9-2020

Particle Dynamics in Anti-de Sitter Space by EIH Method

Jiusi Lei
The Graduate Center, City University of New York

Particle Dynamics in Anti-de Sitter Space by EIH Method

by

Jiusi Lei

A dissertation submitted to the Graduate Faculty in Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2020

Particle Dynamics in Anti-de Sitter Space by EIH Method

by

Jiusi Lei

This manuscript has been read and accepted by the Graduate Faculty in Physics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy

Date

V. Parameswaran Nair

Chair of Examining Committee

Date

Alexios Polychronakos

Executive Officer

Supervisory Committee:

Dimitra Karabali

Alexios Polychronakos

Daniel Kabat

Sebastian Franco

The City University of New York

Abstract

Particle Dynamics in Anti-de Sitter Space by EIH Method

by

Jiusi Lei

Advisor: Professor V. Parameswaran Nair

Following the work of Einstein, Infeld and Hoffmann, we show that particle dynamics in Anti-de Sitter spacetime can be built up by regarding singularities in spacetime manifold as the source of particles.

Since gauge fields play a foundational role in the action, the singularities are chosen to be point-like instantons. Their winding number, defined by an integration on the spheres surrounding those singularities, will turn out to be related to their masses. And their action, derived from the Chern-Simons forms, will be a co-adjoint orbit action, with group element $g \in SO(4,2)$ describing the collective coordinates of the particle.

We also consider bringing the more standard gravitational interactions into the system, which is achieved by combining Chern-Simons action with Einstein-Hilbert action. Interactions are proposed to be small perturbations in gauge fields as well as in metric tensors. The equation of motion for perturbed field is obtained and solved. The action, with the solution of perturbed field inserted, approaches Newtonian gravity in non-relativistic limit.

Acknowledgments

I would like to first thank Prof. Parameswaran Nair, my PhD advisor, for the help and guidance provided in my PhD career. His enthusiasm for science and swift mind has deeply impressed me over the years. I especially thank his kind suggestions, encouragement and patience whenever I need any help from him.

I also want to appreciate our physics community at CUNY, especially in City College. I want to thank the members of my supervisory committee: Prof. Alexios Polychronakos, Prof. Dimitra Karabali, Prof. Daniel Kabat and Prof. Sebastian Franco for all their suggestions about my research and mind-broadening discussions.

I am also sincerely grateful to Azeem, Amol, Cody, Arthur, Ariana, Alexandra, Marcelo, Xiang, Rong and those who have had great and intriguing discussions with me while I'm in City College.

I want to thank my friends for their support, advice and all joyful memories during these years in New York City.

In the end, I express my deepest gratefulness to my family for always standing by me throughout my life. Especially to my wife Dr. Caiyue Xu, your intelligence, righteousness and passion of life has always been motivating me to become a better man.

This research was supported by NSF grants 1213380, 1519449, 1820721.

Contents

Chapter 1

Introduction

Particle is one of the most basic objects and ideas in physics. In classical mechanics, particles are considered as the most fundamental blocks of all matter. Even with the acknowledgement that fields are playing a more fundamental role in modern physics, we still prefer to use particle as a nice approach when the internal effect of an object is negligible in many cases nowadays. Many decades ago, Einstein, Infeld and Hoffmann made the observation that particles could be treated as singularities of the spacetime manifold and their dynamics would then be determined by the gravitational field equations in vacuum. Beyond being a matter of principle, this has led to equations of motion in a formalism which can be applied to practical calculations in astrophysics. Recently, more general theories of gravity have generated a lot research interest. Among them, Lovelock theories, and theories where the action is in the form of Chern-Simons forms, form a special class. Inspired by the EIH work, in this thesis, we analyze a similar approach to these more general theories, focusing on particle dynamics in AdS_5 spacetime. This thesis is organized as follows.

Sec. II of this thesis will provide some background knowledge as the building blocks of our theory. EIH method will be briefly explained. It shows that particles can be regarded as singularities in spacetime manifold. And by wisely choosing boundary conditions surrounding those singularities, single particle dynamics as well as interactions can be successfully recovered from field equations. Then we have a basic introduction about instantons. They are self-dual or self-antidual configurations of the gauge field. They serve to characterize different sectors of the configuration space and naturally, being a solution to field equations, they minimize the Yang-Mills action. At last, the coadjoint orbit method will be introduced. It is used to describe particle actions in AdS_5 spacetime.

Sec. III will first discuss how coadjoint orbit method could be used to construct particle action in AdS_5 as well as its twistor form, and even string actions. Then a brief introduction about the most general gravity theory, Lovelock action, will be given. Based on the topological nature of Chern-Simons action, it is most likely that a similar trick as EIH method can help to derive particle actions in Anti-de Sitter space. Then we will give a detailed description about how single particle dynamics is constructed in AdS_5 spacetime by considering particles as singularities. In our case, point-like gauge instantons will serve as the sources. Their winding numbers, the most characteristic property of instantons, will be related to particle masses. The particle action is shown to be in the right form for the use of the coadjoint orbit method.

Sec. IV is devoted to exploring particle interactions in our model. Although the CS action could generate particle dynamics, interaction between particles is absent essentially because CS action is topological. Thus an extra term in the action is needed to introduce non-trivial interaction between particles. We select Einstein-Hilbert action because it shares the same vacuum solution as CS action and preserves parity invariance. Interactions are proposed to be fluctuations in metric tensor and the corresponding gauge fields. By expanding the actions up to the second order, the equation of motion of perturbation is solved. Now behaving as source terms, the perturbation term successfully generates non-trivial interactions between particles. In the non-relativisitc limit, we show that these terms reduce to Coulomb potential as expected.

There is a short summary of all the results in the last section, with discussions about current difficulties in the model. These difficulties require careful further investigations and hopefully receive attention from researchers in this area.

Chapter 2

Background

This chapter will give a brief review on the tools and ideas that are the fundamentals of this thesis: the Einstein-Infeld-Hoffmann method, Gauge Instantons and the coadjoint orbit method. They are not a very detailed review, but the most important idea of each topic, which play a crucial role in later development, will be discussed.

2.1 Einstein-Infeld-Hoffmann (EIH) Method

The starting point of the EIH idea is a simple question: Can the motion of particles be determined by equations for the fields which they generate? The reason why one might expect this is the following. Consider, for example, the electromagnetic field. If we have charge particle sources, the field configuration at a given time t is a solution of the field equations. If we remove the points corresponding to the locations of the charges from the manifold under consideration, then we have a solution of the free field equations. Now consider the same situation at a later time t' when the sources may have moved to new locations. This field configuration at t' is also a solution of the free field equations. Can we describe this new configuration as the time-evolved version of the earlier one, with evolution just controlled by the free field equations? The answer in general is no, because the energy and momentum of the particles are not just what is contained in the energy-momentum of the fields they carry, so additional dynamics is needed to complete the time-evolution. The only possible exception would be gravity, because energy and momentum are themselves defined by the asymptotic behavior of the gravitational fields or spacetime curvatures. So one would expect that in a theory of gravity the vacuum field equations on a manifold with a number of points removed, corresponding to locations of the particles viewed as singularities of the fields, should be describing particle dynamics. The work of Einstein, Infeld and Hoffmann demonstrates that this is indeed the case. They showed that the motion of particles, even including gravitational interactions, can be derived from the field equations of General Relativity in empty space. Around each singularity, a small region of spacetime (say, denoted by C_k for the k-th singularity) has to be removed and appropriate boundary conditions have to be imposed, with the vacuum field equations holding everywhere else, i.e. on spacetime $\mathcal{M}$ minus the regions $\{C_k\}$. The boundary conditions define the key characteristics of particles, e.g., their masses, very similar to what is done in the ADM (Arnowitt-Deser-Misner) formalism (which came much later). EIH were able to construct a systematic expansion starting with the Newtonian limit and including relativistic corrections. Beyond the theoretical interest, EIH method is even useful practically and has been applied to astrophysical cases such as binary and multiple star systems [11] [28], where relativistic corrections (at least to the post-Newtonian approximation) are important.

Here a very brief review of EIH method will be given. For the detailed calculation and discussion, one can consult their original paper [14] and a later review by Infeld [17]. Starting with vacuum, we expand the metric around its flat background as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.1}$$

Introduce the following tensor constructed from $h_{\mu\nu}$,

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{2.2}$$

Then the vacuum Einstein equation can be written as the following set of equations:

$$\partial_i^2\gamma_{00} = 2\,\Lambda_{00}, \quad \partial_i^2\gamma_{0n} = 2\,\Lambda_{0n}, \quad \partial_i^2\gamma_{mn} = 2\Lambda_{mn} \tag{2.3}$$

The Latin indices refer to the spatial components. Λ's are quantities constructed from non-linear terms in the Einstein equation. Instead of the harmonic coordinate condition, EIH have used the following conditions to reduce extra degrees of freedom due to coordinate transformations:

$$\partial_i\gamma_{0i} - \gamma_{00,0} = 0, \quad \partial_i\gamma_{mi} = 0 \tag{2.4}$$

From these we construct the following two relations:

$$\partial_i(\partial_i\gamma_{0n} - \partial_n\gamma_{0i}) = 2\Lambda_{0n} - \partial_0\partial_n\gamma_{00}, \quad \partial_i(\partial_i\gamma_{mn} - \partial_n\gamma_{mi}) = 2\Lambda_{mn} \tag{2.5}$$

Notice that the quantities on the left hand side can be written as curls of 3-vectors.

$$\begin{aligned}
(\nabla\times B)_n &= \partial_i(\partial_i\gamma_{0n} - \partial_n\gamma_{0i}), \qquad \partial_i\gamma_{0n} - \partial_n\gamma_{0i} = \epsilon_{nsa}B_a \\
(\nabla\times \tilde{B}_m)_n &= \partial_i(\partial_i\gamma_{mn} - \partial_n\gamma_{mi}), \qquad \partial_i\gamma_{mn} - \partial_n\gamma_{mi} = \epsilon_{nsa}\tilde{B}_{ma}
\end{aligned} \tag{2.6}$$

We can integrate these on small spheres surrounding the singularities, with the fields not singular on the surfaces, to get

$$\oint dS^n\,[\Lambda_{0n} - \partial_n\partial_0\gamma_{00}] = 0, \qquad 2\oint dS^n\,\Lambda_{mn} = 0 \tag{2.7}$$

These integral conditions will be the key ingredient for the EIH analysis in addition to the vacuum field equations in (2.3). By virtue of the identity $\nabla \cdot (\nabla \times A) = 0$, (2.5) also lead to

$$\partial_i(\partial_i\partial_0\gamma_{00} - 2\Lambda_{0n}) = 2\partial_0\Lambda_{00} - 2\partial_i\Lambda_{0i} = 0, \quad 2\partial_i\Lambda_{mi} = 0 \tag{2.8}$$

These may be viewed as the reduced version of the Bianchi identity for the Einstein field equations.

The strategy used by EIH is then the following. Consider solving the field equations (2.3) in a series expansion, starting with the nonrelativistic limit. The energy-momentum tensor for a point-particle is of the form $T^{\mu\nu} \sim \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$, and so, for small velocities, we have the hierarchy,

$$|T^{00}| \gg |T^{0i}| \gg |T^{ij}| \tag{2.9}$$

Based on this, and since the field equations have $T^{\mu\nu}$ as the source, EIH proposed to solve the field equations, in the vacuum on $\mathcal{M} - \{C_k\}$, by a series expansion

$$\begin{aligned} \gamma_{00} &= \lambda^2\,\gamma_{00}^{(2)} + \lambda^4\,\gamma_{00}^{(4)} + \cdots \\ \gamma_{0n} &= \lambda^3\,\gamma_{0n}^{(3)} + \lambda^5\,\gamma_{0n}^{(5)} + \cdots \\ \gamma_{mn} &= \lambda^4\,\gamma_{mn}^{(4)} + \lambda^6\,\gamma_{mn}^{(6)} + \cdots \end{aligned} \tag{2.10}$$

(The parameter λ is just to keep track of different orders in solving the field equations, it can be set to 1 at the end.) In the lowest order, the field equations (2.3) give $\gamma_{00,ss} \approx 0$ (Laplace equation), so that we can take

$$\gamma_{00} = 2\varphi, \quad \varphi = -\sum_k \frac{m_k}{|x - x_k|} \tag{2.11}$$

where m_k can be identified as the masses and x_k are the positions of the singularities. If Λ_{0n}

is evaluated using this approximation, the integral conditions (2.7) will not hold, i.e.,

$$\oint_k dS^n \left[\Lambda_{0n} - \partial_n \partial_0 \gamma_{00}\right] \neq 0 \tag{2.12}$$

for surfaces around each singularity labeled by k. EIH then take an ansatz of the form

$$\oint_k dS^n \left[\Lambda_{0n} - \partial_n \partial_0 \gamma_{00}\right] = (c_0)_k, \quad 2 \oint_k dS^n \, \Lambda_{mn} = (c_m)_k \tag{2.13}$$

where $(c_0)_k$, $(c_m)_k$ are taken to be functions of the positions x_k (which are functions of time) and their derivatives. By using the Dirac δ-function, this is equivalent to the ansatz

$$\partial_i(\partial_i \partial_0 \gamma_{00} - 2\Lambda_0) = 4\pi \sum_k (c_0)_k \delta^{(3)}(x - x_k) \tag{2.14}$$

$$2\partial_i \Lambda_{mi} = 4\pi \sum_k (c_m)_k \delta^{(3)}(x - x_k) \tag{2.15}$$

According to (2.3), this may be expressed in terms of the γ's as

$$\partial_0 \gamma_{00} - \partial_i \gamma_{0i} = -\sum_k \frac{(c_0)_k}{|x - x_k|} \tag{2.16}$$

$$\partial_i \gamma_{mi} = -\sum_k \frac{(c_m)_k}{|x - x_k|} \tag{2.17}$$

There is some freedom in these ansatze. Notice that extra terms can be added to γ_{00} and γ_{0n} without affecting the first two equations in (2.3), again related to the freedom of coordinate transformations. These equations are expected to be true everywhere on the manifold. Since a small region around the singularities are removed, additional terms proportional to δ-function at singularities will not break this relation outside singularities. Then by choosing terms appropriately, we can eliminate $(c_0)_k/|x - x_k|$ terms. In summary, the field equations are reduced to the following set:

$$\partial_i^2 \gamma_{\mu\nu} = 2\Lambda_{\mu\nu}, \quad \partial^\mu \gamma_{0\mu} = 0, \quad \partial_i \gamma_{mi} = -\sum_k \frac{(c_m)_k}{|x - x_k|} \tag{2.18}$$

These reduced field equations can be solved in a series using (2.10). Once a solution is obtained, one can then impose the integral conditions (2.7) (which must hold true as an identity) to obtain constraints on the coefficients $(c_m)_k$. Explicitly, this amounts to requiring

$$(c_m)_k = \sum_l \lambda^l (c_m)_k^{(l)} = 0 \tag{2.19}$$

This becomes the equation of motion for the positions of the particles. There are exactly three equations for each particle as expected. Starting with (2.11) and eliminating $(c_0)_k$ and solving to the next order, EIH show that one gets the integral

$$(c_m)_1^{(4)} = \frac{1}{4\pi} \oint S^n \; 2\Lambda_{mn}^{(4)} = 4m_1 \left(\ddot{x}_1^m - \partial_m \left(\frac{m_2}{r_2} \right) \right) \tag{2.20}$$

(We consider a two-particle case for simplicity.) Setting this to zero as required by (2.7), we get

$$m_1 \ddot{\vec{x}}_1 = \nabla \left(\frac{m_1 m_2}{r} \right) \tag{2.21}$$

which is recognized as the Newtonian law of gravitation. As mentioned in the beginning, a systematic expansion to higher orders can be generated, but we will not go into the details.

It is also useful to think about this with the benefit of hindsight from the ADM formalism [2]. Consider the metric in the ADM splitting,

$$ds^2 = N^2 \, dt^2 - \sigma_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{2.22}$$

where σ_{ij} is the metric for the spatial slice and N, N^i are the lapse and shift functions, respectively. The phase space version of the Einstein-Hilbert action is

$$\begin{aligned} S &= \frac{1}{16\pi G}\int d^d x\,\left[\Pi^{ij}\dot{\sigma}_{ij} - \sqrt{\sigma}\left(N\,H + N^i H_i\right)\right] + S_{ADM} \\ S_{ADM} &= \int dt\left[\oint dS^i\,N(\partial_j\sigma_{ij} - \partial_i\sigma_{jj}) + 2\oint dS^i\,N^m\sigma_{mn}\Pi^n_i\right] \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} H &= -R^{(d-1)} + \frac{1}{\sigma}\left(\Pi^{ij}\Pi_{ij} - \frac{1}{2}\Pi^2\right) \\ H_i &= -2\,\sigma_{im}D_j\left(\frac{\Pi^{jm}}{\sqrt{\sigma}}\right) \end{aligned} \tag{2.24}$$

Here $R^{(d-1)}$ is the Ricci scalar for the spatial slice and D_j denotes the covariant derivative.

