# An example of cross-polarized standing gravitational waves

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**Abstract**

We use a cosmological counterpart of the cylindrical Halilsoy solution to illustrate properties of cross-polarized standing gravitational waves.

## 1 Introduction

The linearized gravitational waves are well understood within general relativity. In the era of gravitational wave astronomy, it is especially interesting to extend our intuitions beyond linear regime where unexplored phenomena may be hidden. One of the most basic settings are standing gravitational waves. The studies of this type has been initiated by Bondi [2] and Stephani [7]. More recently, the problem has been investigated in the articles [8], [9].

The aim of this work is to clarify the role of polarization in the context of standing gravitational waves. As a toy-model, we examine a class of exact solutions to Einstein equations which are cosmological counterparts of the Halilsoy cylindrical spacetimes [4]. (The Halilsoy solutions are cross-polarized Einstein-Rosen waves.) These solutions correspond to three-torus Gowdy models. In Gowdy models (contrary to the original cylindrical Halilsoy spacetime), a privileged class of stationary observers exists (at antinodes) and one can examine properties of waves relative to this class of observers.

The solution presented in [9] is a special case of the class studied here. It corresponds to '+' polarization of the waves.

## 2 Setting

We consider the line element of the Gowdy form (its relation to the Halilsoy solution is described in Appendix A)

$$g = e^f \left(-dt^2 + dz^2\right) + te^p(dx + \omega dy)^2 + te^{-p}dy^2 \ , \tag{1}$$

where $t > 0$, $0 \leq x, y, z < 2\pi$ and $x^\alpha = (t, z, x, y)$. The metric functions $f$, $p$ and $\omega$ depend on $t$ and $z$ only and are given by

$$\begin{aligned}
e^{-p} =& t\left(\cosh(\alpha)\cosh[2\beta\sqrt{\lambda}J_0(\frac{t}{\lambda})\sin(\frac{z}{\lambda})] - \sinh[2\beta\sqrt{\lambda}J_0(\frac{t}{\lambda})\sin(\frac{z}{\lambda})]\right) \ , \\
f =& -\ln t - p + \frac{\beta^2}{\lambda}t^2\left[J_0^2(\frac{t}{\lambda}) + J_1^2(\frac{t}{\lambda}) - 2\frac{\lambda}{t}J_0(\frac{t}{\lambda})J_1(\frac{t}{\lambda})\sin^2(\frac{z}{\lambda})\right] \ , \\
\omega =& 2\beta\sqrt{\lambda}\sinh(\alpha)tJ_1(\frac{t}{\lambda})\cos(\frac{z}{\lambda}) \ ,
\end{aligned} \tag{2}$$

where $\alpha$ is a constant and $J_0$, $J_1$ are the Bessel functions of the first kind and orders 0, 1, respectively. The number of waves on the torus is given by $1/\lambda$ where $\lambda$ is a parameter such that $1/\lambda \in \mathbb{N}$. For $\alpha = 0$, the solution (2) reduces to the '+' polarized case studied in [9]. For $\beta = 0$ it corresponds to the three-torus identified Minkowski metric in coordinates expanding along $\partial_y$. Only non-diagonal components of the metric depend on the sign of $\alpha$ (the function $\omega$ is odd in $\alpha$), so without the loss of generality we assume $\alpha \geq 0$. The solution (2) corresponds to the three-torus Gowdy model.

In this article, in addition to the coordinate basis $\partial_\alpha$ we use two sets of non-holonomic bases. They are given by

- the orthonormal tetrad $e_{\hat{\alpha}}$

$$\begin{aligned}
e_{\hat{0}} =& e^{-f/2}\,\partial_t \ , \quad e_{\hat{1}} = e^{-f/2}\,\partial_z \ , \\
e_{\hat{2}} =& \frac{e^{-p/2}}{\sqrt{t}}\,\partial_x \ , \quad e_{\hat{3}} = \frac{e^{p/2}}{\sqrt{t}}\,(-\omega\,\partial_x + \partial_y) \ ,
\end{aligned} \tag{3}$$

- and the null tetrad $w_{\breve{\alpha}} = \{k, l, m, \bar{m}\}$

$$\begin{aligned}
k =& \frac{1}{\sqrt{2}}(e_{\hat{0}} + e_{\hat{1}}) \ , \quad l = \frac{1}{\sqrt{2}}(e_{\hat{0}} - e_{\hat{1}}) \ , \\
m =& \frac{1}{\sqrt{2}}(e_{\hat{2}} - ie_{\hat{3}}) \ , \quad \bar{m} = \frac{1}{\sqrt{2}}(e_{\hat{2}} + ie_{\hat{3}}) \ .
\end{aligned} \tag{4}$$

  where $k \cdot l = -1$ , $m \cdot \bar{m} = 1$ , $k \cdot k = l \cdot l = m \cdot m = \bar{m} \cdot \bar{m} = 0$, and $g_{\breve{\alpha}\breve{\beta}} = -2k_{(\breve{\alpha}}l_{\breve{\beta})} + 2m_{(\breve{\alpha}}\bar{m}_{\breve{\beta})}$.

