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Analogies using non-identical equations

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Iowa State College

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ANALOGIES USING NON-IDENTICAL EQUATIONS

by

Robert Eugene Uhrig

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Theoretical and Applied Mechanics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State College
1954

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INTRODUCTION

The solution of many engineering problems by direct mathematical means is frequently so complicated that the work necessary to obtain a direct answer is prohibitive, and other methods of dealing with these problems must be found. The use of dissimilar models or analogies is one of these methods that has come into large scale use during the past ten years. The fact that many different physical phenomena can be represented by similar or equivalent equations often permits the results found in one field of science or engineering to be used or applied in some other field. The most common example of this is the use of electrical models, particularly analog computers that are used as simulators, to represent mechanical, acoustical or other electrical systems.

The usual procedure for establishing an analogy is to match directly terms of equations having identical form or to use linear relationships between corresponding terms of the analogous equations. The purpose of this dissertation is to investigate the possibility of using functional relationships between the variables of the equations of the analogous systems in order to liberalize the design conditions and to establish analogies between two phenomena having characteristic equations which do not have identical form.

BACKGROUND OF ANALOGIES

An analogy is based upon the similarity between the characteristic equations of two phenomena. This may lead to the use of a dissimilar model in which the model bears no obvious physical resemblance to the prototype. In many cases the similarity between the characteristic equations of two different phenomena was pointed out by mathematicians many years before practical applications were made and techniques developed to give more reliable results.

A typical example presented by Higgins[6] is the well known soap film or membrane analogy used to solve torsion problems. This analogy is based on the similarity between the differential equation for equilibrium of a membrane subjected to pressure on one side and the differential equation for a member subjected to torsion. The analogy was first presented by Prandtl[18] about 1903, but the first practical application was not made until 1912 by Anthes.[1] In 1917, A. A. Griffith and G. I. Taylor[5] used the analogy to determine the torsional strength and stiffness of airplane propeller blades and structural members having various cross sections. In 1931, Biezeno and Rademaker[2] devised an electrical contact system to determine the contour lines more accurately. Two years later, Quest[19] designed an optical system to determine the contour lines without contact, and this system was incorporated in Reichenbacher's apparatus for automatically recording the contour lines on a photographic plate. The difficulties associated with the production and maintenance of the soap film were avoided by the use of the "liquid surface" proposed by Piccard and Beas[17] in 1926, and improved by Sunatani, Matuyama and Hatamura[23] a decade later.

Murphy[14] gives a complete coverage of the fundamental analogies normally encountered in engineering practice, including the development of the characteristic equations and the establishment of the analogy. The following abstracts summarize several of the more important analogies described by Murphy:

> Column Analogy. Hardy Cross[3] has shown that if any statically consistent set of moments is assumed to act in a planar structure that is fixed at both ends, the difference between the true moments and the assumed moments is given by an equation of the same form as the equation for the stress at any point in an eccentrically loaded short compression block. This analogy is of value in determining the reactions on statically indeterminate structures.
>
> Area Moments Analogy. Greene[4] and Mohr[10] independently devised this procedure for dealing with statically indeterminate structures. The model used in an analysis of this type was called a "conjugate

beam" by Westergaard,[26] who generalized the previous work of Greene and Mohr.

Conjugate Frame Analogy. The analogy between a deflection curve and a moment diagram for a beam has been extended by Murphy[11] to assist in the determination of deflections in planar structures, or, indirectly, the solution of statically indeterminate structures.

Slab Analogy. In the slab analogy, the characteristic equation of an unloaded slab has the same form as that of the Airy Stress Function. Application of the slab analogy to evaluate the stresses in Hoover Dam was indicated by Westergaard,[27] and experimental and analytical investigations for a section of the dam were conducted by Jensen[8] and Murphy.[13]

Electrical Analogy. In this analogy, the differential equation for flow of current in a thin homogeneous plate of variable thickness has the same form as the differential equation for torsional stresses in a circular shaft of variable cross section. Jacobsen[7] used this analogy to evaluate stress concentrations near fillets and grooves in shafts.

Electron Tube Analogy. Kleynen[9] indicated that the equations of motion of an electron in a two dimensional field are identical in form with those of a particle sliding (without friction) on a membrane, the deflection of which is proportional to the potential of the electrodes.

Olson[16] gives a detailed outline of dynamical analogies in which differential equations are used to show the basis for the analogies between electrical, mechanical, and acoustical systems. His primary purpose was to apply the tremendous amount of study which has been directed towards the solution of electrical circuits to the solution of vibration problems in the mechanical and acoustical fields.

In all of the analogies previously indicated, the characteristic equations of the two phenomena have been identical, and the design and prediction equations have been merely linear scalar relationships between the corresponding quantities in the two phenomena.

RELATION OF THE TRANSFORMATION TO THE ANALOGY

The solution of many types of differential equations can be obtained by the substitution of one variable or quantity for another. This procedure often changes the differential equation into a form that is familiar and that can be solved by the use of standard methods. An example of this procedure is the substitution of $y = e^z$ or $z = \ln y$ to reduce a homogeneous linear equation to a linear equation with constant coefficients.

In the solution of a differential equation by this substitution method, the differential equation is changed by the use of particular transformations into a form that can be readily solved. The boundary conditions of the original problem must be transformed into terms of the new variables in order to evaluate the constants of integration. Then the transformations are used inversely to obtain the solution of the original equation. This general procedure can be adapted to aid in establishing analogies between phenomena having characteristic equations which do not have identical forms.

The method of establishing analogies used throughout this dissertation is the "Indirect Procedure" described by Murphy.[14] A simple example will illustrate this "Indirect Procedure" and how it can be modified to establish analogies between phenomena in which the characteristic equations are not identical.

The characteristic equation of an elastic column[12] is

$$E\,I\,\frac{d^2y}{dx^2} = -\,P\,y \qquad (1)$$

which can be rewritten as

$$\frac{d^2y}{dx^2} + \frac{P}{E\,I}\,y = 0 \qquad (2)$$

where P, E, and I are axial load, modulus of elasticity of the column material, and moment of inertia of the column cross section. An electrical model in which electrical charge (Q) varies with time (t) can be designed for the column. Prediction equations (sometimes called transformations or substitutions) are assumed to relate the variables of the two systems. These prediction equations are assumed to be

$$y = n\,Q \qquad \text{(Dependent variables)} \quad (3)$$

$$x = n_1\,t \qquad \text{(Independent variables)} \quad (4)$$

where n and n_1 are dimensional scales. If these transformations and their derivatives

$$\frac{dy}{dx} = \frac{n}{n_1}\,\frac{dQ}{dt} \qquad (5)$$

$$\frac{d^2y}{dx^2} = \frac{n}{n_1^2} \frac{d^2Q}{dt^2} \qquad (6)$$

are substituted into the characteristic equation (2) of the column, the resultant equation after simplification is

$$\frac{d^2Q}{dt^2} + \frac{n_1^2 P}{E I} Q = 0 \qquad (7)$$

which has the same form as the original characteristic equation (2) but is expressed in terms of the new variables Q and t. Equation (7) can be matched term by term with the characteristic equation of an electrical system (the model) having a mathematically identical form, and the equating of coefficients gives the design equations for the electrical model.

The characteristic equation of the electrical circuit shown in Figure 1 is

$$L \frac{d^2Q}{dt^2} + \frac{1}{C} Q = 0 \qquad (8)$$

which can be rewritten as

$$\frac{d^2Q}{dt^2} + \frac{1}{L C} Q = 0 \; . \qquad (9)$$

Equation (9) is of the same form as equation (2), and hence matching the coefficients will give the design condition

$$\frac{P\,n_1^{\,2}}{E\,I} = \frac{1}{L\,C}\;. \tag{10}$$

However, if one or both of the prediction equations are functional rather than merely a linear relationship, the resultant equation after substitution of the prediction equations and their derivatives will not have the same form as the original characteristic equation of the column. An example illustrated later having one functional prediction equation and one linear prediction equation is

$$y = n\,e^{\alpha t}\,Q \tag{11}$$

$$x = n_1 t \tag{12}$$

where n, n_1 and α are dimensional constants. If the prediction equations (11) and (12) and their derivatives

$$\frac{dy}{dx} = \frac{n}{n_1}\;e^{\alpha t}\left(\frac{dQ}{dt} + \alpha\,Q\right) \tag{13}$$

$$\frac{d^2y}{dx^2} = \frac{n}{n_1^{\,2}}\;e^{\alpha t}\left(\frac{d^2Q}{dt^2} + 2\,\alpha\,\frac{dQ}{dt} + \alpha^2 Q\right) \tag{14}$$

are substituted into the characteristic equation (2) of the column, the resultant equation after simplification is

$$\frac{d^2Q}{dt^2} + 2\,\alpha\,\frac{dQ}{dt} + \left(\alpha^2 + \frac{P\,n_1^{\,2}}{E\,I}\right) Q = 0\;. \tag{15}$$

The resultant equation (15) can be matched term by term with the characteristic equation of an electrical circuit having a mathematically identical form and the equating of coefficients will give the design equations for the electrical model.

The equation of the series electrical circuit shown in Figure 2 after division by L is

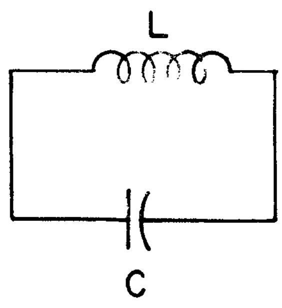
$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{L\,C}\,Q = 0 \tag{16}$$

which has the same form as the resultant equation (15). Matching of the coefficients of equations (15) and (16) gives the design equations

$$R/L = 2\,\alpha \tag{17}$$

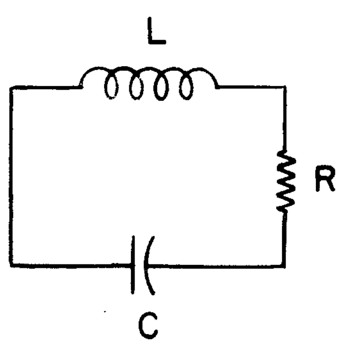
$$\frac{1}{L\,C} = \alpha^2 + \frac{P\,n_1^{\,2}}{E\,I}\,. \tag{18}$$

Hence an analogy has been established between two phenomena in which the characteristic equations (2) and (16) do not have the same form. (Throughout this dissertation transformed or resultant equations such as equation (15) will not be considered as characteristic equations, even though they were derived from characteristic equations.)



ELECTRICAL ANALOG CIRCUIT FOR A COLUMN

FIG. 1.



MODIFIED ELECTRICAL ANALOG CIRCUIT FOR A COLUMN

FIG. 2

The boundary conditions of the column must be transformed into boundary conditions of the electrical model by using the prediction equations, and these transformed boundary conditions must be satisfied in the electrical model if the analogy is to be valid.

It is easily seen that this procedure is dependent upon the change in form of a differential equation when it is transformed by a functional prediction equation. Therefore, a short review of the mathematical basis of some of these transformations will give an insight into the types of transformations desired and the limitations involved.

