

GABOR FRAMES AND DIRECTIONAL TIME-FREQUENCY ANALYSIS

LOUKAS GRAFAKOS AND CHRISTOPHER SANSING

ABSTRACT. We introduce a directionally sensitive time-frequency decomposition and representation of functions. The coefficients of this representation allow us to measure the “amount” of frequency a function (signal, image) contains in a certain time interval, and also in a certain direction. This has been previously achieved using a version of wavelets called ridgelets [2, 3] but in this work we discuss an approach based on time-frequency or Gabor elements. For such elements, a Parseval formula and a continuous frame-type representation together with boundedness properties of a semi-discrete frame operator are obtained. Spaces of functions tailored to measure quantitative properties of the time-frequency-direction analysis coefficients are introduced and some of their basic properties are discussed. Applications to image processing and medical imaging are presented.

1. INTRODUCTION

In this work we discuss certain topics of time-frequency analysis with the additional element of direction. A fundamental tool in time-frequency analysis, as developed in the classical references [1, 6, 7, 13, 14, 19, 22, 23] is the short-time Fourier transform which contains localized time and frequency information of a function. The short-time Fourier transform of a function f on \mathbb{R}^n with respect to a window function g is defined as

$$(1) \quad V_g(f)(t, m) = \int_{\mathbb{R}^n} f(x) \overline{g(x-t)} e^{-2\pi i m \cdot x} dx,$$

for $m, t \in \mathbb{R}^n$. Here, the window function g is usually a nice bump and is used for spacial localization. Square integrable functions f can be retrieved from their short-time Fourier transform in the following way:

$$(2) \quad f = \frac{1}{\langle \psi, g \rangle} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} V_g(f)(t, m) \psi^{m,t} dm dt,$$

where $g, \psi \in L^2(\mathbb{R}^n)$ and $\langle \psi, g \rangle \neq 0$. Throughout this paper we will consistently use the complex inner product notation $\langle h, k \rangle = \int h(x) \overline{k(x)} dx$ for functions h, k defined on \mathbb{R}^n .

We want to extend these concepts to allow for localization in direction. To achieve this, we utilize the directionally-sensitive Radon transform defined for functions f in

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the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ as follows:

$$(3) \quad R(f)(u, s) = R_u(f)(s) = \int_{u \cdot x = s} f(x) dx,$$

where $u \in \mathbb{S}^{n-1}$ and $s \in \mathbb{R}$. This transform can be easily extended to a continuous operator that maps $L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R})$ uniformly in $u \in \mathbb{S}^{n-1}$. The Radon transform is an even operator, i.e. $R_{-u}(f)(-s) = R_u(f)(s)$, a fact that will be useful to us later.

We introduce an operator R^* associated with the Radon transform R , as follows:

$$R^*(g)(x) = \int_{\mathbb{S}^{n-1}} g(u, u \cdot x) du,$$

for functions $g(u, s)$ defined on the cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$. This operation is the adjoint operator to the Radon transform, in some sense, and when applied to the Radon transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$, yields the back-projection:

$$\mathcal{B}(f)(x) = R^*(R(f))(x) = |\mathbb{S}^{n-2}| \int_{\mathbb{R}^n} |x - y|^{-1} f(y) dy = |\mathbb{S}^{n-2}| (|x|^{-1} * f).$$

For an introduction to the Radon transform and some of its properties the reader may consult [8, 9, 20, 25, 26, 27]. The inversion formula

$$(4) \quad f = \frac{1}{2} (2\pi)^{1-n} R^* \left\{ H^{n-1} \left[\frac{d^{n-1} R(f)}{ds^{n-1}} \right] \right\},$$

holds for suitable functions f , where H is the one-dimensional Hilbert transform (in the variable s). We also recall the Fourier slice theorem saying that the n -dimensional Fourier transform of an integrable function f is related to the 1-dimensional Fourier transform of its Radon transform $R_u(f)$ in the following way:

$$\widehat{R_u(f)}(\sigma) = \widehat{f}(\sigma u).$$

Here $\sigma \in \mathbb{R}$ and $u \in \mathbb{S}^{n-1}$ and in this article we use the notation

$$\widehat{h}(\xi) = \int_{\mathbb{R}^n} h(x) e^{-2\pi i x \cdot \xi} dx$$

for the Fourier transform of an integrable function h and $h^\vee(\xi) = \widehat{h}(-\xi)$ for its inverse Fourier transform. If both \widehat{h} and h are integrable we have $(\widehat{h})^\vee = (h^\vee)^\wedge = h$. Throughout this paper we will be working on \mathbb{R}^n for some $n \geq 2$.

2. GABOR RIDGE FUNCTIONS

In this section, we introduce the Gabor ridge functions which can be viewed as time-frequency analysis elements in the Radon domain. We build upon them to develop a directionally-sensitive time-frequency analysis. This is related to work of Walnut [30], where the idea was presented to study combinations of wavelet theory and time-frequency analysis using directional sensitivity.

Definition 1 (Gabor Ridge Function). *Let $g \in \mathcal{S}(\mathbb{R})$ be a real-valued (non-zero) window function. We construct a Gabor element on \mathbb{R} associated with g as follows:*

$$g^{m,t}(s) = e^{2\pi i m(s-t)} g(s-t), \quad \text{for } s, t, m \in \mathbb{R}.$$

We call the following function a Gabor ridge function

$$(5) \quad g_{m,t,u}(x) = g^{m,t}(u \cdot x), \quad \text{for } x \in \mathbb{R}^n,$$

where $u \in \mathbb{S}^{n-1}$ and $m, t \in \mathbb{R}$.

These functions are constant along the hyperplanes perpendicular to u and modulate like one-dimensional Gabor wavelets in the direction of u . One may wonder what sort of information we are actually collecting when we calculate the coefficients $\langle f, g_{m,t,u} \rangle$ and also how can we use this information to reconstruct the given signal f .

When we pair f and $g_{m,t,u}$ we pick up time-frequency information of f in the direction of u because of the directional modulations of $g_{m,t,u}$ while the parameters m and t measure the modulations and translations as in the classical time-frequency theory. It is important to note that the function $g_{m,t,u}$ is not in $L^2(\mathbb{R}^n)$, so we need to be careful when we form its inner product with f .

It is natural to wonder whether the function f coincides with the representation

$$(6) \quad \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle f, g_{m,t,u} \rangle \psi_{m,t,u} \, dm \, dt \, du$$

for a suitable choice of window functions $g, \psi \in \mathcal{S}(\mathbb{R}^n)$. This is actually not the case, although this representation yields an object close to the original function. This object is a multiple of the back-projection $\mathcal{B}(f) = R^*(R(f))$.

Theorem 1. *Let $f \in L^1(\mathbb{R}^n)$ and suppose $\hat{f} \in L^p(\mathbb{R}^n)$ for some $1 < p < n$. For given functions $g, \psi \in \mathcal{S}(\mathbb{R})$ with $\langle g, \psi \rangle \neq 0$, we have the following identity:*

$$(7) \quad \mathcal{B}(f) = R^*(R(f)) = \frac{1}{\langle g, \psi \rangle} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle f, g_{m,t,u} \rangle \psi_{m,t,u} \, dm \, dt \, du.$$

A lemma is needed that surfaces a relationship with the Radon transform.

Lemma 1. *For $f \in L^1(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R})$, we have the following equality:*

$$\langle f, g_{m,t,u} \rangle = \langle R_u(f), g^{m,t} \rangle.$$

Proof. First fix a direction $u \in \mathbb{S}^{n-1}$. Then

$$\begin{aligned} \langle f, g_{m,t,u} \rangle &= \int_{\mathbb{R}^n} f(x) e^{-2\pi i m(u \cdot x - t)} \overline{g(u \cdot x - t)} \, dx \\ &= \int_{\mathbb{R}} \left(\int_{u \cdot x = s} f(x) e^{-2\pi i m(u \cdot x - t)} \overline{g(u \cdot x - t)} \, dx \right) \, ds \\ &= \int_{\mathbb{R}} e^{-2\pi i m(s-t)} \overline{g(s-t)} \left(\int_{u \cdot x = s} f(x) \, dx \right) \, ds \\ &= \int_{\mathbb{R}} R_u(f)(s) \overline{g^{m,t}(s)} \, ds \\ &= \langle R_u(f), g^{m,t} \rangle. \end{aligned}$$

which is well-defined as $R_u(f) \in L^1(\mathbb{R})$ and $g^{m,t} \in \mathcal{S}(\mathbb{R})$. □

This lemma shows that the Radon transform arises naturally in this setting, because of the dependence of the Gabor ridge functions on the direction $u \in \mathbb{S}^{n-1}$. Now Theorem 1 may be proved.

