Linear Programming Duality

Design and Analysis of Algorithms
Andrei Bulatov

Linear Programming

Linear Programming

Instance

Objective function $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Constraints:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2$$
$$\vdots$$
$$a_{m1}x1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m$$

Objective

Find values of the variables that satisfy all the constraints and maximize the objective function

Lower Bounds

Maximize $7x_1 + x_2 + 5x_3$

Constraints
$$x_1 - x_2 + 3x_3 \le 10$$
$$5x_1 + 2x_2 - x_3 \le 1$$
$$-x_3 \le -1$$
$$x_2 \le 1$$
$$x_1, x_2, x_3 \ge 0$$

Can we show that the optimum is greater than 7?

Yes, check (2/5, 0, 1)

Upper Bounds

Maximize $7x_1 + x_2 + 5x_3$

Constraints
$$x_1 - x_2 + 3x_3 \le 10$$
$$5x_1 + 2x_2 - x_3 \le 1$$
$$-x_3 \le -1$$
$$x_2 \le 1$$
$$x_1, x_2, x_3 \ge 0$$

Can we show that the optimum is less than 23?

Use linear combinations of constraints

Dual Program

Maximize $7x_1 + x_2 + 5x_3$

y_1 $\quad x_1 - x_2 + 3x_3 \le 10$

y_2 $\quad 5x_1 + 2x_2 - x_3 \le 1$

y_3 $\quad -x_3 \le -1$

y_4 $\quad x_2 \le 1$

$\quad x_1, x_2, x_3 \ge 0$

Minimize $10y_1 + y_2 - y_3 + y_4$

$$y_1 + 5y_2 \ge 7$$
$$-y_1 + 2y_2 + y_4 \ge 1$$
$$3y_1 - y_2 - y_3 \ge 5$$
$$y_1, y_2, y_3, y_4 \ge 0$$

We get the dual program.

Dual Program

Maximize

$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2$$
$$\vdots$$
$$a_{m1}x1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m$$

primal program

Minimize

$b_1y_1 + b_2y_2 + \ldots + b_my_m$

$$a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \ge c_1$$
$$a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \ge c_2$$
$$\vdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \ge c_n$$

dual program

Weak Duality

Theorem (Weak Duality)

If $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_m)$ are feasible primal and dual solutions, then

$$\sum_{i=1}^{n} c_i x_i \le \sum_{j=1}^{m} b_i y_i$$

Proof

$$\sum_{i=1}^{n} c_i x_i \le \sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ji} y_j\right) x_i$$
$$= \sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ji} x_i\right) y_j \le \sum_{j=1}^{m} b_j y_i$$

Network Flow

Flow on edge (i,j): f_{ij}

Maximize $\sum_{(s,i)\in E} f_{si}$

d_{ij} $\quad f_{ij} \le c_{(i,j)}$ (i,j) ∈ E

p_i $\quad \sum_{j:(j,i)\in E} f_{ij} - \sum_{j:(i,j)\in E} f_{ji} = 0$

i ∈ V – {s,t}

Feasible solution: a flow

Optimum: max flow

Minimize $\sum_{(i,j)\in E} c_{ij} d_{ij}$

$$d_{ij} + p_i - p_j \ge 0, \quad i \ne s, j \ne t$$
$$d_{sj} + p_j \ge 1, \quad j \ne t$$
$$d_{it} - p_i \ge 0, \quad i \ne s$$

A cut is a feasible solution

Strong Duality

Theorem (Strong Duality)

If $(x_1,\ldots,x_n)$ is an optimal solution of the primal program, and $(y_1,\ldots,y_m)$ is an optimal solution of the dual problem, then

$$\sum_{i=1}^{n} c_i x_i = \sum_{j=1}^{m} b_i y_i$$