REVIEW OF MATHEMATICAL BACKGROUND OF TRANSFORMATIONS

Basis of Transformations

Method of ordinary differential equations

Change of dependent variable. The general second order differential equation

$$y'' + G(x)\,y' + H(x)\,y = 0 \tag{19}$$

can be put into the Normal Form (i.e. the form without the term involving the first derivative) by the transformation

$$y = U\,e^{-\frac{1}{2}\int G\,dx} . \tag{20}$$

The resulting equation is

$$U'' + I\,U = 0 \tag{21}$$

where I is the invariant of equation (19) and is defined by

$$I = H - \frac{1}{2} G' - \frac{1}{4} G^2 \ . \tag{22}$$

Rainville[20] gives a proof that I is an invariant. If G, H, G_1 and H_1 are so related that equation (19) can be transformed into

$$y_1'' + G_1(x)\, y_1' + H_1(x)\, y_1 = 0 \tag{23}$$

by the transformation

$$y = f(x)\, y_1 \tag{24}$$

where f is a function of x alone, it can be easily shown that I is an invariant and that equations (19) and (23) have the same normal form.

The first two derivatives of y are

$$y' = f y_1' + f' y_1 \tag{25}$$

$$y'' = f y_1'' + 2f' y_1' + f'' y_1 \ . \tag{26}$$

Substitution of (24), (25), and (26) into equation (19) and division by f gives

$$y_1'' + y_1'\left(2\,\frac{f'}{f} + G\right) + y_1\left(\frac{f''}{f} + \frac{f'}{f}\,G + H\right) = 0 \ . \tag{27}$$

Hence, comparison of equations (23) and (27) gives

$$G_1 = 2\,\frac{f'}{f} + G \tag{28}$$

$$H_1 = \frac{f''}{f} + \frac{f'}{f}\,G + H \ . \tag{29}$$

By definition

$$I_1 = H_1 - 1/2\ G_1' - 1/4\ G_1^{\ 2} \quad . \tag{30}$$

Substitution of (28) and (29) into (30) gives

$$I_1 = \frac{f''}{f} + \frac{f'}{f} G + H - 1/2\left[2\left(\frac{f''f - f'^2}{f^2}\right) + G'\right]$$
$$- 1/4\left[2\frac{f'}{f} + G\right]^2 \tag{31}$$

which can be simplified to

$$I_1 = H - 1/2\ G' - 1/4\ G^2 = I \tag{32}$$

proving that I is an invariant.

Equation (28) can be solved to give

$$f = e^{\frac{1}{2}\int (G_1 - G)dx} \quad . \tag{33}$$

This can be substituted into (24) to give

$$y = y_1\ e^{\frac{1}{2}\int (G_1 - G)dx} \quad . \tag{34}$$

By this "normalizing process" the term involving the first derivative can be eliminated from any equation of the form of equation (19) by the transformation (20). Similarly, equation (19) can be changed into the form of equation (23) by the transformation (34). Therefore, it is possible to use transformations (20) and (34) to establish analogies between phenomena in which the characteristic equations do not have identical form.

<u>Change of independent variable.</u> The procedure described above involved a change of dependent variable. Another possible type of transformation is a change of independent variable. The independent variable x in the equation

$$y'' + G(x)\, y' + H(x)\, y = 0 \tag{19}$$

can be changed to a new independent variable z by the relationships

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \tag{35}$$

$$\frac{d^2y}{dx^2} = \frac{d^2z}{dx^2}\frac{dy}{dz} + \left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dz^2} \quad . \tag{36}$$

Substitution of (35) and (36) into (19) and division by H gives

$$\left[\frac{1}{H}\left(\frac{dz}{dx}\right)^2\right] \frac{d^2y}{dz^2} + \left[\frac{1}{H}\left(\frac{d^2z}{dx^2} + G\,\frac{dz}{dx}\right)\right]\frac{dy}{dz} + y = 0 \tag{37}$$

If this equation is to be an equation with constant coefficients, then

$$\frac{1}{H}\left(\frac{dz}{dx}\right)^2 = K_1{}^2 \tag{38}$$

$$\frac{1}{H}\left(\frac{d^2z}{dx^2} + G\,\frac{dz}{dx}\right) = K_2 \tag{39}$$

where $K_1{}^2$ and K_2 are constants. Solving equation (38) for $\frac{dz}{dx}$ gives

$$\frac{dz}{dx} = K_1 H^{1/2} . \tag{40}$$

Equation (40) can be integrated to give

$$z = K_1 \int H^{1/2} dx \tag{41}$$

and differentiated to give

$$\frac{d^2z}{dx^2} = \frac{K_1}{2} H^{-1/2} \frac{dH}{dx} . \tag{42}$$

Equations (40) and (42) can be substituted into (39) and simplified to give

$$\frac{K_1}{2} \left(\frac{H' + 2H G}{H^{3/2}} \right) = K_2 \tag{43}$$

Therefore, if

$$\frac{H' + 2H G}{H^{3/2}} = K_3 \tag{44}$$

where K_3 is a constant, then

$$z = \int H^{1/2} dx \tag{45}$$

will reduce equation (19) to an equation with constant coefficients. This change of independent variable has indicated another method that can be used to establish analogies between phenomena in which the characteristic equations are not identical.

Change of both dependent and independent variables. The two procedures described above may be combined to change both the dependent and the independent variables.

The equation

$$\frac{d^2y}{dx^2} + I(x)\ y = 0 \tag{46}$$

which can be obtained from equation (19) by the "normalizing process" may be transformed into

$$\frac{d^2U}{dz^2} + J(z)\ U = 0 \tag{47}$$

by using the relationships

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \tag{35}$$

$$\frac{d^2y}{dx^2} = \frac{d^2z}{dx^2}\frac{dy}{dx} + \left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dz^2} \tag{36}$$

to change the independent variable x to the new independent variable z, and by using

$$y = f\ U \tag{48}$$

to change the dependent variable y, where f is a function of the independent variable z, and U is the new dependent variable. Substitution of (35) and (36) into equation (46) and division by $\left(\frac{dz}{dx}\right)^2$ gives

$$\frac{d^2y}{dz^2} + \frac{dy}{dz}\left[\frac{\frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2}\right] + y\left[\frac{I}{\left(\frac{dz}{dx}\right)^2}\right] = 0 \tag{49}$$

which has the form of equation (19) when

$$G = \frac{\dfrac{d^2z}{dx^2}}{\left(\dfrac{dz}{dx}\right)^2} = \frac{d^2z}{dx^2}\left(\frac{dx}{dz}\right)^2 \tag{50}$$

$$H = \frac{I}{\left(\dfrac{dz}{dx}\right)^2} = I\left(\frac{dx}{dz}\right)^2 . \tag{51}$$

The conditions under which equations (47) and (49) are equivalent are that

(a) the invariants of the normal forms of these two equations must be equal,

(b) the dependent variables are related by

$$y = U\, e^{-\frac{1}{2}\int G\, dz} . \tag{20}$$

By definition

$$\text{Invariant} = H - 1/2\, G' - 1/4\, G^2 . \tag{22}$$

Therefore,

$$\text{Inv. } (47) = J \tag{52}$$

$$\text{Inv. } (49) = \frac{I}{\left(\dfrac{dz}{dx}\right)^2} - 1/2\,\frac{d}{dz}\left[\frac{\dfrac{d^2z}{dx^2}}{\left(\dfrac{dz}{dx}\right)^2}\right] - 1/4\left[\frac{\dfrac{d^2z}{dx^2}}{\left(\dfrac{dz}{dx}\right)^2}\right]^2 \tag{53}$$

which simplifies to

$$\text{Inv. (49)} = \frac{1}{\left(\frac{dz}{dx}\right)^2}\left[I - 1/2 \quad \frac{d^3z}{dx^3}\,\frac{dx}{dz} + 3/4\left(\frac{d^2z}{dx^2}\right)^2\left(\frac{dx}{dz}\right)^2\right. \quad . \quad (54)$$

Equating these invariants gives

$$J = \frac{1}{\left(\frac{dz}{dx}\right)^2}\left[I - 1/2\,\frac{d^3z}{dx^3}\,\frac{dx}{dz} + 3/4\left(\frac{d^2z}{dx^2}\right)^2\left(\frac{dx}{dz}\right)^2\right]. \quad (55)$$

or

$$\left(\frac{dz}{dx}\right)^2 J = I - 1/2\,\left\{z,x\right\} \quad (56)$$

where $\{z,\chi\}$ is the Schwarzian derivative of z with respect to x and is defined as

$$\left\{z,x\right\} = \frac{z'''}{z'} - 3/2\left(\frac{z''}{z'}\right)^2 \quad . \quad (57)$$

Substitution of (50) into equation (20) gives

$$y = U\,e^{\frac{1}{2}\int\frac{d^2z}{dx^2}\left(\frac{dx}{dz}\right)^2 dz} \quad . \quad (58)$$

Since

$$\frac{d^2z}{dx^2}\left(\frac{dx}{dz}\right)^2 = \frac{d}{dz}\left(\ln\frac{dz}{dx}\right) \quad (59)$$

equation (43) can be integrated to give

$$y = U\, e^{\frac{1}{2}\int \frac{d}{dz}\left(\ln \frac{dz}{dx}\right) dz} \tag{60}$$

which simplifies to

$$y = U \left(\frac{dx}{dz}\right)^{\frac{1}{2}}. \tag{61}$$

The conditions for equivalence between equations (47) and (49) can now be stated mathematically as

$$\text{(a)} \quad \left(\frac{dz}{dx}\right)^2 J = I - 1/2 \left\{z,x\right\} \tag{62}$$

$$\text{(b)} \quad y = U \left(\frac{dx}{dz}\right)^{\frac{1}{2}}. \tag{63}$$

These three change of variable procedures indicate how a few specific second order differential equations can be transformed into equations which have constant coefficients or into equations which do not contain first derivatives. These transformations presented above are subject to the severe limitations indicated, but when these conditions are satisfied, the transformations will give the results indicated. However, it is not always desirable to have a transformed equation that has constant coefficients or an equation that has no first derivative. Sometimes a non-linear equation or an equation with different variable coefficients is the desired result, since many phenomena in engineering or science have equations with these general forms. Theoretically, an infinite number of transformations could be applied to any equation of a physical phenomenon. Practically speaking, only a small number of these transformations would result in transformed equations that has the same form as an equation of an actual phenomenon, and hence could be used to establish an analogy. No attempt has been made in this dissertation to catalog all the possible transformations. A selected few have been worked through in detail, and one has been investigated experimentally.

Method of functions

The example of a functional prediction equation given in equation (11), namely

$$y = n_1 \, e^{\alpha t} \, Q \tag{11}$$

is the product of two functions. Since this is one of the most useful prediction equations, it will be of great interest to develop this equation to see the assumptions and limitations involved. In this development, the dependent variable is assumed to be the product of the functions of two variables, while the independent variable is assumed to be a function of only one variable.