Proof. Since $\langle R_u(f), g^{m,t} \rangle = \langle f, g_{m,t,u} \rangle$, by Lemma 1 we have

$$\begin{aligned}
& \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle f, g_{m,t,u} \rangle \psi_{m,t,u} \, dm \, dt \, du \\
&= \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle R_u(f), g^{m,t} \rangle \psi_{m,t,u} \, dm \, dt \, du \\
&= \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \widehat{R_u(f)}(\sigma) \overline{\widehat{g^{m,t}}(\sigma)} \, d\sigma \right) e^{2\pi i m(u \cdot x - t)} \psi(u \cdot x - t) \, dm \, dt \, du \\
&= \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{R_u(f)}(\sigma) \overline{\widehat{g}(\sigma - m)} e^{2\pi i \sigma t} e^{2\pi i m(u \cdot x - t)} \psi(u \cdot x - t) \, d\sigma \, dm \, dt \, du,
\end{aligned}$$

where we used that $\widehat{g^{m,t}}(\sigma) = \widehat{g}(\sigma - m) e^{-2\pi i \sigma t}$. Via the change of variables $\sigma - m \rightarrow m$ we rewrite the previous expression as

$$\int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{R_u(f)}(\sigma) \overline{\widehat{g}(m)} e^{2\pi i \sigma t} e^{2\pi i (\sigma - m)(u \cdot x - t)} \psi(u \cdot x - t) \, d\sigma \, dm \, dt \, du.$$

Combining the exponentials yields

$$\int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{R_u(f)}(\sigma) \overline{\widehat{g}(m)} e^{2\pi i \sigma u \cdot x} e^{-2\pi i m u \cdot x} e^{2\pi i m t} \psi(u \cdot x - t) \, d\sigma \, dm \, dt \, du.$$

Next we integrate over m , noting that $\int \overline{\widehat{g}(m)} e^{2\pi i m(t - u \cdot x)} \, dm = \overline{\widehat{g}(u \cdot x - t)}$.

$$\int_{\mathbb{R}} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \widehat{R_u(f)}(\sigma) \overline{\widehat{g}(u \cdot x - t)} e^{2\pi i \sigma u \cdot x} \psi(u \cdot x - t) \, d\sigma \, du \, dt.$$

Finally, we integrate over the translation variable t and recognize the inverse Fourier transform of the Radon transform at the point $u \cdot x$ to obtain

$$\begin{aligned}
& \langle \psi, g \rangle \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \widehat{R_u(f)}(\sigma) e^{2\pi i \sigma u \cdot x} \, d\sigma \, du \\
&= \langle \psi, g \rangle \int_{\mathbb{S}^{n-1}} R_u(f)(u \cdot x) \, du \\
&= \langle \psi, g \rangle \mathcal{B}(f)(x).
\end{aligned}$$

The Fourier inversion can be justified since $\int_{\mathbb{S}^{n-1}} \widehat{R_u(f)} \, du \in L^1(\mathbb{R})$. Indeed, by the Fourier slice theorem we have $\widehat{R_u(f)}(\sigma) = \widehat{f}(\sigma u)$ and

$$\begin{aligned}
& \int_{\mathbb{R}} \int_{\mathbb{S}^{n-1}} |\widehat{R_u(f)}(\sigma)| \, du \, d\sigma \\
&= 2 \int_{\mathbb{S}^{n-1}} \int_0^1 |\widehat{f}(\sigma u)| \, d\sigma \, du + 2 \int_{\mathbb{S}^{n-1}} \int_1^\infty |\widehat{f}(\sigma u)| \sigma^{n-1} \frac{d\sigma}{\sigma^{n-1}} \, du \\
&\leq 2 |\mathbb{S}^{n-1}| \|f\|_{L^1} + 2 \int_{|\xi|>1} |\widehat{f}(\xi)| \frac{d\xi}{|\xi|^{n-1}} \\
&\leq C_n (\|f\|_{L^1} + \|\widehat{f}\|_{L^p}) < \infty,
\end{aligned}$$

in view of Hölder's inequality and the fact that $p'(n-1) > n$. \square

Thus we have a reconstruction formula for the back-projection $\mathcal{B}(f)$ of f which, as we saw, is f convolved with a certain weight. Note that when $n > 2$, the condition $\widehat{f} \in L^p(\mathbb{R}^n)$ of Theorem 1 can be replaced by the slightly stronger, but more natural, condition that $f \in L^2(\mathbb{R}^n)$.

2.1. Weighted Gabor Ridge Functions. We have seen that to obtain exact reproduction of a signal using the Gabor ridge functions, the straightforward representation in (6) does not suffice. A similar phenomenon appears in the related actual inversion of the Radon transform, which requires a filtered form of back-projection, as was originally shown by Radon [27]. We will therefore need to modify our class of Gabor ridge functions to adapt them to the presence of the weight that appears in the back-projection.

In what follows we will use the differential operator D_α acting on functions on the line for $\alpha > 0$, defined by

$$D_\alpha(h) = (\widehat{h}(\xi)|\xi|^\alpha)^\vee.$$

Definition 2 (Weighted Gabor Ridge Functions). *Let $g \in \mathcal{S}(\mathbb{R})$ be some real-valued (non-zero) window function. Recall the functions $g^{m,t}$ and $g_{m,t,u}$ introduced in Definition 1. For $m, t \in \mathbb{R}$ we define weighted functions*

$$(8) \quad G^{m,t}(s) = D_{\frac{n-1}{2}}(g^{m,t})(s) = \left(\widehat{g^{m,t}}(\sigma) |\sigma|^{\frac{n-1}{2}} \right)^\vee(s), \quad \text{for } s \in \mathbb{R},$$

and for $u \in \mathbb{S}^{n-1}$ we introduce the weighted Gabor ridge functions

$$(9) \quad G_{m,t,u}(x) = G^{m,t}(u \cdot x), \quad \text{for } x \in \mathbb{R}^n.$$

We will denote weighted Gabor ridge functions with the upper case letter corresponding to the letter of window. For instance, $\Psi_{m,t,u}$ will denote the weighted Gabor ridge function associated with a window ψ . To understand the behavior of the $G_{m,t,u}$'s, it will be useful to compute and analyze their Fourier transform.

Theorem 2. *The Fourier transform of $G_{m,t,u}$ on \mathbb{R}^n is the following tempered distribution*

$$(10) \quad \widehat{G_{m,t,u}}(\xi) = \widehat{g}(u \cdot \xi - m) e^{-2\pi i(u \cdot \xi)t} |u \cdot \xi|^{\frac{n-1}{2}} \delta(\xi - (u \cdot \xi)u),$$

where $\delta(\xi - (u \cdot \xi)u)$ denotes the Dirac distribution along the line through the origin that contains the vector u .

Proof. We first prove that for any integrable function h on \mathbb{R} , the Fourier transform of the n -dimensional function $H(x) = h(u \cdot x)$ is the distribution

$$\widehat{H}(\xi) = \widehat{h}(u \cdot \xi) \delta(\xi - (u \cdot \xi)u).$$

Indeed, let L_u be the line through the origin that contains the unit vector u . Introducing coordinates

$$\begin{aligned} x &= (u \cdot x)u + x', & x' &\in u^\perp, \\ \xi &= (u \cdot \xi)u + \xi', & \xi' &\in u^\perp, \end{aligned}$$

for a test function ϕ on \mathbb{R}^n we have

$$\langle \widehat{H}, \phi \rangle = \langle H, \widehat{\phi} \rangle = \int_{\mathbb{R}^n} h(u \cdot x) \widehat{\phi}(x) dx = \int_{u^\perp} \int_{L_u} h(\lambda) \widehat{\phi}(\lambda u + x') d\lambda dx'.$$

Denoting by $\mathcal{F}_{\xi'}$ the Fourier transform in ξ' we write

$$\widehat{\phi}(\lambda u + x') = \int_{u^\perp} \int_{L_u} \phi(\mu u + \xi') e^{-2\pi i(\lambda \mu + x' \cdot \xi')} d\mu d\xi' = \int_{L_u} \mathcal{F}_{\xi'}(\phi(\mu u + \xi'))(x') e^{-2\pi i \lambda \mu} d\mu$$

and using the Fubini-Tonelli theorem justified from the fact that

$$\int_{u^\perp} \int_{L_u} \int_{L_u} |h(\lambda)| |\mathcal{F}_{\xi'}(\phi(\mu u + \xi'))(x')| d\lambda d\mu dx' < \infty,$$

we obtain that

$$\begin{aligned} \langle \widehat{H}, \phi \rangle &= \int_{u^\perp} \int_{L_u} \int_{L_u} h(\lambda) \mathcal{F}_{\xi'}(\phi(\mu u + \xi'))(x') e^{-2\pi i \lambda \mu} d\lambda dx' d\mu \\ &= \int_{u^\perp} \int_{L_u} \widehat{h}(\mu) \mathcal{F}_{\xi'}(\phi(\mu u + \xi'))(x') d\mu dx' \\ &= \int_{L_u} \widehat{h}(\mu u) \phi(\mu u) d\mu = \langle \widehat{h}(u \cdot \xi) \delta(\xi - (u \cdot \xi)u), \phi(\xi) \rangle. \end{aligned}$$

This leads to the identity

$$\widehat{H}(\xi) = \widehat{h}(u \cdot \xi) \delta(\xi') = \widehat{h}(u \cdot \xi) \delta(\xi - (u \cdot \xi)u),$$

where δ is the Dirac delta distribution. Now set $h(s) = G^{m,t}(s)$ to obtain (10). \square

This calculation shows that the Fourier transforms of the weighted Gabor ridge function $G_{m,t,u}$ is supported on the line $L_u = \{\lambda u : \lambda \in \mathbb{R}\}$ and when $u \cdot \xi = \lambda$ we have $\widehat{G_{m,t,u}}(\xi) = \widehat{g}(\lambda - m) e^{-2\pi i \lambda t} |\lambda|^{\frac{n-1}{2}}$. This reflects the one-dimensional nature of the Gabor ridge functions.

