Dynamic Programming

Design and Analysis of Algorithms

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Knapsack

The Knapsack Problem

Instance:

A set of n objects, each of which has a positive integer value v_i and a positive integer weight w_i. A weight limit W.

Objective:

Select objects so that their total weight does not exceed W, and they have maximal total value

Idea

A simple question: Should we include the last object into selection?

Let OPT(n,W) denote the maximal value of a selection of objects out of {1, …, n} such that the total weight of the selection doesn't exceed W

More general, OPT(i,U) denote the maximal value of a selection of objects out of {1, …, i} such that the total weight of the selection doesn't exceed U

Then

$$\text{OPT}(n,W) = \max\{ \text{OPT}(n-1, W),\ \text{OPT}(n-1, W - w_n) + v_n \}$$

Algorithm (First Try)

Knapsack(n,W)

set V1:=Knapsack(n-1,W)

set V2:=Knapsack(n-1,W- w_n)

output max(V1,V2+v_n)

Is it good enough?

Example

Let the values be 1,3,4,2, the weights 1,1,3,2, and W = 5

Recursion tree

Another Idea: Memoization

Let us store values OPT(i,U) as we find them

We need to store (and compute) at most $n \times W$ numbers

We'll do it in a regular way:

Instead of recursion, we will compute those values starting from smaller ones, and fill up a table

Algorithm (Second Try)

Knapsack(n,W)

```
array  M[0..n,0..W]
set M[0,w]:=0 for each w=0,1,...,W
for i=1 to n do
   for w=0 to W do
      set M[i,w]:= max{M[i-1,w],M[i-1,w-wᵢ]+vᵢ}
   endfor
endfor
```

Example

Example

Let the values be 1,3,4,2, the weights 1,1,3,2, and W = 5

i \ w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	3	4	4	4	4
3	0	3	4	4	7	8
4	0	3	4	5	7	8

$M[i,w] = \max\{ M[i-1, w], M[i-1, w - w_i] + v_i\}$

Shortest Path

Suppose that every arc e of a digraph G has length (or cost, or weight, or …) len(e)

But now we allow negative lengths (weights)

Then we can naturally define the length of a directed path in G, and the distance between any two nodes

The s-t-Shortest Path Problem

Instance:

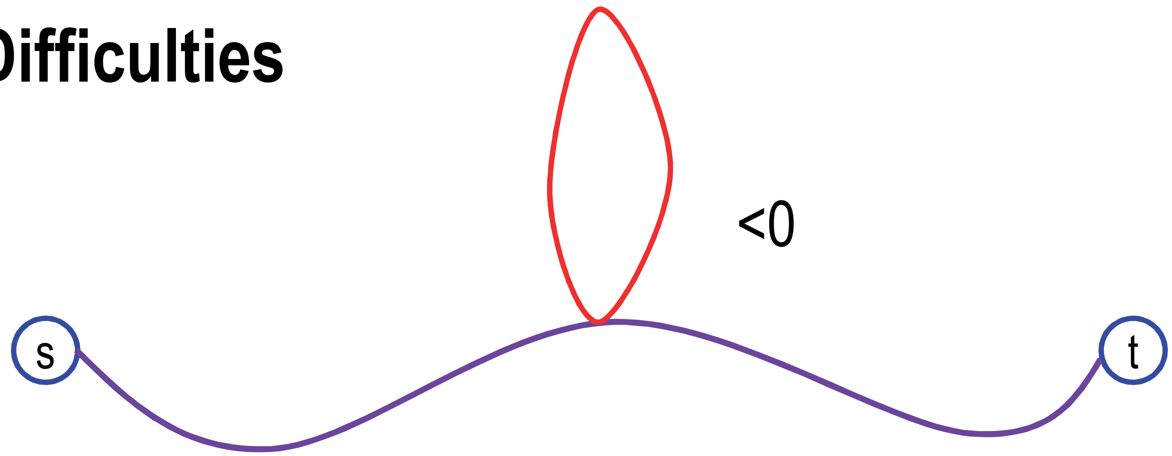
Digraph G with lengths of arcs, and nodes s,t

Objective:

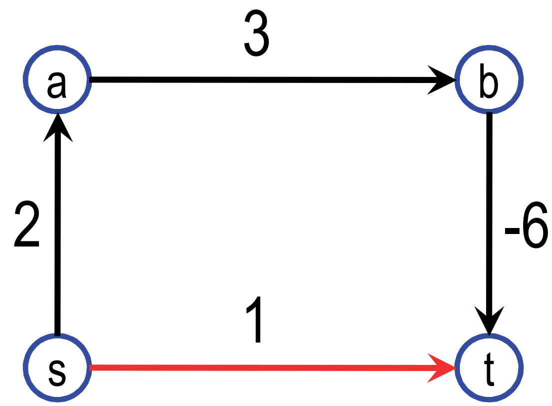
Find a shortest path between s and t

Shortest Path: Difficulties

Negative Cycles.