The generalized transformations for this set of prediction equations are

$$y = f(Q) \cdot g(t) = f \, g \tag{64}$$

$$x = h(t) = h \tag{65}$$

where f is a function of Q, and g and h are functions of t alone. The first and second derivatives after simplification are

$$\frac{dy}{dx} = \frac{\frac{df}{dQ}\frac{dQ}{dt} g + \frac{dg}{dt} f}{\frac{dh}{dt}} \tag{66}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d^2f}{dQ^2}\left(\frac{dQ}{dt}\right)^2 g}{\left(\frac{dh}{dt}\right)^2} + \frac{\frac{df}{dQ}\frac{d^2Q}{dt^2} g}{\left(\frac{dh}{dt}\right)^2} - \frac{\frac{df}{dQ}\frac{dQ}{dt}\frac{d^2h}{dt^2}}{\left(\frac{dh}{dt}\right)^3}$$

$$+ 2\frac{\frac{df}{dQ}\frac{dQ}{dt}\frac{dg}{dt}}{\left(\frac{dh}{dt}\right)^2} + \frac{f\frac{d^2g}{dt^2}}{\left(\frac{dh}{dt}\right)^2} - \frac{f\frac{dg}{dt}\frac{d^2h}{dt^2}}{\left(\frac{dh}{dt}\right)^3} . \tag{67}$$

Substitution of (64), (65), (66) and (67) into the general second order differential equation

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + B\,y = K\cos\omega x \tag{68}$$

where A, B, and K are constants, gives after simplification

$$\frac{d^2Q}{dt^2} + \frac{dQ}{dt}\left[\frac{2\frac{dg}{dt}}{g} - \frac{\frac{d^2h}{dt^2}}{\frac{dh}{dt}} + A\frac{dh}{dt}\right]$$

$$+ \frac{dQ}{dt}^{2} \left[\frac{\frac{d^2f}{dQ^2}}{\frac{df}{dQ}} \right] + f \left[\frac{\frac{d^2g}{dt^2}}{g\,\frac{df}{dQ}} - \frac{\frac{dg}{dt}\,\frac{d^2h}{dt^2}}{g\,\frac{dh}{dt}\,\frac{df}{dQ}} \right.$$

$$\left. + \frac{A\,\frac{dg}{dt}\,\frac{dh}{dt}}{g\,\frac{df}{dQ}} + \frac{B\left(\frac{dh}{dt}\right)^2}{\frac{df}{dQ}} \right] = \left[\frac{K\left(\frac{dh}{dt}\right)^2}{\frac{df}{dQ}\,g} \right] \cos \omega h. \quad (69)$$

At this point, some assumptions must be made if the above equation is to be of any practical use. It is desirable to keep the cosine function as simple as possible, and hence, an assumption is made that

$$h = \beta t \quad (70)$$

where β is a constant. The first and second derivatives are

$$\frac{dh}{dt} = \beta \quad (71)$$

$$\frac{d^2h}{dt^2} = 0 \quad . \quad (72)$$

Substitution of (70), (71) and (72) into equation (69) gives

$$\frac{d^2Q}{dt^2} + \frac{dQ}{dt}\left[2\,\frac{\frac{dg}{dt}}{g} - \beta A\right] + \left(\frac{dQ}{dt}\right)^2 \left[\frac{\frac{d^2f}{dQ^2}}{\frac{df}{dQ}}\right]$$

$$+ f\left[\frac{\frac{d^2g}{dt^2}}{g\,\frac{df}{dQ}} + \frac{A\,\frac{dg}{dt}\,\beta}{g\,\frac{df}{dQ}} + \frac{B\,\beta^2}{\frac{df}{dQ}}\right] = \left[\frac{K\,\beta^2}{\frac{df}{dQ}\,g}\right]\cos\omega\beta t. \tag{73}$$

If the assumption is made that

$$f = \delta\,Q \tag{74}$$

where δ is a constant, the first and second derivatives of f with respect to t are

$$\frac{df}{dQ} = \delta \tag{75}$$

$$\frac{d^2f}{dQ^2} = 0 \quad . \tag{76}$$

Substitution of (74), (75) and (76) into equation (73) gives after simplification

$$\frac{d^2Q}{dt^2} + \frac{dQ}{dt}\left[\frac{2\frac{dg}{dt}}{g} - A\beta\right] + Q\left[\frac{\frac{d^2g}{dt^2}}{g} + \frac{A\,\frac{dg}{dt}\,\beta}{g} + B\,\beta^2\right]$$

$$= \frac{K\beta^2}{\delta\,g}\cos\omega\beta t\ . \tag{77}$$

It appears that equation (77) is the equation of an electrical circuit. It is most convenient to work with electrical circuits where the parameters are constant. Therefore, this assumption that the coefficients of equation (77) are constant gives

$$2\,\frac{\frac{dg}{dt}}{g} + A\beta = K_1 \tag{78}$$

$$\frac{\frac{d^2g}{dt^2}}{g} + \frac{A\beta\,\frac{dg}{dt}}{g} + B\beta^2 = K_2 \quad . \tag{79}$$

The solution of equation (78) is

$$g = K_3\, e^{\left(\frac{K_1 - \beta}{2}\right)t} \quad . \tag{80}$$

The solution of equation (64) is

$$g = K_4\, e^{\left\{-\frac{A\beta}{2} + \left[\left(\frac{A\beta}{2}\right)^2 - B\beta^2 + K_2\right]^{1/2}\right\}t} + K_5\, e^{\left\{-\frac{A}{2} - \left[\left(\frac{A\beta}{2}\right)^2 - B\beta^2 + K_2\right]^{1/2}\right\}t} \tag{81}$$

which can be shown to be equivalent to equation (80) if

$$K_3 = K_4 \tag{82}$$

$$K_5 = 0 \tag{83}$$

and the exponents are equal. When (78), (79) and (80) are substituted into equation (77), it becomes

$$\frac{d^2Q}{dt^2} + K_1 \frac{dQ}{dt} + K_2 Q = K_6 e^{-K_7 t} \cos \omega \beta t \tag{84}$$

where K_6 and K_7 are defined as

$$K_6 = \frac{K \beta^2}{K_3 \delta} \tag{85}$$

$$K_7 = \frac{K_1 - \beta}{2} . \tag{86}$$

Substitution of (70), (74) and (80) into equations (64) and (65) gives the prediction equations

$$y = K_3 e^{K_7 t} Q \tag{87}$$

$$x = \beta t . \tag{88}$$

These prediction equations have transformed equation (68) into equation (84) which are not of identical form because of the $e^{-K_7 t}$ on the right side of equation (84). However, an analogy can be established between two phenomena having equations of the form of equations (68) and (84). In this particular example, one design equation has been complicated by the introduction of $e^{-K_7 t}$, but some of the others have been liberalized because K_1 and K_2 are sums of several terms.

Equation (84) is the equation of an electrical circuit having an inductance, a resistance, and a capacitance connected in series with a voltage generator producing a cosine wave having an amplitude which decreases exponentially with time.

The assumptions made in this development were directed at a specific result. At any point, different assumptions could have been made or different limitations imposed, thereby changing the transformation, the resultant equation and the type of analogy that could be established. It is important to note that other types of generalized transformations could have been assumed at the very beginning, i.e. addition, division, derivatives, etc. instead of multiplication.

Types of Transformations

Ordinary differential equations

Product of functions. Two examples of transformations that are products of functions have already been developed. Equations (34) and (87) both are transformations that are products of a dependent variable and an exponential term. The uses and limitations of this type of transformation have already been described. In equation (80), the exponential factor was obtained by assuming that the coefficients of the transformed equation were constants as defined in equations (78) and (79). If these limitations are modified so that the coefficients of the transformed equation are functions of the independent variable alone, then an analogy can be established between a linear differential equation with constant coefficients and a linear differential equation of the same order with variable coefficients. Two simple examples will illustrate this procedure.

The prediction equations

$$y = n\ t\ Q \tag{89}$$

$$x = n_1\ t \tag{90}$$

transform the general second order differential equation with constant coefficients

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + B\ y = 0 \tag{91}$$

into

$$\frac{d^2Q}{dt^2} + \frac{dQ}{dt}\left(\frac{2}{t} + A\ n_1\right) + Q\left(\frac{A\ n_1}{t} + B\ n_1^{\ 2}\right) = 0, \tag{92}$$

and hence an analogy can be established between two phenomena having characteristic equations of the forms of equations (91) and (92) if the boundary conditions can be satisfied.

If the "normalizing" transformation (20) is applied to Bessel's equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{N^2}{x^2}\right)y = 0 \tag{93}$$

where N is the order, the transformation (20) simplifies to

$$y = Q\, x^{-\frac{1}{2}} \quad . \tag{94}$$

The prediction equations corresponding to (94) are

$$y = n\, Q\, t^{-\frac{1}{2}} \tag{95}$$

$$x = n_1\, t \tag{96}$$

which transform Bessel's equation into

$$\frac{d^2Q}{dt^2} + \left(n_1^{\ 2} + \frac{1 - 4N^2}{4t^2} \right) Q = 0 \ . \tag{97}$$

The electrical circuit represented in equation (97) does not have any resistance, and the capacitance is a function of t. A method of introducing resistance into the circuit and varying the capacitance is described in detail later.

<u>Sum of functions</u>. Another type of transformation that may be quite useful in establishing analogies is the type in which the dependent variable is assumed to be the sum of two different functions of the new dependent variable, and the independent variable is assumed to be a function of the new independent variable only. The generalized transformation for this type of prediction equations are

$$y = f(Q) + g(Q) = f + g \tag{98}$$

$$x = h(t) = h \quad . \tag{99}$$

The first and second derivatives are

$$\frac{dy}{dx} = \frac{\frac{dQ}{dt}}{\frac{dh}{dt}} \left(\frac{df}{dQ} + \frac{dg}{dQ} \right) \tag{100}$$

$$\frac{d^2y}{dx^2} = \left[\frac{\frac{d^2Q}{dt^2}}{\left(\frac{dh}{dt}\right)^2} \right] \left(\frac{d^2f}{dQ^2} + \frac{d^2g}{dQ^2} \right) + \left(\frac{df}{dQ} + \frac{dg}{dQ} \right) \left\{ \frac{1}{\left(\frac{dh}{dt}\right)^2} \left[\frac{d^2Q}{dt^2} - \frac{\frac{dQ}{dt}}{\frac{dh}{dt}} \right] \right\}. \tag{101}$$