# 3 Geodesic deviation

Our research extends results obtained previously [9] to the cross-polarized standing gravitational waves (the polarization '$\times$'). An example of such waves is provided by the solution (1), (2) with the non-zero value of the parameter $\alpha$. In this section, we clarify how this parameter alters the effect of standing gravitational waves on test particles.

The behavior of test particles is investigated relative to a freely falling observer. For a standing wave spacetime, a preferred family of coordinate systems exist in which nodes and antinodes are labelled by one of coordinates. Our coordinates (1) satisfy this criterion. In Appendix C, we show that, similarly as in the '+' polarized model, observers at antinodes are stationary, namely, the curves $\gamma_{k\in\mathbb{Z}}$: $z = \lambda\pi(1/2 + k)$, $x = x_0$ $y = y_0$ with constants $x_0$, $y_0$ are future-directed timelike geodesic. The vector tangent to this geodesic corresponds to $e_{\hat{0}}$. Thus $e_{\hat{0}}$ is parallelly transported along $\gamma_k$: $\nabla_{e_{\hat{0}}} e_{\hat{0}} = 0$. Moreover, we have along $\gamma_k$: $\nabla_{e_{\hat{0}}} e_{\hat{\alpha}} = \omega^{\hat{\beta}}_{\ \hat{\alpha}}(e_{\hat{0}})e_{\hat{\beta}} = 0$, where $\omega^{\hat{\beta}}_{\ \hat{\alpha}}$ are connection one forms given in Appendix E. Therefore, the remaining orthonormal basis vectors $e_{\hat{i}}$ are also parallelly transported along $\gamma_k$ which implies that $e_{\hat{\alpha}}$ [given by (3)] is a freely falling frame at antinodes.

Let $\xi$ be a deviation vector, then along $\gamma_k$ in a freely falling frame

$$\frac{d^2\xi^{\hat{\alpha}}}{d\tau^2} = -R^{\hat{\alpha}}_{\ \hat{0}\hat{\beta}\hat{0}}\xi^{\hat{\beta}} \ , \tag{5}$$

where $\tau$ is a proper time of the observer. In vacuum, the Riemann and Weyl tensors are equal so $R^{\hat{\alpha}}_{\ \hat{0}\hat{\beta}\hat{0}} = C^{\hat{\alpha}}_{\ \hat{0}\hat{\beta}\hat{0}}$. The non-zero components of the Riemann tensor may be calculated from the curvature two-forms presented in Appendix F using the relation $\Omega^{\hat{\alpha}}_{\ \hat{\beta}} = \frac{1}{2}R^{\hat{\alpha}}_{\ \hat{\beta}\hat{\sigma}\hat{\delta}}\theta^{\hat{\sigma}} \wedge \theta^{\hat{\delta}}$ where the cobasis $\theta^{\hat{\alpha}}$ is given in Appendix D. However, it is instructive to rewrite the geodesic deviation equation in terms of the complex Weyl coefficients (defined in a standard way [9]). (The appropriate formulas for Weyl coefficients are too large to be usefully cited here, but one may evaluate them with the help of computer system MATHEMATICA.) The relevant non-zero components of the

Riemann tensor are

$$\begin{aligned} R^{\hat{1}}{}_{\hat{0}\hat{1}\hat{0}} &= 2\Re(\Psi_2) \ , \\ R^{\hat{2}}{}_{\hat{0}\hat{2}\hat{0}} &= \frac{1}{2}\Re\left[-2\Psi_2 + \Psi_0 + \Psi_4\right] \ , \\ R^{\hat{3}}{}_{\hat{0}\hat{3}\hat{0}} &= \frac{1}{2}\Re\left[-2\Psi_2 - \Psi_0 - \Psi_4\right] \ , \\ R^{\hat{2}}{}_{\hat{0}\hat{3}\hat{0}} &= \frac{1}{2}\Im\left[\Psi_0 - \Psi_4\right] \ , \end{aligned} \tag{6}$$

