						
1 995	10	26	40	43	46	46							
1 996	9	23	33	39	41								
1 997	6	25	32	38									
1 998	8	26	33										
1 999	2	12											
2 000	12												

Table 2. Claim payments by reporting+valuation delay

Paid claims (N€)	Delay d+t												
Accident year	-	1	2	3	4	5	6	7	8	9	10	11	12
1 988	0	0	106	185	205	250	250	250	250	250	259	290	296
1 989	-	-	151	151	152	179	575	581	581	816	816	819	
1 990	-	-	49	292	435	447	1 004	1 004	1 022	1 022	1 022		
1 991	-	5	54	323	1 206	2 747	2 941	3 012	3 013	3 023			
1 992	-	11	34	40	133	199	273	294	334				
1 993	-	0	24	530	594	897	898	928					
1 994	-	0	237	332	487	549	613						
1 995	-	62	298	733	1 297	2 277							
1 996	-	0	14	147	501								
1 997	-	15	55	490									
1 998	0	43	137										
1 999	0	1											
2 000	0												

Table 3. Reported incurred claims by reporting+valuation delay

Reported incurred claims (N€)	Delay d+t												
Accident year	-	1	2	3	4	5	6	7	8	9	10	11	12
1 988	60	30	230	235	245	250	350	250	270	350	359	360	460
1 989	500	20	151	211	312	299	615	681	641	830	818	821	
1 990	110	166	355	848	1 143	1 055	1 023	1 028	1 040	1 040	1 054		
1 991	20	60	794	2 093	2 858	3 087	3 001	3 075	3 085	3 097			
1 992	30	141	245	547	916	944	1 522	1 547	1 574				
1 993	100	496	1 066	1 021	1 038	1 156	1 115	1 394					
1 994	62	669	494	661	912	1 009	1 151						
1 995	420	1 808	1 961	2 604	2 733	2 788							
1 996	22	136	425	627	655								
1 997	318	800	811	836									
1 998	104	665	1 061										
1 999	28	149											
2 000	294												

The reporting pattern $\{\pi_d : d = 0,1,\cdots\}$ was estimated by the standard chain ladder procedure, which involves calculation of year-to-year development factors in Table 1, smoothing the development factors and appending a tail beyond the observed data, and converting the cumulative development factors to probabilities. Graph 1 shows the estimated reporting pattern.

The payment pattern $\{\upsilon_t : t = 0,1,\cdots\}$ was estimated by the same type of procedure, using the triangle of accumulated payments by reporting year and valuation delay that is shown in Table 4. Graph 2 shows the estimated payment pattern.

Table 4. Claim payments by valuation delay

Paid claims (N€)	Delay t												
Reporting year	-	1	2	3	4	5	6	7	8	9	10	11	12
1 988	2	18	151	153	153	153	153	153	153	153	153	153	153
1 989	-	0	118	138	138	138	138	138	138	138	138	138	
1 990	106	187	189	358	650	650	1 143	1 143	1 153	1 153	1 153		
1 991	151	308	769	826	1 267	1 267	1 267	1 267	1 267	1 267			
1 992	0	78	98	251	431	431	482	484	515				
1 993	9	391	1 111	2 252	2 408	2 469	2 469	2 490					
1 994	0	9	927	1 083	1 145	1 148	1 158						
1 995	0	207	368	955	1 019	1 990							
1 996	61	243	715	1 316	1 331								
1 997	0	81	234	813									
1 998	10	51	345										
1 999	31	129											
2 000	1												

The claim revaluation pattern $\{\omega_t : t = 0,1,\cdots\}$ was estimated in the same way, using the triangle of accumulated reported incurred claims by reporting year and valuation delay that is shown in Table 5. One can see a number of substantial upward revaluations. Graph 3 shows the estimated claim revaluation pattern. A notional tail was appended to that pattern to allow for claims being re-opened.