If (98) through (101) are substituted into

$$\frac{d^2y}{dx^2} + A \frac{dy}{dx} + B\, y = 0 \tag{102}$$

where A and B are constants, it is transformed into

$$\frac{d^2Q}{dt^2} \left[1 - \frac{\frac{d^2h}{dt^2}}{\frac{dh}{dt}} \right] + \left(\frac{dQ}{dt} \right)^2 \left[\frac{\frac{d^2f}{dQ^2} + \frac{d^2g}{dQ^2}}{\frac{df}{dQ} + \frac{dg}{dQ}} \right]$$

$$+ \frac{dQ}{dt}\left[B \quad \frac{dh}{dt}\right] + \left[\frac{B\ f + g\left(\frac{dh}{dt}\right)^2}{\frac{df}{dQ} + \frac{dg}{dQ}}\right] = 0 \qquad (103)$$

after simplification. As in equation (70) the assumption is made that

$$h = \beta t \qquad (104)$$

where β is a constant. The first and second derivatives are

$$\frac{dh}{dt} = \beta \qquad (105)$$

$$\frac{d^2h}{dt^2} = 0 \ . \qquad (106)$$

Substitution of (104) through (106) into equation (103) gives

$$\frac{d^2Q}{dt^2}\left(\frac{df}{dQ} + \frac{dg}{dQ}\right) + \frac{dQ}{dt} \ A\beta\left(\frac{df}{dQ} + \frac{dg}{dQ}\right) + \left(\frac{dQ}{dt}\right)^2\left(\frac{d^2f}{dQ^2} + \frac{d^2g}{dQ^2}\right)$$

$$+ \quad B\beta^2\left(f + g\right) = 0 \ . \qquad (107)$$

If the assumptions are made that

$$f = \gamma \ln Q \qquad (108)$$

$$g = \delta Q \qquad (109)$$

where γ and δ are constants, the first and second derivatives are

$$\frac{df}{dQ} = \frac{\gamma}{Q} \qquad (110)$$

$$\frac{d^2 f}{dQ^2} = -\frac{\gamma}{Q^2} \tag{111}$$

$$\frac{dg}{dQ} = \delta \tag{112}$$

$$\frac{d^2 g}{dQ^2} = 0 \quad . \tag{113}$$

Substitution of (108) through (113) into equation (107) gives after simplification

$$\frac{d^2Q}{dt^2}\left(\frac{\gamma}{Q}+\delta\right) + \frac{dQ}{dt}\, A\beta\left(\frac{\gamma}{Q}+\delta\right) - \left(\frac{dQ}{dt}\right)^2 \frac{1}{Q^2} + \beta^2 B\delta Q$$

$$+ \gamma\beta^2 B \ln Q = 0 \quad . \tag{114}$$

If the phenomenon could be found having the non-linear differential equation (114), an analogy could be established between this phenomenon and a phenomenon having the characteristic equation of the form of equation (102). It appears unlikely that such a phenomenon can be found, but a procedure has been illustrated that will be shown to be very useful later.

Derivative of functions. The substitution

$$y = \frac{dQ}{dt} \tag{115}$$

is used to solve a differential equation in which the dependent variable is absent. This substitution can be adapted to establish an analogy. The prediction equations and their derivatives corresponding to equation (115) are

$$y = n \frac{dQ}{dt} \tag{116}$$

$$x = n_1 t \tag{117}$$

$$\frac{dy}{dx} = \frac{n}{n_1} \frac{d^2Q}{dt^2} \tag{118}$$

$$\frac{d^2y}{dx^2} = \frac{n}{n_1^2} \frac{d^3Q}{dt^3} \quad . \tag{119}$$

Substitution of (116) through (119) into the equation

$$\frac{d^2y}{dx^2} + A \frac{dy}{dx} + B\, y = 0 \tag{120}$$

where A and B are constants gives after simplification

$$\frac{d^3Q}{dt^3} + n_1 A \frac{d^2Q}{dt^2} + n_1^2 B \frac{dQ}{dt} = 0 \tag{121}$$

This transformation has effectively raised the order of each derivative by unity. Hence, an analogy may be established between two phenomena having characteristic equations of the form of equations (120) and (121), which are of different order, if the boundary conditions of the prototype can be satisfied in the model.

Combination of product, sum, and derivative of functions. The three procedures given previously can be combined in many different ways to give analogies that are more useful or more easily established than the previous examples. A set of prediction equations that combines the three methods given above is

$$y = e^{\alpha t} \left[n_1 \frac{dQ}{dt} + n_2 Q \right] \tag{122}$$

$$x = n t \quad . \tag{123}$$

The first and second derivatives are

$$\frac{dy}{dx} = \frac{e^{\alpha t}}{n} \left[n_1 \frac{d^2Q}{dt^2} + \frac{dQ}{dt} (n_2 + \alpha n_1) + \alpha n_2 Q \right] \tag{124}$$

$$\frac{d^2y}{dx^2} = \frac{e^{\alpha t}}{n^2} \left[n_1 \frac{d^3Q}{dt^3} + \frac{d^2Q}{dt^2} (n_2 + 2 \alpha n_1) + \frac{dQ}{dt} (2 \alpha n_2 + \alpha^2 n_1) + \alpha^2 n_2 Q \right] \quad . \tag{125}$$

Substitution of (122) through (125) into the equation

$$\frac{d^2y}{dx^2} + A \frac{dy}{dx} + B y = 0 \tag{126}$$

where A and B are constants, gives after simplification

$$\frac{d^3Q}{dt^3} + \frac{d^2Q}{dt^2} \left[2 \alpha + \frac{n_2}{n_1} + A n \right] + \frac{dQ}{dt} \left[\alpha^2 + 2\alpha \frac{n_2}{n_1} + A n \left(\alpha + \frac{n_2}{n_1} \right) + B n^2 \right] + Q \left[\alpha^2 \frac{n_2}{n_1} + A n \alpha \frac{n_2}{n_1} + B n^2 \frac{n_2}{n_1} \right] = 0 \; . \tag{127}$$

The transformation (122) has changed a second order differential equation into a third order differential where both equations have constant coefficients. Here again, an analogy may be established between phenomena having characteristic equations of the form of equations (126) and (127) using the prediction equations (122) and (123) if the boundary conditions can be satisfied in the model.

Another interesting set of prediction equations of this general type

$$y = n \frac{\frac{dQ}{dt}}{H\ Q} \tag{128}$$

$$x = n_1\ t \tag{129}$$

which will transform the first order non-linear differential equation

$$\frac{dy}{dx} = G(x)\ y + H(x)\ y^2 = R(x) \tag{130}$$

into

$$\frac{d^2Q}{dt^2} + \left(n_1\ G - \frac{1}{H}\ \frac{dH}{dt}\right) \frac{dQ}{dt} - \left(\frac{n_1}{n}\ R\ H\right) Q + \left(\frac{dQ}{dt}\right)^2 \frac{1}{Q}\ (n\ n_1 - 1) = 0\ . \tag{131}$$

If

$$(n\,n_1 - 1) = 0 \quad \text{or } n\,n_1 = 1, \tag{132}$$

then equation (131) reduces to the second order linear equation

$$\frac{d^2Q}{dt^2} + \frac{dQ}{dt}\left(n\,G - \frac{1}{H}\,\frac{dH}{dt}\right) - \left(\frac{n_1}{n}\,R\,H\right)Q = 0 \quad . \tag{133}$$

Here again, an analogy can be established between two phenomena whose characteristic equations have the forms of equations (130) and (133) if the boundary conditions can be satisfied in the model.

Partial differential equations

The general procedures previously illustrated for ordinary differential equations may also be used with partial differential equations. Since there is more than one independent variable in a partial differential equation, there will have to be as many prediction equations as there are variables.

Since the product of functions type of transformation in which one of the terms was an exponential factor proved to be the most useful in ordinary differential equations, it seems reasonable to try this type of transformation on partial differential equations. If the prediction equations are assumed to be

$$U = n\, e^{\alpha t + \beta z}\, Q \tag{134}$$

$$x = n_1\, t \tag{135}$$

$$y = n_2\, z \tag{136}$$

where U and Q are the dependent variables and x, y, t, and z are the independent variables, then the first and second partial derivatives are

$$\frac{\partial U}{\partial x} = \frac{n}{n_1}\, e^{\alpha t + \beta z} \left(\frac{\partial Q}{\partial t} + \alpha Q \right) \tag{137}$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{n}{n_1^2}\, e^{\alpha t + \beta z} \left(\frac{\partial^2 Q}{\partial t^2} + 2\alpha \frac{\partial Q}{\partial t} + \alpha^2 Q \right) \tag{138}$$

$$\frac{\partial U}{\partial y} = \frac{n}{n_2}\, e^{\alpha t + \beta z} \left(\frac{\partial Q}{\partial z} + \beta Q \right) \tag{139}$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{n}{n_2^2}\, e^{\alpha t + \beta z} \left(\frac{\partial^2 Q}{\partial z^2} + 2\beta \frac{\partial Q}{\partial z} + \beta^2 Q \right) . \tag{140}$$

If equations (138) and (140) are substituted into Laplace's equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \tag{141}$$

it is transformed into

$$\frac{\partial^2 Q}{\partial t^2} + \left(\frac{n_1}{n_2}\right)^2 \frac{\partial^2 Q}{\partial z^2} + 2\alpha \frac{\partial Q}{\partial t} + 2\beta\left(\frac{n_1}{n_2}\right)^2 \frac{\partial Q}{\partial z} + Q\left[\alpha^2 + \left(\frac{n_1}{n_2}\right)^2 \beta^2\right] = 0 \tag{142}$$

Unfortunately, there are only three parameters, namely α, β, and $\frac{n_1}{n_2}$, to match four coefficients in equation (142). However, there are still many possibilities of using this transformation to establish an analogy by a judicious choice of physical quantities in the design. If

$$n_1 = n_2 \tag{143}$$

equation (127) reduces to

$$\frac{\partial^2 Q}{\partial t^2} + \frac{\partial^2 Q}{\partial z^2} + 2\alpha \frac{\partial Q}{\partial t} + 2\beta \frac{\partial Q}{\partial z} + Q\,(\alpha^2 + \beta^2) = 0 \tag{144}$$

which has only three coefficients to match, but only two parameters, α and β.

APPLICATIONS OF ANALOGIES USING NON-IDENTICAL EQUATIONS

Mathematical Examples

A general procedure for establishing analogies between phenomena in which the characteristic equations are not identical has been developed, and several sets of prediction equations have been presented, as well as a short review of the mathematical background of transformations. The following examples will indicate how this type of analogy can be used advantageously when applied to three familiar phenomena.

Electrical analogy for a column

The equation of an elastic column was previously given as

$$\frac{d^2y}{dx^2} + \frac{P}{E\,I}\,y = 0\,. \tag{2}$$

The boundary conditions sometimes are

$$x = 0, \quad y = 0 \tag{145}$$

$$x = \ell, \quad y = 0\,. \tag{146}$$

If the prediction equations

$$y = n\,e^{\alpha t}\,Q \tag{11}$$

$$x = n_1 t \tag{12}$$

and their derivatives

$$\frac{dy}{dx} = \frac{n}{n_1} e^{\alpha t} \left(\frac{dQ}{dt} + \alpha Q \right) \tag{13}$$

$$\frac{d^2y}{dx^2} = \frac{n}{{n_1}^2} e^{\alpha t} \left(\frac{d^2Q}{dt^2} + 2\alpha \frac{dQ}{dt} + \alpha^2 Q \right) \tag{14}$$

are substituted into equation (2), the resultant equation after simplification is

$$\frac{d^2Q}{dt^2} + 2\alpha \frac{dQ}{dt} + \left(\alpha^2 + \frac{{n_1}^2 P}{E I} \right) Q = 0 \quad . \tag{15}$$

The equation of the series electrical circuit shown in Figure 2 after division by L is

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{L C} Q = 0 \quad . \tag{16}$$

Matching equations (15) and (16) gives the design conditions

$$R/L = 2\alpha \tag{17}$$

$$\frac{1}{L C} = \alpha^2 + \frac{{n_1}^2 P}{E I} \quad . \tag{18}$$

When the prediction equations (11) and (12) are applied to the boundary conditions (145) and (146), they become

$$t = 0, \quad Q = 0 \tag{147}$$

$$t = \frac{\ell}{n_1}, \quad Q = 0 \ . \tag{148}$$

Another boundary condition inherent in this analogy is that the solution of equation (16) must be of an oscillatory form, i.e. damping in the model must not be over critical. The electrical circuit shown in Figure 2 can be built and used to determine the deflection of the column at any point by observing the charge Q in the electrical circuit at the proper time and then applying the prediction equations.