where $\Re$, $\Im$ denote real and imaginary parts. At antinodes $\Psi_1 = \Psi_3 = 0$ and $\Psi_4 = \Psi_0$, hence $R^{\hat{2}}{}_{\hat{0}\hat{3}\hat{0}} = R^{\hat{3}}{}_{\hat{0}\hat{2}\hat{0}} = 0$. For $\alpha = 0$, $\Psi_0$ and $\Psi_2$ are real (not only at antinodes). The geodesic deviation equation can be written at antinodes as

$$\begin{aligned} \frac{d^2\xi^{\hat{0}}}{d\tau^2} =& 0 \ , \\ \frac{d^2\xi^{\hat{1}}}{d\tau^2} =& -2\Re(\Psi_2)\xi^{\hat{1}} \ , \\ \frac{d^2\xi^{\hat{2}}}{d\tau^2} =& \Re(\Psi_2 - \Psi_0)\xi^{\hat{2}} \ , \\ \frac{d^2\xi^{\hat{3}}}{d\tau^2} =& \Re(\Psi_2 + \Psi_0)\xi^{\hat{3}} \ . \end{aligned} \tag{7}$$

The polarization of waves is not directly observable at antinodes because the metric function $\omega$ equals zero there. The vanishing of $R^{\hat{2}}{}_{\hat{0}\hat{3}\hat{0}}$, $R^{\hat{3}}{}_{\hat{0}\hat{2}\hat{0}}$ imply that $\frac{d^2\xi^{\hat{\alpha}}}{d\tau^2} \sim \xi^{\hat{\alpha}}$. The equations decouple and the standard effect of cross-polarization is not visible at antinodes. However, the parameter $\alpha$ changes the global evolution of spacetime and in this sense, it alters the trajectories of test particles at antinodes (via $\Psi_0$, $\Psi_2$). The Tissot diagrams are very similar to those presented in [9] and we do not include them here.

# 4 The interpretation of the $\alpha$ parameter

The role of the $\alpha$ parameter follows from the field equations [5]. A standing wave may be seen as a 'superposition' of two identical gravitational waves moving in the opposite directions. The amplitudes of these waves may be split between two polarizations. The oscillations of the metric function $\omega$

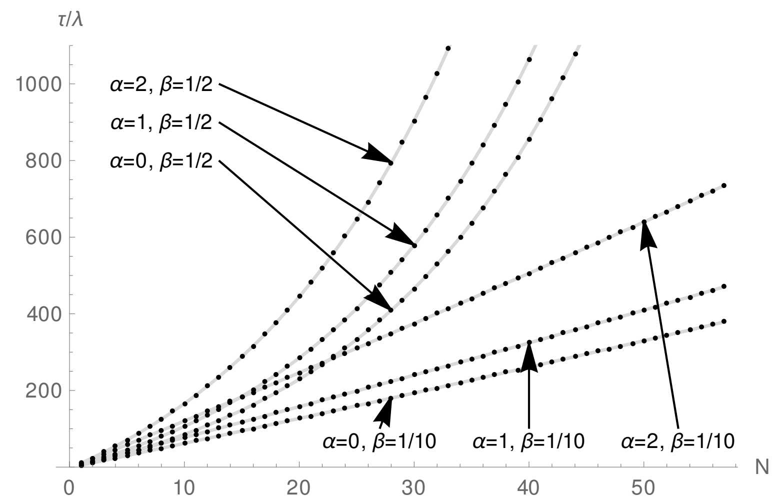


Figure 1: The photon's flight time between subsequent antinodes (numbered by $N$) meassured in the proper time of stationary observers at antinodes. The photon flies along $\partial_z$ direction. The remaining parameters: $t_0 = 1$, $\lambda = 1/10$.

correspond to the rotation of polarization between + and × modes. This is a gravitational analogue of the Faraday rotation [5]. Since $\omega = 0$ at antinodes, then the trajectories of test particles at antinodes depend only indirectly on $\alpha$. One may try to understand this dependency in terms of the high frequency limit of standing waves (e.g. the article [10]) where the energy of the gravitational waves alters global expansion of spacetime, but such attempts leads to counter-intuitive results.

In order to evaluate the effect of $\alpha$ parameter on the global expansion explicitly, we calculated numerically a flight time of a photon between stationary observers at antinodes in terms of the proper time of these observers. We note that for $\beta = 0$ the metric (1) describes the three-torus identified Minkowski spacetime in expanding coordinate system. Thus, any expansion for $\beta = 0$ is of artificial nature and depends on the choice of coordinates. Standing waves apear already for small $\beta$ and a position of stationary observers at antinodes is distinguished by the geometry of spacetime. We show in Figure 1 that in this setting the time flight of photons between subsequent observers (measured in their proper time) grows initially almost linearly with the slope controlled by $\alpha$ parameter. Therefore, $\alpha$ describes not only polarization of waves, but also encodes initial conditions how fast the distance between antinodes should grow. Whether this is merely a coincidence (or artifact of the model studied) needs further investigation. For larger $\beta$ an exponential expansion of spacetime becomes evident (Figure 1), but the role of the $\alpha$ parameter remains unchanged: larger $\alpha$ implies faster expansion.

