Matrix Solutions to Linear Equations

Augmented matrices can be used as a simplified way of writing a system of linear equations. In an **augmented matrix**, a vertical line is placed inside the matrix to represent a series of equal signs and dividing the matrix into two sides.

Thus, for example, the matrix

$$\left[\begin{array}{ccc|c} 2 & -3 & 4 & -3 \\ -1 & 1 & 2 & 1 \\ 5 & -2 & -3 & 7 \end{array}\right]$$

would represent the set of linear equations

$$\begin{array}{rrrrrrr} 2x & - & 3y & + & 4z & = & -3 \\ -x & + & y & + & 2z & = & 1 \\ 5x & - & 2y & - & 3z & = & 7 \end{array}$$

In the above augmented matrix, each row represents an equation. The numbers in the left side of the matrix represent the coefficients of the variables in the set of equations. The numbers in the right side of the matrix represent the constant values to the right of the equal signs. Note that if one or more of the variables did not exist in a particular equation, the coefficient associated with that variable would be zero, and a 0 would appear at that position in the matrix.

We can now use the elimination method of solving a system of linear equations on our augmented matrix. Row operations will be performed on the matrix to reduce it to a simpler form containing the solutions to the equations. The matrix can be reduced and solved by the two different methods – Gaussian Elimination with back-substitution (row-echelon form) or Gauss-Jordan elimination (reduced row-echelon form).

The matrix is reduced through a series of row operations that can include any of the following procedures:

1) Interchange any two of the rows.

    Example: Switching rows 1 and 2

$$\left[\begin{array}{ccc|c} -2 & -1 & 6 & -4 \\ 3 & 0 & -3 & 9 \\ 3 & 4 & -1 & 10 \end{array}\right] \begin{array}{l} R_1 \leftrightarrow R_2 \\ \\ \text{would give} \end{array} \left[\begin{array}{ccc|c} 3 & 0 & -3 & 9 \\ -2 & -1 & 6 & -4 \\ 3 & 4 & -1 & 10 \end{array}\right]$$

2) Multiply any row by a nonzero constant.

Example: Multiplying row 1 by one-third

$1/3R_1 = 1/3[3 \; 0 \; \text{-3} \; 9] = [1/3(3) \; 1/3(0) \; 1/3(\text{-3}) \; 1/3(9)] = [1 \; 0 \; \text{-1} \; 3]$

$$\begin{bmatrix} 3 & 0 & -3 & | & 9 \\ -2 & -1 & 6 & | & -4 \\ 3 & 4 & -1 & | & 10 \end{bmatrix} \frac{1}{3}R_1 \quad \text{would give} \quad \begin{bmatrix} 1 & 0 & -1 & | & 3 \\ -2 & -1 & 6 & | & -4 \\ 3 & 4 & -1 & | & 10 \end{bmatrix}$$

3) Replace any row by the sum of that row and a constant multiple of any other row.

Example: Multiplying row 1 by two and adding it to row 2

$2[1 \; 0 \; \text{-1} \; 3] = [2 \; 0 \; \text{-2} \; 6]$
$[2 \; 0 \; \text{-2} \; 6] + [\text{-2} \; \text{-1} \; 6 \; \text{-4}] = [(2-2) \; (0-1) \; (\text{-2}+6) \; (6-4)] = [0 \; \text{-1} \; 4 \; 2]$

$$\begin{bmatrix} 1 & 0 & -1 & | & 3 \\ -2 & -1 & 6 & | & -4 \\ 3 & 4 & -1 & | & 10 \end{bmatrix} 2R_1 + R_2 \quad \text{would give} \quad \begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & -1 & 4 & | & 2 \\ 3 & 4 & -1 & | & 10 \end{bmatrix}$$

To solve a system of linear equations using Gaussian elimination with back-substitution the goal is to reduce the augmented matrix into the form where there are 1's going down the main diagonal (upper left corner to the lower right corner) and 0's below and to the left of the main diagonal.

$$\begin{bmatrix} 1 & * & * & | & * \\ 0 & 1 & * & | & * \\ 0 & 0 & 1 & | & * \end{bmatrix}$$

The steps for Gaussian elimination are:

1) Write the system of equations in an augmented matrix
2) Get a 1 in the first row of the first column
3) Use row 1 to get 0's in the first column of rows 2 and 3
4) Get a 1 in the second row of the second column
5) Use row 2 to get a 0 in the second column of row 3
6) Get a 1 in the third row of the third column
7) Change the augmented matrix back into a system of linear equations
8) Use back-substitution to solve for the variables