Therefore, an analogy has been established between a mechanical system (the column) whose equation contained only the second derivative and the variable y, and an electrical circuit whose equation contained the second derivative, the first derivative and the variable Q.

Using a set of direct or linear prediction equations such as equations (3) and (4) instead of the exponential prediction equations (11) and (12) was shown to give a circuit design with an inductance and a capacitance but no resistance. Since any inductance inherently has some resistance, it is necessary to introduce some resistance into the design of the analog, or to compensate for this inherent resistance by using "negative resistances". Therefore, the use of the exponential prediction equation introduces resistance into the design of the analog circuit of a column.

The validity of this analogy can be shown by solving equation (15) assuming an oscillatory form of solution to obtain

$$Q = e^{-\alpha t}\left(K_1 \sin \sqrt{\frac{P}{E\,I}}\, n_1\, t + K_2 \cos \sqrt{\frac{P}{E\,I}}\, n_1\, t\right) \qquad (149)$$

where K_1 and K_2 are constants of integration. The transformed boundary condition (147) will indicate that

$$K_2 = 0 \;. \qquad (150)$$

Substitution of (12), (149) and (150) into the prediction equation (11) will give

$$y = n\, K_1 \sin \sqrt{\frac{P}{E\,I}}\; x \qquad (151)$$

which is the equation of an elastic column where $n\, K_1$ represents the maximum displacement of the column. If boundary condition (148) is applied to equation (151), then

$$\sin \sqrt{\frac{P}{E\,I}}\, \ell = 0 \qquad (152)$$

and hence

$$\sqrt{\frac{P}{E\,I}}\, \ell = n_2\, \pi \qquad (153)$$

where n_2 is any integer. If n_2 is assumed to be 1, equation (153) becomes

$$P = \frac{\pi^2 E I}{\ell^2} \tag{154}$$

which is the well known Euler equation for a column.

Electrical analogy for a mechanical vibrating system

The exponential prediction equation used in the electrical analogy for a column may also be used to an advantage in establishing an electrical analogy of a mechanical vibrating system. This example will serve to show how the design conditions may be made more flexible, but at the expense of losing the steady state condition. It will also illustrate that the electrical analogy will predict an infinite amplitude of vibration at the natural frequency for a vibrating system without damping, even though the electrical circuit contains resistance.

The equation of motion of the simple vibrating system shown in Figure 3 is

$$\frac{W}{g}\frac{d^2y}{dt^2} + c\frac{dy}{dt} + K y = F \cos \omega t \quad . \tag{155}$$

The prediction equations

$$y = n e^{\alpha t_m} Q \tag{156}$$

$$t = n_1 \ t_m \qquad (157)$$

and their derivatives

$$\frac{dy}{dt} = \frac{n}{n_1} \ e^{\alpha t_m} \left(\frac{dQ}{dt_m} + \alpha Q \right) \qquad (158)$$

$$\frac{d^2y}{dt^2} = \frac{n}{{n_1}^2} \ e^{\alpha t_m} \left(\frac{d^2Q}{{dt_m}^2} + 2\alpha \frac{dQ}{dt_m} + \alpha^2 Q \right) \qquad (159)$$

can be substituted into equation (155) to give

$$\frac{d^2Q}{{dt_m}^2} + \frac{dQ}{dt_m} \left(2\alpha + \frac{g \ c \ n_1}{W} \right) + Q \left(\alpha + \frac{c \ n_1 \ \alpha \ g}{W} + \frac{K \ {n_1}^2 \ g}{W} \right)$$

$$= \frac{F \ {n_1}^2 \ g}{W \ n} \ e^{-\alpha t_m} \cos \omega \, n_1 \ t_m \qquad (160)$$

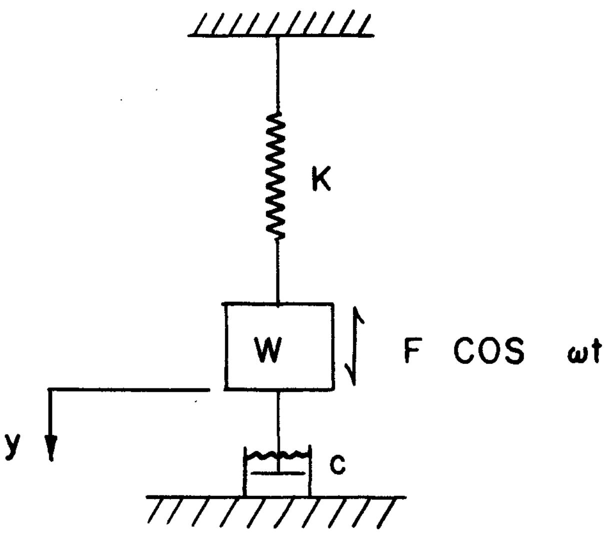
after simplification.

Applying Kirchoff's voltage law to the circuit shown in Figure 4 and dividing by L gives

$$\frac{d^2Q}{{dt_m}^2} + \frac{R}{L} \frac{dQ}{dt_m} + \frac{1}{L \ C} Q = \frac{\mathcal{E}}{L} \cos \beta \ t_m . \qquad (161)$$

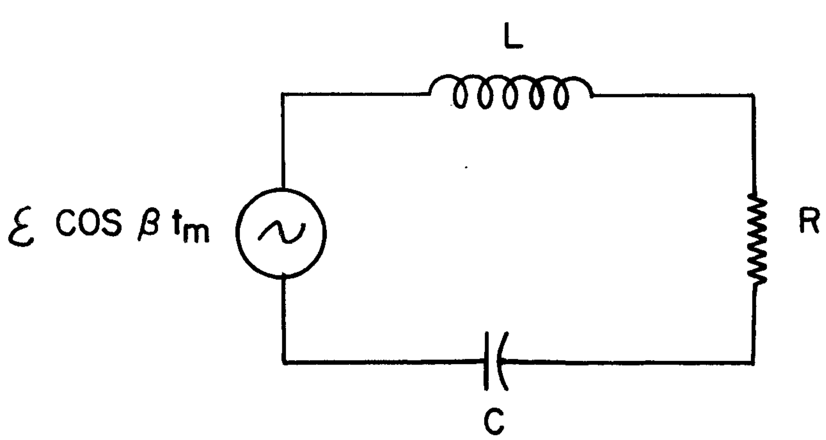
Equations (160) and (161) can be matched term by term to obtain the design conditions

$$R/L = 2\alpha + \frac{g \ c \ n_1}{W} \qquad (162)$$



SCHEMATIC DIAGRAM OF A MECHANICAL VIBRATING SYSTEM

FIG. 3



ELECTRICAL ANALOG CIRCUIT FOR THE MECHANICAL VIBRATING SYSTEM OF FIG.

FIG. 4

$$\frac{1}{L\ C} = \alpha^2 + \frac{c\ n_1 \alpha g}{W} + \frac{K\ n_1^2\ g}{W} \tag{163}$$

$$\mathcal{E}/L = \frac{F\ n_1^2\ g}{n\ W} e^{-\alpha t_m} \tag{164}$$

$$\beta = n_1 \omega\ . \tag{165}$$

Design equation (164) means that the amplitude of the cosine wave generated in the electrical circuit is decaying exponentially with model time, and therefore, the steady state condition in the electrical model is lost. Equation (162) shows that the electrical model may have a large resistance, even though the damping factor c in the mechanical system is quite small, if a proper value of α is chosen. The principal limitation on this value of resistance is that the motion of the charge in the circuit must remain oscillatory.

The analog circuit of Figure 4 can be built using a cosine wave generator with some provision for decreasing the amplitude exponentially, or by using an arbitrary function generator that can be adjusted to give any desired voltage wave shape. It is possible to start this exponentially decaying wave at properly timed intervals so that it appears as a steady state phenomenon on an oscilloscope.

Here again it is possible to verify the analogy by solving the transformed equation, inversely applying the prediction equations, and then comparing the results with the known solution of the original system.

The general solution of equation (160) assuming oscillatory motion of the charge is shown in the appendix to be

$$Q = e^{-\alpha t_m}\left\{\frac{\frac{F}{W}\frac{g}{n}\left[\left(\frac{K\,g}{W} - \omega^2\right)\cos\omega n_1\, t_m - \frac{g\,\omega\, c}{W}\sin\omega n_1\, t_m\right]}{\left(\frac{K\,g}{W} - \omega^2\right)^2 + \left(\frac{\omega\, g\, c}{W}\right)^2}\right.$$

$$+ e^{-\left(\alpha + \frac{g\,c\,n_1}{2W}\right)t_m}\left\{K_3 \cos\left[\frac{g}{W}\left(K - \frac{g\,c^2}{4W}\right)\right]^{\frac{1}{2}} n_1\, t_m\right.$$

$$\left.\left. + K_4 \sin\left[\frac{g}{W}\left(K - \frac{g\,c^2}{4W}\right)\right]^{\frac{1}{2}} n_1\, t_m\right\}\right. \qquad (166)$$

If the prediction equation

$$t = n_1\, t_m \qquad (157)$$

and equation (166) are substituted into the other prediction equation

$$y = n\, e^{\alpha t_m}\, Q \qquad (156)$$

it becomes after simplification

$$y = \frac{\frac{F\,g}{W}\left[\left(\frac{K\,g}{W} - \omega^2\right) \cos \omega t - \frac{g\,\omega\,c}{W} \sin \omega t\right]}{\left(\frac{K\,g}{W} - \omega^2\right)^2 + \left(\frac{\omega\,g\,c}{W}\right)^2}$$

$$+ e^{-\frac{g\,c}{2W} t}\left\{K_3' \cos\left[\frac{K\,g}{W} - \left(\frac{g\,c}{2W}\right)^2\right]^{\frac{1}{2}} t + K_4' \sin\left[\frac{K\,g}{W} - \left(\frac{g\,c}{2W}\right)^2\right]^{\frac{1}{2}} t\right\}$$

(167)

which agrees with the known results of elementary theory of vibrations as given by Myklestad.[15]

When there is no damping in the mechanical system and the shaking force has the same frequency as the natural frequency of the vibrating system, i.e.,

$$c = 0 \qquad (168)$$

$$\omega = \omega_n = \left(\frac{K\,g}{W}\right)^{\frac{1}{2}} \qquad (169)$$

the solution of equation (160) for the charge Q is shown in Appendix I to be

$$Q = e^{-\alpha t_m}\left[K_3 \cos n_1 \omega_n t_m + K_4 \sin n_1 \omega_n t_m + \frac{F\,\omega_n\,n_1}{2K\,n} t_m \sin n_1 \omega_n t_m\right]. \qquad (170a)$$

At first glance, it appears that Q becomes infinite with time because of the t_m in the last term. However, the limit of $t_m \, e^{-\alpha t_m}$ is zero as t_m approaches infinity, and the charge in the electrical circuit does not become infinite.