# 5 The Poynting vector

Stephani suggested [7] that gravitational analogue of the Poynting vector may play a fundamental role in understanding of standing gravitational waves. In his approach, he derived the Poynting vector from the Lagrangian that leads to the Ernst equations. We follow here another approach. We construct the gravitational analogue of the Poynting vector with the help of the Bel-Robinson tensor $T_{\alpha\beta\gamma\delta}$ [1]. The Bel-Robinson tensor is given by

$$T_{\alpha\beta\gamma\delta} = C_{\alpha\mu\gamma}{}^{\nu} C_{\delta\nu\beta}{}^{\mu} + \star C_{\alpha\mu\gamma}{}^{\nu} \star C_{\delta\nu\beta}{}^{\mu} \; . \tag{8}$$

$C$ is the Weyl tensor and $\star$ denotes the Hodge dual, namely

$$\star C_{\alpha\beta\gamma\delta} = \frac{1}{2} \eta_{\alpha\beta\mu\nu} C^{\mu\nu}{}_{\gamma\delta} \; ,$$

where $\eta$ is the canonical volume form. The supermomentum $P$ is defined relative to an observer with four-velocity $u$: $P^{\alpha} = -T^{\alpha}{}_{\beta\gamma\delta} u^{\beta} u^{\gamma} u^{\delta}$. It might be decomposed into the superenergy density $W$ and the super-Poynting vector $S$: $P^{\alpha} = W u^{\alpha} + S^{\alpha}$. The superenergy density and the super-Poynting vector can be written in terms of the Bel-Robinson tensor

$$\begin{aligned} W =& T_{\alpha\beta\gamma\delta} u^{\alpha} u^{\beta} u^{\gamma} u^{\delta} \geq 0 \; , \\ S^{\alpha} =& - T^{\mu}{}_{\beta\gamma\delta} (\delta^{\alpha}_{\ \mu} + u^{\alpha} u_{\nu}) u^{\beta} u^{\gamma} u^{\delta} \; . \end{aligned} \tag{9}$$

In the orthonormal frame, relative to the observer with the four-velocity $e_{\hat{0}}$, these formulas take a form ($S^{\hat{0}} = 0$)

$$\begin{aligned} W =& T_{\hat{0}\hat{0}\hat{0}\hat{0}} \; , \\ S^{\hat{i}} =& - T_{\hat{i}\hat{0}\hat{0}\hat{0}} \; . \end{aligned} \tag{10}$$

The expressions for the superenergy $W$ and the super-Poyting vector for solutions studied in this paper are too long to be usefully cited here. The analysis of these formulas with the help of the computer algebra system MATHEMATICA reveals