Table 5. Reported incurred claims by valuation delay

Reported incurred claims (N€)	Delay t												
Reporting year	-	1	2	3	4	5	6	7	8	9	10	11	12
1 988	132	158	151	153	153	153	153	153	153	153	153	153	153
1 989	600	40	128	138	138	138	138	138	138	138	138	138	
1 990	520	427	519	658	1 050	1 050	1 143	1 143	1 153	1 153	1 153		
1 991	257	499	850	1 246	1 267	1 267	1 267	1 267	1 267	1 267			
1 992	171	398	537	569	449	443	496	498	529				
1 993	1 104	2 063	2 796	2 761	2 571	2 631	2 631	2 911					
1 994	818	1 326	1 212	1 179	1 191	1 178	1 206						
1 995	1 416	1 680	1 942	3 239	3 316	3 530							
1 996	1 557	1 480	1 630	1 692	2 106								
1 997	800	897	979	1 059									
1 998	1 103	708	918										
1 999	1 174	1 198											
2 000	675												

For the claim frequencies the Bühlmann-Straub model was used, and its parameters τ and λ were estimated with the iterative procedure of De Vylder (1981), treating the previously estimated reporting pattern as a given. The volume of risks exposed had been stable and was set to one throughout the period, so that Θ_j expresses the expected number of accidents in year *j*. De Vylder's procedure returned estimates of $\tau^* = 50$ and $\lambda^* = 162$.

The means $\{\xi_d : d = 0,1,\cdots\}$ and variances $\{\sigma_d^2 : d = 0,1,\cdots\}$ in the severity distribution were calculated using individual claim data that had been adjusted with expected future revaluations using the previously estimated claim revaluation pattern. Graph 4 shows the estimated means as a function of the reporting delay. The variances were estimated on the basis of all claims and linked to the means by assuming that the coefficients of variation

σ_d / ξ_d were independent of *d*. The estimated coefficient of variation, using all claims, was $(\sigma_d / \xi_d)^* = 3.58$.

The parameter α in the Dirichlet distribution of partial payments was estimated on the basis of the regression equations

$$\text{(7.1)} \quad \mathrm{E}\left(U_{jdt}^2 \mid N_{jd}, \Xi_{jd}\right) = \left(N_{jd}\Xi_{jd}\right)^2 \left(\frac{\upsilon_t(1-\upsilon_t)}{\alpha+1} + \upsilon_t^2\right),$$

where the ultimate claim cost $N_{jd}\Xi_{jd}$ had been approximated by reported incurred claims adjusted for expected revaluations, and the υ_t had been replaced by their estimated values. The estimate that came out of the procedure was $\alpha^* = 3.37$.

To estimate the sequence $\{\eta_t : t = 0,1,\cdots\}$ in the compound Poisson distributions for reported incurred claims, another simplifying assumption was made, being that $\eta_t = \eta\omega_t$. The parameter η was then estimated on the basis of the regression equations

$$\text{(7.2)} \quad \mathrm{E}\left(W_{jdt}^2 \mid N_{jd}, \Xi_{jd}\right) = N_{jd}\Xi_{jd}\eta_t + \left(N_{jd}\Xi_{jd}\right)^2 \omega_t^2 = N_{jd}\Xi_{jd}\eta\omega_t + \left(N_{jd}\Xi_{jd}\right)^2 \omega_t^2,$$

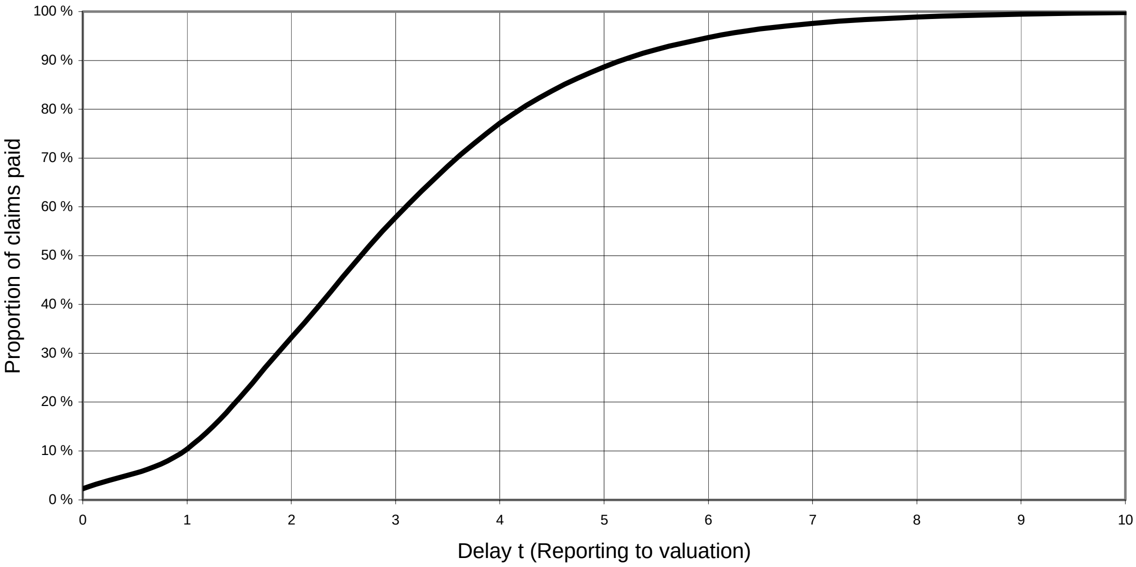
again replacing unknown quantities with the estimates at hand and ignoring correlations. The procedure returned an estimate of $\eta^* = 176$, but admittedly the result was highly uncertain – a number of outliers in (7.2) had to be eliminated. The level of censoring has significant influence on the resulting estimate.