Substitution of (157) and (170a) into prediction equation (156) gives

$$y = n\,K_3 \cos \omega_n t + n\,K_4 \sin \omega_n t + \frac{F\,\omega_n t}{2K} \sin \omega_n t \qquad (170b)$$

which indicates that the amplitude of vibration does become infinite as time t approaches infinity. This result agrees with the theoretical results given by Myklestad.[15]

It is of interest to note that it is possible to predict an infinite amplitude of vibration in a mechanical system by using an electrical circuit which does contain resistance and in which the electrical charge does not become infinite. Hence, a condition of instability can be predicted using a stable electrical model.

Electrical analogy for a Bessel's equation system

An equation that occurs quite frequently in problems expressed in terms of cylindrical coordinates is Bessel's equation, usually written in the form

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{N^2}{x^2}\right) y = 0 \qquad (93)$$

where N is the order of the equation. Previously, the prediction equations

$$y = n\ Q\ t^{-1/2} \qquad (95)$$

$$x = n_1\ t \qquad (96)$$

were used, but the transformed equation did not have a first derivative term which means that the electrical model did not contain resistance. Since the use of an exponential prediction equation was effective in introducing resistance into the analog circuit for a column, it seems reasonable that this type of prediction equation should do the same for a system having Bessel's equation as its characteristic equation. Therefore, the prediction equations and their first and second derivatives are assumed to be

$$y = n\, t^{-\frac{1}{2}}\, e^{\alpha t}\ Q \qquad (171)$$

$$x = n_1\ t \qquad (172)$$

$$\frac{dy}{dx} = \frac{n}{n_1}\, t^{-\frac{1}{2}}\, e^{\alpha t}\left[\frac{dQ}{dt} + Q\left(\alpha - \frac{1}{2t}\right)\right] \qquad (173)$$

$$\frac{d^2y}{dx^2} = \frac{n}{{n_1}^2}\, t^{-\frac{1}{2}}\ e^{\alpha t}\left[\frac{d^2Q}{dt^2} + \left(2\alpha - \frac{1}{t}\right)\frac{dQ}{dt} + \left(\alpha^2 - \frac{\alpha}{t} + \frac{3}{4t^2}\right)Q\right] \qquad (174)$$

Substitution of (171), (172), (173) and (174) into Bessel's equation gives after simplification

$$\frac{d^2Q}{dt^2} + 2\alpha\frac{dQ}{dt} + Q\left(\alpha^2 - {n_1}^2 + \frac{1 - 4N^2}{t^2}\right) = 0 \qquad (175)$$

which does contain a first derivative term with a constant coefficient. This equation can be matched term by term with the equation of the electrical circuit shown in Figure 2 which is

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{L\,C}\ Q = 0 \qquad (16)$$

to give the design conditions

$$R/L = 2\alpha \qquad (176)$$

$$\frac{1}{L\,C} = \alpha^2 + n_1^{\,2} + \frac{1 - N^2}{4t^2} \quad . \tag{177}$$

Since design equation (176) shows that the resistance and inductance are constant, design equation (177) can be satisfied by using a variable condenser constructed so that the capacitance

$$C = \frac{1}{L\left(\alpha^2 + n_1^{\,2} + \dfrac{1 - 4N^2}{t^2}\right)} \tag{178}$$

when the rotor blades of the condenser rotate at a constant angular velocity. At time t equals zero, the capacitance C is zero. As time t approaches infinity, the capacitance C approaches a value

$$C = \frac{1}{L\left(\alpha^2 + n_1^{\,2}\right)} \tag{179}$$

asymptotically. At the end of one revolution of the condenser rotor, the initial conditions could be set into the analog circuit again and the cycle repeated so that the solution could be observed on an oscilloscope having a sweep frequency equal to, or some multiple of, the frequency of the rotating condenser.

If

$$N = \pm 1/2$$

equation (175) reduces to

$$\frac{d^2Q}{dt^2} + 2\alpha \frac{dQ}{dt} + (\alpha^2 + n_1^2)\, Q = 0 \qquad (180)$$

and design equation (177) reduces to

$$\frac{1}{L\,C} = \alpha^2 + n_1^2 \qquad (179)$$

i.e., the capacitance is constant.

Since Bessel's equations of the ± 1/2 order do have solutions of closed form, the validity of this transformation can be shown easily. The solution of equation (180) (see appendix for general method of solution) is

$$Q = e^{-\alpha t}\left(K_1 \cos n_1 t + K_2 \sin n_1 t\right) \qquad (181)$$

where K_1 and K_2 are arbitrary constants that must be evaluated from the boundary conditions. Substitution of equations (172) and (181) into prediction equation (171) gives

$$y = n\, n_1^{1/2}\, x^{-1/2}\left(K_1 \cos x + K_2 \sin x\right) \qquad (182)$$

This solution for Bessel's equation of the ± 1/2 order is entirely compatible with the known solutions indicated by

Reddick and Miller[22] and given in equations (183) and (184), because the boundary conditions which originally established the order of the equation will determine the proper value of the constants. For

$$N = + 1/2$$

$$y = J_{1/2}(x) = \left(\frac{2}{x\pi}\right)^{1/2} \sin x \tag{183}$$

For

$$N = - 1/2$$

$$y = J_{-1/2}(x) = \left(\frac{2}{x\pi}\right)^{1/2} \cos x \quad . \tag{184}$$

Experimental Example

Electrical analogy for a beam column

The electrical analogy for a column previously described can be modified to include lateral loads, i.e. establish an electrical analogy for a beam column.

The sketch of Figure 5 represents a beam column of length ℓ with axial loads P and b lateral loads F_1, F_2, ---F_b at distances from the left end of a_1, a_2, ---a_b, respectively. The loads are applied in an upward direction so that the displacement of the beam column can be shown in a positive direction. The end reactions can be calculated by summation of moments and shown to be

$$F_L = \sum_{i=1}^{i=b} F_i \left(1 - \frac{a_i}{\ell}\right) \tag{185}$$

$$F_R = \sum_{i=1}^{i=b} F_i \frac{a_i}{\ell} \tag{186}$$

The boundary conditions at the ends of the beam column sometimes are

$$x = 0, \quad y = 0 \tag{187}$$

$$x = \ell, \quad y = 0 \tag{188}$$

The equation of this beam column is

$$E\,I \frac{d^2y}{dx^2} = M \tag{189}$$

where E, I and M are modulus of elasticity of the beam column material, moment of inertia of the cross section, and bending moment in the beam column at any distance x from the origin, respectively. The equation for this bending moment, and hence the equation of the beam column, must be written in b + 1 parts, one for each portion of the beam column between the concentrated loads.

Figure 6 is a free body diagram of the left portion of a beam column in Figure 5 that has been cut between loads F_m and F_{m+1}. For the interval

$$a_m \leq x \leq a_{m+1} \tag{190}$$

summation of moments gives

$$M = - P\,y - \left[\sum_{i=1}^{i=b} F_i \left(1 - \frac{a_i}{\ell}\right)\right] x + \sum_{i=1}^{i=m} F_i \,(x - a_i). \qquad (191)$$

Substitution of (191) into equation (189) gives

$$E\,I\,\frac{d^2y}{dx^2} = - P\,y - \left[\sum_{i=1}^{i=b} F_i \left(1 - \frac{a_i}{\ell}\right)\right] x + \sum_{i=1}^{i=m} F_i \,(x - a_i) \qquad (192)$$

which can be simplified to

$$\frac{d^2y}{dx^2} + \frac{P}{E\,I}\,y = - \frac{1}{E\,I}\left[\sum_{i=1}^{i=b} F_i \left(1 - \frac{a_i}{\ell}\right)\right] x + \frac{1}{E\,I}\sum_{i=1}^{i=m} F_i \,(x - a_i). \qquad (193)$$

The prediction equations used to obtain the design equations for the electrical model are the same as those used in the electrical analogy for a column, namely

$$y = n\,e^{\alpha t}\,Q \qquad (11)$$

$$x = n_1\,t \qquad (12)$$

and their derivatives

$$\frac{dy}{dx} = \frac{n}{n_1}\,e^{\alpha t}\left(\frac{dQ}{dt} + \alpha Q\right) \qquad (13)$$

$$\frac{d^2y}{dx^2} = \frac{n}{{n_1}^2} \; e^{\alpha t}\left(\frac{d^2Q}{dt^2} + 2\alpha\frac{dQ}{dt} + \alpha^2 Q\right) \; . \qquad (14)$$

Substitution of (11) through (14) into equation (193) gives after simplification

$$\frac{d^2Q}{dt^2} + 2\alpha\frac{dQ}{dt} + \left(\alpha^2 + \frac{P\,{n_1}^2}{E\,I}\right)Q =$$

$$e^{-\alpha t}\left\{\left[-\sum_{i=1}^{i=b} F_i\left(1 - \frac{a_i}{\ell}\right) + \sum_{i=1}^{i=m} F_i\right]\frac{{n_1}^3}{n\,E\,I}\,t - \left[\sum_{i=1}^{i=m} F_i\,a_i\right]\frac{{n_1}^2}{n\,E\,I}\right\}$$

(194)

This is the general equation for the portion of a beam column between loads F_m and F_{m+1} when it has b lateral loads. It can be seen that the left side of this equation is the same for all portions of the beam column and is also the same as the transformed equation (15) for the column. As in the case of the column, this portion of the equation can be matched term by term with the equation of an electrical circuit containing capacitance, inductance and resistance in series, and the design equations will be the same as for the column, namely

$$R/L = 2\alpha \qquad (17)$$

$$\frac{1}{L\,C} = \alpha^2 + \frac{P\,n_1^2}{E\,I} \quad . \tag{18}$$

To complete this analog circuit, an arbitrary voltage generator must be added to the circuit. The complete analog circuit is shown in Figure 7. The characteristic equation of this circuit after division by L is