Greediness fails



Adding constant weight to all arcs fails

Shortest Path: Observations

Assumption

There are no negative cycles

Lemma

If graph G has no negative cycles, then there is a shortest path from s to t that is simple (i.e. does not repeat nodes), and hence has at most $n - 1$ arcs

Proof

If a shortest path P from s to t repeats a node v, then it also include a cycle C starting and ending at v.

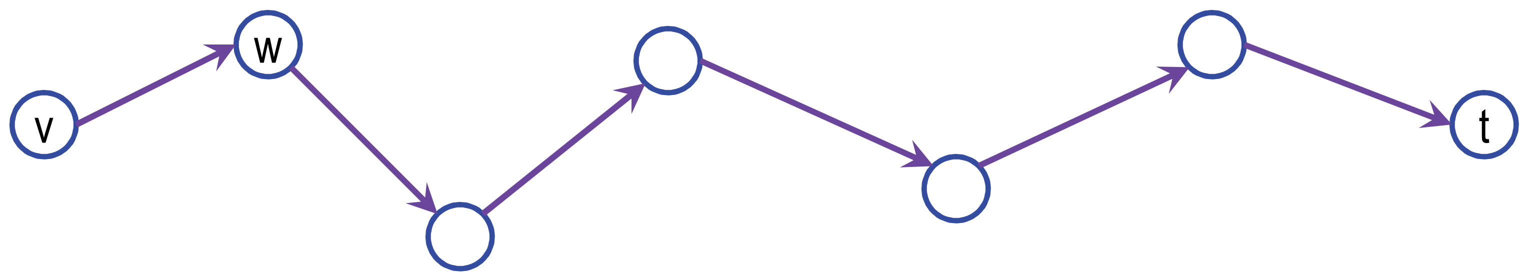
The weight of the cycle is non-negative, therefore removing the cycle makes the path shorter (no longer).

QED

Shortest Path: Dynamic Programming

We will be looking for a shortest path with increasing number of arcs

Let OPT(i,v) denote the minimum weight of a path from v to t using at most i arcs



Shortest v – t path can use i – 1 arcs. Then OPT(i,v) = OPT(i – 1,v)

Or it can use i arcs and the first arc is vw. Then

OPT(i,v) = len(vw) + OPT(i – 1,w)

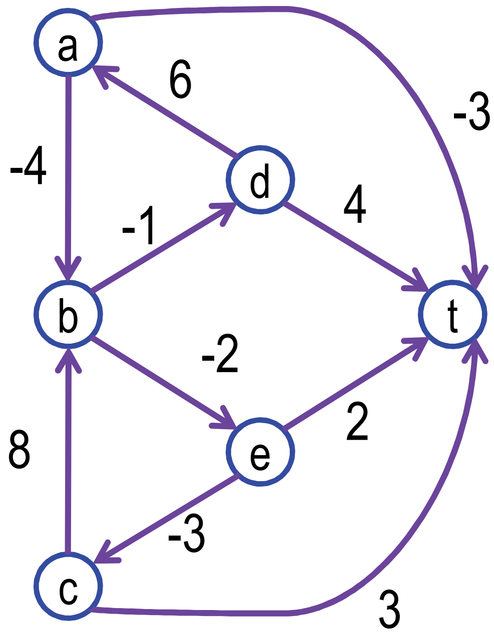
$$OPT(i,v) = \min\{OPT(i-1,v), \min_{w \in V}\{OPT(i-1,w) + len(vw)\}\}$$

Shortest Path: Algorithm

Shortest-Path(G,s,t)

```
set n:=|V|            /*number of nodes in G
array  M[0..n-1,V]
set M[0,t]:=0 and M[0,v]:=∞ for each v∈V-{t}
for i=1 to n-1 do
   for v∈V do
    set M[i,v]:=min{M[i-1,v],min_{w∈V}{M[i-1,w]+len(vw)}}
   endfor
endfor
return M[n-1,s]
```

Example



	t	a	b	c	d	e
0	0	∞	∞	∞	∞	∞
1	0	-3	∞	3	4	2
2	0	-3	0	3	3	0
3	0	-4	-2	3	3	0
4	0	-6	-2	3	2	0
5	0	-6	-2	3	0	0

$$M[i,v] = \min\{ M[i-1, v], \min_{w \in V}\{ M[i-1, w] + len(vw) \}\}$$

Shortest Path: Soundness and Running Time

Theorem

The ShortestPath algorithm correctly computes the minimum cost of an s-t path in any graph that has no negative cycles, and runs in O(n^3) time

Proof.