The bulk equations of motion will reduce the bulk terms in the action (2.23) to zero. Consider evaluating the boundary terms for a single particle. N, N^i can be viewed as arising from a coordinate transformation. For a point-particle solution of the field equations, if we take the values of N, N^i at spatial infinity as functions of time, independent of angular coordinates, then we find

$$\begin{aligned} S_{ADM} &= dZ^\mu\,P_\mu \qquad N^\mu = \frac{dZ^\mu}{dt} \\ P_0 &= \oint dS^i\,(\partial_j\sigma_{ij} - \partial_i\sigma_{jj}) \\ P_i &= 2\oint dS^i\,\sigma_{mn}\,\Pi^n_i \end{aligned} \tag{2.25}$$

We see that we get the correct point-particle action with the ADM energy and momentum for the particle. This suggest that we can directly work with the action rather than the equations of motion as EIH did. The ADM energies are picked up at the surface at spatial infinity, but if we have sufficiently localized curvatures, we should be able to do this for small surfaces surrounding the singularities of the curvatures. This is what we will do for the Chern-Simons gravity with the inclusion of the EH action.

2.2 Instanton

For our purpose, instantons for a nonabelian gauge field will play an important role. As an example, let us consider the Euclidean action of a gauge field in $SU(2)$ group. The action is given by

$$S = \frac{1}{4g^2} \int d^4x F^a_{\mu\nu} F^a_{\mu\nu} \tag{2.26}$$

where g is the coupling constant. Define the dual field strength as

$$\tilde{F}^a_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\lambda} F^a_{\rho\lambda} \tag{2.27}$$

Then the action can be rewritten as

$$g^2\, S = \int d^4x \left[\pm \frac{1}{4} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} + \frac{1}{8} (F^a_{\mu\nu} \mp \tilde{F}^a_{\mu\nu})^2 \right] \geq \pm \int d^4x \frac{1}{4} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} \tag{2.28}$$

We notice that the action has a lower bound given by the integral of $F\tilde{F}$, which is saturated only when the field strength is self-dual or antiself-dual, i.e., $F^a_{\mu\nu} = \pm \tilde{F}^a_{\mu\nu}$. It is also worth mentioning that, the self-dual/antiself-dual field actually is a solution of the field equation, by the virtue of Bianchi identity:

$$D_\mu F^a_{\mu\nu} = \pm D_\mu \tilde{F}^a_{\mu\nu} = \pm \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} D_\mu F^a_{\alpha\beta} = 0 \tag{2.29}$$

Such self-dual solutions are then called instantons, the antiself-dual solutions being anti-instantons. The instanton number is defined to be

$$Q = \frac{1}{32\pi^2} \int d^4x\ F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} \tag{2.30}$$

And in terms of this the bound on the action (2.28) is

$$S \geq \frac{8\pi^2}{g^2} Q \tag{2.31}$$

The instanton number Q is a topological quantity, i.e., invariant under small deformations of the field A. The statement is clearer if we notice the quantity is actually a surface integral:

$$F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = \partial_\mu \left[\varepsilon_{\mu\nu\rho\lambda} \left(2A^a_\nu \partial_\rho A^a_\lambda + \frac{2}{3} g \varepsilon^{abc} A^a_\nu A^b_\rho A^c_\lambda \right) \right] \tag{2.32}$$

In terms of differential forms, the gauge fields are

$$A = (-it_a) A^a_\mu dx^\mu, \qquad F = (-it_a)\, \frac{1}{2} F^a_{\mu\nu}\, dx^\mu \wedge dx^\nu \tag{2.33}$$

where t_a are 2×2 matrices which are generators of $SU(2)$, with $\mathrm{Tr}(t_a t_b) = \frac{1}{2}\delta_{ab}$. ($t_a = \frac{1}{2}\sigma_a$, where σ_a are the Pauli matrices.) In terms of differential forms, (2.32) can be written as

$$\mathrm{Tr}(F \wedge F) = d\, \mathrm{Tr} \left(AdA + \frac{2}{3} A^3 \right) = d\ CS(A) \tag{2.34}$$

The right hand side is the famous Chern-Simos forms in three dimensions. To show the invariance of Q, take a small variation $A \to A + a$, and keep only first order terms to get

$$\begin{aligned} CS(A+a) &= CS(A) + \mathrm{Tr}(adA + dAa + 2aA^3) \\ &= CS(A) + \mathrm{Tr}\left[2a(dA + A^2)\right] \end{aligned} \tag{2.35}$$

If $F = dA + A^2 = 0$, which must be satisfied by our instanton configuration at the boundary of spatial infinity (otherwise the integration will not be finite), the instanton number is proven to be invariant. Later in this thesis, we will use instantons as the source of particles. These instantons are taken to be in their point-like limit, i.e., we will eventually shrink the size of instantons to zero. To be more specific, it is useful to consider the form of the exact instanton solutions. One way to display the $SU(2)$ one-instanton solution is given by the so-called singular gauge [26] [27]:

$$\begin{aligned} A^a_\mu &= -\frac{1}{g}\bar{\eta}^a_{\mu\nu}\partial_\nu \ln\left[1+\frac{\rho^2}{(x-x_0)^2}\right] \\ &= \frac{2}{g}\bar{\eta}^a_{\mu\nu}(x-x_0)_\nu \frac{\rho^2}{(x-x_0)^2[(x-x_0)^2+\rho^2]} \end{aligned} \tag{2.36}$$

where $\bar{\eta}^a_{\mu\nu} = -\bar{\eta}^a_{\nu\mu}$ is the 't Hooft symbol given by

$$\bar{\eta}^a_{ij} = \epsilon_{aij}, \qquad \eta^a_{i0} = \delta_{ai} \tag{2.37}$$

The parameter ρ in (2.36) is called the instanton size. The field A behaves singularly at point x_0 but the integral of $F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}$ over the whole space is finite, which is the instanton number. Equivalently, according to (2.32), the integral is actually on the boundary surface, so the singularity behavior at point x_0 is avoided. Another convenient form, which is a gauge transformation of (2.36), is given by

$$\bar{A}^a_\mu = \frac{2}{g}\eta^a_{\mu\nu}\frac{(x-x_0)_\nu}{(x-x_0)^2+\rho^2} \tag{2.38}$$

In this parametrization, the bad behavior of gauge field A at point x_0 no longer appears, because the transformation matrix is singular at x_0, or we could say that the singularity has been tranferred to infinity:

$$\bar{A} = UAU^{-1} - dUU^{-1}, \quad U = \frac{i(\vec{\sigma}\cdot\vec{X})+X_4}{\sqrt{X^2}}, \quad X_\mu = (x-x_0)_\mu \tag{2.39}$$

To show why the parameter ρ is referred to as instanton size, it is useful to consider the field strenth F or the instanton density $\mathrm{Tr}(F\wedge F)$, because the density is gauge-invariant. From

(2.38) or (2.36), the instanton density is easily calculated:

$$\mathrm{Tr}(F \wedge F) = F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} \sim \frac{\rho^4}{[(x-x_0)^2+\rho^2]^4} \tag{2.40}$$

In zero-size approach, using the following relation:

$$\delta(x-x_0) = \frac{1}{\pi}\lim_{\epsilon\to 0}\frac{\epsilon}{(x-x_0)^2+\epsilon^2} \tag{2.41}$$

we see that the instanton density is replaced by $\delta^{(4)}(x-x_0)$. More rigorously, we can carry out the integral $\int \mathrm{Tr}(F \wedge F)$ over a small ball containing the point x_0, or equivalently integrate the Chern-Simons form on the sphere of the ball. In the limit $\rho \to 0$, the result will be indepentend of the radius of the small ball and gives us the instanton number as expected. So no matter what kind of gauge is chosen, in the point-like limit, we can always say that the instanton behaves like a Dirac-δ function, and the integration over any surface surrounding the instanton will give us the instanton number, or the winding number.

Besides gauge instanton, there are also gravitational instantons. They are defined in a similar manner. Analogous to the case in Yang-Mills theory, the curvature 2-form is self-dual or antiself-dual, and again by using the Bianchi identity, one can show that the vacuum field equation (Einstein equation) is automatically satisfied, i.e.,

$$R_{ab} = \pm\tilde{R}_{ab} = \pm\frac{1}{2}\varepsilon_{abcd}R_{cd} \tag{2.42}$$

In the case of gravity, the above condition can be reduced to the self-duality or antiself-duality of connection 1-form:

$$R_{ab} = \pm\tilde{R}_{ab} \quad \Leftrightarrow \quad \omega_{ab} = \pm\tilde{\omega}_{ab} = \pm\frac{1}{2}\varepsilon_{abcd}\omega_{cd} \tag{2.43}$$

with the following definition of curvature 2-form in terms of connection 1-form:

$$R_{ab} = d\omega_{ab} + \omega_{ac}\omega_{cb} \tag{2.44}$$

An example of the gravitational instanton solution is the one given by Eguchi and Hanson [13]:

$$ds^2 = \frac{dr^2}{1-\left(\frac{a}{r}\right)^4} + r^2\left\{\sigma_x^2 + \sigma_y^2 + \left[1-\left(\frac{a}{r}\right)^4\right]\sigma_z^2\right\} \tag{2.45}$$

where σ_i's are defined in terms of the polar coordinates as

$$\sigma_x = \frac{1}{2}(\sin\psi d\theta - \sin\theta\cos\psi d\phi),\ \ \sigma_y = -\frac{1}{2}(\cos\psi d\theta + \sin\theta\sin\psi d\phi),\ \ \sigma_z = \frac{1}{2}(d\psi + \cos\theta d\phi) \tag{2.46}$$

It is easily checked that the curvature is self-dual. For a list of other gravitational instanton solutions, one can refer to [12].

This brief discussion about instantons should suffice for this thesis, especially the argument about zero-size limit. The general solution of any $SU(N)$ instanton is discussed in [9]. More detailed discussion about gravitational instantons is given by Gibbons and Hawking [15].

2.3 Coadjoint Orbit Method

The coadjoint orbit method gives a process for constructing unitary irreducible representations of a Lie group. This has its origin in the works of Borel and Weil and Bott in the 1950s and in the geometric quantization work of Kostant, Souriau and Kirillov [20]. The key question, from a physics point of view, is the following: Is there a classical action which upon quantization leads to exactly one unitary irreducible representation of a Lie group? A physical system such as a rigid rotor does lead to representations of the rotation group, but it gives all representations, not just one specified irreducible representation. The focus here is on getting exactly one irreducible representation. For example, asking for a classical theory which, upon quantization, describes the spin states of a particle is a clear case of such a problem. In quantum mechanics, more generally, particle states can be considered, actually defined, by a unitary irreducible representation of the spacetime group, such as the Poincaré group. So naturally the quantization of coadjoint orbits can be utilized to describe particle dynamics.

To state the key theorem behind this approach, we start with a few preliminary results. Consider a semisimple Lie group G, with T as its maximal torus. Thus if g denotes an element of G, in a matrix realization, we can use $\{T_a\}$ as a basis for the generators of G. The set of diagonal generators $\{t_\alpha\}$, $\alpha = 1, 2, \cdots, r$, where r is the rank of the group, forms the Cartan subalgebra and group elements of the form $h = \exp(it_\alpha \varphi^\alpha)$ form the maximal torus of G. The space G/T is defined by identifying elements of G which differ by the action of the diagonal elements, i.e., $g \sim g\,h$. This space admits a symplectic structure, the symplectic potential is of the form

$$\mathcal{A} = i \sum_\alpha w_\alpha \,\mathrm{Tr}(t_\alpha g^{-1} dg) \tag{2.47}$$

Here w_α are a set of real numbers. The action is then taken as the integral of $\mathcal{A}$ along a trajectory,

$$S = \int \mathcal{A}\, d\tau = i \sum_\alpha w_\alpha \int d\tau \;\mathrm{Tr}\left(t_\alpha\, g^{-1} \frac{dg}{d\tau} \right) \tag{2.48}$$

The key theorem then states that, upon quantization, the theory defined by this action gives a Hilbert space of states which is the carrier space for a unitary irreducible representation (UIR) of G characterized by $(w_1, w_2, \cdots, w_r)$ as the highest weight state of the representation. Notice that, under the transformation $g \to g\, \exp(-it_\alpha \varphi^\alpha)$, $\mathcal{A} \to \mathcal{A} + df$, $f = \sum_\alpha w_\alpha \varphi^\alpha$. (We use the normalization $\mathrm{Tr}(t_\alpha t_\beta) = \delta_{\alpha\beta}$.) This shows that $\Omega = d\mathcal{A}$ is actually defined on

G/T, although $\mathcal{A}$ by itself is not. The transformation $\mathcal{A} \to \mathcal{A} + df$ also shows that the wave functions should transform as $\psi(g) \to e^{if}\,\psi(g)$. The periodicity conditions of the angular parameters φ^α will impose quantization conditions on the weights $\{w_\alpha\}$. This will be in accordance with what is expected for the UIRs of the group G.

As an example, one can apply this method to the Poincaré group to obtain free particle states in relativistic quantum mechanics. This has been discussed by many authors, including the recent paper [19] where applications to fluids were also considered. The Poincaré group does not have a well defined finite dimensional matrix representation. To bypass the difficulty of defining traces, one has to either do a regularization on infinite dimensional representation, or choose to obtain Poincaré group as a contraction over de Sitter group, which has a finite-dimensional representation. We will follow the latter one. The generators of de Sitter group are given by:

$$
\begin{aligned}
P_\mu &= \gamma_\mu / r_0 \\
J_{\mu\nu} &= \gamma_{\mu\nu} = (i/4)[\gamma_\mu, \gamma_\nu]
\end{aligned}
\tag{2.49}
$$

where γ_μ's are the standard 4×4 Dirac matrices and the parameter r_0 is the curvature of de Sitter space. In the end, we will take $r_0 \to \infty$ to recover the Poincaré group. The group element g can be parametrized as:

$$
g = e^{iP_\mu x_\mu} \Lambda \tag{2.50}
$$

where Λ is generated by $J_{\mu\nu}$, corresponding to Lorentz transformation. The de Sitter group has rank equal to 2, so we choose P_0 and J_{12} to be the elements of its Cartan subgroup.

Their corresponding weights will be mass and spin. Λ can be furtherly parametrized as:

$$\Lambda = B(p)R$$
$$B(p) = \frac{1}{\sqrt{2m(p_0+m)}} \begin{bmatrix} p_0+m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & p_0+m \end{bmatrix} \tag{2.51}$$

$B(p)$ is the boost matrix and R is purely spatial rotations and m denotes $\sqrt{p^2}$. Then we can write down the canonical one-form:

$$\mathcal{A} = ir_0^2\sqrt{p^2}\,\mathrm{Tr}\left(\frac{\gamma_0}{r_0}g^{-1}dg\right) + is\,\mathrm{Tr}(J_{12}g^{-1}dg) \tag{2.52}$$

In the limit $r_0 \to \infty$ this leads to

$$\mathcal{A} = -p_\mu dx^\mu + i\frac{s}{2}\,\mathrm{Tr}(\Sigma_3\Lambda^{-1}d\Lambda), \quad \Sigma_a = \begin{bmatrix} \sigma_a & 0 \\ 0 & \sigma_a \end{bmatrix} \tag{2.53}$$

The Poincaré limit has been successfully recovered.

Chapter 3

Single Particle Action

In this chapter, we will utilize the idea of the EIH method, treating particles as singularities on the spacetime manifold. By a proper reduction from the gravitational Chern-Simons action, we show that the point-particle action for single particles can be constructed. The action will be of the coadjoint orbit form in AdS$_5$ space. So we will start by writing down the specific form of the action for AdS$_5$ before turning to the Chern-Simons gravity.

3.1 Coadjoint Orbit Action in AdS$_5$

We want to specialize the general discussion of the coadjoint orbit method from the introduction to the case of AdS$_5$ and write down the symplectic one-form of a point particle. Towards this, we start with the observation that anti-de Sitter spacetime (in 4+1 dimensions) can be realized as the coset space of $SO(4,2)/SO(4,1)$. A parametrization of the Lie group $SO(4,2)$, for a group element g, which is particularly convenient for us, is given by

$$g = \begin{pmatrix} \sqrt{z} & i\tilde{X}/\sqrt{z} \\ 0 & 1/\sqrt{z} \end{pmatrix} \Lambda, \qquad \tilde{X} = x^0 - \vec{x} \cdot \vec{\sigma} \tag{3.1}$$

where $\Lambda \in SO(4,1)$. We will be using the 4×4 spinorial representation of these groups, so $SO(4,1)$ is generated by the commutators $\Sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$. Here we have selected the following representation for the 4×4 Dirac matrices,

$$\gamma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \gamma_i = \begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{bmatrix}, \ \gamma_5 = -i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.2}$$

They are normalized as $\mathrm{Tr}(\gamma_\mu \gamma_\nu) = 4\, \eta_{\mu\nu}$, $\eta_{\mu\nu} = \mathrm{Diag}\{1, -1, -1, -1, -1\}$. The parametrization of the group element in (3.1) leads to the following coset space metric:

$$ds^2 = -\frac{R^2}{4} \eta^{\mu\nu}\, \mathrm{Tr}(\gamma_\mu g^{-1} dg)\, \mathrm{Tr}(\gamma_\nu g^{-1} dg) = \frac{R^2}{z^2}(dx^2 - dz^2) \tag{3.3}$$

This is the well known Poincaré patch metric of AdS_5.