The presence of the weight in the weighted Gabor ridge functions leads to the exact reconstruction of a signal. The next theorem is the backbone of this theory as it shows how it is possible to obtain reconstruction from directional Gabor elements. We will later discuss some discrete versions that are more suitable for applications.

Theorem 3 (Continuous Representation). *Let $G_{m,t,u}, \Psi_{m,t,u}$ be weighted Gabor ridge functions, as defined in (9), associated with two window functions g and ψ satisfying $\langle g, \psi \rangle \neq 0$. Given a function $f \in L^1(\mathbb{R}^n)$ such that $\widehat{f} \in L^1(\mathbb{R}^n)$ we have*

$$(11) \quad f = \frac{1}{2\langle \psi, g \rangle} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle f, G_{m,t,u} \rangle \Psi_{m,t,u} dm dt du.$$

Proof. Starting with the right-hand side of (11) and using Lemma 1, we have

$$\int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle f, G_{m,t,u} \rangle \Psi_{m,t,u} dm dt du = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle R_u(f), G^{m,t} \rangle \Psi_{m,t,u} dm dt du$$

where the inner product is well-defined since $R_u(f) \in L^1(\mathbb{R})$ and $G^{m,t} \in L^\infty(\mathbb{R})$. So, we may use Plancherel's theorem to arrive at

$$\begin{aligned} & \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle \widehat{R_u(f)}, \widehat{G^{m,t}} \rangle \Psi_{m,t,u} dm dt du \\ &= \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \widehat{R_u(f)}(\sigma) \overline{\widehat{g^{m,t}}(\sigma)} |\sigma|^{\frac{n-1}{2}} d\sigma \right) \Psi^{m,t}(u \cdot x) dm dt du. \end{aligned}$$

Noting that $\overline{\widehat{g^{m,t}}(\sigma)} = e^{2\pi i t \sigma} \widehat{g}(\sigma - m)$, we are left with

$$\int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{R_u(f)}(\sigma) e^{2\pi i t \sigma} \overline{\widehat{g}(\sigma - m)} |\sigma|^{\frac{n-1}{2}} (\widehat{\psi^{m,t}}(s) |s|^{\frac{n-1}{2}})^\vee (u \cdot x) d\sigma dm dt du.$$

We expand out the inverse Fourier transform, which yields

$$\begin{aligned} & \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{R_u(f)}(\sigma) \overline{\widehat{g}(\sigma - m)} |\sigma|^{\frac{n-1}{2}} e^{2\pi i t \sigma} \\ & \quad \left\{ \int_{\mathbb{R}} e^{2\pi i (u \cdot x) \eta} \widehat{\psi}(\eta - m) e^{-2\pi i \eta t} |\eta|^{\frac{n-1}{2}} d\eta \right\} d\sigma dm dt du. \end{aligned}$$

The expression inside the curly brackets above is the Fourier transform of the function $\eta \rightarrow e^{2\pi i (u \cdot x) \eta} \widehat{\psi}(\eta - m) |\eta|^{\frac{n-1}{2}}$ at the point $t \in \mathbb{R}$, so the t -integral of the expression inside the curly brackets multiplied by $e^{2\pi i t \sigma}$ is equal to $e^{2\pi i (u \cdot x) \sigma} \widehat{\psi}(\sigma - m) |\sigma|^{\frac{n-1}{2}}$ by Fourier inversion. Thus the previously displayed expression is equal to

$$\begin{aligned} & \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{R_u(f)}(\sigma) \overline{\widehat{g}(\sigma - m)} |\sigma|^{\frac{n-1}{2}} e^{2\pi i (u \cdot x) \sigma} \widehat{\psi}(\sigma - m) |\sigma|^{\frac{n-1}{2}} d\sigma dm du \\ &= \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{R_u(f)}(\sigma) |\sigma|^{n-1} e^{2\pi i (u \cdot x) \sigma} \overline{\widehat{g}(\sigma - m)} \widehat{\psi}(\sigma - m) d\sigma dm du. \end{aligned}$$

Integrating over m yields

$$\langle \psi, g \rangle \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \widehat{R_u(f)}(\sigma) |\sigma|^{n-1} e^{2\pi i (u \cdot x) \sigma} d\sigma du$$

which is equal to

$$\langle \psi, g \rangle \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{+\infty} \widehat{f}(\sigma u) |\sigma|^{n-1} e^{2\pi i (u \cdot x) \sigma} d\sigma du$$

by the Fourier slice theorem. We divide the integral over $(-\infty, +\infty)$ into the integrals over $(-\infty, 0)$ and over $[0, +\infty)$ and we make a double change of variables in the second integral ($\sigma \rightarrow -\sigma$, $u \rightarrow -u$). We obtain that the previously displayed expression is

$$2 \langle \psi, g \rangle \int_{\mathbb{S}^{n-1}} \int_0^\infty \widehat{f}(\sigma) |\sigma|^{n-1} e^{2\pi i (u \cdot x) \sigma} d\sigma du$$

By a change of variables ($k = \sigma u$), we arrive at

$$2 \langle \psi, g \rangle \int_{\mathbb{R}^n} \widehat{f}(k) e^{2\pi i x \cdot k} dk = 2 \langle \psi, g \rangle f(x).$$

□

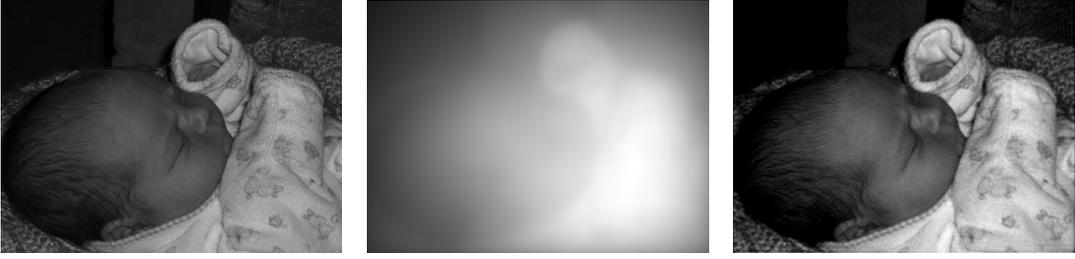


FIGURE 1. Left: original image. Middle: reconstructed image using Gabor ridge functions. Right: reconstructed image using weighted Gabor ridge functions.

The example presented in Figure 1 shows that the back-projection yields a blurry reconstruction of the original image while the “correct reconstruction” is that obtained using the weighted Gabor ridge functions [15, 16, 21, 30, 31, 32, 33].

2.2. A Parseval Formula. As one may expect, there is a Parseval-type formula that accompanies the reconstruction (11) of Theorem 3. We will prove that

$$\|f\|_{L^2(\mathbb{R}^n)}^2 = C_g \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} |\langle f, G_{m,t,u} \rangle|^2 dm dt du$$

where $G_{m,t,u}$ is associated with a nonzero $g \in \mathcal{S}(\mathbb{R})$ and C_g is a constant. This can be viewed as an energy conservation identity for the weighted Gabor ridge functions. The following lemma tells us for what sort of functions the expressions $D_{\frac{n-1}{2}}(R_u(f))$ and $\langle f, G_{m,t,u} \rangle$ are defined. As before, the $G_{m,t,u}$'s are constructed from a fixed Schwartz function g .

Lemma 2. *Given a function $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we have that*

$$D_{\frac{n-1}{2}}(R_u(f)) \in L^2(\mathbb{R})$$

for almost every $u \in \mathbb{S}^{n-1}$. Moreover $|\langle f, G_{m,t,u} \rangle|$ is finite for all $m, t \in \mathbb{R}$ and $u \in \mathbb{S}^{n-1}$.

Proof. Since $f \in L^1(\mathbb{R}^n)$ we have that $R_u(f)$ is well-defined. We begin with

$$\begin{aligned} \infty > \|f\|_{L^2(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 d\xi \\ &= \int_{\mathbb{S}^{n-1}} \int_0^\infty |\widehat{R_u(f)}(\sigma)| |\sigma|^{\frac{n-1}{2}}|^2 d\sigma du \\ &= \frac{1}{2} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} |D_{\frac{n-1}{2}}(R_u(f))(s)|^2 ds du, \end{aligned}$$

where we have used the fact that $R_u(f)$ is even. It follows that for almost every $u \in \mathbb{S}^{n-1}$, $D_{\frac{n-1}{2}}(R_u(f)) \in L^2(\mathbb{R})$.