The above sequence of operations would find 1's and 0's in the following order:

$$\begin{bmatrix} 1 & * & * & | & * \\ * & * & * & | & * \\ * & * & * & | & * \end{bmatrix} \begin{bmatrix} 1 & * & * & | & * \\ 0 & * & * & | & * \\ 0 & * & * & | & * \end{bmatrix} \begin{bmatrix} 1 & * & * & | & * \\ 0 & 1 & * & | & * \\ 0 & * & * & | & * \end{bmatrix} \begin{bmatrix} 1 & * & * & | & * \\ 0 & 1 & * & | & * \\ 0 & 0 & * & | & * \end{bmatrix} \begin{bmatrix} 1 & * & * & | & * \\ 0 & 1 & * & | & * \\ 0 & 0 & 1 & | & * \end{bmatrix}$$

Example 1: Solve the system of equations with augmented matrices using the Gaussian elimination with back-substitution method.

$$\begin{array}{rrrrrr} x & - & 2y & - & z & = & 2 \\ 2x & - & y & + & z & = & 4 \\ -x & + & y & - & 2z & = & -4 \end{array}$$

Solution:

Step 1: Write the system of equations in an augmented matrix

$$\begin{bmatrix} 1 & -2 & -1 & | & 2 \\ 2 & -1 & 1 & | & 4 \\ -1 & 1 & -2 & | & -4 \end{bmatrix}$$

Step 2: Get a 1 in the first row of the first column

This is already done so we can skip to the next step.

Step 3: Use row 1 to get 0's in the first column of rows 2 and 3

For the second row we can obtain a zero by multiplying row 1 by -2 and adding it to row 2.

$$\begin{bmatrix} 1 & -2 & -1 & | & 2 \\ 2 & -1 & 1 & | & 4 \\ -1 & 1 & -2 & | & -4 \end{bmatrix} -2R_1 + R_2 \qquad \begin{array}{l} -2\begin{bmatrix} 1 & -2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 2 & | & -4 \end{bmatrix} \\ +\begin{bmatrix} 2 & -1 & 1 & | & 4 \end{bmatrix} \\ =\begin{bmatrix} 0 & 3 & 3 & | & 0 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 & -2 & -1 & | & 2 \\ 0 & 3 & 3 & | & 0 \\ -1 & 1 & -2 & | & -4 \end{bmatrix}$$

Example 1 (Continued):

For the third row we can simply add row 1 to row 3.

$$\begin{bmatrix} 1 & -2 & -1 & | & 2 \\ 0 & 3 & 3 & | & 0 \\ -1 & 1 & -2 & | & -4 \end{bmatrix} R_1 + R_3 \qquad \begin{aligned} & \begin{bmatrix} 1 & -2 & -1 & | & 2 \end{bmatrix} \\ + & \begin{bmatrix} -1 & 1 & -2 & | & -4 \end{bmatrix} \\ \hline = & \begin{bmatrix} 0 & -1 & -3 & | & -2 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & -1 & | & 2 \\ 0 & 3 & 3 & | & 0 \\ 0 & -1 & -3 & | & -2 \end{bmatrix}$$

Step 4: Get a 1 in the second row of the second column

To get the 1, we can multiply row 2 by one-third

$$\begin{bmatrix} 1 & -2 & -1 & | & 2 \\ 0 & 3 & 3 & | & 0 \\ 0 & -1 & -3 & | & -2 \end{bmatrix} \tfrac{1}{3}R_2 \qquad \begin{aligned} \tfrac{1}{3}\begin{bmatrix} 0 & 3 & 3 & | & 0 \end{bmatrix} &= \begin{bmatrix} \tfrac{1}{3}(0) & \tfrac{1}{3}(3) & \tfrac{1}{3}(3) & | & \tfrac{1}{3}(0) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 & | & 0 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & -1 & | & 2 \\ 0 & 1 & 1 & | & 0 \\ 0 & -1 & -3 & | & -2 \end{bmatrix}$$

Step 5: Use row 2 to get a 0 in the second column of row 3

To make the second column of row 3 a zero, we can add row 2 to row 3

$$\begin{bmatrix} 1 & -2 & -1 & | & 2 \\ 0 & 1 & 1 & | & 0 \\ 0 & -1 & -3 & | & -2 \end{bmatrix} R_2 + R_3 \qquad \begin{aligned} & \begin{bmatrix} 0 & 1 & 1 & | & 0 \end{bmatrix} \\ + & \begin{bmatrix} 0 & -1 & -3 & | & -2 \end{bmatrix} \\ \hline = & \begin{bmatrix} 0 & 0 & -2 & | & -2 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & -1 & | & 2 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & -2 & | & -2 \end{bmatrix}$$