To write down an orbit action in AdS_5, we will need three elements in the Cartan subgroup of $SO(4,2)$, which will be mutually commuting. One of them will correspond to a constraint on momentum while the other two will describe spin. We need two for the latter since the spin part of the isotropy group $SO(4,1)$ is $SO(4)$, which has rank equal to 2. The Cartan elements of the algebra will be chosen to be $(\gamma_0, \gamma_1\gamma_2, \gamma_3\gamma_5)$ or $(\gamma_0, \Sigma_{12}, \Sigma_{35})$. The corresponding weights will give the mass and spins. So the symplectic one-form of the particle action in AdS_5 is obtained as:

$$\mathcal{A} = -i\frac{mR}{2}\, \mathrm{Tr}(\gamma_0 g^{-1} dg) + \frac{s_1}{2}\, \mathrm{Tr}(\gamma_1\gamma_2 g^{-1} dg) + \frac{s_2}{2}\, \mathrm{Tr}(\gamma_3\gamma_5 g^{-1} dg) \tag{3.4}$$

As for now, we will mainly focus on the part without spins, so we can consider the simpler version,

$$\mathcal{A} = -i\frac{mR}{2}\, \mathrm{Tr}(\gamma_0 g^{-1} dg) = mR\, \eta_{\mu\nu}\, \Lambda^\mu_0\, \frac{dx^\nu}{z} \tag{3.5}$$

where, for the second expression, we have used the vector representation of the Lorentz group defined by $\Lambda\gamma_\mu\Lambda^{-1} = \gamma_\nu\Lambda^\nu{}_\mu$. This shows that the particle momentum can be identified as

$$p_\mu = mR\,\eta_{\mu\nu}\frac{\Lambda^\nu{}_0}{z} \tag{3.6}$$

Using the property of Lorentz transformations, namely that $\eta_{\mu\nu}\Lambda^\mu_0\Lambda^\nu_0 = \eta_{00} = 1$, we can verify that this momentum p_μ obeys the correct mass shell condition in AdS$_5$ spacetime, namely,

$$g^{\mu\nu}p_\mu p_\nu = m^2 \tag{3.7}$$

It is also possible to write the action in terms of a set of twistor variables. Although this is not directly relevant to our analysis using the EIH method, it does provide another perspective, and we note that twistors in AdS space have been of some research interest recently [3] . For this, we consider another set of Dirac matrices given by

$$\Gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Gamma_i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \quad \Gamma_5 = -i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.8}$$

These are related to γ-matrices by the following similarity transformation:

$$\Gamma_\mu = S\gamma_\mu S^{-1}, \quad S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{3.9}$$

We also define the transformed group element $N = SgS^{-1}$. It is easy to verify that $N^\dagger\Gamma_0 N = \Gamma_0$. From the form of Γ_0, we see that transformations which preserve it are elements of $SU(2,2)$. Thus $N \in SU(2,2)$. This representation shows explicitly the realization of $SO(4,2)$ as $SU(2,2)$ for the spinorial representation. Using this version of the group element, as in (3.5), the symplectic one-form is:

$$\begin{aligned}
\mathcal{A} &= -i\frac{mR}{2}\,\mathrm{Tr}(\Gamma_0 N^{-1}dN) \\
&= -i\frac{mR}{2}\,\mathrm{Tr}[(1+\Gamma_0)N^{-1}dN] \\
&= -imRN^\dagger_{ar}(\Gamma_0)_{rs}dN_{sa} \qquad (3.10)\\
&= -imR\left[N^\dagger_{ab}dN_{ba} - N^\dagger_{a\tilde{b}}dN_{\tilde{b}a}\right]
\end{aligned}$$

where a, b run from 1 to 2 and $\tilde{b} = 1, 2$ corresponds to $r, s = 3, 4$. We now define the following 2×2 matrices:

$$\xi_{ba} = \sqrt{2mR}\,N_{ba}, \quad \zeta_{ba} = \sqrt{2mR}\,N_{\tilde{b}a} \qquad (3.11)$$

The symplectic potential can now be written as

$$\mathcal{A} = -\frac{i}{2}\,\mathrm{Tr}(\xi^\dagger d\xi - \zeta^\dagger d\zeta) \qquad (3.12)$$

with the constraint

$$\xi^\dagger\xi - \zeta^\dagger\zeta = 2mR \qquad (3.13)$$

This constraint is just the statement $N^\dagger\Gamma_0 N = \Gamma_0$ expressed in terms of ξ_{ba}, ζ_{ba}; it is again the statement that ξ_{ba}, ζ_{ba} describe an element of $SU(2,2)$. Further, in order to write the action in a more recognizable twistor form, we define U and W by

$$\xi = \frac{U - iW}{\sqrt{2}}, \quad \zeta = \frac{W - iU}{\sqrt{2}} \qquad (3.14)$$

Then the action written in terms of these new variables is

$$S = \frac{1}{2}\int d\tau\,\mathrm{Tr}(W^\dagger\dot{U} - U^\dagger\dot{W}) \qquad (3.15)$$

and the constraint becomes:

$$U^\dagger W - W^\dagger U = i(2mR) \tag{3.16}$$

The action (3.15) is in agreement with the twistor description of massive particles in AdS_5 proposed in [3].

The action for a string in AdS_5 can also be derived using the orbit method. String action, as is well known, can be viewed as the area of a world surface tracing out a two-dimensional surface in spacetime. Therefore a timelike one-form and a spacelike one-form are needed to construct the area element of string world-sheet. We can choose the timelike one-form as in (3.5), and choose the following spacelike one-form:

$$\mathcal{B} = -i\frac{\tilde{m}R}{2}\,\mathrm{Tr}(\gamma_3 g^{-1}dg) = mR\,\eta_{\mu\nu}\,\Lambda^\mu{}_3\,\frac{dx^\nu}{z} \tag{3.17}$$

The action is then defined as the integral of the two-form:

$$\begin{aligned} S &= \int \mathcal{A}\wedge\mathcal{B} = \int V_{\alpha\beta}dx^\alpha \wedge dx^\beta \\ V_{\alpha\beta} &= M^2\frac{R^2}{z^2}\eta_{\mu\alpha}\eta_{\nu\beta}(\Lambda^\mu{}_0\Lambda^\nu{}_3 - \Lambda^\mu{}_3\Lambda^\nu{}_0) \end{aligned} \tag{3.18}$$

with $M^2 = m\tilde{m}/2$. $V_{\alpha\beta}$ obeys the following constraints:

$$V^{\alpha\beta}V_{\alpha\beta} = -2M^4, \quad \epsilon^{\alpha\beta\mu\nu}V_{\alpha\beta}V_{\mu\nu} = 0 \tag{3.19}$$

Alternatively, we could treat $V_{\alpha\beta}$ as free variables in the action at first, and then impose the above constraints via Lagrange multipliers. This leads to the action:

$$S = \int V_{\alpha\beta}\partial_a\xi^\alpha\partial_b\xi^\beta d\xi^a\wedge d\xi^b - \frac{1}{2}\int d^2\xi\sqrt{-\sigma}(V_{\alpha\beta}V^{\alpha\beta} + 2M^4) \tag{3.20}$$

where we have written x^μ as a function of world sheet coordinate $\xi^i (i = 1, 2)$. σ is the determinant of world sheet metric. Variation with respect to σ will give the constraint and by putting the solution back to the action, we get the Nambu-Goto action:

$$S = -2M^2 \int d^2\xi \sqrt{-\det \rho} \tag{3.21}$$

where $\rho_{ab} = \eta_{\alpha\beta}\partial_a x^\alpha \partial_b x^\beta$ is the induced metric on the world sheet.

3.2 Gravity Models in AdS Space

Since we will be focusing on Chern-Simons gravity along with the Einstein-Hilbert action, here we give a brief review of gravity models in anti-de Sitter spacetime. The most general gravity model in d dimensions that gives equations of motion which are at most of the second order in time-derivatives is described by the Lovelock action [32]:

$$I = \kappa \int \sum_{p=0}^{[d/2]} \alpha_p L^{(p)} \tag{3.22}$$

Here α_p's are arbitrary constants and $L^{(p)}$ are differential forms constructed as products of the vielbein and curvature,

$$L^{(p)} = \epsilon_{a_1 a_2 \cdots a_d} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_d} \tag{3.23}$$

As usual, the curvature 2-form is defined in terms of the spin connection ω^{ab} as

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \tag{3.24}$$

Different choices of the coefficients α_p in (3.22) will lead to different theories. The equations of motion are derived via independent variations with respect to frame fields and spin connection. The variation of R^{ab} with respect to ω^{ab} is:

$$\delta R^{ab} = d\delta\omega^{ab} + \delta\omega^{a}{}_{c}\omega^{cb} + \omega^{a}{}_{c}\delta\omega^{cb} = D(\delta\omega^{ab}) \tag{3.25}$$

Thus by integration by parts and noticing $T^a = De^a = de^a + \omega^{a}{}_{b}e^b$ is the torsion 2-form, the variation of the action (3.22) takes the form

$$\delta I = \int (\delta e^a \mathcal{E}_a + \delta\omega^{ab}\mathcal{E}_{ab}) \tag{3.26}$$

$$\mathcal{E}_a = \sum_{p=0}^{[\frac{d-1}{2}]} a_p(d-2p)\mathcal{E}_a^{(p)} \tag{3.27}$$

$$\mathcal{E}_{ab} = \sum_{p=0}^{[\frac{d-1}{2}]} a_p(d-2p)\, p\, \mathcal{E}_{ab}^{(p)} \tag{3.28}$$

where

$$\mathcal{E}_a^{(p)} = \epsilon_{ab_2\cdots b_d}R^{b_2b_3}\cdots R^{b_{2p}b_{2p+1}}e^{b_{2p+2}}\cdots e^{d} \tag{3.29}$$

$$\mathcal{E}_{ab}^{(p)} = \epsilon_{abb_3\cdots b_d}R^{b_3b_4}\cdots R^{b_{2p-1}b_{2p}}T^{b_{2p+1}}e^{a_{2p+2}}\cdots e^{b_d} \tag{3.30}$$

The equations of motion for Lovelock gravity are thus given by

$$\mathcal{E}_a = 0, \qquad \mathcal{E}_{ab} = 0 \tag{3.31}$$

These equations do not necessarily imply the vanishing of the torsion T^a, except in 2+1 dimensions. But $T^a = 0$ is obviously a solution automatically satisfying the second equation, and in many cases this choice is made. When the choice of zero torsion ($T = 0$) is made, i.e., for a Riemannian manifold, e^a and $\omega^{a}{}_{b}$ are no longer independent but are related as

$$\omega_\mu^{ab} = e_\lambda^a\,\partial_\mu e^{\lambda b} + e_\lambda^a\,\Gamma^\lambda_{\mu\rho}\,e^{\rho b} \tag{3.32}$$

Here $\Gamma^\lambda_{\mu\rho}$ is the standard Christoffel symbol given by

$$\Gamma^\lambda_{\mu\rho} = \frac{1}{2} g^{\lambda\sigma}\left[-\frac{\partial g_{\mu\rho}}{\partial x^\sigma} + \frac{\partial g_{\sigma\mu}}{\partial x^\rho} + \frac{\partial g_{\rho\sigma}}{\partial x^\mu}\right] \tag{3.33}$$

To understand the equation of motion better, we group all terms in (3.27) into polynomials involving powers of the curvature 2-form and write them as

$$\epsilon_{ab_1\cdots b_{d-1}}\left(R^{b_1b_2} + \beta_1 e^{b_1}e^{b_2}\right)\cdots(R^{b_{2k-1}b_{2k}} + \beta_k e^{b_{2k-1}}e^{b_k})e^{b_{2k+1}}\cdots e^{b_{d-1}} = 0 \tag{3.34}$$

where the β_i are given in terms of the α_p's. This equation can have several different solutions, characterized by the coefficients β_i. They are all spaces with constant curvatures. So for the general equation, there could be different asymptotic behaviors along different spatial directions, which could make the theory rather involved. In many cases, one makes a simple choice setting all β's equal, $\beta_i = 1/R^2$. This is equivalent to making specific choices for the coefficients α_p in (3.22) given by

$$\alpha_p = \begin{cases} \frac{R^{2(p-k)}}{d-2p} C^k_p, & p \leq k \\ 0, & p > k \end{cases} \tag{3.35}$$

Here $1 \leq k \leq [d-1]/2$. The Einstein-Hilbert action with negative cosmological constant is given by $k = 1$ and Chern-Simons gravity is given by $k = [(d-2)/2]$. These theories admit the same vacuum solution, with the cosmological constant related to the scale factor as

$$\Lambda = -\frac{(d-1)(d-2)}{R^2} \tag{3.36}$$

Later in this thesis, we will consider the Chern-Simons action with the addition of the Einstein-Hilbert action. To avoid conflict between the vacuum solutions allowed by the two theories, we will choose the cosmological constant to be the same for both.

3.3 Single Particle Dynamics from the EIH Method

In this section, we will construct the single particle action for AdS_5 by following the idea of Einstein-Infeld-Hoffmann. Particles will be treated as singularities in spacetime, considered as the point-particle limit of instantons in four dimensions, relating the boundary integral surrounding these singularities to their masses.

We shall start with Chern-Simons theory. The Chern-Simons action in 5d spacetime $\mathcal{M}$ is given by:

$$\begin{aligned} CS(A) &= -\frac{ik}{24\pi^2}\int_{\mathcal{M}} \mathrm{Tr}\left(AF^2 - \frac{1}{2}A^3F + \frac{1}{10}A^5\right) \\ &= -\frac{ik}{24\pi^2}\int_{\mathcal{M}} \mathrm{Tr}\left(AdAdA + \frac{3}{2}A^3dA + \frac{3}{5}A^5\right) \end{aligned} \tag{3.37}$$

Under variation of A, the change in the action is given by

$$\delta S = -\frac{ik}{8\pi^2}\int \mathrm{Tr}(\delta AFF) + \text{boundary term} \tag{3.38}$$

So the bulk equation of motion is $FF = 0$. Clearly $F = 0$ is a solution of this equation, although it may not be the most general solution. In the case of Chern-Simons gravity, the gauge field can be related to frame fields and spin connection as

$$A = -\frac{i}{2}e^a\gamma_a - \frac{i}{2}\omega^{ab}\Sigma_{ab} \tag{3.39}$$

The field strength is thus given by

$$F = dA + A^2 = (R^{ab} + e^ae^b)\left(-\frac{i}{2}\Sigma_{ab}\right) + T^a\left(-\frac{i}{2}\gamma_a\right) \tag{3.40}$$

The solution $F = 0$ is equivalent to $R = -e^2$, $T = 0$. With the scaling of $e^a \to \sqrt{12/R^2}\, e^a$, this corresponds to the AdS vacuum as expected.

Following Einstein-Infeld-Hoffmann, we now consider field configurations with a number of singularities. These will be a number of points on a spatial slice, i.e., a number of lines in spacetime. We consider the manifold after removing a small neighborhood of each singularity, thus a small sphere for the spatial slice M, or a small tubular neighborhood for the spacetime. The vacuum field equations are imposed on this manifold. Since this implies zero curvature, the gauge field is effectively a pure gauge. The Chern-Simons form for a pure gauge, reduced to one dimension, will be of the coadjoint orbit form. So we expect that the point-particle action can be easily obtained for Chern-Simons gravity by reducing it to the vacuum manifold with the singularities removed. This is also in line with the fact that the Chern-Simons terms and the coadjoint orbit actions are essentially topological in nature.

Since we are discussing gauge fields in five dimensions, the natural candidate for a configuration which can have point-like properties and nontrivial integrals on the surfaces surrounding the singular points is the instanton. So we will consider instantons, taking the scale size to be infinitesimal, effectively shrinking them to zero size. The field strength will be localized to a point in this limit. This can be viewed as the point-like singularity in the EIH language.

Specifically for $SO(4, 2)$, we will consider the instantons in the $SO(4)$ subgroup. It will represented as the field configuration $A = a$. This is a static, time-independent configuration in 5 dimensions. (Essentially, the instanton is a soliton from the 5d point of view.) Collective coordinates for the motion of this "particle" can be introduced by an $SO(4, 2)$ transformation since the latter will correspond to Lorentz transformations and translations. Thus the configuration of interest should be the gauge transformation of the singular solution a, i.e.,

$$A = g^{-1}ag + g^{-1}dg \tag{3.41}$$

The group element g will be taken to be time-dependent and this captures the dynamics of the particle.

We will actually consider a number of singular points, not just one, corresponding to instantons localized at several spatial points $\{x_\alpha\}$. We remove small spheres $\{C_\alpha\}$ surrounding these points from the manifold, so that the spatial slice is $M - \{C_\alpha\}$. Again, away from these points, the field strength is well approximated by the vacuum configuration $F = 0$, so we can take $F = 0$ on $M - \{C_\alpha\}$.