Soundness follows by induction from the recurrent relation for the optimal value.

DIY.

Running time:

We fill up a table with n^2 entries. Each of them requires O(n) time

Shortest Path: Soundness and Running Time

Theorem

The ShortestPath algorithm can be implemented in O(mn) time

A big improvement for sparse graphs

Proof.

Consider the computation of the array entry M[i,v]:

$$M[i,v] = \min\{ M[i-1, v], \min_{w \in V}\{ M[i-1, w] + len(vw) \}\}$$

We need only compute the minimum over all nodes w for which v has an edge to w

Let n_v denote the number of such edges

Shortest Path: Running Time Improvements

It takes O(n_v) to compute the array entry M[i,v].

It needs to be computed for every node v and for each i, $1 \le i \le n$.

Thus the bound for running time is

$$O\left(n \sum_{v \in V} n_v \right) = O(nm)$$

Indeed, n_v is the outdegree of v, and we have the result by the Handshaking Lemma.

QED

Shortest Path: Space Improvements

The straightforward implementation requires storing a table with n^2 entries

It can be reduced to O(n)

Instead of recording M[i,v] for each i, we use and update a single value M[v] for each node v, the length of the shortest path from v to t found so far

Thus we use the following recurrent relation:

$$M[v] = \min\{ M[v], \min_{w \in V}\{ M[w] + len(vw) \}\}$$

Shortest Path: Space Improvements (cntd)

Lemma

Throughout the algorithm M[v] is the length of some path from v to t, and after i rounds of updates the value M[v] is no larger than the length of the shortest from v to t using at most i edges

Shortest Path: Finding Shortest Path

In the standard version we only need to keep record on how the optimum is achieved

Consider the space saving version.

For each node v store the first node on its path to the destination t

Denote it by first(v)

Update it every time M[v] is updated

Let P be the pointer graph $P = (V, \{(v, first(v)): v \in V\})$

Shortest Path: Finding Shortest Path

Lemma

If the pointer graph P contains a cycle C, then this cycle must have negative cost.

Proof

If w = first(v) at any time, then M[v] ≥ M[w] + len(vw)

Let $v_1, v_2, \ldots, v_k$ be the nodes along the cycle C, and (v_k, v_1) the last arc to be added

Consider the values right before this arc is added

We have $M[v_i] \geq M[v_{i+1}] + len(v_i v_{i+1})$ for i = 1,…, k – 1 and

$M[v_k] > M[v_1] + len(v_k v_1)$

Adding up all the inequalities we get $0 > \sum_{i=1}^{k-1} len(v_i v_{i+1}) + len(v_k v_1)$

Shortest Path: Finding Shortest Path (cntd)

Lemma

Suppose G has no negative cycles, and let P be the pointer graph after termination of the algorithm. For each node v, the path in P from v to t is a shortest v-t path in G.

Proof

Observe that P is a tree.

Since the algorithm terminates we have M[v] = M[w] + len(vw), where w = first(v).

As M[t] = 0, the length of the path traced out by the pointer graph is exactly M[v], which is the shortest path distance.

QED

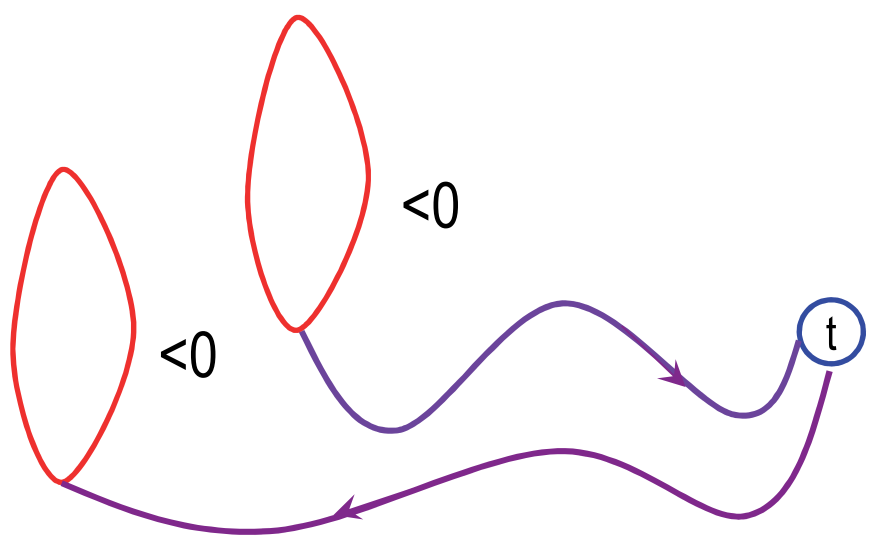
Shortest Path: Finding Negative Cycles

Two questions:

- how to decide if there is a negative cycle?
- how to find one?