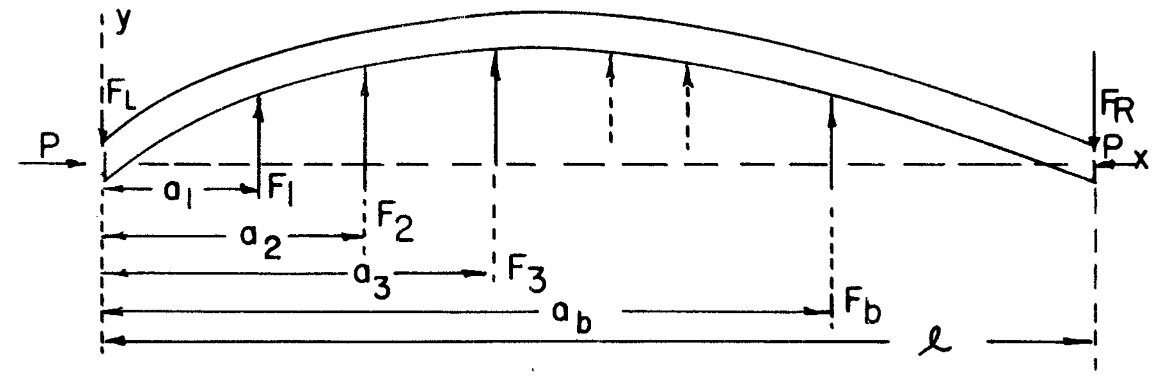
$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{L\,C}\,Q = \frac{\mathcal{E}(t)}{L} \tag{195}$$

where $\mathcal{E}(t)$ is the output of the arbitrary voltage function generator. The third design equation is obtained by matching the right hand sides of equations (194) and (195). The right hand side of equation (194) represents a straight line for each interval

$$a_m \leq x \leq a_{m+1}$$

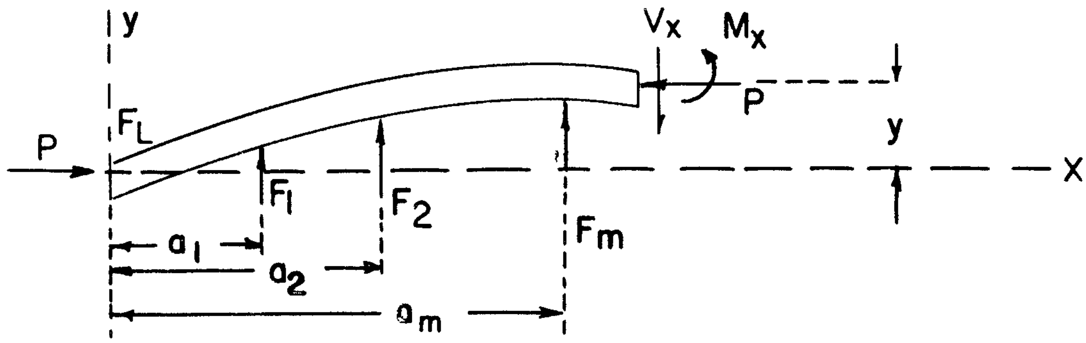
of the beam column, modified by an exponential decay factor $e^{-\alpha t}$. It can be easily seen by referring to equations (189), (191) and (193) that this series of straight lines is proportional to the moment diagram of the lateral loading on the beam column transformed by the linear prediction equation

$$x = n_1\,t \tag{12}$$



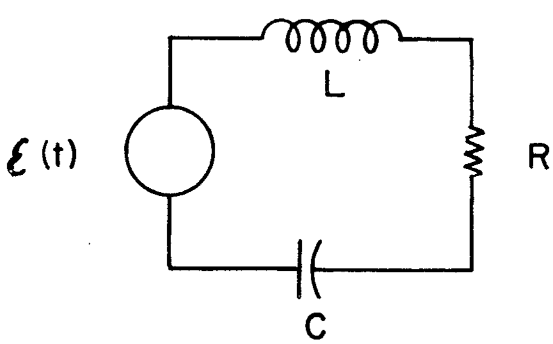
BEAM COLUMN WITH b LATERAL LOADS

FIG. 5



FREE BODY DIAGRAM OF A BEAM COLUMN CUT BETWEEN LOADS F_m AND F_{m+1}

FIG. 6



ANALOG CIRCUIT OF A BEAM COLUMN

FIG. 7

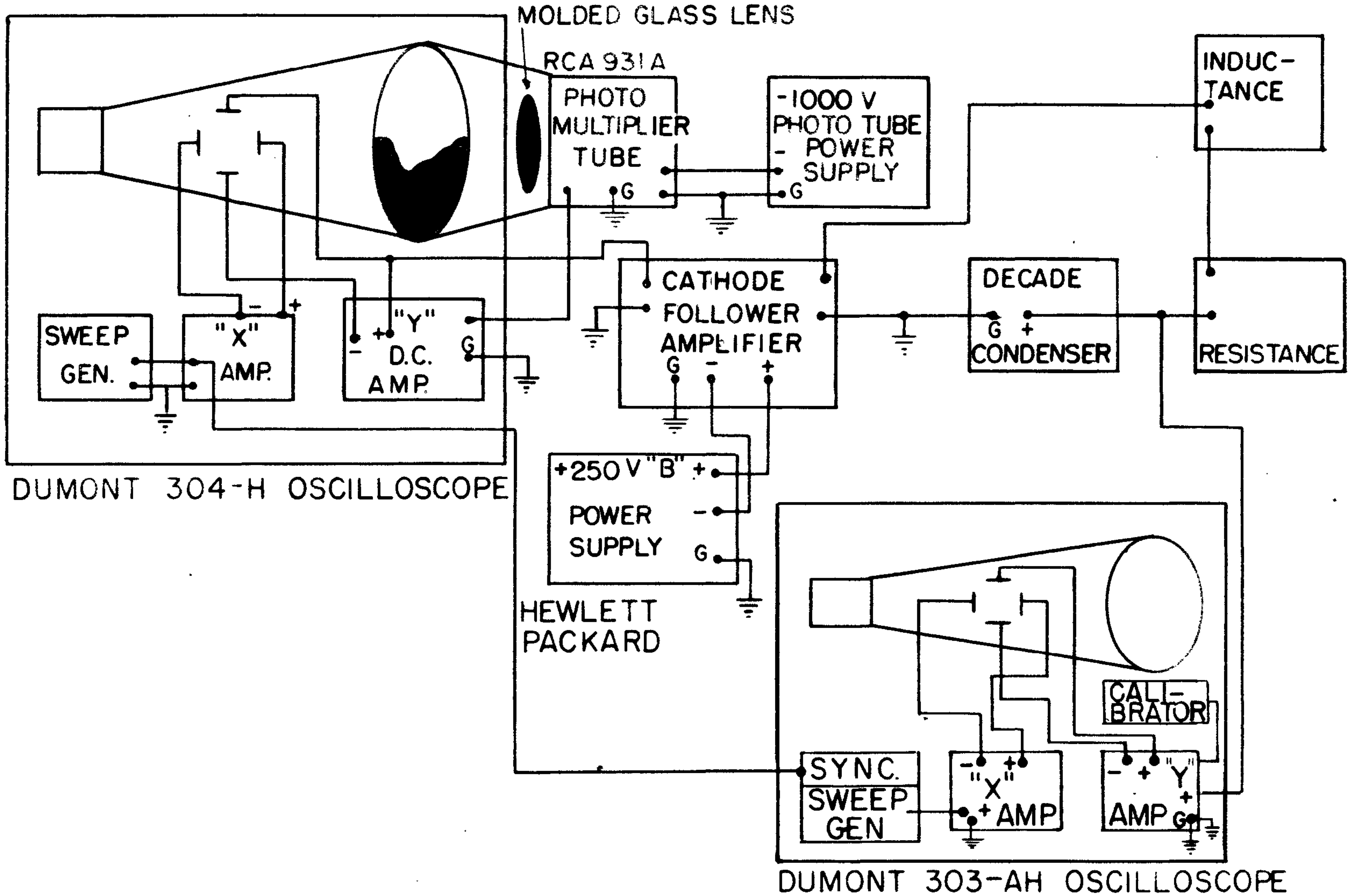
because this moment diagram is only a function of x. Therefore, the shape of the voltage wave output of the arbitrary voltage function generator used in the analog circuit must have the same shape (when viewed on an oscilloscope) as the moment diagram of the lateral loading modified by the exponential decay factor $e^{-\alpha t}$. Hence, the third design condition can be expressed as

$$\frac{\mathcal{E}(t)}{L} = \frac{n_1^2 \, e^{-\alpha t} \, M(n_1 t)}{n E I} \qquad (196)$$

where $M(n_1 t)$ represents the moment at any point x in the beam which corresponds to any time $(n_1 t)$ in the analog circuit.

Description of equipment

The analog circuit consisted of four units, a resistance, a capacitance, an inductance, and an arbitrary function generator. The resistance, capacitance and inductance were commercial units, but the arbitrary function generator was built. The type of arbitrary function generator chosen as the most practical from the standpoint of availability of equipment was a photoformer described by Sunstein.[24] Parts remaining from a photoformer built by Rector[21] were obtained from the Electrical Engineering Department and used to build the photoformer employed in this investigation. Figure 8



BLOCK DIAGRAM OF THE ANALOG SYSTEM
FIG. 8

is the block diagram of the entire analog system, and Figure 9 shows the equipment in use.

The photoformer is essentially an electronic servo-system employing feedback to make the electron beam of a cathode ray oscilloscope follow a desired path as the spot is moved horizontally across the face of the tube by the sweep generator of the oscilloscope.

A Dumont 304-H cathode ray oscilloscope was the basic unit of this photoformer. A mask of the desired shape was made of black electrical tape and attached directly to the face of the cathode ray tube. A RCA 931-A photomultiplier tube was mounted in front of the cathode ray tube using the oscilloscope camera mounting bracket. Figure 10 shows the photomultiplier tube circuit. A three-inch molded glass lens was placed between the cathode ray tube and the photomultiplier tube and was adjusted so that the light beam was focused on the light sensitive cathode. The output of the photomultiplier tube was connected to the direct coupled "Y" amplifier of the 304-H oscilloscope, and the output of the amplifier was connected to the vertical deflecting plates of the cathode ray tube. The feedback voltage applied to the vertical deflecting plates was of such polarity that increasing values of light intensity at the photomultiplier tube caused the spot to move downward. If the gain around the feedback loop is sufficient, the spot will be partially