The Chern-Simons action can now be written in terms of singular gauge field a and the group element g as

$$\begin{aligned} CS(A) = CS(a) &- \frac{ik}{240\pi^2}\int \mathrm{Tr}(dgg^{-1})^5 \\ &+ \frac{ik}{48\pi^2}\oint_{\partial M}\mathrm{Tr}\left(dgg^{-1}(ada+daa+a^3)+a(dgg^{-1})^3-\frac{1}{2}(dgg^{-1}a)^2\right) \\ &- \frac{ik}{48\pi^2}\oint_{C_\alpha}\mathrm{Tr}\left(dgg^{-1}(ada+daa+a^3)+a(dgg^{-1})^3-\frac{1}{2}(dgg^{-1}a)^2\right) \end{aligned} \tag{3.42}$$

This equation reveals a certain difficulty with using the Chern-Simons action. It is not gauge-invariant if the manifold has a boundary. The starting action, before we introduce the instantons, can be made gauge-invariant by adding a boundary term $S_b(A,\psi)$ (which may depend on the gauge fields and on some matter-type fields ψ) for the boundary of the whole manifold. We refer to this as the outer boundary term to distinguish it from the boundaries around each singularity when we consider point-like instantons. The particular form of this outer boundary term is not important, except that it should be taken to transform nontrivially under a gauge transformation in such a way as to cancel the outer boundary gauge anomaly as well as the term $(dgg^{-1})^5$ in (3.42). Thus we postulate that there is a boundary term, defined on $\partial M \times \mathbb{R}$, with the property

$$S_b(A,\psi^g) \;=\; S_b(a,\psi) + \frac{ik}{240\pi^2}\int \mathrm{Tr}(dgg^{-1})^5$$

$$-\frac{ik}{48\pi^2}\oint_{\partial M}\mathrm{Tr}\left(dgg^{-1}(ada+daa+a^3)+a(dgg^{-1})^3-\frac{1}{2}(dgg^{-1}a)^2\right) \tag{3.43}$$

Examples of such terms are provided by the effective actions for field theories with anomalies in four dimensions. We may even replace $S_b(A,\psi)$ by an effective action

$$e^{iS_{\mathrm{eff}}(A)}=\int[\mathcal{D}\psi]\,e^{iS_b(A,\psi)} \tag{3.44}$$

defined by a field theory with action $S_b(A,\psi)$.

Based on the arguments given above, we modify the starting point of our analysis to be given by the action

$$S=CS(A)+S_b(A,\psi) \tag{3.45}$$

This ensures gauge invariance of the action in general, before we introduce the instantons. It should be emphasized that this gauge invariance is the sole reason for the term $S_b(A,\psi)$. Evaluating this action on the configurations $A=g^{-1}ag+g^{-1}dg$, we find

$$S(A,\psi^g)=S(a,\psi)-\frac{ik}{48\pi^2}\oint_{C_\alpha}\mathrm{Tr}\left(dgg^{-1}(ada+daa+a^3)+a(dgg^{-1})^3-\frac{1}{2}(dgg^{-1}a)^2\right) \tag{3.46}$$

The contribution of the Chern-Simons term in (3.42) is proportional to $\mathrm{Tr}(a)^5$ because a is a pure gauge outside the singularities, with $da=-a^2$. And by our assumption, a is a spatial instanton solution at fixed time t, thus a^5 must vanish since $a_t=0$. The only nonvanishing relevant term is the last one in (3.42), or (3.46), which simplifies as

$$S=-\frac{ik}{48\pi^2}\oint_{C_\alpha}\mathrm{Tr}\left(-dgg^{-1}a^3+a(dgg^{-1})^3-\frac{1}{2}(dgg^{-1}a)^2\right) \tag{3.47}$$

Since g describes collective coordinates and so does not change the instanton number, g should contain no singularity (or it should be topologically trivial). This requirement eliminates the last two terms because the angular derivatives will vanish while we shrink the size of instantons to zero to ensure regularity. So finally the action is reduced to

$$S = -\frac{ik}{48\pi^2} \sum_{\alpha} \oint_{C_\alpha} \mathrm{Tr}\left(-\partial_0 g g^{-1} a^3\right) \tag{3.48}$$

We now need the more specific form of the instanton configuration. In the spinorial representation, $SO(4)$ splits into two $SU(2)$'s corresponding to the upper and lower 2×2 blocks. We thus have $SU(2) \times SU(2)$ instantons, for which we can choose the following parametrization, which is valid on $M - \{C_\alpha\}$,

$$a = t_1 U^{-1} dU,\ U = \phi^0 + i\sigma_i \phi^i,\ \phi_0^2 + \phi_i^2 = 1 \tag{3.49}$$

σ_i are the Pauli matrices. The matrix t_1 is a projection operator which identifies the $SU(2)$ subgroup for the instantons; i.e.,

$$t_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.50}$$

With this choice of the field configuration, a^3 is worked out to be:

$$a^3 = t_1 \epsilon_{\mu\nu\alpha\beta} \phi^\mu d\phi^\nu d\phi^\alpha d\phi^\beta \tag{3.51}$$

Note this expression is proportional to the standard winding number definition of a $S^3 \to SU(2)$ map so its integration will lead to the instanton number Q_α.

$$\begin{aligned} S &= -\frac{ik}{4} \sum_{\alpha} \oint_{C_\alpha} \mathrm{Tr}(-t_1 \partial_0 g g^{-1} dt\ \frac{1}{12\pi^2} \epsilon_{\mu\nu\alpha\beta} \phi^\mu d\phi^\nu d\phi^\alpha d\phi^\beta) \\ &= -\frac{ik}{4} \sum_{\alpha} \int dt\, \mathrm{Tr}(t_1 g^{-1} \partial_0 g) \cdot Q_\alpha \end{aligned} \tag{3.52}$$

In the second line we also made a change $g \to g^{-1}$ for easy comparison with the actions quoted earlier in this chapter.

For the two different possibilities of t_1 (of course there are various ways of embedding $SU(2)$ in $SO(4)$ but the simplest ways are sufficient for our purposes now), we can take different instanton solutions but we would like them to reside on the same set of $\{x_\alpha\}$ only with differences in winding numbers $\{Q^1_\alpha, Q^2_\alpha\}$. In this case, combining the two solutions, we get

$$\begin{aligned} S &= -\frac{ik}{4}\sum_\alpha \int dt\, \mathrm{Tr}\left(\frac{1+\Gamma_0}{2}Q^1_\alpha + \frac{1-\Gamma_0}{2}Q^2_\alpha\right) g^{-1}\partial_0 g \\ &= -\frac{ik}{8}\sum_\alpha (Q^1_\alpha - Q^2_\alpha)\int dt\, \mathrm{Tr}(\Gamma_0 g^{-1} dg) \end{aligned} \tag{3.53}$$

Γ matrices are given in (3.8). Then we can obsorb the matrix S which relates Γ matrices to γ matrics into the group element g:

$$\Gamma_\mu = S\gamma_\mu S^{-1} \tag{3.54}$$

Then choosing g as in (3.1), we get the multi-particle action, from the reduction of the CS action to instantons, as

$$S = \sum_\alpha \int \frac{k}{4}(Q^1_\alpha - Q^2_\alpha)\eta_{\mu\nu}{\Lambda^\mu}_0 \frac{dx^\nu}{z} \tag{3.55}$$

So far we have focused on the Chern-Simons form for gauge fields, which have the form (3.39) in terms of the vielbein and the spin connection. But for the case of parity-preserving AdS gravity, two copies of Chern-Simons action are required, and the action is given by

$$S = CS(A_L) - CS(A_R) + S_b(A_L, \psi) - S_b(A_R, \psi') \tag{3.56}$$

where A_L and A_R are related to vielbein and spin connection as:

$$\begin{aligned} A_L &= -\frac{i}{2}\left(\omega^{ab}\Sigma_{ab} + e^a\gamma_a\right) \\ A_R &= -\frac{i}{2}\left(\omega^{ab}\Sigma_{ab} - e^a\gamma_a\right) \end{aligned} \tag{3.57}$$

In this case, the second gauge field A_R can be parametrized as follows:

$$\begin{aligned} &\tilde{A} = \tilde{g}^{-1}a\tilde{g} + \tilde{g}^{-1}d\tilde{g}, \\ &\tilde{g} = \begin{bmatrix} 1/\sqrt{z} & 0 \\ -i\tilde{X}/\sqrt{z} & \sqrt{z} \end{bmatrix}\Lambda, \quad \tilde{X} = x^0 + \vec{x}\cdot\vec{\sigma} \end{aligned} \tag{3.58}$$

Although g and $\tilde{g}$ have different parametrizations, it can be proved that e and ω in A_L and A_R are the same. One could follow the same path as before and get a coadjoint orbit action for A_R part as

$$\tilde{S} = -\sum_\alpha \int \frac{k}{4}(Q^1_\alpha - Q^2_\alpha)\eta_{\mu\nu}{\Lambda^\mu}_0 \frac{dx^\nu}{z} \tag{3.59}$$

Here we require the two copies of gauge field to have the same instanton positions and instanton numbers. The combination of $CS(A_L)$ and $CS(A_R)$ then leads to the action

$$S = \sum_\alpha \int \frac{k}{2}(Q^1_\alpha - Q^2_\alpha)\eta_{\mu\nu}{\Lambda^\mu}_0 \frac{dx^\nu}{z} \tag{3.60}$$

which is the correct particle action for the coadjoint orbit in AdS_5 (for the translational degrees of freedom) if we make the following identification:

$$mR = \frac{k}{2}(Q^1_\alpha - Q^2_\alpha) \tag{3.61}$$

In summary, we have shown that the action (3.56) for gravity in five dimensions leads to the coadjoint orbit action for the dynamics of particles defined as field configurations which are point-like singularities. At this stage, the particles are noninteracting. We expect this is tied to the topological nature of the Chern-Simons form. Clearly a key question of interest would be to add to the action other terms, such as the Einstein-Hilbert action, and see if that would lead to an interacting multi-particle system. This is the subject of the next chapter.

Chapter 4

Particle Interaction

Chern-Simons gravity when reduced to point-particle configurations along the lines of the EIH method led to the coadjoint orbit action for particles. But as mentioned at the end of the last chapter, the particles were noninteracting. We may envisage a modification of the background due to the point-particles or instantons, in other words, the back reaction of the instantons. But, at least to the lowest (i.e., second) order in perturbation theory, the Chern-Simons action does not lead to propagating fields in the bulk away from the singularities. This is the reason for the absence of the interactions. The addition of other terms to the action can presumably change this. In this chapter, we will modify the starting action by the addition of the standard Einstein-Hilbert term. The most important reason for choosing this particular modification is that both CS and EH actions admit vacuum AdS_5 space as the solution to their field equations. This requirement makes EH action rather special.

Besides the possibility of a common vacuum solution, parity is another reason why Einstein-Hilbert action is preferred in combination with Chern-Simons action to generate particle interactions. When we construct particle dynamics in last chapter, even if we do not add the $CS(A_R)$ piece, the theory still makes sense. However, it is not parity-preserving. Once we have the parity-preserving combination $CS(A_L)-CS(A_R)$, the EH action is natural if we want to modify the theory. We will see that the addition of Einstein-Hilbert action can successfully generate interactions, by treating the interaction as a perturbation in the metric tensor. In the non-relativistic limit, not surprisingly, we will see that the interaction will reduce to the usual Coulomb potential between particles.

4.1 Parity Operation

The parity operation will play a crucial role in what follows, so we will start by defining the parity properties carefully.

The group $SO(4,2)$ has two chiral components obeying the same Lie algebra relations. Denote one set of generators by t_a with the Lie algebra commutation rules,

$$[t_a, t_b] = i f_{abc} t_c \tag{4.1}$$

Its chiral conjugate (Lie algebra conjugate) is given by $-t_a^T$, where the superscript T denotes the matrix-transpose. We then define the parity conjugation matrix as

$$C = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tag{4.2}$$

In terms of this matrix C, the parity conjugation operation is defined by

$$P(t_a) = t_a^P = C(-t_a^T)C \tag{4.3}$$

For our choice of γ_a matrices, and the corresponding $\Sigma_{ab} = [\gamma_a, \gamma_b]/4i$, they behave under parity operation as:

$$P(\gamma_a) = -\gamma_a, \quad P(\Sigma_{ab}) = \Sigma_{ab} \tag{4.4}$$

The parity operation acting on the gauge fields is therefore given by

$$\begin{aligned} P(A_L) &= -\frac{i}{2}\left(\omega^{ab}P(\Sigma_{ab}) + e^a P(\gamma_a)\right) \\ &= -\frac{i}{2}\left(\omega^{ab}\Sigma_{ab} - e^a\gamma_a\right) = A_R \end{aligned} \tag{4.5}$$

The parity conjugation of the Chern-Simons action is then obtained as

$$\begin{aligned} P(CS(A_L)) &= (-1)\left[-\frac{ik}{24\pi^2}\right]\mathrm{Tr}\left[A_L^P dA_L^P dA_L^P + \frac{3}{2}(A_L^P)^3 dA_L^P + \frac{3}{5}(A_L^P)^5\right] \\ &= \frac{ik}{24\pi^2}\,\mathrm{Tr}\left[A_R dA_R dA_R + \frac{3}{2}A_R^3 dA_R + \frac{3}{5}A_R^5\right] \\ &= -CS(A_R) \end{aligned} \tag{4.6}$$

The Chern-Simons action defined in the previous chapter, namely, equation (3.37) has an overall minus sign. Using that expression for A_L and carrying out a parity transformation, we get the transformations of the Lie algebra elements and an overall sign change (explicitly indicated in the first line) due to the change of orientation for the volume element. This is how $P(CS(A_L))$ is obtained above. Equation (4.6) tells us that the parity-even Chern-Simons action should have a minus sign between left- and right-handed parts,

$$S = CS(A_L) + CS(A_L^P) = CS(A_L) - CS(A_R) \tag{4.7}$$

This is exactly the form we used in the last chapter. Now we will show that Einstein-Hilbert action is also parity even under this conjugation. From (3.57) we can solve for e and ω in terms of gauge fields A_L, A_R as

$$e^a = \frac{i}{4}\eta^{ac}\,\mathrm{Tr}\left[\gamma_c(A_L - A_R)\right], \quad \omega^{ab} = \frac{i}{2}\eta^{ac}\eta^{bd}\,\mathrm{Tr}\left[\Sigma_{cd}(A_L + A_R)\right] \tag{4.8}$$

Correspondingly, the curvature and torsion 2-forms are given in terms of field strengths as

$$T^a = \frac{i}{4}\eta^{ac}\,\mathrm{Tr}\left[\gamma_c(F_L + F_R)\right] \tag{4.9}$$

$$(R - e^2)^{ab} = \frac{i}{2}\eta^{ac}\eta^{bd}\,\mathrm{Tr}\left[\Sigma_{cd}(F_L - F_R)\right] \tag{4.10}$$

With these expressions, the Einstein-Hilbert action can be written using the gauge fields and the field strengths as

$$S_{EH} = \frac{i}{192\pi G}\int \mathrm{Tr}(F_L + F_R)(A_L - A_R)^3 \tag{4.11}$$

It is then straightforward to see that S_{EH} is invariant under parity transformation as we have defined above,

$$\begin{aligned} P(S_{EH}) &= -\frac{i}{192\pi G}\int \mathrm{Tr}(F_L^P + F_R^P)(A_L^P - A_R^P)^3 \\ &= -\frac{i}{192\pi G}\int \mathrm{Tr}(F_R + F_L)(A_R - A_L)^3 \\ &= S_{EH} \end{aligned} \tag{4.12}$$

The parity operation and the transformation of various fields and actions will be useful in working out the perturbation expansions. We can work out results for the one chiral component and obtain the results for the other by the parity transformation. This will simplify the calculations significantly.

4.2 Perturbative Expansion

We expect to see interactions between particles. When interactions are present, assuming they are small, there will be small deviations in the metric from the AdS-flat background.