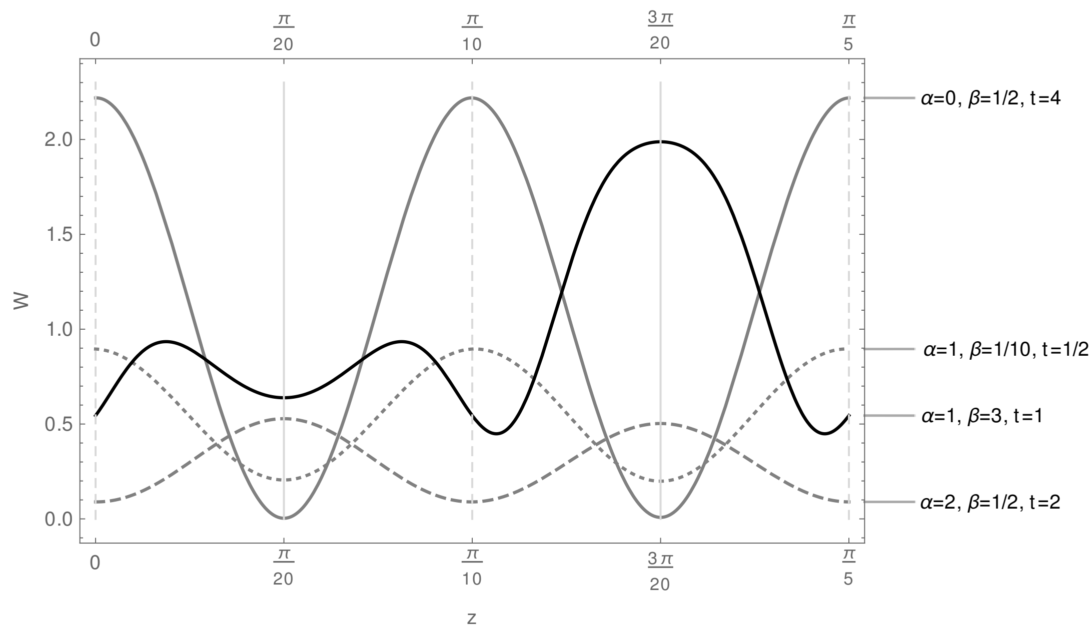


Figure 2: The superenergy density for $\lambda = 1/10$ and several sets of remaining parameters. The amplitude of the solid black function was multiplied by a factor 300. The anitnodes and nodes are indicated with solid and dashed vertical lines, respectively.

- $W$ oscillates periodically with extrema at antinodes,
- $S^{\hat{1}}$ oscillates periodically with zeros at antinodes (no energy transfer at antinodes),
- $S^{\hat{1}}$ averages to zero over hypersurfaces $t = const$,
- $S^{\hat{2}} = S^{\hat{3}} = 0$.

We remind here that beyond antinodes observers with the four-velocity $e_{\hat{0}}$ do not move on geodesics. The superenergy and the z-component of the super-Poynting vector are presented in Figures 2, 3 for a particular set of parameters.

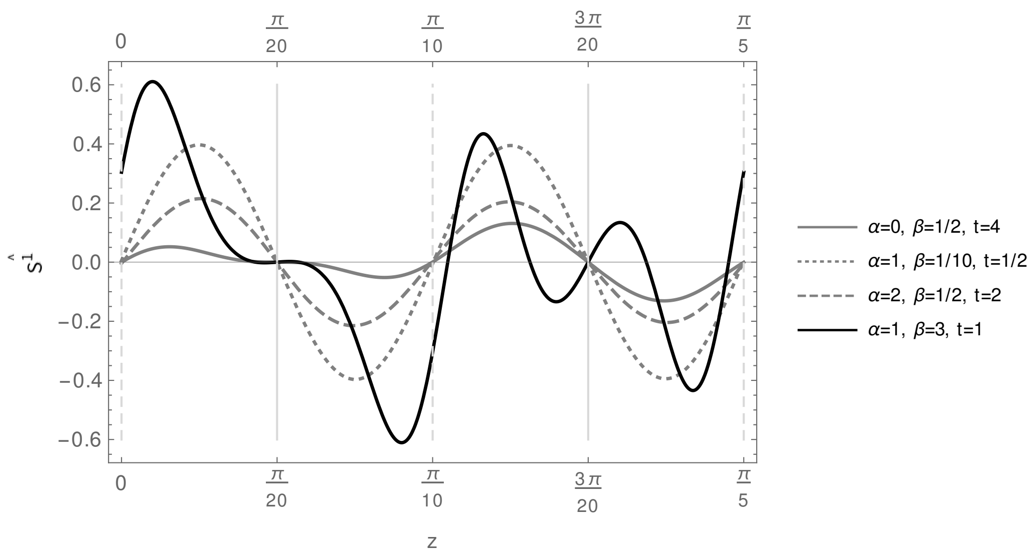


Figure 3: The $S^{\hat{1}}$ component of the super-Poyting vector for $\lambda = 1/10$ and several sets of remaining parameters. It shows the superenergy transfer in $\partial_z$ direction. The amplitude of the solid black function was multiplied by a factor 300. The anitnodes and nodes are indicated with solid and dashed vertical lines, respectively.

# 6 Summary

We constructed, starting from the Halilsoy cylindrical spacetime [4], the cosmological model of the Gowdy form with standing gravitational waves. This solution generalizes the spacetime studied in [9] (with '+' polarization) to cross-polarized waves (denoted with '×'). Similarly to the '+' case, in the '×' polarized model there exist stationary observers at antinodes. The trajectories of neighbouring test particles for these observers differ trivially from the trajectories in the '+' polarized model. The Tissot diagrams do not reveal standard effects of cross-polarization although the '×' polarized model expands differently than the '+' polarized model. In order to evaluate the expansion rate of the spacetime, we studied the time of flight of photons between antinodes measured in the proper time of stationary observers at antinodes. We showed that the expansion rate of the model depends on a parameter which describes the deviation of the model from the '+' polarized model and controls the polarization. Finally, we used the Bel-Robinson tensor to calculate the superenergy density and the gravitational analogue of the Poynting vector. We showed that both quantities have expected properties and are useful tools in the studies of the standing gravitational waves.