Example 1 (Continued):

Step 6: Get a 1 in the third row of the third column

To make the -2 a 1, we can multiply row 3 by a negative one-half

$$\begin{bmatrix} 1 & -2 & -1 & | & 2 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & -2 & | & -2 \end{bmatrix} -\tfrac{1}{2}R_3$$

$$-\tfrac{1}{2}[0 \quad 0 \quad -2 \; | \; -2] = \left[-\tfrac{1}{2}(0) \; -\tfrac{1}{2}(0) \; -\tfrac{1}{2}(-2) \; \middle| \; -\tfrac{1}{2}(-2) \right]$$

$$= [0 \quad 0 \quad 1 \; | \; 1]$$

$$\begin{bmatrix} 1 & -2 & -1 & | & 2 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

Step 7: Change the augmented matrix back into a system of equations

$$\begin{array}{rcrcrcl} x & - & 2y & - & z & = & 2 \\ & & y & + & z & = & 0 \\ & & & & z & = & 1 \end{array}$$

Step 8: Use back-substitution to solve for the variables

$z = 1$ so we can substitute it into the second equation to find y

$y + z = 0$
$y + 1 = 0$
$y = -1$

Now we can substitute the values for y and z into the first equation to find x

$x - 2y - z = 2$
$x - 2(-1) - (1) = 2$
$x + 2 - 1 = 2$
$x + 1 = 2$
$x = 1$

The solution set for this system of equations is (1, -1, 1).

The simplest matrix containing the solutions to the linear equations is called a ***reduced* row-echelon *matrix***. Normally, we can solve a system of linear equations if the number of variables is equal to the number of independent equations. For example, if there are three variables in a system of linear equations, we will need three of these independent equations in order to obtain a unique solution for each of the variables.

Our reduced row-echelon matrix will look like

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$$

where a, b, and c are the solutions for the three variables.

The reduced row-echelon matrix will generally be in the form above, containing only 1's and 0's to the left of the vertical line, with the 1's in a diagonal pattern extending from the upper left to the lower right. This is also called a *square* matrix, where the number of rows and columns to the left of the vertical line are equal.

A more general rule for the reduced row-echelon matrix specifies that it meet the following conditions:

- In each row, the leftmost nonzero element is 1.
- In each row, the column containing the leftmost 1 has 0's above and below it.
- In each row, the leftmost 1 is to the right of the leftmost 1 in the row above it.
- Any row consisting entirely of 0's is below any row having a nonzero element.

In order to transform a matrix representing a system of linear equations into a reduced row-echelon matrix, we will need to apply a sequence of row operations to the original matrix. A sequence that we can follow to obtain the reduced row-echelon matrix is called *Gauss-Jordan elimination.* The steps are as follows:

1) Write the system of linear equations in an augmented matrix
2) Get a 1 in the first row of the first column.
3) Use row 1 to get 0's in the first column of rows 2 and 3.
4) Get a 1 in the second row of the second column
5) Use row 2 to get 0's in the second column of rows 1 and 3.
6) Get a 1 in the third row of the third column
7) Use row 3 to get 0's in the third column of rows 1 and 2.
8) If any row appears containing all 0's, move that row to the bottom.

The above sequence of operations would find 1's and 0's in the following order:

$$\begin{bmatrix} 1 & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & * & * & * \end{bmatrix} \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \end{bmatrix} \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$$

where the system's solution set would be (a, b, c).

Example 2: Solve the system of equations with augmented matrices using the Gauss-Jordan elimination method.

$$\begin{array}{rrrrr} x & & - & 3z & = -2 \\ 2x & + 2y & + & z & = 4 \\ 3x & + y & - & 2z & = 5 \end{array}$$

Solution:

Step 1: Write the system of equations in an augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 2 & 2 & 1 & 4 \\ 3 & 1 & -2 & 5 \end{array}\right]$$

Always enter a 0 for any missing variables such as the y in the first equation.

Step 2: Get a 1 in the first row of the first column

This is already done so we can skip to the next step.

Step 3: Use row 1 to get 0's in the first column of rows 2 and 3

For the second row we can obtain a zero by multiplying row 1 by -2 and adding it to row 2.