Thus we propose to consider the following modification to the field, $A \rightarrow A'$ with

$$A' = g^{-1}(a + \delta a)g + g^{-1}dg \tag{4.13}$$

The corresponding deviation in the metric tensor and its inverse are defined by

$$g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \quad g'^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} \tag{4.14}$$

δa is written in term of δe and $\delta\omega$, and the relation between the change in the frame field and the change in metric is given by:

$$e'^a_\mu = e^a_\mu + f^a_\mu, \quad h_{\mu\nu} = \eta_{ab}(e^a_\mu f^b_\nu + e^a_\nu f^b_\mu + f^a_\mu f^b_\nu) \tag{4.15}$$

In the perturbative expansion up to quadratic order in $h_{\mu\nu}$, we can safely use the first order relation of $h_{\mu\nu}$, which is approximated by:

$$e_{a\mu} f^a_\nu = e_{a\nu} f^a_\mu = \frac{1}{2} h_{\mu\nu} \tag{4.16}$$

With this preliminary items, we are ready to write down the perturbative expansion of the Einstein-Hilbert action and the Chern-Simons action. The EH action expanded up to 2nd order in $h_{\mu\nu}$ is given by (see Appendix for detailed calculation),

$$S(g+h) = S^{(0)} + S^{(1)} + S^{(2)} + \cdots \tag{4.17}$$

$$S^{(0)}_{EH} = \int (R - 2\Lambda)\sqrt{-g}\, d^5x \tag{4.18}$$

$$S^{(1)}_{EH} = -\int h^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda \right) \sqrt{-g}\, d^5x \tag{4.19}$$

$$\begin{aligned}S^{(2)}_{EH} = \int \Bigg\{ & \frac{1}{4}h^{\mu\nu}(\nabla^2+2\Lambda)h_{\mu\nu} - \frac{1}{8}h(\nabla^2+2\Lambda)h + \frac{1}{2}\left(\nabla^\alpha h_{\alpha\mu} - \frac{1}{2}\nabla_\mu h\right)^2 \\ & + \frac{1}{2}h^{\mu\lambda}h^{\nu\sigma}R_{\mu\nu\lambda\sigma} + \frac{1}{2}\left(h^{\mu\lambda}h^{\nu}_{\lambda} - hh^{\mu\nu}\right)R_{\mu\nu} + \frac{1}{8}\left(h^2 - 2h^{\mu\nu}h_{\mu\nu}\right)R\Bigg\}\sqrt{-g}\,d^5x\end{aligned} \tag{4.20}$$

The first order term actually gives the field equation. If we choose our background as AdS flat, its bulk contribution is zero, so that we can reduce the above equations to

$$S_{EH} = S^{(0)}_{EH} + S^{(2)}_{EH} \tag{4.21}$$

Boundary terms have been omitted. First order boundary term can be regulated by adding the Gibbons-Hawking-York term,

$$S_b = \oint K\sqrt{\gamma}d^{D-1}x \tag{4.22}$$

Second order boundary terms actually will be of higher order for the interactions, because they will involve two bulk propagators, so we will not consider them for now. (See comments later.) So we will only focus on the bulk terms in the expansion of EH action. The expansion of Chern-Simons action is straightforward, and the result is:

$$CS(A+\delta A) = CS^{(0)}(A) + CS^{(1)}(\delta A, A) + CS^{(2)}(\delta A, A) \tag{4.23}$$

$$CS^{(0)}(A) = -\frac{ik}{24\pi^2}\int \mathrm{Tr}\left(AdAdA + \frac{3}{2}A^3dA + \frac{3}{5}A^5\right) \tag{4.24}$$

$$CS^{(1)}(\delta A, A) = -\frac{ik}{8\pi^2}\int \mathrm{Tr}(\delta AFF) - \frac{ik}{24\pi^2}\oint \mathrm{Tr}\left[\delta A\left(dAA + AdA + \frac{3}{2}A^3\right)\right] \tag{4.25}$$

$$\begin{aligned}CS^{(2)}(\delta A, A) = & -\int \frac{ik}{24\pi^2}\,\mathrm{Tr}\left[3D\delta A\delta AF - \delta A^2(AF+FA)\right] \\ & - \oint \frac{ik}{24\pi^2}\,\mathrm{Tr}\left(\delta AFD\delta A - \delta A^2F + \frac{1}{2}\delta A^2A^2 - \frac{1}{4}\delta AA\delta AA\right)\end{aligned} \tag{4.26}$$

where D is the gauge covariant derivative. Again, we will take the AdS vacuum (with the same cosmological constant as for the EH part) as the background, so $F = 0$. With this choice, the bulk terms in the first and second order expansions both vanish. (The vanishing of the bulk second order term is the reason for there being no bulk propagator just from the CS action, and hence no interactions to this order just from the CS action, as mentioned at the beginning of this chapter.) And the boundary term of 2nd order will be dropped for reasons similar to the case of the EH action, namely, it will involve two bulk propagators and so will be of higher order than the lowest corrections we are interested in. So only the boundary term from the 1st order expansion contributes, so that to the order we are interested in,

$$\begin{aligned} CS(A+\delta A) = &-\frac{ik}{24\pi^2}\int \mathrm{Tr}\left(AdAdA + \frac{3}{2}A^3dA + \frac{3}{5}A^5\right) \\ &- \frac{ik}{24\pi^2}\oint \mathrm{Tr}\left[\delta A\left(dAA + AdA + \frac{3}{2}A^3\right)\right] \end{aligned} \tag{4.27}$$

When the background gauge field A is a pure gauge $dA = -A^2$, the above result reduces to:

$$CS(A+\delta A) = -\frac{ik}{240\pi^2}\int \mathrm{Tr}(A^5) + \frac{ik}{48\pi^2}\oint \mathrm{Tr}(\delta A A^3) \tag{4.28}$$

4.3 Equation of Motion

By perturbative expansion, we have an action up to quadratic order in the metric variation $h_{\mu\nu}$. Treating $h_{\mu\nu}$ as dynamic variables, we can solve for it via its equations of motion, obtained by taking a variation of S with respect to $h_{\mu\nu}$:

$$\frac{\delta S}{\delta h_{\mu\nu}} = 0 \quad \Rightarrow \quad \text{solution of } h_{\mu\nu} \quad \Rightarrow \quad \text{interaction} \tag{4.29}$$

One has to keep in mind that in deriving (4.20), the second order expansion of Einstein-Hilbert action, zero torsion condition is always implicitly kept. This assumption has to be obeyed by δA, meaning that δe and $\delta\omega$ are not independent perturbations. With the relation between ω and e given in (3.32), we can solve for $\delta\omega$ in terms of δe as

$$\delta\omega_{\mu}^{ab} = -e^{\rho a}e^{\lambda b}\nabla_{\rho}h_{\lambda\mu} \tag{4.30}$$

Notice that our construction of S_{EH} involves both left- and right-handed gauge fields. So we need to be careful to ensure the perturbations do not break parity. To take account of this, the parametrization of gauge fields will be slightly altered to

$$\begin{aligned} A_L &= g^{-1}(a_L + \delta a_L)g + g^{-1}dg \\ A_R &= g^{-1}(a_R + \delta a_R)g + g^{-1}dg \end{aligned} \tag{4.31}$$

Unlike in last chapter, here the same group element g is used for parametrization of A_L and A_R. And the explicit form of g is also altered:

$$g = S^{-1}\Lambda V, \quad V = \begin{bmatrix} \sqrt{z} & \frac{i\tilde{X}}{\sqrt{z}} \\ 0 & \frac{1}{\sqrt{z}} \end{bmatrix} \tag{4.32}$$

With this parametrization, the gauge field A can be written as

$$\begin{aligned} A &= g^{-1}ag + g^{-1}dg \\ &= V^{-1}\left[\Lambda^{-1}SaS^{-1}\Lambda + (\Lambda^{-1}S)d(S^{-1}\Lambda)\right]V + V^{-1}dV \\ &= V^{-1}\tilde{a}V + V^{-1}dV \end{aligned} \tag{4.33}$$

We have absorbed the $S^{-1}\Lambda$ factor into the instanton part, now named as $\tilde{a}$. The Chern-Simons action is similar to (3.42):

$$CS(A) = CS(\tilde{a}) - \frac{ik}{48\pi^2}\oint_{C_\alpha} \mathrm{Tr}\left(dVV^{-1}(\tilde{a}d\tilde{a} + d\tilde{a}\tilde{a} + \tilde{a}) + \tilde{a}(dVV^{-1})^3 - \frac{1}{2}(dVV^{-1}\tilde{a})^2\right) \tag{4.34}$$

Upon using the new parametrization, we can show that the result obtained in last section does not change. Starting from (3.55),

$$S = -\frac{ik}{48\pi^2}\oint_{C_\alpha} \mathrm{Tr}\left(-dVV^{-1}\tilde{a}^3\right) \tag{4.35}$$

The instanton part will be simplified similarly:

$$\mathrm{Tr}(\tilde{a}^3) = \mathrm{Tr}\left[\Lambda^{-1}SaS^{-1}\Lambda + (\Lambda^{-1}S)d(S^{-1}\Lambda)\right]^3 = \mathrm{Tr}\left[a + d(S^{-1}\Lambda)(\Lambda^{-1}S)\right]^3 \tag{4.36}$$

The winding number is a topological quantity. Any two gauge fields related by continuous and regular gauge transformations will give the same result. According to our assumption, the group element g is regular, so is its Lorentz transformation part Λ. Therefore, we conclude that

$$\int_{S^3} \mathrm{Tr}\left[a + d(S^{-1}\Lambda)(\Lambda^{-1}S)\right]^3 = \int_{S^3} \mathrm{Tr}(a^3) \tag{4.37}$$

Using the same argument as in last chapter, we conclude that a^3 is the instanton density, which gives the instanton numbers upon integration over the spatial S^3 spheres. For $\tilde{a}^3$, before taking the trace, we have to keep track of the $S^{-1}\Lambda$ factors. This gives

$$\begin{aligned}\tilde{a}^3 &= 12\pi^2\Lambda^{-1}S\left[\frac{1}{2}(1+\Gamma_0)Q^1_\alpha + \frac{1}{2}(1-\Gamma_0)Q^2_\alpha\right]S^{-1}\Lambda \\ &= 12\pi^2\Lambda^{-1}\left[\frac{1}{2}(1+\gamma_0)Q^1_\alpha + \frac{1}{2}(1-\gamma_0)Q^2_\alpha\right]\Lambda\end{aligned} \tag{4.38}$$

Consequently, the instanton part remains the same as in (3.53):

$$\frac{ik}{48\pi^2}\oint(V^{-1}dV\tilde{a}^3)=\frac{k}{4}\left(Q_\alpha^1-Q_\alpha^2\right)\Lambda_0^\mu\eta_{\mu\nu}\frac{dx^\nu}{z} \tag{4.39}$$

The right-handed part can be obtained without explicit calculation by using the parity transformation. In $\mathrm{Tr}(V^{-1}dV\tilde{a}^3)$, we need odd numbers of γ-matrices to ensure non-zero trace. The γ matrix will flip sign after parity transformation, while Σ's stay the same. Keeping in mind that there is an overall sign change in volume element, the final result is parity-even and reads

$$\frac{ik}{48\pi^2}\oint(V^{-1}dV\tilde{a_L}^3)=\frac{ik}{48\pi^2}\oint(V^{-1}dV\tilde{a_R}^3)=\frac{k}{4}\left(Q_\alpha^1-Q_\alpha^2\right)\Lambda_0^\mu\eta_{\mu\nu}\frac{dx^\nu}{z} \tag{4.40}$$

This just reproduces (3.60).

When the perturbation $\delta\tilde{a}$ enters the Chern-Simons action, $CS(\tilde{a})$ is no longer zero. From the second line in (4.27), it acquires a boundary term propotional to the variation:

$$\begin{aligned}CS^{(1)}&=-\frac{ik}{24\pi^2}\oint\mathrm{Tr}\left[\delta\tilde{a}\left(d\tilde{a}\tilde{a}+\tilde{a}d\tilde{a}+\frac{3}{2}\tilde{a}^3\right)\right]=\frac{ik}{48\pi^2}\oint\mathrm{Tr}\left(\delta\tilde{a}\tilde{a}^3\right)\\&=\frac{ik}{8}(Q_\alpha^1-Q_\alpha^2)\,\mathrm{Tr}(\delta a\Lambda^{-1}\gamma_0\Lambda)\\&=\frac{ik}{8}(Q_\alpha^1-Q_\alpha^2)\,\mathrm{Tr}\left[-\frac{i}{2}(\delta e^c\gamma_c+\delta^{cd}\Sigma_{cd})\Lambda_0^a\gamma_a\right]\\&=\frac{k}{4}(Q_\alpha^1-Q_\alpha^2)\Lambda_0^a\eta_{ac}\delta e^c\end{aligned} \tag{4.41}$$

We must also include the parity transform of above result. Again, We need odd numbers of γ-matrices in total so that the trace product does not vanish. Taking into account the overall sign change from the volume element, we conclude that the parity transform of $CS^{(1)}$

is the same:

$$CS^{(1)}(a_L) + CS^{(1)}(a_R) = \frac{k}{2}(Q^1_\alpha - Q^2_\alpha)\Lambda^a{}_0\eta_{ac}\delta e^c = mR\Lambda^a{}_0\eta_{ac}\delta e^c \tag{4.42}$$

This result can be further simplified as follows. Consider the usual action for a single particle given by

$$S = -m\int ds = -m\int\sqrt{-g_{\mu\nu}dx^\mu dx^\nu} \tag{4.43}$$

The energy-momentum tensor corresponding to this is obtained by varying S with respect to the metric tensor and gives

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = m\int d\tau \frac{\delta^{(5)}(x - x(\tau))}{\sqrt{-g}}\frac{dx^\mu}{ds}\frac{dx^\nu}{d\tau} \tag{4.44}$$

Multiplying this by $e_{\nu a}\delta e^a_\mu$ and integrating we find

$$\int\sqrt{-g}\,T^{\mu\nu}e_{\nu a}\delta e^a_\mu = m\int\frac{dx^\nu}{ds}e_{\nu a}\delta e^a_\mu dx^\mu \tag{4.45}$$

Notice that $\frac{dx^\nu}{ds}e_{\nu a}$ behaves as a Lorentz group element since

$$\eta_{ab}\frac{dx^\nu}{ds}e^a_\nu\frac{dx^\mu}{ds}e^b_\mu = 1 \quad \Rightarrow \quad \frac{dx^\nu}{ds}e^a_\nu = \Lambda^a{}_0 \tag{4.46}$$

Using this result,

$$\int\sqrt{-g}\,T^{\mu\nu}e_{\nu a}\delta e^a_\mu = m\int\Lambda^a{}_0\eta_{ab}\delta e^b \tag{4.47}$$

We now see that the correction term (4.42) can be viewed as an integral involving the energy-momentum tensor for a free particle, with the identification of Λ^a_0 from g as the quantity $\frac{dx^\nu}{ds} e^a_\nu$. Thus we can write

$$CS^{(1)}(a_L) + CS^{(1)}(a_R) = R \int \sqrt{-g}\, T^{\mu\nu} e_{\nu a} \delta e^a_\mu = \frac{1}{2} \int \sqrt{-g}\, T^{\mu\nu} h_{\mu\nu} \tag{4.48}$$

Here we shall notice that the vieblein we write down before actually has absorbed a factor of $1/R$. After restoring the factor, there's no R dependence in the integral. In addition to this, there will be more terms in $CS(\tilde{a})$ involving both $V^{-1}dV$ and δa. Setting these aside for the moment, we turn to the EH action. Schematically, we can write the second order perturbation of the EH action as

$$S^{(2)}_{EH} = \frac{1}{2} \int \sqrt{-g}\; h_{\mu\nu}\, \mathbf{L}^{\mu\nu\alpha\beta}\, h_{\alpha\beta} \tag{4.49}$$

The operator $\mathbf{L}^{\mu\nu\alpha\beta}$ is what is known as the Lichnerowicz operator. The variation of $S^{(2)}_{EH}$ with respect to $h_{\mu\nu}$ is:

$$\delta_h S^{(2)}_{EH} = \int \sqrt{-g}\, \delta h_{\mu\nu}\, \mathbf{L}^{\mu\nu\alpha\beta} h_{\alpha\beta} \tag{4.50}$$

To the order we are calculating, we can solve for the perturbation $h_{\mu\nu}$ using its equation from this variation, obtained as

$$\begin{aligned} \delta_h S = \frac{1}{2} \int \sqrt{-g}\, T^{\mu\nu} \delta h_{\mu\nu} \quad &+ \quad \int \sqrt{-g}\, \delta h_{\mu\nu}\, \mathbf{L}^{\mu\nu\alpha\beta} h_{\alpha\beta} = 0 \\ \text{or} \qquad \mathbf{L}^{\mu\nu\alpha\beta} h_{\alpha\beta} + \frac{1}{2} T^{\mu\nu} \quad &= \quad 0 \end{aligned} \tag{4.51}$$

We now define the Green's function for the Lichnerowicz operator by

$$\mathbf{L}^{\mu\nu\rho\lambda} G_{\rho\lambda\alpha\beta}(x, x') = \frac{\delta^{(5)}(x - x')}{\sqrt{-g}}\, \frac{1}{2} \left(\delta^\mu_\alpha \delta^\nu_\beta + \delta^\mu_\beta \delta^\nu_\alpha \right) \tag{4.52}$$

Indeed there're extra gauge degrees of freedom and they must be addressed before defining the Green's function otherwise there'll be zero modes which are not invertible. This will be taken care of later, for example, by choosing the harmonic coordinate condition. The solution of this equation for $h_{\mu\nu}$ and the action evaluated on this solution are given by

$$h_{\mu\nu}(x) = -\frac{1}{2}\int dy\sqrt{-g}\; G_{\mu\nu\rho\lambda}(x,y)\, T^{\rho\lambda}(y) \tag{4.53}$$

$$S^{(2)}(h) = -\frac{1}{16}\int dx\sqrt{-g}\int dy\sqrt{-g}\; G_{\mu\nu\alpha\beta}(x,y)\, T^{\mu\nu}(x)\, T^{\alpha\beta}(y) \tag{4.54}$$

This term of the action should contain the gravitational interaction between the particles to the order we are calculating. To simplify things further, we will need an explicit formula for the Green's function. An explicit expression has been derived in [10], but for our purpose an approximation to the complete formula will suffice.