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# A From Halilsoy solution to cross-polarized Gowdy standing waves

Halilsoy showed [4] that the Einstein-Rosen waves [3, 6] may be extended to include the second polarization. The exact solution to Einstein equations has a form

$$g = e^{2(\gamma-\psi)}\left(-dt^2 + d\rho^2\right) + \rho^2 e^{-2\psi} d\varphi^2 + e^{2\psi}(dz + \omega d\varphi)^2 \,, \tag{11}$$

where $\rho > 0$, $-\infty < t, z < \infty$, $0 \le \varphi < 2\pi$. The metric functions $\psi$, $\gamma$ and $\omega$ depend on $t$ and $\rho$ only and are given by

$$\begin{aligned} e^{-2\psi} =& e^{AJ_0 \cos\sigma t} \sinh^2 \frac{\alpha}{2} + e^{-AJ_0 \cos\sigma t} \cosh^2 \frac{\alpha}{2} \,, \\ \omega =& -(A \sinh\alpha)\rho J_1 \sin\sigma t \,, \\ \gamma =& \frac{1}{8} A^2 \left(\sigma^2\rho^2(J_0^2 + J_1^2) - 2\sigma\rho J_0 J_1 \cos^2\sigma t\right) \,, \end{aligned} \tag{12}$$

where $J_0 = J_0(\sigma\rho)$, $J_1 = J_1(\sigma\rho)$ are the Bessel functions of the first kind and orders 0, 1, respectively.

The Einstein equations for the metric 11 are form-invariant under the complex substitution

$$t \mapsto iz, \,, \qquad \rho \mapsto it \,, \qquad \varphi \mapsto iy \,, \qquad z \mapsto z \,, \qquad \omega \mapsto -i\omega$$

which brings the metric (11) into the form (1) where the remaining metric functions are related by $f = 2(\gamma - \psi)$, $p = -\ln t + 2\psi$. Therefore, any cylindrical solution has its cosmological counterpart. The Halilsoy solution (12) corresponds to (2) with additional trivial redefinitions: $\sigma \mapsto 1/\lambda$ followed by $z \mapsto z + \lambda\pi/2$ and $A \mapsto 2\beta\sqrt{\lambda}$.

## B Geodesic equation

The geodesic equation in coordinates $x^\alpha = (t, z, x, y)$ for the metric (1) has a form

$$
\begin{aligned}
&\ddot{t} + \frac{1}{2}\left\{f_{,t}\dot{t}^2 + 2f_{,z}\dot{t}\dot{z} + f_{,t}\dot{z}^2 + e^{-f}\left[e^p\dot{x}^2(1+p_{,t}t) + e^{-p}\dot{y}^2(1-p_{,t}t)\right]\right\} \\
&\quad + e^{p-f}\left(t\omega_{,t}\dot{x}\dot{y} + \omega\dot{y}[(1+p_{,t}t) + t\omega_{,t}] + \frac{1}{2}\omega^2\dot{z}(1+p_{,t}t)\right) = 0\ , \\
&\ddot{z} + \frac{1}{2}\left[f_{,z}\dot{z}^2 + 2f_{,t}\dot{t}\dot{z} + f_{,z}\dot{t}^2 + tp_{,z}e^{-f}(-e^p\dot{x}^2 + e^{-p}\dot{y}^2)\right] \\
&\quad - e^{p-f}\dot{y}\left(t\dot{x}\omega_{,z} + \omega\dot{x}p_{,z} + \dot{y}\omega_{,z} + \frac{1}{2}\omega^2 t\dot{y}p_{,z}\right) = 0\ , \\
&\ddot{x} + \dot{x}(\dot{t}/t + p_{,t}\dot{t} + p_{,z}\dot{z}) + \dot{y}(\dot{t}\omega_{,t} + \dot{z}\omega_{,z}) + \omega\left[2\dot{y}(\dot{t}p_{,t} + \dot{z}p_{,z}) - e^{2p}\dot{x}(\dot{t}\omega_{,t} + \dot{z}\omega_{,z})\right] \\
&\quad - e^{2p}\omega^2\dot{y}(\dot{t}\omega_{,t} + \dot{z}\omega_{,z}) = 0\ , \\
&\ddot{y} + \dot{y}(\dot{t}/t - p_{,t}\dot{t} - p_{,z}\dot{z}) + e^{2p}(\dot{x} + \omega\dot{y})(\dot{t}\omega_{,t} + \dot{z}\omega_{,z}) = 0\ ,
\end{aligned}
\tag{13}
$$