For the other conclusion of the lemma we observe that

$$\begin{aligned}\langle \widehat{f}, \widehat{G_{m,t,u}} \rangle &= \int_{\mathbb{R}^n} \widehat{f}(\xi) \overline{\widehat{g}(u \cdot \xi - m)} e^{2\pi i(u \cdot \xi)t} |u \cdot \xi|^{\frac{n-1}{2}} \delta(\xi - (u \cdot \xi)u) d\xi \\ &= \int_{\mathbb{R}} \widehat{f}(\lambda u) \overline{\widehat{g}(\lambda - m)} e^{2\pi i\lambda t} |\lambda|^{\frac{n-1}{2}} d\lambda,\end{aligned}$$

which is finite since $|\widehat{f}(\lambda u)| \leq \|f\|_1$ and $\widehat{g}(\lambda - m)$ decays rapidly at infinity. \square

We introduce the reflection \widetilde{g} of g by setting $\widetilde{g}(x) = g(-x)$. We have another lemma.

Lemma 3. *Given $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R})$ we have the following identity:*

$$\begin{aligned}\langle f, G_{m,t,u} \rangle &= (R_u(f) * D_{\frac{n-1}{2}}(\overline{(\widetilde{g})^{m,0}}))(t) \\ &= (D_{\frac{n-1}{2}}(R_u(f)) * \overline{(\widetilde{g})^{m,0}})(t) \\ &= D_{\frac{n-1}{2}}(R_u(f) * \overline{(\widetilde{g})^{m,0}})(t).\end{aligned}$$

Proof. We have

$$\begin{aligned}\langle R_u(f), G^{m,t} \rangle &= \int_{\mathbb{R}} R_u(f)(s) \overline{(D_{\frac{n-1}{2}}(g^{m,t}))(s)} ds \\ &= \int_{\mathbb{R}} R_u(f)(s) \left\{ \int_{\mathbb{R}} e^{2\pi i s y} \widehat{g}(y - m) e^{-2\pi i y t} |y|^{\frac{n-1}{2}} dy \right\} ds \\ &= \int_{\mathbb{R}} \widehat{R_u(f)}(y) \overline{\widehat{g}(y - m)} e^{2\pi i y t} |y|^{\frac{n-1}{2}} dy \\ &= R_u(f) * D_{\frac{n-1}{2}}(\overline{(\widetilde{g})^{m,0}})(t).\end{aligned}$$

\square

The following result on orthogonality shows that the Gabor ridge transform possesses properties similar to those of the ordinary Fourier transform and the short-time Fourier transform.

Theorem 4 (Orthogonality Relation). *Given two functions $f, h \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and a window function $g \in \mathcal{S}(\mathbb{R})$, we have*

$$(12) \quad 2 \|g\|_{L^2(\mathbb{R})}^2 \langle f, h \rangle = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle f, G_{m,t,u} \rangle \langle h, G_{m,t,u} \rangle dm dt du.$$

Proof. By Lemma 3 we have,

$$\begin{aligned}& \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle f, G_{m,t,u} \rangle \langle h, G_{m,t,u} \rangle dm dt du \\ &= \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle D_{\frac{n-1}{2}}(R_u(f)) * \overline{(\widetilde{g})^{m,0}}, D_{\frac{n-1}{2}}(R_u(h)) * \overline{(\widetilde{g})^{m,0}} \rangle dt dm du \\ &= \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{f}(\xi u) \overline{\widehat{h}(\xi u)} |\xi|^{n-1} |\overline{(\widetilde{g})}(\xi + m)|^2 d\xi dm du.\end{aligned}$$

Integrating over m yields

$$\|g\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \widehat{f}(\xi u) \overline{\widehat{h}(\xi u)} |\xi|^{n-1} du d\xi = 2\|g\|_{L^2(\mathbb{R})}^2 \langle f, h \rangle.$$

□

Corollary 1 (A Parseval Formula). *For $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and g nonzero in $\mathcal{S}(\mathbb{R}^n)$ we have the following identity:*

$$(13) \quad \|f\|_{L^2(\mathbb{R}^n)}^2 = (2\|g\|_{L^2(\mathbb{R})}^2)^{-1} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}} |\langle f, G_{m,t,u} \rangle|^2 dm dt du.$$

2.3. Remark. It is interesting to note the similarities and differences of Gabor based time-frequency-direction analysis presented here with that of standard time-frequency analysis. Although in this paper the underlying dimension n is at least 2, it is worth observing that these notions coincide when $n = 1$ in which case \mathbb{S}^0 is the set $\{-1, 1\}$. The previous analysis gives

$$\int_{\mathbb{S}^0} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle f, G_{m,t,u} \rangle \Psi_{m,t,u} dm dt du = 2 \int_{\mathbb{R}} \int_{\mathbb{R}} \langle f, g^{m,t} \rangle \psi^{m,t} dm dt = \frac{2}{\langle \psi, g \rangle} f.$$

This is because the integral over \mathbb{S}^0 is just a sum of two integrals corresponding to x and $-x$ and the differential operator $D_{\frac{n-1}{2}}$ is now just $D_0 = I$.

3. A SEMI-DISCRETE REPRODUCTION FORMULA

We will now show how the continuous reproduction formula can be turned into a semi-discrete one, which in our setting, means discrete in the modulation and translation variables and continuous in spherical variable u . In applications, even the spherical integral in (11) is replaced by a finite sum over “enough” directions u but we will not pursue such a discretization here. We will utilize classical time-frequency analysis and frame theory [4, 5] tools to obtain the semi-discrete discretization of the reproduction formula.

Theorem 5 (Semi-discrete Reproduction). *There exist $g, \psi \in \mathcal{S}(\mathbb{R})$ and $\alpha, \beta > 0$ such that for all $f \in L^1 \cap L^2(\mathbb{R}^n)$ we have*

$$A\|f\|_{L^2(\mathbb{R}^n)}^2 \leq \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} |\langle f, G_{\alpha m, \beta t, u} \rangle|^2 du \leq B\|f\|_{L^2(\mathbb{R}^n)}^2$$

with A, B depending on g, α , and β . (It is possible that $A = B$.) Also for this choice of g and ψ we have,

$$f = \frac{1}{2} \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \langle f, G_{\alpha m, \beta t, u} \rangle \Psi_{\alpha m, \beta t, u} du.$$

Proof. First note that we may move the weight from the Gabor element to the Radon transform of f :

$$(14) \quad \langle f, G_{m,t,u} \rangle = \langle R_u(f), G^{m,t} \rangle = \langle D_{\frac{n-1}{2}}(R_u(f)), g^{m,t} \rangle.$$

Now this just becomes a standard pairing of an element of $L^2(\mathbb{R})$ (Lemma 2) with a Gabor element and there is a general theory to work with this. First of all, let us recall the definition of the Wiener class: A function g belongs to the Wiener class $W(L^1)$ if

$$\|g\|_{W(L^1)} = \sum_{n \in \mathbb{Z}} \operatorname{ess\,sup}_{x \in [0,1]} |g(x+n)| < \infty.$$

It is already known [19] that if α and β are “small enough”, and $g \in W(L^1)$, we have a Gabor frame for $L^2(\mathbb{R})$. As a consequence, if g is some Gaussian function in $L^2(\mathbb{R})$, there exist $\alpha, \beta > 0$, such that the following holds for almost every $u \in \mathbb{S}^{n-1}$:

$$A \|D_{\frac{n-1}{2}}(R_u(f))\|_{L^2(\mathbb{R})}^2 \leq \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} |\langle D_{\frac{n-1}{2}}(R_u(f)), g^{\alpha m, \beta t} \rangle|^2 \leq B \|D_{\frac{n-1}{2}}(R_u(f))\|_{L^2(\mathbb{R})}^2$$

where A and B depend on g , α , and β .