Table 6 shows the estimation of outstanding claims, using the development patterns. The model estimate of outstanding claim payments is N€ 10 935, of which N€ 4 895 is outstanding case estimates, N€ 1 879 is for expected revaluation of claims RBNS and N€ 4 161 is for claims IBNR. The table also shows the square root of the MSEP as computed by (5.20), (4.11) and (4.18), which of course does not include the effect of parameter estimation error. Inflation and discounting have been ignored.

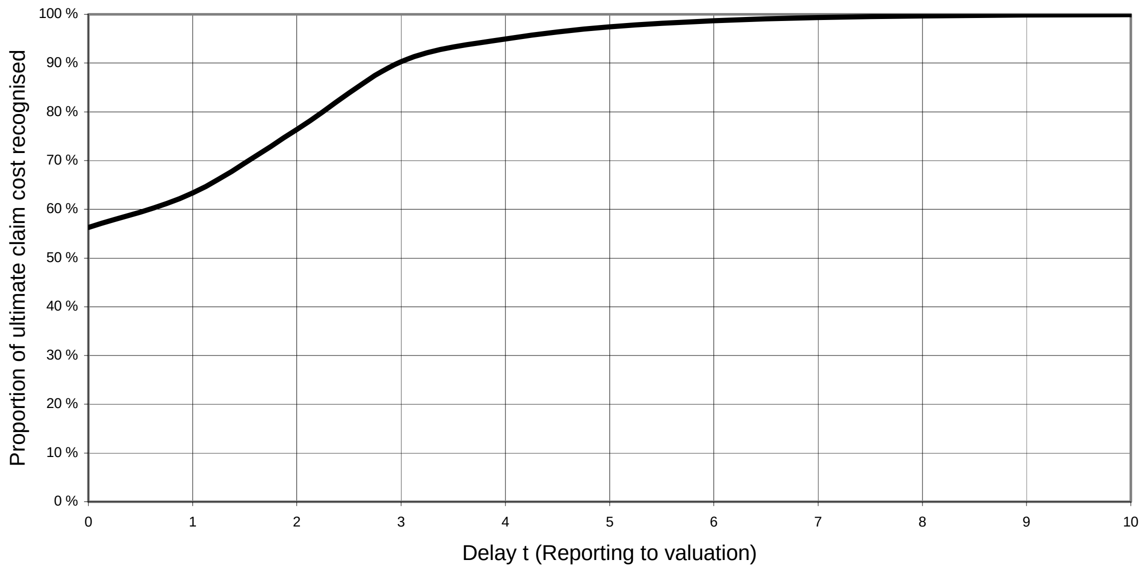
From a statistical point of view, one could argue that the model is over-parametrised, considering the small volume of data and the length of the development delays involved. That is probably true. The dataset was chosen for the example mainly because it was well-organised and clearly illustrates the problems that need to be addressed – long reporting delays, slow payments and unreliable case estimates.