where a dot denotes differentiation in the proper time $\tau$ or the affine parameter for timelike or null geodesics, respectively. The normalization of the four-velocity/wave vector gives rise to the first integral of the form

$$
-\epsilon = e^f(-\dot{t}^2 + \dot{z}^2) + t(e^p(\dot{x} + \omega\dot{y})^2 + e^{-p}\dot{y}^2)\ .
$$

where the constant $\epsilon$ is equal to 1 or 0 for timelike or null geodesics, respectively. The Killing fields $\partial_x$, $\partial_y$ give two more quantities $c_x$, $c_y$ that are conserved along geodesics

$$
\begin{aligned}
(\dot{x} + \omega\dot{y})e^p t &= c_x\ , \\
\dot{y}e^{-p}t + (\dot{x} + \omega\dot{y})\omega e^p t &= c_y\ .
\end{aligned}
\tag{14}
$$

## C Observers at antinodes

In this Appendix, we show that the observers at antinodes are stationary. The geodesic equation is presented in Appendix B. For stationary solutions $\dot{z} = \dot{x} = \dot{y} = 0$ it takes the same form as for $\alpha = 0$, thus this part of the analysis mimics calculations in [9]. For readers convenience we repeat it below. We have

$$
\begin{aligned}
0 &= \ddot{t} + \frac{1}{2} f_{,t} \dot{t}^2 \ , && (15) \\
0 &= f_{,z} \ , \\
\dot{t} &= e^{-\frac{1}{2}f} \ , && (16)
\end{aligned}
$$

where (16) is the first integral of (15). The condition $f_{,z} = 0$ corresponds to antinodes. Using (2) we have $f_{,z} = (\dots)\cos(z/\lambda)$, thus $z = \lambda\pi(1/2 + k)$, where $k \in \mathbb{Z}$, implies $f_{,z} = 0$. Therefore, the curve $\gamma_k$

$$
x^\mu = [t(\tau), \lambda\pi(1/2 + k), x_0, y_0] \ , \tag{17}
$$

with $k \in \mathbb{Z}$ and $t(\tau)$ determined by (16) is a future-directed timelike geodesic and a stationary solution to the geodesic equation.

# D Orthonormal cobasis and its external derivative

$$
\begin{aligned}
\theta^{\hat{0}} &= e^{f/2}\, dt \ , \\
\theta^{\hat{1}} &= e^{f/2}\, dz \ , \\
\theta^{\hat{2}} &= \sqrt{t}\, e^{p/2} \, (dx + \omega\, dy) \ , \\
\theta^{\hat{3}} &= \sqrt{t}\, e^{-p/2}\, dy \ , \\
d\theta^{\hat{0}} &= -\frac{1}{2} e^{-f/2}\, f_{,z}\, \theta^{\hat{0}} \wedge \theta^{\hat{1}} \ , \\
d\theta^{\hat{1}} &= \frac{1}{2} e^{-f/2}\, f_{,t}\, \theta^{\hat{0}} \wedge \theta^{\hat{1}} \ , \\
d\theta^{\hat{2}} &= \frac{1}{2} e^{-f/2} \left( \left( \frac{1}{t} + p_{,t} \right) \theta^{\hat{0}} \wedge \theta^{\hat{2}} + p_{,z}\, \theta^{\hat{1}} \wedge \theta^{\hat{2}} + 2e^{p} \left( \omega_{,t}\, \theta^{\hat{0}} \wedge \theta^{\hat{3}} + \omega_{,z}\, \theta^{\hat{1}} \wedge \theta^{\hat{3}} \right) \right) \ , \\
d\theta^{\hat{3}} &= \frac{1}{2} e^{-f/2} \left( \left( \frac{1}{t} - p_{,t} \right) \theta^{\hat{0}} \wedge \theta^{\hat{3}} - p_{,z}\, \theta^{\hat{1}} \wedge \theta^{\hat{3}} \right) \ .
\end{aligned} \tag{18}
$$