Recall that

$$\|D_{\frac{n-1}{2}}(R_u(f))\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |\widehat{R_u(f)}(\xi)|^2 |\xi|^{\frac{n-1}{2}} d\xi = \int_{\mathbb{R}} |\widehat{f}(\xi u)|^2 |\xi|^{n-1} d\xi,$$

so integrating the entire identity over the sphere \mathbb{S}^{n-1} gives $2\|f\|_{L^2(\mathbb{R}^n)}^2$. Using (14) we therefore obtain

$$2A \|f\|_{L^2(\mathbb{R}^n)}^2 \leq \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} |\langle f, G_{\alpha m, \beta t, u} \rangle|^2 du \leq 2B \|f\|_{L^2(\mathbb{R}^n)}^2.$$

Appealing to standard properties of Gabor (or Weyl-Heisenberg) frames (see [19]) we deduce the following reproduction formula for almost every $u \in \mathbb{S}^{n-1}$:

$$D_{\frac{n-1}{2}}(R_u(f)) = \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \langle D_{\frac{n-1}{2}}(R_u(f)), g^{\alpha m, \beta t} \rangle \psi^{\alpha m, \beta t}$$

where ψ is the dual function of g . Applying our differential operator to both sides of the equation gives

$$D_{n-1}(R_u(f)) = D_{\frac{n-1}{2}} D_{\frac{n-1}{2}}(R_u(f)) = \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \langle D_{\frac{n-1}{2}}(R_u(f)), g^{\alpha m, \beta t} \rangle D_{\frac{n-1}{2}}(\psi^{\alpha m, \beta t}).$$

We know that if we evaluate the left hand side of the equation at the point $u \cdot x$ and integrate over the sphere we recover the function f , since this is the filtered back-projection inversion for the Radon transform.

$$f = \frac{1}{2} \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \langle D_{\frac{n-1}{2}}(R_u(f)), g^{\alpha m, \beta t} \rangle D_{\frac{n-1}{2}}(\psi^{\alpha m, \beta t})(u \cdot x) du.$$

We finally conclude that

$$f = \frac{1}{2} \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \langle f, G_{\alpha m, \beta t, u} \rangle \Psi_{\alpha m, \beta t, u}$$

for a pair of dual functions g, ψ . □

3.1. Analysis Operator. We will begin with some general aspects of frame theory and see how our weighted Gabor frames relate. First, we will define the analysis and reconstruction operators associated with a fixed window function g on \mathbb{R} .

Definition 3. *The analysis (or coefficient) operator is given by*

$$(15) \quad \mathcal{C}_g(f) = \{\langle f, G_{m,t,u} \rangle : m \in \mathbb{Z}, t \in \mathbb{Z}, u \in \mathbb{S}^{n-1}\}.$$

for functions $f \in L^1(\mathbb{R}^n)$.

The analysis operator acts on integrable functions and produces a semi-discrete sequence.

Definition 4. *The synthesis operator is given by*

$$(16) \quad \mathcal{D}_g(\{c_{m,t,u}\}) = \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{m,t,u} G_{m,t,u} du.$$

The synthesis operator acts on a semi-discrete sequence $\{c_{m,t,u}\}$ and produces a function defined on \mathbb{R}^n .

Proposition 1. *The operators \mathcal{C}_g and \mathcal{D}_g are adjoint to each other.*

Proof. Indeed, we have

$$\begin{aligned} \langle \mathcal{C}_g^*(\{c_{m,t,u}\}), f \rangle &= \langle \{c_{m,t,u}\}, \mathcal{C}_g(f) \rangle \\ &= \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{m,t,u} \overline{\langle f, G_{m,t,u} \rangle} du \\ &= \left\langle \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{m,t,u} G_{m,t,u}, f \right\rangle \\ &= \langle \mathcal{D}_g(\{c_{m,t,u}\}), f \rangle. \end{aligned}$$

□

These operators are the building blocks of what we call the Gabor ridge frame operator which is defined in the following way.

Definition 5 (Gabor ridge frame operator). *Given window functions $g, \psi \in \mathcal{S}(\mathbb{R})$, we call the linear operator acting on functions $f \in L^1(\mathbb{R}^n)$,*

$$S_{g,\psi}(f) = \mathcal{D}_\psi(\mathcal{C}_g(f)) = \int_{\mathbb{S}^{n-1}} \sum_{t \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \langle f, G_{m,t,u} \rangle \Psi_{m,t,u} du,$$

the semi-discrete Gabor ridge frame operator.

3.2. Representation Formula and Half-filtered Operators. In the general Gabor theory there is a nice representation formula due to Walnut [31] which brings a different perspective to the problem. We will seek an analogous formula for our semi-discrete frame. Before we undertake this task we need to define the notion of half-filtered operators.

Definition 6 (Half-filtered Radon Transform). *We define the half-filtered Radon transform as an operator taking functions on $L^1(\mathbb{R}^n)$ to functions on the cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$ in the following way:*

$$(17) \quad \mathcal{R}(f)(u, t) = D_{\frac{n-1}{2}}(R_u(f))(t) = D_{\frac{n-1}{2}}(R(f))(u, t)$$

where the differentiation operator $D_{\frac{n-1}{2}}$ is taken with respect to the affine parameter t of the Radon transform.

Definition 7 (Half-filtered Back-projection Operator). *The half-filtered back-projection operator takes functions on $\mathbb{S}^{n-1} \times \mathbb{R}$ to functions on \mathbb{R}^n and is defined as follows:*

$$(18) \quad \mathcal{R}^*(F)(x) = \int_{\mathbb{S}^{n-1}} D_{\frac{n-1}{2}}(F(u, \cdot))(u \cdot x) du.$$

We have the following concerning the semi-discrete Gabor ridge frame operator.

Theorem 6 (Representation Formula). *Let f be in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then the semi-discrete Gabor ridge frame operator*

$$(19) \quad S_{g,\psi}(f) = \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \langle f, G_{m,t,u} \rangle \Psi_{m,t,u} du$$

can be written as

$$(20) \quad S_{g,\psi}(f) = \mathcal{R}^*Q(f),$$

where

$$(21) \quad Q(f)(u, s) = \sum_{r \in \mathbb{Z}} G_r(s) \mathcal{R}(f)(u, s - r)$$

and

$$(22) \quad G_r(s) = \sum_{t \in \mathbb{Z}} \psi(s - t) \overline{g(s - r - t)}$$

is the standard correlation function of Gabor theory.

Proof. The proof of this theorem will be a consequence of the identity

$$(23) \quad Q(f)(u, s) = \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \langle f, G_{m,t,u} \rangle \psi^{m,t}(s).$$

If (23) is true, then

$$\mathcal{R}^*(Q(f)) = \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \langle f, G_{m,t,u} \rangle \Psi_{m,t,u} du,$$

which yields the required conclusion.

So, to prove (23), we start with the left hand side,

$$\begin{aligned}
& \sum_{r \in \mathbb{Z}} G_r(s) \mathcal{R}(f)(u, s-r) \\
&= \sum_{r \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \psi(s-t) \overline{g(s-r-t)} \int_{\mathbb{R}} e^{2\pi i(s-r)\sigma} \widehat{R_u(f)}(\sigma) |\sigma|^{\frac{n-1}{2}} d\sigma \\
&= \sum_{r \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \psi(s-t) \overline{g(s-r-t)} \int_{\mathbb{R}} e^{2\pi i s \sigma} e^{-2\pi i r \sigma} \widehat{R_u(f)}(\sigma) |\sigma|^{\frac{n-1}{2}} d\sigma \\
&= \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \psi(s-t) \overline{\widehat{g}(\sigma-m)} e^{-2\pi i(s-t)(\sigma-m)} \int_{\mathbb{R}} e^{2\pi i s \sigma} \widehat{R_u(f)}(\sigma) |\sigma|^{\frac{n-1}{2}} d\sigma,
\end{aligned}$$

where we are using the following simple consequence of the Poisson summation formula [18]

$$\sum_{r \in \mathbb{Z}} \overline{g(s-r-t)} e^{-2\pi i r \sigma} = \sum_{m \in \mathbb{Z}} \overline{\widehat{g}(\sigma-m)} e^{-2\pi i(s-t)(\sigma-m)}.$$

Simplifying the last line of the previous calculation leads to

$$\begin{aligned}
& \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \int_{\mathbb{R}} \psi(s-t) \overline{\widehat{g}(\sigma-m)} e^{2\pi i s m} e^{2\pi i t \sigma} e^{-2\pi i t m} \widehat{R_u(f)}(\sigma) |\sigma|^{\frac{n-1}{2}} d\sigma \\
&= \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \left(\int_{\mathbb{R}} \widehat{R_u(f)}(\sigma) \overline{\widehat{g}(\sigma-m)} e^{2\pi i t \sigma} |\sigma|^{\frac{n-1}{2}} d\sigma \right) \psi(s-t) e^{2\pi i m(s-t)} \\
&= \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \langle f, G_{m,t,u} \rangle \psi^{m,t}.
\end{aligned}$$

□

The operator \mathcal{R} forms an isometry between $L^2(\mathbb{S}^{n-1} \times \mathbb{R})$ and $L^2(\mathbb{R}^n)$. Indeed, let $[\ , \]$ denote inner product on the cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$. Then for functions f, h in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ we have

$$\begin{aligned}
[\mathcal{R}(f), \mathcal{R}(h)] &= [D_{\frac{n-1}{2}}(R_u(f)), D_{\frac{n-1}{2}}(R_u(h))] \\
&= [R_u(f), D_{n-1}(R_u(h))] \\
&= \langle f, R^*(D_{n-1}(R_u(h))) \rangle \\
&= c_n \langle f, h \rangle,
\end{aligned}$$

where c_n is a constant depending on the constant that appears in the inversion formula for the Radon transform in (4), as we have that $H^{n-1} \circ \frac{d^{n-1}}{ds^{n-1}}$ coincides with a constant multiple of D_{n-1} .

3.3. Some spaces of functions. We will now look at some new functional spaces that measure the quantitative properties of the Gabor ridge coefficients and classify functions (signals) according to the size of their associated coefficients.