4.4 Particle Interaction

For the AdS vacuum background we have the relations

$$\begin{gathered} R_{\mu\nu\alpha\beta} = -\frac{1}{R^2}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}), \quad R_{\mu\nu} = -\frac{d-1}{R^2}g_{\mu\nu} \\ R = -\frac{1}{R^2}d(d-1), \quad \Lambda = -\frac{1}{R^2}(d-1)(d-2) \end{gathered} \tag{4.55}$$

Thus $S^{(2)}_{EH}$ reduces to

$$\begin{aligned} S^{(2)}_{EH} = \int \sqrt{-g}\,\Bigg\{ & \frac{1}{2R^2}(h^{\mu\nu}h_{\mu\nu} + h^2) + \frac{1}{4}h^{\mu\nu}\nabla^2 h_{\mu\nu} - \frac{1}{8}h\nabla^2 h \\ & + \frac{1}{2}\left(\nabla^\nu h_{\nu\mu} - \frac{1}{2}\nabla_\mu h\right)^2 \Bigg\} \end{aligned} \tag{4.56}$$

Further, there still exist extra redundant (gauge) degrees of freedom in the system, which will allow us to choose a gauge condition to simplify the action. One such condition, which is commonly used and is convenient for our purpose, is harmonic coordinate condition or de Donder gauge condition given by

$$\nabla^{\nu} h_{\nu\mu} - \frac{1}{2}\nabla_{\mu} h = 0 \tag{4.57}$$

To simplify things further and bring out easily recognizable formulae, we consider a cluster of particles. The scale of the cluster is assumed to be relatively small compared to the scale of AdS background, or equivalently, we take R to be large. Thus, within the cluster, the $1/R^2$ factor in the equation can be neglected. The gauge condition and this approximation will help us reduce the action to

$$\begin{aligned} S_{EH}^{(2)} &= \frac{1}{8}\int \sqrt{-g}\,(2h^{\mu\nu}\nabla^2 h_{\mu\nu} - h\nabla^2 h) \\ &= \frac{1}{8}\int \sqrt{-g}\; h_{\mu\nu}\,(2g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})\nabla^2 h_{\alpha\beta} \\ \mathbf{L}^{\mu\nu\alpha\beta} &= \frac{1}{4}(2g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})\nabla^2 \end{aligned} \tag{4.58}$$

So within the approximation mentioned, the Green's function (4.52) for the Lichenrowicz operator is easily obtained as

$$G_{\alpha\beta\rho\lambda}(x,x') = -\frac{\delta(t-r)}{4\pi^2 r^2}\left(2g_{\alpha\rho}g_{\beta\lambda} - \frac{2}{3}g_{\alpha\beta}g_{\rho\lambda}\right) \tag{4.59}$$

where the quantity r is the spatial distance between two particles $r = |\vec{x}-\vec{y}|$ and $t = x^0 - y^0$, and we have chosen the retarded function as usual. Using the solution of $h_{\mu\nu}$ and $G_{\alpha\beta\mu\nu}$ in (4.54) we get

$$\begin{aligned} S &= -\frac{1}{8}\int dx\sqrt{-g}\int dy\sqrt{-g}\;\; G_{\mu\nu\alpha\beta}(x,y)T^{\mu\nu}(x)T^{\alpha\beta}(y) \\ &= \frac{1}{12}\int dx\sqrt{-g}\int dy\sqrt{-g}\;\frac{\delta\left(x^0 - y^0 - |\vec{x}-\vec{y}|\right)}{4\pi^2|\vec{x}-\vec{y}|^2}\Big[3\,T^{\mu\nu}(x)T_{\mu\nu}(y) - T^{\mu}_{\;\;\mu}(x)T^{\alpha}_{\;\;\alpha}(y)\Big] \end{aligned} \tag{4.60}$$

Since we are considering a cluster of particles, there should be a summation over individual particles in the expression of energy-momentum tensor, i.e.,

$$T^{\mu\nu} = \sum_i m_i \int d\tau \frac{\delta^{(5)}(x_i - x_i(\tau))}{\sqrt{-g}} \frac{dx^\mu}{ds} \frac{dx^\nu}{d\tau} \tag{4.61}$$

The trace of $T^{\mu\nu}$ which is needed for (4.60) is given by

$$\begin{aligned} T^\mu_\mu &= \sum_i m_i \int d\tau \frac{\delta^{(5)}(x_i - x_i(\tau))}{\sqrt{-g}} \frac{dx^\mu}{ds} \frac{dx_\mu}{d\tau} \\ &= \sum_i m_i \int ds \; \frac{\delta^{(5)}(x_i - x_i(\tau))}{\sqrt{-g}} \end{aligned} \tag{4.62}$$

So the perturbed action, after solving the equation of motion of $h_{\mu\nu}$ and using its value back in the action, is given by

$$\begin{aligned} S =& \frac{1}{12} \int d^5x d^5y \; \frac{\delta(x^0 - y^0 - |\vec{x} - \vec{y}|)}{4\pi^2 |\vec{x} - \vec{y}|^2} \int d\tau d\xi \sum_{i,j} m_i m_j \\ & \delta^{(5)}(x_i - x_i(\tau)) \delta^{(5)}(x_j - x_j(\xi)) \left(3 \frac{dx^\mu}{ds} \frac{dx^\nu}{d\tau} \frac{dy_\mu}{ds} \frac{dy_\nu}{d\xi} - 1 \right) \\ =& \frac{1}{12} \sum_{i,j} m_i m_j \int d\tau d\xi \; \frac{\delta(x_i^0 - x_j^0 - |\vec{x}_i - \vec{x}_j|)}{4\pi^2 |\vec{x}_i - \vec{x}_j|^2} \left(3 \frac{dx_i^\mu}{ds} \frac{dx_i^\nu}{d\tau} \frac{dx_{j\mu}}{ds} \frac{dx_{j\nu}}{d\xi} - \frac{ds}{d\tau} \frac{ds}{d\xi} \right) \end{aligned} \tag{4.63}$$

In non-relativistic limit, at zeroth order, only the time-components will be important in the 4-velocities. The coordinate time t can be taken as equal to the proper time τ, so that

$$3 \frac{dx_i^\mu}{ds} \frac{dx_i^\nu}{d\tau} \frac{dx_{j\mu}}{ds} \frac{dx_{j\nu}}{d\xi} - \frac{ds}{d\tau} \frac{ds}{d\xi} \approx 3 \frac{dx_i^0}{d\tau_i} \frac{dx_i^0}{d\tau_i} \frac{dx_j^0}{d\tau_j} \frac{dx_j^0}{d\tau_j} - \frac{ds_i}{d\tau_i} \frac{ds_j}{d\tau_j} = 2 \tag{4.64}$$

And finally, in this limit, the perturbation to the action becomes

$$S = \frac{1}{6} \sum_{i \neq j} \int dx_i^0 \; \frac{m_i m_j}{4\pi^2 r_{ij}^2}, \qquad x_j^0 = x_i^0 + |\vec{x}_i - \vec{x}_j| \tag{4.65}$$

This corresponds to an attractive Coulomb potential between two particles, if the retardation condition is also neglected.

In above discussion, we have ignored terms involving both dgg^{-1} and δa. Such terms emerge from the boundary action we used to derive single particle action:

$$\begin{aligned} S(dgg^{-1}, \delta a) &= -\frac{ik}{48\pi^2} \int \mathrm{Tr} \left\{ dgg^{-1} \left[(a+\delta a) d(a+\delta a) + d(a+\delta a)(a+\delta a) + (a+\delta a)^3 \right] \right\} \\ &= \text{orbit action} - \frac{ik}{48\pi^2} \int \mathrm{Tr} \left[dgg^{-1} (da\delta a + \delta a da + a\delta a a) \right] \end{aligned} \tag{4.66}$$

These terms in general do not vanish. In analyzing non-vanishing trace products, we note that there are two possibilities. We can have a term where δe must be coupled with Σ part in dgg^{-1}. This will contain spin-orbit effects since the trace of dgg^{-1} with Σ refers to the spin. We can also have $\delta\omega$ coupled with γ part in dgg^{-1}. Since $\delta\omega$ involves the derivatives of e^a, this will correspond to multipole interactions, which can occur even in the absence of spin. When particles are moving, the Coulomb field is Lorentz-contracted leading to an effect which is not spherically symmetric and, along with retardation effects, generally gives rise to velocity- and acceleration-dependent forces between particles. Such terms should be the GR analog of the Darwin Lagrangian for particles interactions in electrodynamics.

Chapter 5

Summary

As a summary, in this thesis, I used the basic idea from Einstein-Infeld-Hoffmann that a particle in spacetime manifold can be regarded as a singularity. A small region containing the singularity has to be removed to ensure non-singular behavior of physical fields everywhere. The boundary condition on the spheres enclosing singularities will give the physical characteristics of the singularities.

Following this philosophy, the particle action written in coadjoint orbit form in AdS_5 was successfully constructed from Chern-Simons gravity. But this model lacks particle interactions, due to the topological nature of Chern-Simons actions. To bypass the difficulty, we propose the addition of Einstein-Hilbert action to the Chern-Simons action. To avoid ambiguity, the cosmological constant in the two models should match.

Then we have shown that, by expanding the action around its vacuum background up to the second order in perturbations, we can obtain interactions between the particles. The bulk terms in the second order perturbation of the Einstein-Hilbert action leads to the propagator for the gravitational field and this connects in the right way to the perturbations generated by the Chern-Simons term, the latter being proportional to the single point-particle energy-momentum tensor. This is what leads to the interactions. In non-relativistic limit, the classic

Coulomb potential between particles is recovered.

However, there are also emerging spin-orbit couplings and multipole interactions between the particles. We have not investigated these in any detail, these are topics for future work. Also, the expansion beyond 2nd order has not been investigated. Additional nonlinear effects in interactions may be found from higher order terms.

Appendix A

Perturbative Expansion of EH Action

The Einstein-Hilbert action is written as (we are working in 5d spacetime):

$$S = \int \sqrt{-g}\ R\ d^5x \tag{A.1}$$

Variation with respect to the metric $g_{\mu\nu}$ gives the vacuum Einstein equation:

$$\begin{aligned}
\delta S &= \int \delta(\sqrt{-g})R\ d^5x + \int \sqrt{-g}\delta R\ d^5x \\
&= \int \left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\right) Rd^5x + \int \sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu}d^5x \\
&= \int \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\sqrt{-g}\delta g^{\mu\nu}d^5x \\
\delta S &= 0 \quad \implies \quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0
\end{aligned} \tag{A.2}$$

Now we write the action using first order formalism:

$$\begin{aligned}
S &= \frac{1}{6}\int \epsilon_{abcde}R^{ab}\wedge e^c \wedge e^d \wedge e^d = \frac{1}{12}\int \epsilon_{abcde}R^{ab}_{\alpha\beta}e^c_\mu e^d_\nu e^e_\rho \epsilon^{\alpha\beta\mu\nu\rho}d^5x \\
&= \frac{1}{12}\int \epsilon_{abcde}R^{ab}_{\alpha\beta}e^\alpha_m e^\beta_n \epsilon^{mncde}\det(e)d^5x = \int R^{ab}_{\alpha\beta}e^\alpha_a e^\beta_b \det(e)d^5x = \int R\sqrt{-g}d^5x
\end{aligned} \tag{A.3}$$

The variarion with respect to frame field e^a is:

$$\delta_e S = \frac{1}{2}\int \epsilon_{abcde} R^{ab} \wedge e^c \wedge e^d \wedge \delta e^e \quad \Longrightarrow \quad \epsilon_{abcde} R^{ab} e^c e^d = 0 \tag{A.4}$$

The variation with respect to connection ω^{ab} (thus R^{ab}) is:

$$\begin{aligned}\delta_\omega S &= \frac{1}{6}\int \epsilon_{abcde}\delta R^{ab} e^c e^d e^e = \frac{1}{6}\int \epsilon_{abcde}(D\delta\omega^{ab}) e^c e^d e^e \\ &= \frac{1}{6}\int \epsilon_{abcde} D(\delta\omega^{ab} e^c e^d e^e) + \frac{1}{2}\int \epsilon_{abcde}\delta\omega^{ab}(De^c) e^d e^e \\ &= \frac{1}{6}\oint \epsilon_{abcde}\delta\omega^{ab} e^c e^d e^e + \frac{1}{2}\int \epsilon_{abcde}\delta\omega^{ab} T^c e^d e^e\end{aligned} \tag{A.5}$$

If there's no boundary, then the second part gives another equation of motion, which means torsion is zero. When boundary is present, we must have a counter part to cancel the first term. This is the well known Gibbons-Hawking-York boundary term: the trace of extrinsic curvature.

$$S_b = -\frac{1}{6}\oint \epsilon_{abcde}\theta^{ab} e^c e^d e^e, \quad \theta^{ab} = \omega^{ab} - \bar{\omega}^{ab}, \quad \delta\theta^{ab} = \delta\omega^{ab} \tag{A.6}$$

here $\bar{\omega}$ is the connection one-form from the cobordant manifold [21]. In local Gaussian coordinate, near the boundary, metric has the following form:

$$ds^2 = dz^2 + h_{ij}(z,x) dx^i dx^j \tag{A.7}$$

Then the cobordant manifold is defined to be a product manifold at the boundary with its metric given as:

$$d\bar{s}^2 = dz^2 + h_{ij}(z=0,x) dx^i dx^j \tag{A.8}$$

The explicit form of θ^{ab} on the boundary, by its definition, is written as follows in Gaussian coordinate (we use $e^1 = dz$):

$$\theta^{1a} = -K^\alpha_\beta e^a_\alpha dx^\beta \tag{A.9}$$

with all other components vanished. Here i refers to local Lorentz indices, while α, β are coordinates on the boundary.

A.1 First Order Perturbation

Now we want to study how the action behaves under perturbation. Instead of writing $\tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$, we use the frame field:

$$\tilde{e}^a_\mu = e^a_\mu + f^a_\mu \tag{A.10}$$

Define its inverse field as:

$$\tilde{e}^\mu_a = e^\mu_a - f^\mu_a \tag{A.11}$$

We have the following relation

$$(e^a_\mu + f^a_\mu)(e^\nu_a - f^\nu_a) = \delta^\nu_\mu \quad \Longrightarrow \quad f^a_\mu e^\nu_a - e^a_\mu f^\nu_a = 0,\ f^v_a = e^\mu_a f^b_\mu e^\nu_b \tag{A.12}$$

In most cases, changes in metric are defined as:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \quad \tilde{g}^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu}, \quad h^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} h_{\alpha\beta} \tag{A.13}$$

The relations between h and f are:

$$h_{\mu\nu} = \delta g_{\mu\nu} = \eta_{ab}(e^a_\mu f^b_\nu + e^b_\mu f^a_\nu) = f^a_\nu e_{a\mu} + f^a_\mu e_{a\nu},\ h^{\mu\nu} = -\delta g^{\mu\nu} = e^{a\mu} f^\nu_a + e^{a\nu} f^\mu_a \tag{A.14}$$

It is useful to mention that, there's hidden local symmetry in the above relation. If we look at the term $\eta_{ab} e^a_\mu f^b_\nu$, generally it is possible to split it into symmetric and antisymmetric part:

$$e_{a\mu}f^a_\nu = \frac{1}{2}(e_{a\mu}f^a_\nu + e_{a\nu}f^a_\mu) + \frac{1}{2}(e_{a\mu}f^a_\nu - e_{a\nu}f^a_\mu) = \frac{1}{2}(h_{\mu\nu} + t_{\mu\nu}) \tag{A.15}$$

The antisymmetric part is redundant. We can use the local symmetry to eliminate it. The extra freedom comes from local Lorentz invariance:

$$ds^2 = \eta_{ab}(S^a{}_c e^c)(S^b{}_d e^d) = \eta_{cd}e^c e^d \tag{A.16}$$

S is the local Lorentz boost matrix. Infinitesimally, we can write $S^a{}_b = \delta^a{}_b + \theta^a{}_b$. Apply it to the perturbed frame fields:

$$\begin{aligned} S^a{}_b e'^b{}_\mu &= (\delta^a{}_b + \theta^a{}_b)(e^b{}_\mu + f^b{}_\mu) = e^a{}_\mu + \theta^a{}_b f^b{}_\mu + \theta^a{}_b e^b{}_\mu + f^b{}_\mu \\ &= e^a{}_\mu + \theta^a{}_b f^b{}_\mu + e^a{}_\alpha g^{\alpha\lambda}(\theta_{\lambda\mu} + f_{\lambda\mu}) \end{aligned} \tag{A.17}$$

Then we can appropriately choose $\theta_{\lambda\mu}$ to cancel the antisymmetric part in $f_{\lambda\mu}$. Consequently, up to first order, the change in metric is related to change in frame fields as:

$$e_{a\mu}f^a_\nu = e_{a\nu}f^a_\mu = \frac{1}{2}h_{\mu\nu} \tag{A.18}$$

The perturbed Einstein-Hilbert action is:

$$\tilde{S} = \int \tilde{R}^{ab}_{\alpha\beta}\tilde{e}^\alpha_a\tilde{e}^\beta_b \det(\tilde{e})d^5x = \int (R+\delta R)^{ab}_{\alpha\beta}(e-f)^\alpha_a(e-f)^\beta_b \det(e+f)d^5x \tag{A.19}$$

The determinant can be expanded in powers of f:

$$\begin{aligned} \det(e+f) &= \det(e)\cdot\det\big(1+e^{-1}f\big) = \det(e)\cdot\exp\big[\mathrm{tr}\big(\log\big(1+e^{-1}f\big)\big)\big] \\ &\approx \det(e)\cdot\exp\big[\mathrm{tr}\big(e^{-1}f\big)\big] \approx \det(e)\cdot\big[1+\mathrm{tr}\big(e^{-1}f\big)\big] \\ &= \det(e)\cdot(1+e^\mu_a f^a_\mu) = \det(e)\cdot(1+e^a_\mu f^\mu_a) \end{aligned} \tag{A.20}$$