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 2 & 2 & 1 & 4 \\ 3 & 1 & -2 & 5 \end{array}\right] -2R_1 + R_2 \qquad \begin{array}{l} -2\begin{bmatrix} 1 & 0 & -3 & | -2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 6 & | 4 \end{bmatrix} \\ \underline{\hspace{0.5em}\begin{bmatrix} 2 & 2 & 1 & | 4 \end{bmatrix}} \\ = \begin{bmatrix} 0 & 2 & 7 & | 8 \end{bmatrix} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 2 & 7 & 8 \\ 3 & 1 & -2 & 5 \end{array}\right]$$

Example 2 (Continued):

For the third row we can obtain a zero by multiplying row 1 by -3 and adding it to row 3.

$$\begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 0 & 2 & 7 & | & 8 \\ 3 & 1 & -2 & | & 5 \end{bmatrix} -3R_1 + R_3$$

$$-3\begin{bmatrix} 1 & 0 & -3 & | & -2 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 9 & | & 6 \end{bmatrix}$$
$$+\begin{bmatrix} 3 & 1 & -2 & | & 5 \end{bmatrix}$$
$$=\begin{bmatrix} 0 & 1 & 7 & | & 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 0 & 2 & 7 & | & 8 \\ 0 & 1 & 7 & | & 11 \end{bmatrix}$$

Step 4: Get a 1 in the second row of the second column

We can get a 1 in the second row of the second column by multiplying row 2 by one-half. However, doing this will give us a fraction within the matrix. Another option we can use to avoid having to deal with fractions is to switch rows 2 and 3 since there is already a 1 in the second column of row 3.

$$\begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 0 & 2 & 7 & | & 8 \\ 0 & 1 & 7 & | & 11 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 0 & 1 & 7 & | & 11 \\ 0 & 2 & 7 & | & 8 \end{bmatrix}$$

Example 2 (Continued):

Step 5: Use row 2 to get 0's in the second column of rows 1 and 3

The second column of row 1 already contains a 0 so we can move on to row 3. Row 3 has a 2 in the second column so in order to make it a 0 we can multiply row 2 by a -2 and add it to row 3.

$$\begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 0 & 1 & 7 & | & 11 \\ 0 & 2 & 7 & | & 8 \end{bmatrix} -2R_2 + R_3$$

$$\begin{array}{r} -2\begin{bmatrix} 0 & 1 & 7 & | 11 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -14 & | -22 \end{bmatrix} \\ +\begin{bmatrix} 0 & 2 & 7 & | & 8 \end{bmatrix} \\ \hline = \begin{bmatrix} 0 & 0 & -7 & | -14 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 0 & 1 & 7 & | & 11 \\ 0 & 0 & -7 & | & -14 \end{bmatrix}$$

Step 6: Get a 1 in the third row of the third column

We can multiply row 3 by a negative one-seventh to get the 1 that we need.

$$\begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 0 & 1 & 7 & | & 11 \\ 0 & 0 & -7 & | & -14 \end{bmatrix} -\tfrac{1}{7}R_3$$

$$-\tfrac{1}{7}\begin{bmatrix} 0 & 0 & -7 & | -14 \end{bmatrix} = \begin{bmatrix} -\tfrac{1}{7}(0) & -\tfrac{1}{7}(0) & -\tfrac{1}{7}(-7) & | & -\tfrac{1}{7}(-14) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 1 | 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 0 & 1 & 7 & | & 11 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$$

Step 7: Use row 3 to get 0's in the third column of rows 1 and 2

Row 1 has a -3 in the third column. So in order to make it a 0 we can multiply row 3 by 3 and add it to row 1.

$$\begin{bmatrix} 1 & 0 & -3 & | & -2 \\ 0 & 1 & 7 & | & 11 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} 3R_3 + R_1$$

$$\begin{array}{r} 3\begin{bmatrix} 0 & 0 & 1 | 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 | & 6 \end{bmatrix} \\ +\begin{bmatrix} 1 & 0 & -3 & | -2 \end{bmatrix} \\ \hline = \begin{bmatrix} 1 & 0 & 0 & | & 4 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 7 & | & 11 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$$

Example 2 (Continued):

Row 2 has a 7 in the third column. We can, therefore, multiply row 3 by -7 and add it to row 2.

$$\begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 7 & | & 11 \\ 0 & 0 & 1 & | & 2 \end{bmatrix} -7R_3 + R_2$$

$$-7\begin{bmatrix} 0 & 0 & 1 & | & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -7 & | & -14 \end{bmatrix}$$
$$+\begin{bmatrix} 0 & 1 & 7 & | & 11 \end{bmatrix}$$
$$=\begin{bmatrix} 0 & 1 & 0 & | & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & -3 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$$

Now that we have successfully manipulated the matrix into the reduced row-echelon form we can get the solution set as x = 4, y = -3, and z = 2; which can also be written as (4, -3, 2).