We are motivated to introduce such functional spaces where everything is well-behaved by the fact that functions even in very nice spaces do not always act well

under the Radon transform. For example, consider the following function in $L^2(\mathbb{R}^n)$:

$$\psi(x) = (2 + |x|)^{-\frac{n}{2}} (\log(2 + |x|))^{-1}.$$

The Radon transform of ψ is infinity everywhere (for all $u \in \mathbb{S}^{n-1}$ and all $s \in \mathbb{R}$).

There has been some recent work on what spaces f should be in so that the Radon transform is well-defined. A result of Madych [24] says that the Radon transform of $f \in L^p(\mathbb{R}^2)$ functions for $\frac{4}{3} < p < 2$ exists almost everywhere and the Radon inversion formula holds almost everywhere in this case.

Definition 8. For a measurable function F on \mathbb{R}^2 consider the sequence

$$a(j, k) = \operatorname{ess\,sup}_{x, \xi \in [0, 1]} |F(x + j, \xi + k)|.$$

We say that the function F belongs to $W(L^q(L^p))$ with norm

$$\|F\|_{W(L^q(L^p))} = \|a\|_{\ell^q(\ell^p)} = \left(\sum_{k \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} |a(j, k)|^p \right)^{q/p} \right)^{1/q}$$

if the previous expression is finite.

Note that by $L^q(L^p)$ we denote the space of all functions of two variables $h(x, y)$ for which the expression $\|h\|_{L^q(L^p)} = \left(\int \left(\int |h(x, y)|^p dy \right)^{q/p} dx \right)^{1/q} < \infty$.

We recall the definition of modulation spaces on the real line.

Definition 9 (Modulation Spaces $M^{p,q}$). Fix a non-zero Schwartz window function $g \in \mathcal{S}(\mathbb{R})$ and $1 \leq p, q \leq \infty$. The modulation space $M^{p,q}$ is the space of all tempered distributions $h \in S'(\mathbb{R})$ such that the short-time Fourier transform $V_g(h)$ lies in $L^q(L^p)$. The $M^{p,q}$ norm of f is defined as

$$\|h\|_{M^{p,q}} = \|V_g(h)\|_{L^q(L^p)} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\langle h, g^{m,t} \rangle|^p dm \right)^{q/p} dt \right)^{1/q}.$$

When $p = q$ we set $M^p = M^{p,p}$.

This norm measures the time and frequency components of functions. Also M^2 coincides with L^2 . We now look at some important properties of modulation spaces which we will later use. For a proof of the next result we refer to [19] (Theorem 12.2.1).

Theorem A. Let $g \in M^1$ and $h \in M^{p,q}$ for some $1 \leq p, q \leq \infty$. Then we have

$$(24) \quad \|V_g(h)\|_{W(L^q(L^p))} \leq C \|V_g(g)\|_{W(L^1)} \|h\|_{M^{p,q}}.$$

We also mention the following fact which can be easily proved using the relevant definitions. If $F \in W(L^q(L^p))$ is continuous, then $F|_{\mathbb{Z} \times \mathbb{Z}}$ (the restriction of F to the grid $\mathbb{Z} \times \mathbb{Z}$) is in $\ell^q(\ell^p)$, and

$$(25) \quad \|F|_{\mathbb{Z} \times \mathbb{Z}}\|_{\ell^q(\ell^p)} \leq C \|F\|_{W(L^q(L^p))}.$$

3.4. Definition of the Functional Spaces $\Omega^{p,q,r}$. Until the end of this paper we will denote by w a tempered distribution on the line whose Fourier transform is the following function

$$\widehat{w}(\xi) = |\xi|^{\frac{n-1}{2}}.$$

Definition 10 (The Functional Space $\Omega^{p,q,r}$). *For a given function $f \in L^1(\mathbb{R}^n)$, we say that $f \in \Omega^{p,q,r}$ if the following norm is finite:*

$$\|f\|_{\Omega^{p,q,r}} = \left(\int_{\mathbb{S}^{n-1}} \|(\widehat{R_u(f)} \widehat{w})^\vee\|_{M^{p,q}}^r du \right)^{1/r},$$

We also define a semi-discrete version of this space which we denote by $\omega^{p,q,r}$.

Definition 11. *We say that a semi-discrete sequence $\{c_{m,t,u}\}_{m,t,u}$ indexed by the set $\mathbb{S}^{n-1} \times \mathbb{Z} \times \mathbb{Z}$ belongs to $\omega^{p,q,r}$ if the following norm is finite:*

$$\|\{c_{m,t,u}\}\|_{\omega^{p,q,r}} = \left(\int_{\mathbb{S}^{n-1}} \|\{c_{m,t,u}\}\|_{\ell^q(\ell^p)}^r du \right)^{1/r}.$$

In the mixed norm space $\ell^q(\ell^p)$ the ℓ^p is taken in the t variable (taken first) followed by the ℓ^q norm in the m variable.

These functional spaces are specifically tuned to the weighted Gabor ridge functions as they measure the “size” of their coefficients in a way analogous to that in which modulation spaces measure the “size” of time-frequency coefficients. Based on properties of the modulation spaces, one sees that different windows g give rise to comparable norms in $\Omega^{p,q,r}$.

3.5. Boundedness of the Frame Operator on the Spaces $\Omega^{p,q,r}$. An interesting property of the functional spaces $\Omega^{p,q,r}$ is that the semi-discrete frame operator $S_{g,\psi}$ is bounded on them.

Theorem 7 (Boundedness of \mathcal{C}_g). *If $g \in M^1$, then \mathcal{C}_g is bounded from $\Omega^{p,q,r}$ into $\omega^{p,q,r}$ for $1 \leq p, q, r \leq \infty$ and we have*

$$\|\mathcal{C}_g\|_{op} \leq C \|V_g(g)\|_{W(L^1)}$$

independently of p and q .

Proof. Since $(\widehat{R_u(f)} \widehat{w})^\vee \in M^{p,q}$, Theorem A gives that $V_g((\widehat{R_u(f)} \widehat{w})^\vee) \in W(L^q(L^p))$ for $g \in M^1$. Now $V_g((\widehat{R_u(f)} \widehat{w})^\vee)$ is continuous and its restriction on the integer lattice $\mathbb{Z} \times \mathbb{Z}$ coincides with the function $\mathcal{C}_g(f)(\cdot, \cdot, u)$, thus we have

$$\mathcal{C}_g(f)(m, t, u) = V_g((\widehat{R_u(f)} \widehat{w})^\vee)(m, t)$$

for all $m, t \in \mathbb{Z}$. In view of Theorem A, (25), and (24) we deduce

$$\begin{aligned}
\|\mathcal{C}_g(f)\|_{\omega^{p,q,r}}^r &= \int_{\mathbb{S}^{n-1}} \|\mathcal{C}_g(f)\|_{\ell^q(\ell^p)}^r du \\
&= \int_{\mathbb{S}^{n-1}} \|V_g((\widehat{R_u(f)} \widehat{w})^\vee)|_{\mathbb{Z} \times \mathbb{Z}}\|_{\ell^q(\ell^p)}^r du \\
&\leq C^r \int_{\mathbb{S}^{n-1}} \|V_g((\widehat{R_u(f)} \widehat{w})^\vee)\|_{W(L^q(L^p))}^r du \\
&\leq C^r \|V_g(g)\|_{W(L^1)}^r \int_{\mathbb{S}^{n-1}} \|(\widehat{R_u(f)} \widehat{w})^\vee\|_{M^{p,q}}^r du \\
&= C^r \|V_g(g)\|_{W(L^1)}^r \|f\|_{\Omega^{p,q,r}}^r.
\end{aligned}$$

□

We now obtain a similar boundedness property for \mathcal{D}_ψ . To achieve this, we will need the following result that can be found in [19] (Theorem 12.2.4).

Theorem B. *Let $1 \leq p, q \leq \infty$. For fixed $g \in M^1$ we have the estimate*

$$\|\mathcal{D}_g(\{c_{m,t}\})\|_{M^{p,q}} = \|V_g(\mathcal{D}_g(\{c_{m,t}\}))\|_{L^q(L^p)} \leq C \|V_g(g)\|_{W(L^1)} \|\{c_{m,t}\}\|_{\ell^q(\ell^p)}.$$

We now have the following result.