Graph 1. Estimated reporting pattern $\pi^*_{\le d}$

Graph 2. Estimated payment pattern $\upsilon^*_{\le t}$



Graph 3. Estimated claim revaluation pattern $\omega^*_{\leq t}$



Graph 4. Estimated mean severities ξ^*_d

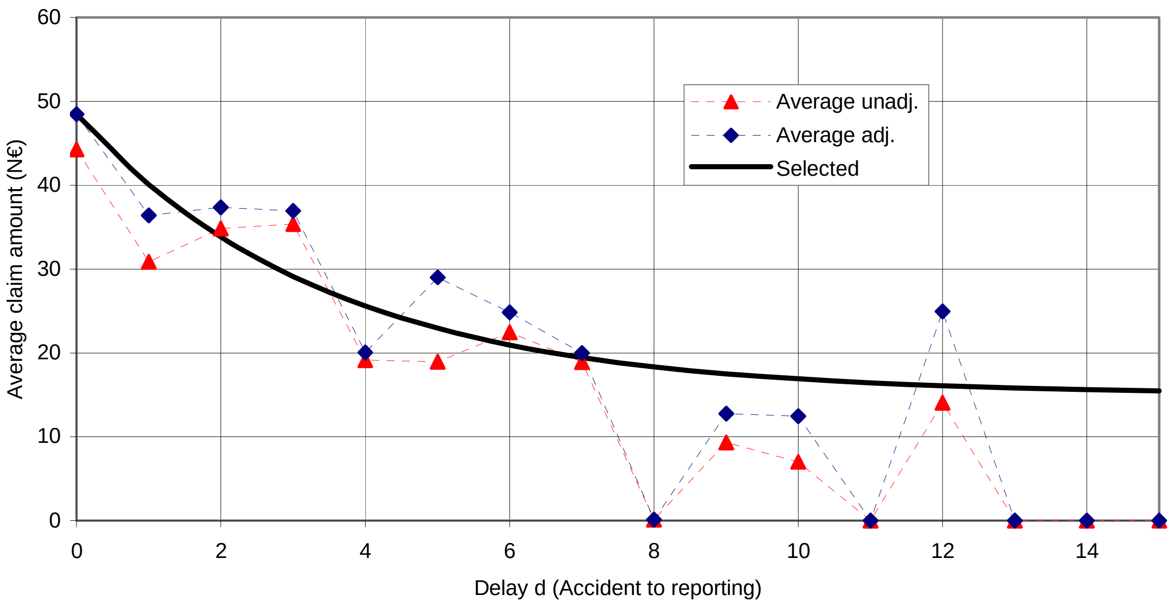


Table 6. Estimation of outstanding claims

						Model estimates					
		Claim statistics				RBNS	IBNR		Accounts	Pricing estimates	
Accident year	Exposure	Reported number of claims	Paid claims	Outst. case estimates	Reported incurred claims	Re-valuations	Number of claims IBNR	Amount of claims IBNR	Outst. claim payments	Total number of claims	Ultimate claim cost
1 988	1	29	296	164	460	21	1	16	201	30	497
1 989	1	30	819	2	821	11	2	25	38	32	856
1 990	1	33	1 022	32	1 054	15	2	38	85	35	1 106
1 991	1	41	3 023	75	3 097	41	4	61	176	45	3 199
1 992	1	45	334	1 241	1 574	39	5	88	1 367	50	1 701
1 993	1	58	928	466	1 394	46	8	142	653	66	1 582
1 994	1	52	613	539	1 151	173	10	172	884	62	1 497
1 995	1	46	2 277	511	2 788	148	12	208	867	58	3 144
1 996	1	41	501	154	655	176	13	254	584	54	1 085
1 997	1	38	490	346	836	295	17	345	986	55	1 476
1 998	1	33	137	925	1 061	530	25	564	2 019	58	2 155
1 999	1	12	1	148	149	149	24	622	918	36	919
2 000	1	12	0	294	294	237	54	1 627	2 158	66	2 158
Sum	13	470	10 440	4 895	15 335	1 879	177	4 161	10 935	647	21 375
					sqrt(MSEP)	1 190	17	1 215	1 701		

The proposed model will not automatically produce more reliable estimates than the traditional models. My point is that by separating claims RBNS from claims IBNR one adds a degree of transparency to the outstanding claim estimates which the traditional models do not have. This transparency makes it much easier to convey the meaning of the estimates and to test alternative assumptions (e.g. in respect of future claim revaluations).

Separate models for the development of reported and unreported claims also facilitate the analysis of claim development, as one can split up the development into its different components: Number of new claims reported (actual vs. predicted), severity of new claims reported (actual vs. predicted) and revaluation of old claims (actual vs. predicted). I'll leave that topic for another paper as it requires heavy notation in theory – in practice it's very easy.

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