Lemma

It suffices to find negative cycles C such that t can be reached from C

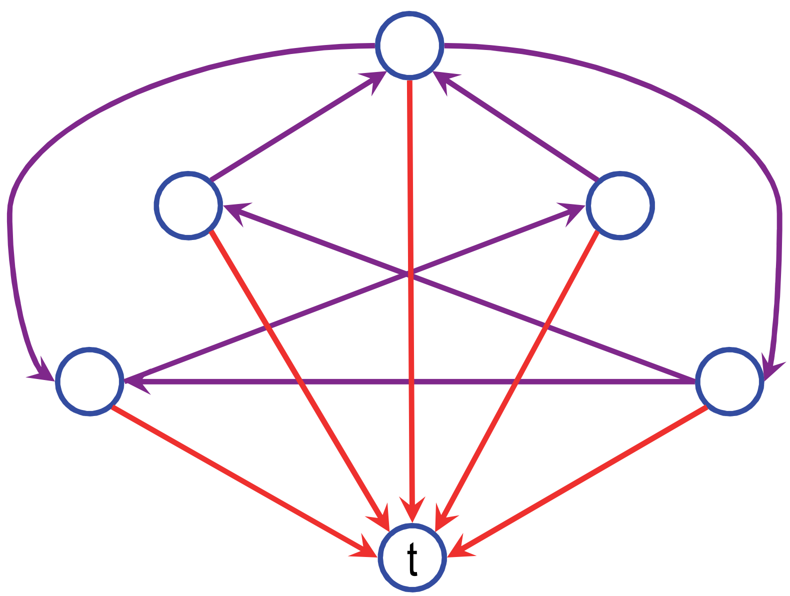


Shortest Path: Finding Negative Cycles

Proof

Let G be a graph

The augmented graph, A(G), is obtained by adding a new node and connecting every node in G with the new node



As is easily seen, G contains a negative cycle if and only if A(G) contains a negative cycle C such that t is reachable from C

QED

Shortest Path: Finding Negative Cycles (cntd)

Extend OPT(i,v) to $i \geq n$

If the graph G does not contain negative cycles then
OPT(i,v) = OPT(n – 1,v) for all nodes v and all $i \geq n$

Indeed, it follows from the observation that every shortest path contains at most n – 1 arcs.

Lemma

There is no negative cycle with a path to t if and only if
OPT(n,v) = OPT(n – 1,v)

Proof

If there is no negative cycle, then OPT(n,v) = OPT(n – 1,v) for all nodes v by the observation above

Shortest Path: Finding Negative Cycles (cntd)

Proof (cntd)

Suppose OPT(n,v) = OPT(n – 1,v) for all nodes v.

Therefore

$$
\begin{aligned}
OPT(n,v) &= \min\{ OPT(n-1,v),\ \min_{w \in V}\{ OPT(n-1,w) + len(vw) \}\} \\
&= \min\{ OPT(n,v),\ \min_{w \in V}\{ OPT(n,w) + len(vw) \}\} \\
&= OPT(n+1,v) \\
&= \ldots
\end{aligned}
$$

However, if a negative cycle from which t is reachable exists, then

$$\lim_{i \to \infty} OPT(i,v) = -\infty$$

QED

Shortest Path: Finding Negative Cycles (cntd)

Let v be a node such that OPT(n,v) $\neq$ OPT(n – 1,v).

A path P from v to t of weight OPT(n,v) must use exactly n arcs

Any simple path can have at most n – 1 arcs, therefore P contains a cycle C

Lemma

If G has n nodes and OPT(n,v) $\neq$ OPT(n – 1,v), then a path P of weight OPT(n,v) contains a cycle C, and C is negative.

Proof

Every path from v to t using less than n arcs has greater weight.

Let w be a node that occurs in P more than once.

Let C be the cycle between the two occurrences of w

Deleting C we get a shorter path of greater weight, thus C is negative