Theorem 8 (Boundedness of \mathcal{D}_ψ). *If $\psi \in M^1$, then \mathcal{D}_ψ is bounded from $\omega^{p,q,r}$ to $\Omega^{p,q,r}$ for $1 \leq p, q, r \leq \infty$ with the following norm estimate*

$$\|\mathcal{D}_\psi\|_{op} \leq \|V_\psi(\psi)\|_{W(L^1)}.$$

Proof. Consider a semi-discrete sequence $\vec{c} = \{c_{m,t,u}\}$. Then we have

$$\|\mathcal{D}_\psi(\vec{c})\|_{\Omega^{p,q,r}}^r = \int_{\mathbb{S}^{n-1}} \|((R_u(\mathcal{D}_\psi(\vec{c})))^\wedge \widehat{w})^\vee\|_{M^{p,q}}^r du.$$

Now, we need to understand what $R_u(\mathcal{D}_\psi(\vec{c}))$ actually is. Let us expand the synthesis operator as follows

$$\begin{aligned}
\mathcal{D}_\psi(\vec{c}) &= \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{m,t,u} \Psi_{m,t,u} du \\
&= \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{m,t,u} D_{\frac{n-1}{2}}(\psi^{m,t})(u \cdot x) du \\
&= \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{m,t,u} D_{\frac{n-1}{2}}(\psi^{m,t}) * w * w^*(u \cdot x) du,
\end{aligned}$$

where w^* is chosen such that $w * w^* \equiv \delta$. This can be achieved by choosing w^* satisfying $\widehat{w^*}(\sigma) = |\sigma|^{\frac{1-n}{2}}$. Then, we can write

$$\mathcal{D}_\psi(\vec{c}) = \mathcal{B} \left(\sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{m,t,u} \psi^{m,t} * w^* \right),$$

where this expression can be viewed as a convolution since $w^*(s) = c|s|^{\frac{n-3}{2}}$, for some constant c , which is integrable over any interval containing zero (since $n \geq 2$) and hence locally integrable. Next, we have

$$\begin{aligned} D_{\frac{n-1}{2}}(R_u(\mathcal{D}_\psi(\vec{c}))) &= D_{\frac{n-1}{2}}\left(R_u\mathcal{B}\left(\sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{m,t,u} \psi^{m,t} * w^*\right)\right) \\ &= \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{m,t,u} D_{\frac{n-1}{2}}(\psi^{m,t}) * w^* \\ &= \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{m,t,u} \psi^{m,t}. \end{aligned}$$

Finally, by Theorem B we obtain

$$\begin{aligned} \|\mathcal{D}_\psi(\vec{c})\|_{\Omega^{p,q,r}}^r &= \int_{\mathbb{S}^{n-1}} \|D_{\frac{n-1}{2}}(R_u(\mathcal{D}_\psi(\vec{c})))\|_{M^{p,q}}^r du \\ &= \int_{\mathbb{S}^{n-1}} \left\| \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \psi^{m,t} \right\|_{M^{p,q}}^r du \\ &= \int_{\mathbb{S}^{n-1}} \left\| V_\psi \left(\sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} c_{m,t,u} \psi^{m,t} \right) \right\|_{L^q(L^p)}^r du \\ &\leq \|V_\psi(\psi)\|_{W(L^1)}^r \int_{\mathbb{S}^{n-1}} \|\{c_{m,t,u}\}\|_{\ell^q(\ell^p)}^r du \\ &= \|V_\psi(\psi)\|_{W(L^1)}^r \|\vec{c}\|_{\omega^{p,q,r}}^r. \end{aligned}$$

□

The following is a simple corollary of the results obtained.

Corollary 2 (Boundedness of $S_{g,\psi} = \mathcal{D}_\psi \mathcal{C}_g$). *If $g, \psi \in M^1$ such that $V_g(\widehat{((R_u(f) \widehat{w})^\vee)})$ lies in $M^{p,q}$, then the semi-discrete frame operator $S_{g,\psi}$ is bounded on $\Omega^{p,q,r}$ for all $1 \leq p, q, r \leq \infty$ and the following norm estimate holds:*

$$\|S_{g,\psi}\|_{op} \leq C \|V_g(g)\|_{W(L^1)} \|V_\psi(\psi)\|_{W(L^1)},$$

with constants independent of p, q , and r .

3.6. Characterization of $\Omega^{p,q,r}$. There is another corollary which extends the concept of frames from $L^1 \cap L^2$ to $\Omega^{p,q,r}$, by characterizing these spaces using the weighted Gabor ridge coefficients. This characterization is in the spirit of the work in [17].

Corollary 3. *Assume $g, \psi \in M^1$ are such that $S_{g,\psi} = I$ on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then the following representation holds*

$$f = \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \langle f, G_{m,t,u} \rangle \Psi_{m,t,u} du = \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \langle f, \Psi_{m,t,u} \rangle G_{m,t,u} du.$$

Also, there are constants $A, B > 0$ such that for all $f \in \Omega^{p,q,r}$ we have

$$A\|f\|_{\Omega^{p,q,r}} \leq \left(\int_{\mathbb{S}^{n-1}} \left(\sum_{m \in \mathbb{Z}} \left(\sum_{t \in \mathbb{Z}} |\langle f, G_{m,t,u} \rangle|^p \right)^{q/p} \right)^{r/q} du \right)^{1/r} \leq B\|f\|_{\Omega^{p,q,r}}.$$

Proof. We have seen that both the coefficient and synthesis operators are bounded on $\Omega^{p,q,r}$ and $\omega^{p,q,r}$ and we know that $f = \mathcal{D}_\psi \mathcal{C}_g(f)$ holds for all $f \in \Omega^{p,q,r}$. The norm equivalence is all just a consequence of the norm estimates for \mathcal{C}_g and \mathcal{D}_ψ .

$$\begin{aligned} \|f\|_{\Omega^{p,q,r}} &= \|\mathcal{D}_\psi \mathcal{C}_g(f)\|_{\Omega^{p,q,r}} \leq \|\mathcal{D}_\psi\|_{op} \|\mathcal{C}_g\|_{\omega^{p,q,r}} \\ &\leq \|\mathcal{D}_\psi\|_{op} \|\mathcal{C}_g\|_{op} \|f\|_{\Omega^{p,q,r}}. \end{aligned}$$

Note: we can choose $A = \|\mathcal{D}_\psi\|_{op}^{-1}$ and $B = \|\mathcal{C}_g\|_{op}^{-1}$. □

3.7. Remark. The results of this section are based on the classical theory of Gabor frames. If the time-frequency-directional transform introduced in this paper can be realized as the coordinate transform of an integrable irreducible representation of a suitable group, then one may be able to explore a connection with the powerful coorbit theory of Feichtinger and Gröchenig [10], [11], [12] to obtain fully discrete frames and deduce deeper properties of the functional spaces $\Omega^{p,q,r}$. This connection will be investigated in future work.

4. APPLICATIONS

4.1. Image Processing. Gabor ridge functions lend themselves to directional frequency information and are quite suitable for image processing applications such as denoising, filtering, enhancement, etc. The added benefit is that they allow performing image processing in a directionally sensitive manner.

4.1.1. Image Enhancement. There are many ways to “enhance” an image. One of the most common ways is to make sharper or more defined the edges or curvilinear singularities of the image. This can be achieved using the general theory of this article. The time-frequency-direction coefficients are largest when there is an edge in our image. The larger the coefficient, the stronger the edge. A smaller coefficient corresponds to a faint edge. In image enhancement the goal is to enhance the faint edges and not disturb the stronger edges. We do this as in [29] by multiplying all the coefficients $c_{m,t,u}$ by

$$\left(\frac{m}{n}\right)^p \quad \text{if } |c| \leq c_{min}, \quad \left(\frac{m}{|c|}\right)^p \quad \text{if } c_{min} < |c| \leq c_{max}, \quad 1 \quad \text{if } |c| > c_{max},$$

where m, n, c_{min} , and c_{max} are suitably chosen constants that depend upon the specific problem.

In Figure 2, we see our original image on the top left along with a display of coefficients corresponding to 0 degrees (top right). We then enhance the coefficients as indicated and reconstruct the function (bottom left) using new coefficients (bottom right). In this reconstruction, the larger new coefficients are not affected as much, while the smaller ones are altered producing an image with much more detail.

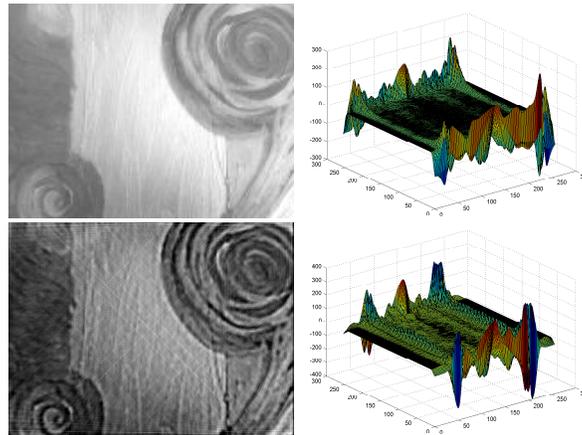


FIGURE 2. Image Enhancement Example

Applying the same enhancement technique to the left picture of Figure 3 we obtain the picture on the right which reveals certain hidden ridges of the planet Mars when the fainter edges are emphasized.