So up to first order, the perturbed EH action is:

$$\tilde{S} = \int \left(R^{ab}_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b} - 2R^{ab}_{\alpha\beta} e^{\alpha}_{a} f^{\beta}_{b} + R^{ab}_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b} e^{c}_{\mu} f^{\mu}_{c} \right) \det(e) d^5x \tag{A.21}$$

Variation with respect to f gives:

$$\delta_f \tilde{S} = -\int \left(2R^{ab}_{\alpha\beta} e^{\alpha}_{a} - R^{mn}_{\mu\nu} e^{\mu}_{m} e^{\nu}_{n} e^{b}_{\beta} \right) \delta f^{\beta}_{b} \det(e) d^5x \tag{A.22}$$

By requiring it to vanish, we get the equation of motion, which is the vacuum Einstein equaiton as expected:

$$\delta_f \tilde{S} = 0 \quad \Longrightarrow \quad R^{ab}_{\alpha\beta} e^{\alpha}_{a} - \frac{1}{2} R^{mn}_{\mu\nu} e^{\mu}_{m} e^{\nu}_{n} e^{b}_{\beta} = 0 \tag{A.23}$$

We have not included the boundary term contained in δR:

$$\delta_f R^{ab} = D\delta_f \omega^{ab} + \delta_f \omega^{a}_{c} \wedge \delta_f \omega^{cb} \tag{A.24}$$

The second term is second order which will be discussed later. The first term gives the boundary action:

$$S_b = \frac{1}{6} \oint \epsilon_{abcde} \delta\omega^{ab} e^c e^d e^e = -2 \oint \delta\omega^{ab}_{\mu} e^{\mu}_{a} e^{\sigma}_{b} \det(e) dS_{\sigma} \tag{A.25}$$

The change in spin-connection, by requiring compatibility with zero-torsion condition, is:

$$\delta\omega^{ab}_{\mu} = \frac{1}{2} \left[e^{\alpha b} (\nabla_{\alpha} f^{a}_{\mu} - \nabla_{\mu} f^{a}_{\alpha}) + e^{\alpha a} (\nabla_{\mu} f^{b}_{\alpha} - \nabla_{\alpha} f^{b}_{\mu}) - e^{\alpha a} e^{\beta b} (\nabla_{\alpha} f^{k}_{\beta} - \nabla_{\beta} f^{k}_{\alpha}) e_{\mu k} \right] \tag{A.26}$$

The fully covariant derivative is defined as:

$$\nabla_\alpha \xi^a_\mu = \partial_\alpha \xi^a_\mu - \Gamma^\lambda_{\alpha\mu}\xi^a_\lambda + \omega^a_{\alpha b}\xi^b_\mu, \quad \nabla_\alpha \xi^\mu_a = \partial_\alpha \xi^\mu_a + \Gamma^\mu_{\alpha\lambda}\xi^\lambda_a - \omega^b_{\alpha a}\xi^\mu_b \tag{A.27}$$

Its action on both metric and frame fields (and their inverse) should vanish, because of zero torsion:

$$\nabla_\mu g^{\alpha\beta} = \nabla_\mu g_{\alpha\beta} = \nabla_\mu e^a_\alpha = \nabla_\mu e^\alpha_a = 0 \tag{A.28}$$

Put (A.26) back into the boundary term (A.25):

$$\begin{aligned} S_b = & -\oint \left[g^{\alpha\sigma}(\nabla_\alpha f^a_\mu - \nabla_\mu f^a_\alpha) e^\mu_a + g^{\alpha\mu}(\nabla_\mu f^b_\alpha - \nabla_\alpha f^b_\mu) e^\sigma_b \right. \\ & \left. - g^{\alpha\mu} g^{\beta\sigma}(\nabla_\alpha f^k_\beta - \nabla_\beta f^k_\alpha) e_{\mu k} \right] \cdot \det(e) dS_\sigma \\ = & -2 \oint \left[g^{\alpha\sigma} \nabla_\alpha (f^a_\mu e^\mu_a) - g^{\alpha\sigma} \nabla_\mu (f^a_\alpha e^\mu_a) \right] \cdot \det(e) dS_\sigma \end{aligned} \tag{A.29}$$

The second term vanishes because the tangential derivative of $\delta g_{\mu\nu}$ on the boundary must vanish:

$$\oint g^{\alpha\sigma} g^{\mu\nu} \nabla_\mu \delta g_{\alpha\nu} dS_\sigma = -\oint \nabla_\mu \delta g^{\mu\sigma} dS_\sigma = 0 \tag{A.30}$$

The first term is the same as we vary the trace of extrinsic curvature:

$$\begin{aligned} S_b = & -2 \oint \nabla_\alpha (f^a_\mu e^\mu_a) \det(e) dS^\alpha = -2 \oint \nabla_\alpha (g^{\mu\nu} f^a_\mu e_{a\nu}) \det(e) dS^\alpha \\ = & -\oint g^{\mu\nu} \nabla_\alpha \delta g_{\mu\nu} \det(e) dS^\alpha \end{aligned} \tag{A.31}$$

So up to first order, this reproduces what we've known: the bulk equation is vacuum Einstein equation and the boundary term is the trace of extrinsic curvature.

A.2 Second Order Perturbation

Up to second order, the perturbation in metric is:

$$
\begin{aligned}
\delta g_{\mu\nu} =& \eta_{ab}(e+f)^a_\mu (e+f)^b_\nu - \eta_{ab} e^a_\mu e^b_\nu = (f^a_\mu e_{\nu a} + f^a_\nu e_{\mu a}) + \eta_{ab} f^a_\mu f^b_\nu \\
\delta g^{\mu\nu} =& \eta^{ab}(e-f)^\mu_a (e-f)^\nu_b - \eta^{ab} e^\mu_a e^\nu_b = -(e^{a\mu} f^\nu_b + e^{a\nu} f^\mu_a) + \eta^{ab} f^\mu_a f^\nu_b \\
h_{\mu\nu} =& (f^a_\mu e_{\nu a} + f^a_\nu e_{\mu a}) + \eta_{ab} f^a_\mu f^b_\nu = h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} \\
h^{\mu\nu} =& (e^{a\mu} f^\nu_b + e^{a\nu} f^\mu_a) - \eta^{ab} f^\mu_a f^\nu_b = (h^{(1)})^{\mu\nu} + (h^{(2)})^{\mu\nu}
\end{aligned}
\tag{A.32}
$$

The perturbed action, up to second order, is:

$$
\begin{aligned}
\tilde{S} =& \frac{1}{6}\int \epsilon_{abcde} \tilde{R}^{ab} \tilde{e}^c \tilde{e}^d \tilde{e}^e \\
=& \frac{1}{6}\int \epsilon_{abcde} (R^{ab} + D\delta\omega^{ab} + \delta\omega^a_k \delta\omega^{kb})(e+f)^c (e+f)^d (e+f)^e \\
=& \frac{1}{6}\int \epsilon_{abcde} \left[R^{ab} \tilde{e}^c \tilde{e}^d \tilde{e}^e + \delta\omega^a_k \delta\omega^{kb} e^c e^d e^e + (D\delta\omega^{ab})(e^c e^d e^e + 3 f^c e^d e^e)\right] \\
=& \frac{1}{6}\int \epsilon_{abcde} \left[R^{ab} \tilde{e}^c \tilde{e}^d \tilde{e}^e + \delta\omega^a_k \delta\omega^{kb} e^c e^d e^e + 3\delta\omega^{ab} (Df)^c e^d e^e\right] \\
& + \frac{1}{6}\oint \epsilon_{abcde} \left[\delta\omega^{ab} e^c e^d e^e + 3\delta\omega^{ab} f^c e^d e^e\right] \\
=& M_1 + M_2 + M_3 + M_4 + M_5
\end{aligned}
\tag{A.33}
$$

We will calculate them seperately. In fact, because M_4 and M_5 represent boundary interactions involving 2 propagators, by the argument in chapter 4, we shall ignore them for now.

Calculate M_1

$$\begin{aligned} M_1 &= \frac{1}{6}\int \epsilon_{abcde} R^{ab} \tilde{e}^c \tilde{e}^d \tilde{e}^e = \frac{1}{12}\int \epsilon_{abcde} R^{ab}_{\mu\nu} \tilde{e}^c_\alpha \tilde{e}^d_\beta \tilde{e}^e_\rho \epsilon^{\mu\nu\alpha\beta\rho} d^5x \\ &= \frac{1}{12}\int \epsilon_{abcde} R^{ab}_{\mu\nu} \tilde{e}^\mu_m \tilde{e}^\nu_n \epsilon^{mncde} \det(\tilde{e}) d^5x \\ &= \int R^{ab}_{\mu\nu} \tilde{e}^\mu_a \tilde{e}^\nu_b \det(\tilde{e}) d^5x = \int R^{ab}_{\mu\nu} (e-f)^\mu_a (e-f)^\nu_b \det(\tilde{e}) d^5x \end{aligned} \tag{A.34}$$

Now, we need the expansion of determinant up to second order:

$$\begin{aligned} \det(\tilde{e}) &= \det(e+f) = \det(e)\cdot\det\bigl(1+e^{-1}f\bigr) \\ &= \det(e)\cdot\exp\left[\operatorname{tr}\log\bigl(1+e^{-1}f\bigr)\right] \\ &\approx \det(e)\cdot\exp\operatorname{tr}\left[e^{-1}f - \frac{1}{2}(e^{-1}f)^2\right] \\ &\approx \det(e)\cdot\left[1+\operatorname{tr}(e^{-1}f) - \frac{1}{2}\operatorname{tr}(e^{-1}f)^2 + \frac{1}{2}\operatorname{tr}^2(e^{-1}f)\right] \end{aligned} \tag{A.35}$$

So:

$$\begin{aligned} M_1 = &\int R^{ab}_{\mu\nu} e^\mu_a e^\nu_b \left[1+\operatorname{tr}(e^{-1}f) - \frac{1}{2}\operatorname{tr}(e^{-1}f)^2 + \frac{1}{2}\operatorname{tr}^2(e^{-1}f)\right]\det(e) d^5x \\ &- \int 2R^{ab}_{\mu\nu} e^\mu_a f^\nu_b [1+\operatorname{tr}(e^{-1}f)]\det(e) d^5x + \int R^{ab}_{\mu\nu} f^\mu_a f^\nu_b \det(e) d^5x \end{aligned} \tag{A.36}$$

At second order, the trace $\operatorname{tr}(e^{-1}f)$ gets a correction:

$$\operatorname{tr}(e^{-1}f) = e^\mu_a f^a_\mu = g^{\mu\nu} f^a_\mu e_{a\nu} = \frac{1}{2} g^{\mu\nu}(h_{\mu\nu} - h^{(2)}_{\mu\nu}) = \frac{1}{2}h - \frac{1}{2} g^{\mu\nu} h^{(2)}_{\mu\nu} \tag{A.37}$$

Since $h^{(2)}$ is already second order, we can use the first order relation (A.12):

$$\begin{aligned} h^{(2)}_{\mu\nu} = &e_{ab} f^a_\mu f^b_\nu \approx e_{ab} e^a_\rho f^\rho_k e^k_\mu f^b_\nu \\ = &f^b_\nu e_{b\rho} e^{k\alpha} f^\rho_k g_{\mu\alpha} \approx \frac{1}{4} g_{\mu\alpha} h^{\alpha\rho} h_{\nu\rho} \end{aligned} \tag{A.38}$$

Terms higher than second order have been ignored. So the trace now is:

$$\mathrm{tr}(e^{-1}f) = \frac{1}{2}h - \frac{1}{2}g^{\mu\nu}f^{(2)}_{\mu\nu} = \frac{1}{2}h - \frac{1}{8}h_{\mu\nu}h^{\mu\nu} \tag{A.39}$$

There are two more trace terms in the expansion of determinant:

$$\begin{aligned}\mathrm{tr}^2(e^{-1}f) &\approx \frac{1}{4}h^2 \\ \mathrm{tr}(e^{-1}f)^2 &= e^\mu_a f^a_\nu e^\nu_b f^b_\mu = g^{\mu\alpha}g^{\nu\beta}f^a_\nu e_{a\alpha}f^b_\mu e_{b\beta} \\ &\approx \frac{1}{4}g^{\mu\alpha}g^{\nu\beta}h_{\nu\alpha}h_{\mu\beta} = \frac{1}{4}h_{\mu\nu}h^{\mu\nu}\end{aligned} \tag{A.40}$$

Then the determinant expanded in $h_{\mu\nu}$ is:

$$\begin{aligned}\det(\tilde{e}) &= \det(e)\cdot\left[1 + \mathrm{tr}(e^{-1}f) - \frac{1}{2}\,\mathrm{tr}(e^{-1}f)^2 + \frac{1}{2}\,\mathrm{tr}^2(e^{-1}f)\right] \\ &= \det(e)\cdot\left(1 + \frac{1}{2}h - \frac{1}{4}h^{\mu\nu}h_{\mu\nu} + \frac{1}{8}h^2\right)\end{aligned} \tag{A.41}$$

The second term in M_1 is:

$$\begin{aligned}2R^{ab}_{\mu\nu}e^\mu_a f^\nu_b[1+\mathrm{tr}(e^{-1}f)] &= 2R^\rho{}_{\sigma\mu\nu}e^a_\rho e^{b\sigma}e^\mu_a f^v_b[1+\mathrm{tr}(e^{-1}f)] \\ &\approx 2R_{\sigma\nu}e^{b\sigma}f^\nu_b(1+\frac{1}{2}h) \\ &= R_{\sigma\nu}(h^{\sigma\nu} + \eta^{ab}f^\sigma_a f^\nu_b)(1+\frac{1}{2}h) \\ &\approx R_{\sigma\nu}h^{\sigma\nu}(1+\frac{1}{2}h) + R_{\sigma\nu}\eta^{ab}f^\sigma_a f^\nu_b\end{aligned} \tag{A.42}$$

Again, use the relation (A.12):

$$\begin{aligned}R_{\sigma\nu}\eta^{ab}f^\sigma_a f^\nu_b &= R_{\sigma\nu}\eta^{ab}e^\mu_a f^k_\mu e^\sigma_k f^\nu_b = R_{\sigma\nu}e^{b\mu}f^\nu_b g^{\sigma\alpha}f^k_\mu e_{k\alpha} \\ &\approx \frac{1}{4}R_{\sigma\nu}h^{\mu\nu}g^{\sigma\alpha}h_{\mu\alpha} = \frac{1}{4}R_{\sigma\nu}h^{\mu\nu}h_\mu{}^\sigma\end{aligned} \tag{A.43}$$

So:

$$2R^{ab}_{\mu\nu}e^{\mu}_{a}f^{\nu}_{b}[1+\mathrm{tr}(e^{-1}f)] \approx R_{\sigma\nu}\left[h^{\sigma\nu}(1+\frac{1}{2}h)+\frac{1}{4}h^{\mu\nu}h_{\mu}^{\ \sigma}\right] \tag{A.44}$$

The last term in M_1 is:

$$R^{ab}_{\mu\nu}f^{\mu}_{a}f^{\nu}_{b} = R_{\rho\sigma\mu\nu}e^{a\rho}f^{\mu}_{a}e^{b\sigma}f^{\nu}_{b} \approx \frac{1}{4}R_{\rho\sigma\mu\nu}h^{\rho\mu}h^{\sigma\nu} \tag{A.45}$$

Collect all terms:

$$\begin{aligned} M_1 = \int \Big\{ & R\left(1+\frac{1}{2}h-\frac{1}{4}h^{\mu\nu}h_{\mu\nu}+\frac{1}{8}h^2\right) - R_{\sigma\nu}\Big[h^{\sigma\nu}(1+\frac{1}{2}h) \\ & +\frac{1}{4}h^{\mu\nu}h_{\mu}^{\ \sigma}\Big] + \frac{1}{4}R_{\rho\sigma\mu\nu}h^{\rho\mu}h^{\sigma\nu}\Big\} \cdot \det(e)d^5x \end{aligned} \tag{A.46}$$

Calculate M_2 and M_3

The second term is:

$$\begin{aligned} M_2 &= \frac{1}{6}\int \epsilon_{abcde}\delta\omega^{a}_{k}\delta\omega^{kb}e^{c}e^{d}e^{e} \\ &= \frac{1}{6}\int \eta_{kl}\epsilon_{abcde}\delta\omega^{al}_{[\mu}\delta\omega^{kb}_{\nu]}e^{c}_{\alpha}e^{d}_{\beta}e^{e}_{\rho}\epsilon^{\mu\nu\alpha\beta\rho}d^5x \\ &= \int \eta_{kl}\epsilon_{abcde}\left(\delta\omega^{al}_{\mu}\delta\omega^{kb}_{\nu}-\delta\omega^{al}_{\nu}\delta\omega^{kb}_{\mu}\right)e^{\mu}_{a}e^{\nu}_{b}\det(e)d^5x \end{aligned} \tag{A.47}$$