# E Connection one-forms

The non-zero connection one-forms

$$
\begin{aligned}
\omega^{\hat{0}}{}_{\hat{1}} =& \frac{1}{2} e^{-f/2} \left( f_{,z}\, \theta^{\hat{0}} + f_{,t}\, \theta^{\hat{1}} \right) \ , \\
\omega^{\hat{0}}{}_{\hat{2}} =& \frac{1}{2} e^{-f/2} \left( \left( \frac{1}{t} + p_{,t} \right) \theta^{\hat{2}} + e^{p}\, \omega_{,t} \theta^{\hat{3}} \right) \ , \\
\omega^{\hat{0}}{}_{\hat{3}} =& \frac{1}{2} e^{-f/2} \left( \left( \frac{1}{t} - p_{,t} \right) \theta^{\hat{3}} + e^{p}\, \omega_{,t} \theta^{\hat{2}} \right) \ , \\
\omega^{\hat{1}}{}_{\hat{2}} =& -\frac{1}{2} e^{-f/2} \left( e^{p}\, \omega_{,z} \theta^{\hat{3}} + p_{,z}\, \theta^{\hat{2}} \right) \ , \\
\omega^{\hat{1}}{}_{\hat{3}} =& \frac{1}{2} e^{-f/2} \left( p_{,z}\, \theta^{\hat{3}} - e^{p}\, \omega_{,z} \theta^{\hat{2}} \right) \ , \\
\omega^{\hat{2}}{}_{\hat{3}} =& -\frac{1}{2} e^{-f/2}\, e^{p} \left( \omega_{,t}\, \theta^{\hat{0}} + \omega_{,z}\, \theta^{\hat{1}} \right) \ .
\end{aligned}
\tag{19}
$$

# F Curvature two-forms

The non-zero curvature two-forms

$$
\begin{aligned}
\Omega^{\hat{1}}{}_{\hat{0}} =& \frac{1}{2} e^{-f/2} \left[ (f_{,tt} - f_{,zz}) \theta^{\hat{0}} \wedge \theta^{\hat{1}} + e^{p} \left( p_{,t}\omega_{,z} - p_{,z}\omega_{,t} \right) \theta^{\hat{2}} \wedge \theta^{\hat{3}} \right] \ , \\
\Omega^{\hat{2}}{}_{\hat{0}} =& \frac{1}{4} e^{-f/2} \left[ \left( (p_{,t} + \frac{1}{t})(p_{,t} - f_{,t} + \frac{1}{t}) + 2(p_{,tt} - \frac{1}{t^2}) - p_{,z} f_{,z} - e^{2p}\omega_{,t}^2 \right) \theta^{\hat{0}} \wedge \theta^{\hat{2}} - \right. \\
& - e^{p} \left( \omega_{,t}(f_{,t} - 4p_{,t} - \frac{2}{t}) - 2\omega_{,tt} + \omega_{,z} f_{,z} \right) \theta^{\hat{0}} \wedge \theta^{\hat{3}} + \\
& + \left( (p_{,z} - f_{,z})(p_{,t} + \frac{1}{t}) + 2p_{,tz} - p_{,z} f_{,t} - e^{2p}\omega_{,t}\omega_{,z} \right) \theta^{\hat{1}} \wedge \theta^{\hat{2}} - \\
& \left. - e^{p} \left( \omega_{,t}(f_{,z} - p_{,z}) - 2p_{,t}\omega_{,z} - 2\omega_{,tz} + \omega_{,z}(f_{,t} - p_{,t} + \frac{1}{t}) \right) \theta^{\hat{1}} \wedge \theta^{\hat{3}} \right] \ ,
\end{aligned}
$$

$$
\begin{aligned}
\Omega^{\hat{3}}{}_{\hat{0}} =& \frac{1}{4}e^{-f/2}\left[-e^{p}\left(\omega_{,t}(f_{,t}-4p_{,t}-\frac{2}{t})-2\omega_{,tt}+f_{,z}\omega_{,z}\right)\theta^{\hat{0}}\wedge\theta^{\hat{2}}+\right.\\
&+\left((f_{,t}+p_{,t}-\frac{1}{t})(p_{,t}-\frac{1}{t})-2(p_{,tt}+\frac{1}{t^{2}})+p_{,z}f_{,z}+3e^{2p}\omega_{,t}^{2}\right)\theta^{\hat{0}}\wedge\theta^{\hat{3}}-\\
&-e^{p}\left(\omega_{,t}(f_{,z}-3p_{,z})-2\omega_{,tz}+\omega_{,z}(f_{,t}-p_{,t}-\frac{1}{t})\right)\theta^{\hat{1}}\wedge\theta^{\hat{2}}+\\
&\left.+\left((f_{,z}+p_{,z})(p_{,t}-\frac{1}{t})-2p_{,tz}+p_{,z}f_{,t}+3e^{2p}\omega_{,t}\omega_{,z}\right)\theta^{\hat{1}}\wedge\theta^{\hat{3}}\right]\ .
\end{aligned}
$$