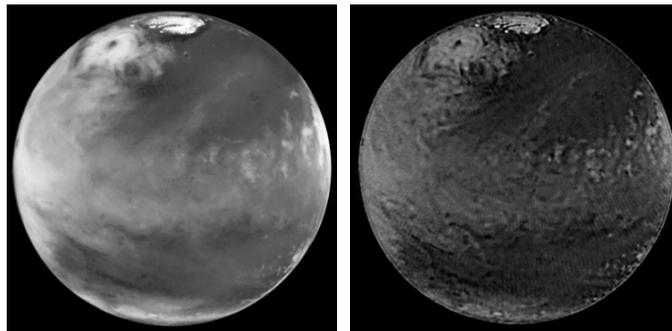


FIGURE 3. Enhancement of a satellite image of Mars.

4.2. Medical Imaging. The Radon transform plays an integral role in current medical imaging modalities. We explore certain perspectives in medical imaging where time-frequency-direction analysis plays a role. One such application is in the area of local tomography (or lambda tomography). The usual process of computerized tomography is global in nature. Local tomography [28], although slightly less accurate, is local and therefore has some advantages over the global process. We are interested to see how the weighted Gabor ridge functions can be used in reconstruction and denoising of medical images.

4.2.1. Applications to Computerized Tomography. Computerized tomography (CT) is achieved by acting the Radon transform on an object (x-rays) and inverting it algorithmically to generate the density image. The weighted Gabor ridge system is

well-tailored for medical imaging because of its inherent relationship with the Radon transform.

As an example we reconstruct a standard phantom image. In Figure 4, we have the original image (left) and the reconstructed image (right) using directions from 0 to 180 degrees. What is reconstructed in the algorithm is the following:

$$CT = \sum_{u=1}^{180} \sum_{m=1}^{256} \sum_{t=1}^{256} \langle original, G_{m,t,u} \rangle \Psi_{m,t,u}.$$

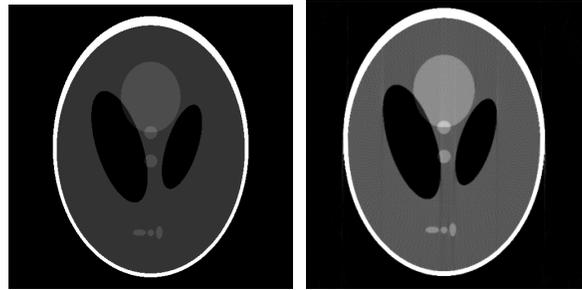


FIGURE 4. CT Reconstruction

4.2.2. *Denoising in Computerized Tomography.* Denoising is possible by setting to zero some of the coefficients which correspond to noise (small coefficients) versus those which correspond to the actual image. As we have seen large coefficients correspond to edges or curvilinear singularities of an image. Small coefficients correspond point singularities such as noise. In Figure 5, we have an example of a denoised phantom image.

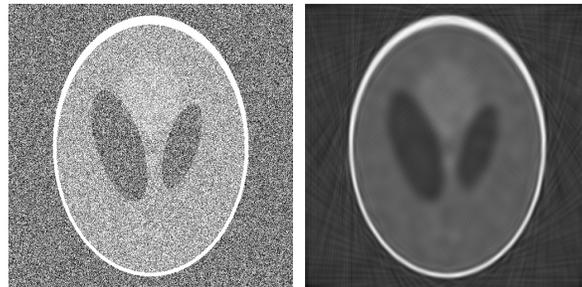


FIGURE 5. Denoised phantom image

4.2.3. *Applications to Local Tomography.* In local tomography (also called lambda tomography) one obtains reconstruction of a function related to the original f instead of itself [28]. We usually reconstruct the function $Lf = \Lambda f + \mu \Lambda^{-1} f$, where $\Lambda = \sqrt{-\Delta}$ (square root of the positive Laplacian) and μ is some constant which may depend upon f . This is a strictly local reconstruction and it has been shown that Lf produces an image that still “looks like” f , i.e. certain important aspects of the function f are

preserved. For instance, since Λf is an invertible elliptic operator, f and Λf have precisely the same singularities. Λf is “cupped”, however, within regions of constant density. The extra term $\mu\Lambda^{-1}f$ is an attempt to neutralize this cupping.

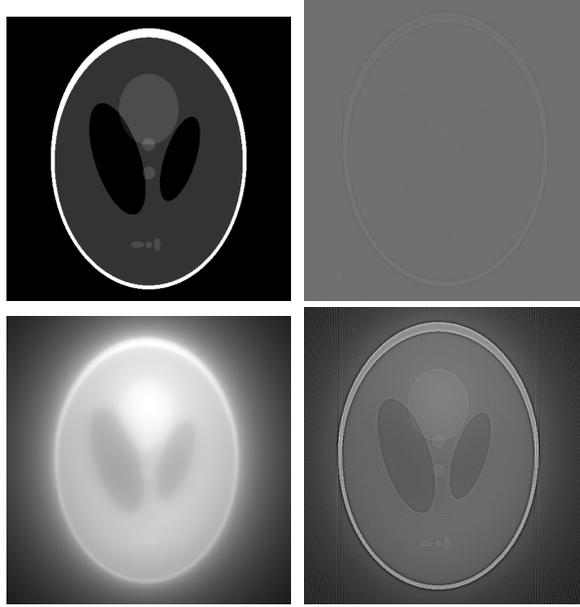


FIGURE 6. From left to right, top to bottom, we have the original head phantom, Λf , $\Lambda^{-1}f$, and the local reconstruction $Lf = \Lambda f + \Lambda^{-1}f$.

Both Λ and its inverse can be defined in terms of their Fourier transform by

$$(\Lambda f)^\wedge(\xi) = |\xi|\widehat{f}(\xi) \quad (\Lambda^{-1}f)^\wedge(\xi) = |\xi|^{-1}\widehat{f}(\xi).$$

Notice that Λ^{-1} is just the non-filtered back-projection (for $n = 2$) that we have previously encountered. This is what produced the blurry image of Figure 1 instead of a good reconstruction. It will now serve as a useful tool in local tomography. This is a local operation as it only back-projects the lines that intersect a point x while one needs the integrals over all lines to obtain a complete reconstruction of the original function; and this explains why the images it produces are blurry.

We can implement a local tomographic reconstruction process using our weighted Gabor ridge functions. The system, however, has to be altered to account for a construction of Lf instead of a reconstruction of f .

Definition 12 (Gabor ridge functions with altered weights). *We define new Lambda Gabor ridge functions with a special weight in the following way:*

$$G_{\beta m, \alpha t, u}^\Lambda = (D_{\frac{n}{2}}(g^{\beta m, \alpha t})) (u \cdot x).$$

As an example we take $n = 2$ that corresponds to

$$G_{\beta m, \alpha t, u}^\Lambda = (D_1(g^{\beta m, \alpha t})) (u \cdot x).$$

This way we create a new representation which allows for analysis of time, frequency, and direction for local tomography. A reconstruction for Lf is now simply

given by:

$$Lf = \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \langle f, G_{\alpha m, \beta t, u}^\Lambda \rangle \Psi_{\alpha m, \beta t, u}^\Lambda du + \mu \int_{\mathbb{S}^{n-1}} \sum_{m \in \mathbb{Z}} \sum_{t \in \mathbb{Z}} \langle f, g_{\beta m, \alpha t, u} \rangle \psi_{\beta m, \alpha t, u} du.$$

In Figure 6 we looked at an example of reconstruction using the method discussed here. We took the original head phantom and constructed the image of Lf .

4.3. Denoising in Local Tomography. We can also apply our denoising algorithm in a local tomographic setting. What one sees in Figure 7 is the local tomographic reconstruction of Lf that has been denoised in the same way as in the CT example (setting to zero coefficients below a certain threshold).

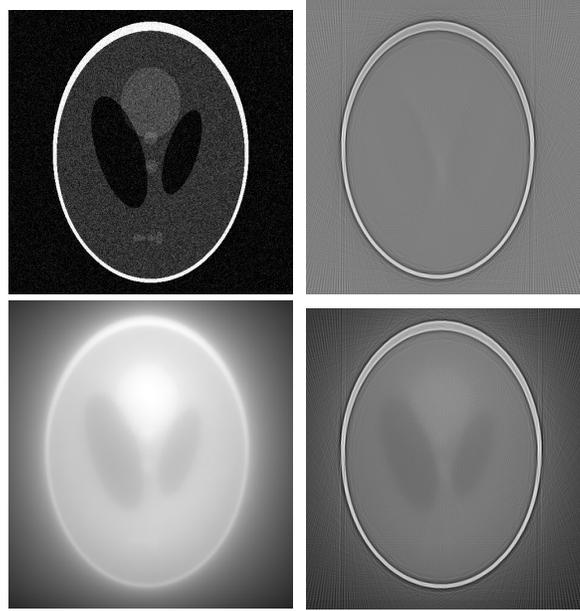


FIGURE 7. From left to right, top to bottom, we have the original head phantom with gaussian noise, Λf , $\Lambda^{-1}f$, and the local reconstruction $Lf = \Lambda f + \Lambda^{-1}f$.

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LOUKAS GRAFAKOS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA,
MO 65211, USA

E-mail address: loukas@math.missouri.edu

CHRISTOPHER SANSING, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA,
MO 65211, USA

E-mail address: sansingc@math.missouri.edu