Using the expression of $\delta\omega$ in (A.26), we have:

$$\begin{aligned} \eta^{kl}\delta\omega^{al}_{\mu}\delta\omega^{kb}_{\nu}e^{\mu}_{a}e^{\nu}_{b} =& 2g^{\alpha\rho}\left(\nabla_\alpha h^{\mu}_{\mu}\nabla_\nu h^{\nu}_{\rho}-\nabla_\alpha h^{\mu}_{\mu}\nabla_\rho h^{\nu}_{\nu}-\nabla_\mu h^{\mu}_{\alpha}\nabla_\nu h^{\nu}_{\rho}\right) \\ \eta^{kl}\delta\omega^{al}_{\nu}\delta\omega^{kb}_{\mu}e^{\mu}_{a}e^{\nu}_{b} =& g^{\alpha\rho}\left(-\frac{1}{2}\nabla_\alpha h^{\mu}_{\nu}\nabla_\rho h^{\nu}_{\mu}+\nabla_\alpha h^{\mu}_{\nu}\nabla_\mu h^{\nu}_{\rho}-\frac{1}{2}\nabla_\nu h^{\mu}_{\alpha}\nabla_\mu h^{\nu}_{\rho}\right) \\ &+\frac{1}{2}g^{\alpha\rho}g^{\beta\mu}g_{\sigma\lambda}\nabla_\alpha h^{\lambda}_{\beta}(\nabla_\mu h^{\sigma}_{\rho}-\nabla_\rho h^{\sigma}_{\mu}) \end{aligned} \tag{A.48}$$

The third term is:

$$\begin{aligned} M_3 &= \frac{1}{2}\int \epsilon_{abcde}\delta\omega^{ab}(Df^c)e^d e^e \\ &= \frac{1}{2}\int \epsilon_{abcde}\delta\omega^{ab}_{\mu} D_{[\alpha}f^c_{\beta]}e^d_\rho e^e_\sigma \epsilon^{\mu\alpha\beta\rho\sigma} d^5x \\ &= 2\int \delta\omega^{ab}_\mu \nabla_{[\alpha}f^c_{\beta]}(e^\mu_a e^\alpha_b e^\beta_c + e^\mu_b e^\alpha_c e^\beta_a + e^\mu_c e^\alpha_a e^\beta_b)\det(e)d^5x \end{aligned} \tag{A.49}$$

After a bit calculation, each term is:

$$\begin{aligned} 2\delta\omega^{ab}e^\mu_a e^\alpha_b \nabla_{[\alpha}f^c_{\beta]}e^\beta_c =& g^{\alpha\rho}(-2\nabla_\rho h^\mu_\mu \nabla_\beta h^\beta_\alpha + \nabla_\rho h^\mu_\mu \nabla_\alpha h^\beta_\beta + \nabla_\mu h^\mu_\rho \nabla_\beta h^\beta_\alpha) \\ 2\delta\omega^{ab}_\mu e^\mu_b e^\beta_a \nabla_{[\alpha}f^c_{\beta]}e^\alpha_c =& g^{\alpha\rho}(-2\nabla_\mu h^\mu_\rho \nabla_\alpha h^\beta_\beta + \nabla_\mu h^\mu_\rho \nabla_\beta h^\beta_\alpha + \nabla_\rho h^\mu_\mu \nabla_\alpha h^\beta_\beta) \\ 2\delta\omega^{ab}e^\alpha_a e^\beta_b \nabla_{[\alpha}f^c_{\beta]}e^\mu_c =& g^{\alpha\rho}(2\nabla_\rho h^\beta_\mu \nabla_\beta h^\mu_\alpha - \nabla_\rho h^\beta_\mu \nabla_\alpha h^\mu_\beta - \nabla_\mu h^\beta_\rho \nabla_\beta h^\mu_\alpha) \\ & - \frac{1}{2}g^{\alpha\rho}g^{\beta\sigma}g_{\mu\lambda}(\nabla_\rho h^\lambda_\sigma - \nabla_\sigma h^\lambda_\rho)(\nabla_\alpha h^\mu_\beta - \nabla_\beta h^\mu_\alpha) \end{aligned} \tag{A.50}$$

We can add up the second and third terms, and it gives:

$$\begin{aligned} M_2 + M_3 = \int \Big\{ & g^{\alpha\rho}(\nabla_\rho h^\mu_\mu \nabla_\alpha h^\beta_\beta - 2\nabla_\rho h^\mu_\mu \nabla_\beta h^\beta_\alpha + \nabla_\mu h^\mu_\rho \nabla_\beta h^\beta_\alpha \\ & + \nabla_\rho h^\beta_\mu \nabla_\beta h^\mu_\alpha - \frac{1}{2}\nabla_\rho h^\beta_\mu \nabla_\alpha h^\mu_\beta - \frac{1}{2}\nabla_\mu h^\beta_\rho \nabla_\beta h^\mu_\alpha) \\ & - g^{\alpha\rho}g^{\beta\sigma}g_{\mu\lambda}\nabla_{[\rho}h^\lambda_{\sigma]}\nabla_{[\alpha}h^\mu_{\beta]} \Big\} \cdot \det(e)d^5x \end{aligned} \tag{A.51}$$

The last term can be simplified as:

$$\begin{aligned} g^{\alpha\rho}g^{\beta\sigma}g_{\mu\lambda}\nabla_{[\rho}h^\lambda_{\sigma]}\nabla_{[\alpha}h^\mu_{\beta]} =& \frac{1}{2}g^{\alpha\rho}g^{\beta\sigma}g_{\mu\lambda}\nabla_\rho h^\lambda_\sigma(\nabla_\alpha h^\mu_\beta - \nabla_\beta h^\mu_\alpha) \\ =& \frac{1}{8}\nabla_\rho h_{\sigma\mu}\nabla^\rho h^{\sigma\mu} - \frac{1}{8}\nabla_\rho h_{\sigma\mu}\nabla^\sigma h^{\rho\mu} \end{aligned} \tag{A.52}$$

The Bulk Action

Now we are ready to write down the bulk action by gathering all terms calculated above:

$$\begin{aligned} M_2 + M_3 =& \frac{1}{4}\nabla^\alpha h \nabla_\alpha h - \frac{1}{2}\nabla_\rho h \nabla_\beta h^{\rho\beta} + \frac{1}{4}\nabla_\mu h^{\alpha\mu}\nabla_\beta h_\alpha{}^\beta + \frac{1}{4}\nabla^\alpha h^{\mu\beta}\nabla_\beta h_{\alpha\mu} \\ & - \frac{1}{8}\nabla^\alpha h^{\mu\beta}\nabla_\alpha h_{\beta\mu} - \frac{1}{8}\nabla^\mu h^{\alpha\beta}\nabla_\beta h_{\alpha\mu} - \frac{1}{8}\nabla_\rho h_{\sigma\mu}\nabla^\rho h^{\sigma\mu} + \frac{1}{8}\nabla_\rho h_{\sigma\mu}\nabla^\sigma h^{\rho\mu} \\ =& \frac{1}{4}\nabla^\alpha h \nabla_\alpha h - \frac{1}{4}\nabla^\alpha h^{\mu\nu}\nabla_\alpha h_{\mu\nu} - \frac{1}{2}\nabla^\mu h \nabla^\nu h_{\nu\mu} + \frac{1}{4}\nabla^\nu h_{\nu\mu}\nabla_\alpha h^{\alpha\mu} \\ & + \frac{1}{4}\nabla_\alpha h^{\mu\nu}\nabla_\nu h^\alpha{}_\mu \end{aligned} \tag{A.53}$$

The last term is:

$$\begin{aligned} \nabla_\alpha h^{\mu\nu}\nabla_\nu h^\alpha{}_\mu =& \nabla_\alpha (h^{\mu\nu}\nabla_\nu h^\alpha{}_\mu) - h^{\mu\nu}\nabla_\alpha\nabla_\nu h^\alpha{}_\mu \\ =& \nabla_\alpha (h^{\mu\nu}\nabla_\nu h^\alpha{}_\mu) - h^{\mu\nu}[\nabla_\alpha, \nabla_\nu] h^\alpha{}_\mu - h^{\mu\nu}\nabla_\nu\nabla_\alpha h^\alpha{}_\mu \\ =& \nabla_\alpha (h^{\mu\nu}\nabla_\nu h^\alpha{}_\mu) - h^{\mu\nu}[\nabla_\alpha, \nabla_\nu] h^\alpha{}_\mu - \nabla_\nu (h^{\mu\nu}\nabla_\alpha h^\alpha{}_\mu) + \nabla_\nu h^{\mu\nu}\nabla_\alpha h^\alpha{}_\mu \\ =& \nabla_\alpha (h^{\mu\nu}\nabla_\nu h^\alpha{}_\mu - h^{\mu\alpha}\nabla_\nu h^\nu{}_\alpha) + \nabla^\nu h_{\nu\mu}\nabla_\alpha h^{\alpha\mu} - h^{\mu\nu}[\nabla_\alpha, \nabla_\nu] h^\alpha{}_\mu \end{aligned} \tag{A.54}$$

The commutator will give terms involving curvature tensor:

$$h^{\mu\nu}[\nabla_\alpha, \nabla_\nu] h^\alpha{}_\mu = h^{\mu\nu}(R^\alpha{}_{\lambda\alpha\nu} h^\lambda{}_\mu - R^\lambda{}_{\mu\alpha\nu} h^\alpha{}_\lambda) = h^{\mu\nu} h^\lambda{}_\mu R_{\lambda\nu} - h^{\mu\nu} h^{\alpha\lambda} R_{\lambda\mu\alpha\nu} \tag{A.55}$$

Put it back into (A.53) and carry out integration by parts:

$$\begin{aligned} M_2 + M_3 =& \frac{1}{4}\nabla^\alpha (h\nabla_\alpha h + h^{\mu\nu}\nabla_\nu h_{\alpha\mu} - h^\mu{}_\alpha \nabla_\nu h^\nu{}_\mu - h^{\mu\nu}\nabla_\alpha h_{\mu\nu}) \\ & - \frac{1}{4} h\nabla^2 h - \frac{1}{2}\nabla^\mu h \nabla^\nu h_{\nu\mu} + \frac{1}{4} h^{\mu\nu}\nabla^2 h_{\mu\nu} + \frac{1}{2}\nabla^\nu h_{\nu\mu}\nabla_\alpha h^{\alpha\mu} \\ & - \frac{1}{4} h^{\mu\nu} h^\lambda{}_\mu R_{\lambda\nu} + \frac{1}{4} h^{\mu\nu} h^{\alpha\lambda} R_{\lambda\mu\alpha\nu} \end{aligned} \tag{A.56}$$

Combined with terms in (A.46), we have the bulk action up to second order in $h_{\mu\nu}$:

$$S_{EH}(g+h) = S^{(0)} + S^{(1)} + S^{(2)} \tag{A.57}$$

$$S^{(0)} = \int R \det(e) d^5x \tag{A.58}$$

$$S^{(1)} = -\int \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) h^{\mu\nu} \det(e) d^5x \tag{A.59}$$

$$\begin{aligned} S^{(2)} = \int d^5x \det(e) \Big\{ &\frac{1}{4} h^{\mu\nu} \nabla^2 h_{\mu\nu} - \frac{1}{4} h \nabla^2 h - \frac{1}{2} \nabla^\mu h \nabla^\nu h_{\nu\mu} + \frac{1}{2} \nabla^\nu h_{\nu\mu} \nabla_\alpha h^{\alpha\mu} \\ &+ \frac{1}{2} h^{\mu\lambda} h^{\nu\sigma} R_{\mu\nu\lambda\sigma} + \frac{1}{2} \left(h^{\mu\lambda} h^\nu{}_\lambda - h h^{\mu\nu}\right) R_{\mu\nu} + \frac{1}{8} \left(h^2 - 2 h^{\mu\nu} h_{\mu\nu}\right) R \Big\} \end{aligned} \tag{A.60}$$

These are in agreement with the result obtained in [6]. Since boundary terms will involve two propagtors, and can be neglected for now, we will omit the calculation here.

Bibliography

[1] A. Achucarro and P. K. Townsend. A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories. *Phys. Lett.*, B180:89, 1986. [,732(1987)].

[2] R. Arnowitt, S. Deser, and C. W. Misner. The Dynamics of General Relativity. may 2004.

[3] Alex S Arvanitakis, Alec E Barns-Graham, and Paul K Townsend. Anti-de Sitter Particles and Manifest (Super)Isometries. *Physical Review Letters*, 118(14), 2017.

[4] A. Ashtekar. New Variables for Classical and Quantum Gravity. *Phys. Rev. Lett.*, 57:2244–2247, 1986.

[5] Vijay Balasubramanian and Per Kraus. A Stress Tensor for Anti-de Sitter Gravity. *Communications in Mathematical Physics*, 208(2):413–428, feb 1999.

[6] Fiorenzo Bastianelli and Roberto Bonezzi. One-loop quantum gravity from a worldline viewpoint. apr 2013.

[7] Sayantani Bhattacharyya, Shiraz Minwalla, Veronika E. Hubeny, and Mukund Rangamani. Nonlinear fluid dynamics from gravity. *Journal of High Energy Physics*, 2008(02):045, 2008.

[8] David G. Boulware and S. Deser. String-Generated Gravity Models. *Physical Review Letters*, 55(24):2656–2660, dec 1985.

[9] Norman H Christ, Erick J. Weinberg, and Nancy K Stanton. General self-dual Yang-Mills solutions. *Physical Review D*, 18(6):2013–2025, 1978.

[10] Eric D'Hoker, Daniel Z Freedman, Samir D Mathur, Alec Matusis, and Leonardo Rastelli. Graviton and gauge boson propagators in AdS+1. *Nuclear Physics B*, 562(1-2):330–352, nov 1999.

[11] D. M. Eardley. Observable effects of a scalar gravitational field in a binary pulsar. *The Astrophysical Journal*, 196:L59–L62, March 1975.

[12] Tohru Eguchi, Peter B. Gilkey, and Andrew J. Hanson. Gravitation, gauge theories and differential geometry. *Physics Reports*, 66(6):213–393, dec 1980.

[13] Tohru Eguchi and Andrew J Hanson. Gravitational Instantons. *General Relativity and Gravitation*, 11(5), 1979.

[14] Albert Einstein, L. Infeld, and B. Hoffmann. The Gravitational equations and the problem of motion. *Annals Math.*, 39:65–100, 1938.

[15] G W Gibbons and S W Hawking. Classification of Gravitational Instanton Symmetries. *Commun. Math. Phys. Mathematical Physics*, 66:291–310, 1979.

[16] Martin Heinze, George Jorjadze, and Luka Megrelidze. Coset construction of AdS particle dynamics. *J. Math. Phys.*, 58(1):012301, 2017.

[17] Leopold Infeld. Equations of Motion in General Relativity Theory and the Action Principle. *Reviews of Modern Physics*, 29(3):398–411, jul 1957.

[18] Lei Jiusi and V. P. Nair. Actions for particles and strings and Chern-Simons gravity. *Phys. Rev.*, D96(6):065019, 2017.

[19] Dimitra Karabali and V. P. Nair. Relativistic Particle and Relativistic Fluids: Magnetic Moment and Spin-Orbit Interactions. *Phys. Rev.*, D90(10):105018, 2014.

[20] A A Kirillov. Merits and demerits of the orbit method. *Bulletin of the American Mathematical Society*, 36(04):433–489, aug 1999.

[21] P Mora, R Olea, R Troncoso, and J Zanelli. Finite action principle for Chern-Simons AdS gravity. *Journal of High Energy Physics*, 2004(06):036–036, jun 2004.

[22] Don N. Page. Some Gravitational Instantons. In *1ST SEMINAR ON QUANTUM GRAVITY Moscow, USSR, December 5-7, 1978*, 2009.

[23] J Samuel. Gravitational instantons from the ashtekar variables. *Classical and Quantum Gravity*, 5(8):L123, 1988.

[24] Joseph Samuel. Gravitational instantons from the Ashtekar variables. *Class. Quantum Grav*, 5:123–125, 1988.

[25] Shozo Uehara. A note on gravitational and SU(2) instantons with Ashtekar variables. *Class. Quantum Grav*, 8, 1991.

[26] Stefan Vandoren and Peter van Nieuwenhuizen. Lectures on instantons. 2008.

[27] A I Vaĭnshteĭn, Valentin I Zakharov, Viktor A Novikov, and Mikhail A Shifman. ABC of instantons. *Soviet Physics Uspekhi*, 25(4):195–215, apr 1982.

[28] C.M. Will. *Theory and Experiment in Gravitational Physics.* Cambridge University Press, 2018.

[29] Edward Witten. (2+1)-Dimensional Gravity as an Exactly Soluble System. *Nucl. Phys.*, B311:46, 1988.

[30] Alexandre Yale and T. Padmanabhan. Structure of Lanczos-Lovelock Lagrangians in Critical Dimensions. *Gen. Rel. Grav.*, 43:1549–1570, 2011.

[31] Jorge Zanelli. Chern-Simons Gravity: from 2+1 to 2n+1 dimensions. *Brazilian Journal of Physics*, 30(2):251–267, jun 2000.

[32] Jorge Zanelli. Lecture notes on Chern-Simons (super-)gravities. Second edition (February 2008). In *Proceedings, 7th Mexican Workshop on Particles and Fields (MWPF 1999): Merida, Mexico, November 10-17, 1999*, 2005.