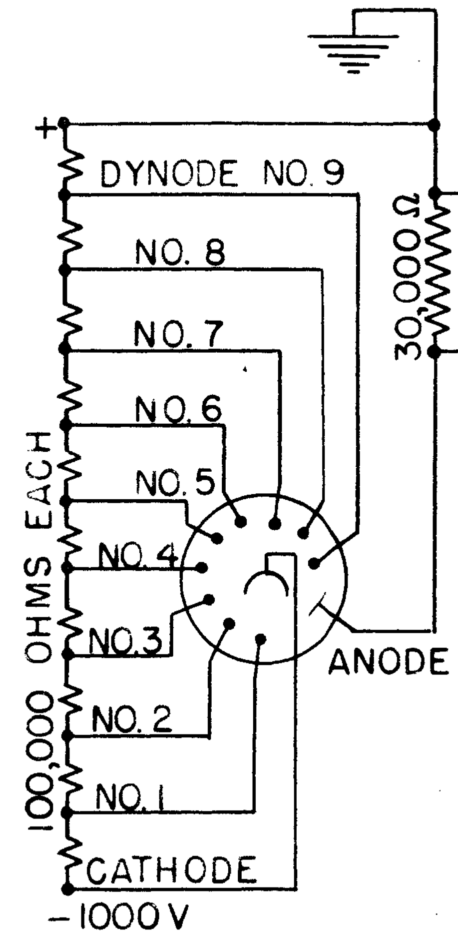
Equipment Used in This Investigation

Fig. 9

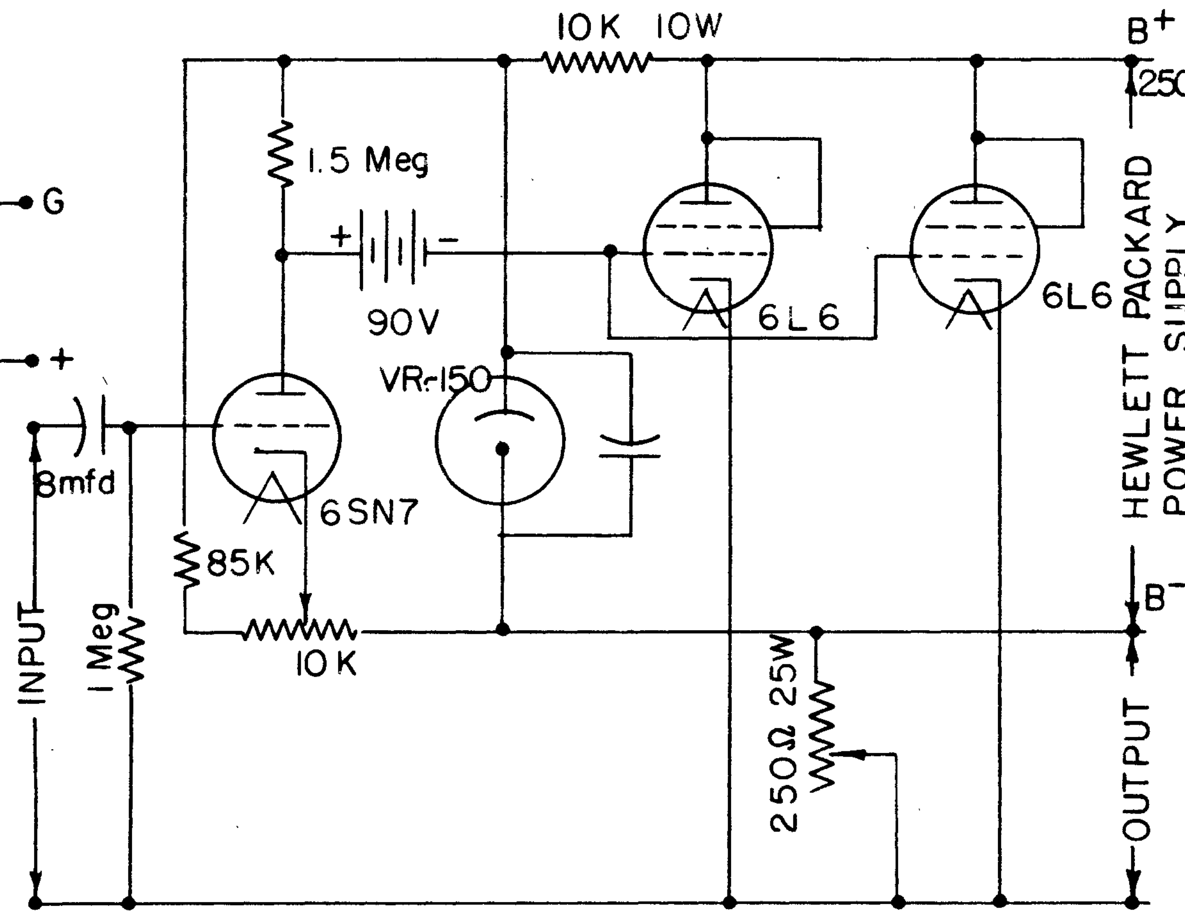
hidden behind the mask, and equilibrium will be maintained at this vertical position. Since the vertical deflection of the spot is proportional to the vertical deflecting voltage, the output is practically proportional to the height of the mask for a given horizontal position. The horizontal position is controlled by the linear sweep generator in the oscilloscope.

The time to establish equilibrium and the exact height of the spot depends upon the total delay time around the feedback loop. This delay time includes the decay time of the phosphor screen plus the delay time of the photomultiplier tube and the amplifier. Of these, the decay time of the phosphor is the most important factor. A cathode ray tube having a persistence of P-11 emitting a blue light was used. Tubes having screens with shorter persistences of P-5 or P-15 may be desirable if high frequencies are used. The RCA 931-A photomultiplier tube is more sensitive to blue or blue-white light, and hence the P-5 or P-11 screens are probably best. The P-1 screen which gives a green trace is unsatisfactory.

The output of this photoformer is a voltage wave of almost the same shape as the mask. The output impedance was very high and the power was quite low; hence, the photoformer could not be connected directly to the analog circuit. The direct coupled cathode follower amplifier shown in Figure 11 was used to match the impedances between the



PHOTOMULTIPLIER
TUBE CIRCUIT
FIG. 10



CIRCUIT DIAGRAM OF CATHODE FOLLOWER AMPLIFIER

FIG. 11

photoformer and the analog circuit, and it amplified the power at the same time. This amplifier was developed specifically for use between the photoformer and the analog circuit employed in this experiment. Two 6L6 vacuum tubes operating in parallel amplify the power. The 6SN7 tube is used as an amplifier in the negative feedback circuit which decreases the output impedance of the cathode follower amplifier to an extremely low value and reduces the distortion of the waveform for large input signals. The VR-150 voltage regulator tube supplies a constant voltage of 150 volts for the plate of the 6SN7 tube. A method of experimentally measuring the output impedance of the cathode follower amplifier is described in Appendix B. A Hewlett-Packard model 710-A power supply was used as a plate voltage source for the cathode follower amplifier. A Hewlett-Packard model 202-D audio frequency sine wave generator was used when adjusting the cathode bias of the cathode follower amplifier to give the largest undistorted output. With proper adjustment of both the cathode bias and the load resistor, it was possible to get an undistorted sine wave of 32 volts peak to peak and still not exceed the power output limitations of the Hewlett-Packard power supply.

The output of the audio frequency sine wave generator was connected to the "Y" terminals of the oscilloscope, and the sweep frequency of the oscilloscope was adjusted to give the desired frequency for the photoformer output.

A Dumont 303-AH oscilloscope used to measure the charge across the capacitance of the analog circuit provides a visual indication of the variation of charge with time. The external synchronizer terminals of this oscilloscope were connected to the sweep signal output terminal of the oscilloscope used in the photoformer in order to synchronize the two oscilloscopes. A Dumont type 295 oscilloscope camera was mounted on the oscilloscope to photograph the variation of voltage across the capacitance with time. The 303-AH oscilloscope had a built-in voltage calibrator which eliminated any voltage calibration difficulties.

The inductance consisted of two 63.7 millihenry coils that had been custom wound for the Electrical Engineering Department's Network Analyzer. The inductance of these coils was checked using an impedance bridge, and the resistance was calculated from the figure of merit ("Q") which was also measured on the impedance bridge.

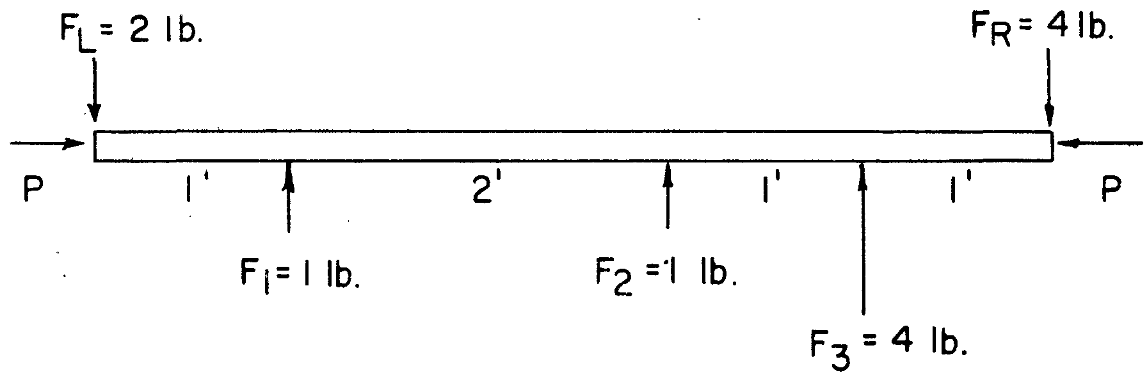
The capacitance across which the charge was measured consisted of a three gang, one microfarad decade capacitor used with five oil filled capacitors which could be connected in series or parallel to give the desired value of capacitance. These capacitances were also calibrated on an impedance bridge.

The resistance element included the effective resistances of the inductances, capacitances, and the output impedance of the cathode follower amplifier, as well as a 500 ohm variable resistor and other fixed resistances which could be used to obtain the desired total resistance.

<u>Design of the analog circuit</u>

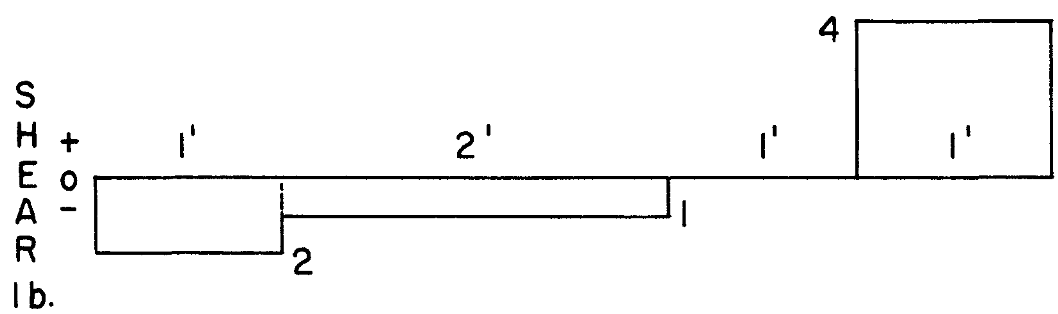
The beam column chosen for representation in the analog circuit was a flat 3/16 inch by 1 inch by 60 inch aluminum alloy bar loaded as shown in Figure 12. The modulus of elasticity (E) of the aluminum alloy bar was assumed to be 10.5×10^6 psi, and the moment of inertia (I) of the bar was calculated to be 7.32×10^{-4} in.^{4}, giving a product (E I) of 40.1 lb. ft^2. The slenderness ratio of the bar was 1109 and the Euler column buckling load was calculated to be 15.8 lb. The shear and moment diagrams of the lateral loads on the beam column are given in Figures 13 and 14.

The output of the photoformer has the shape of the moment diagram of the lateral loading multiplied by the exponential decay factor $e^{-\alpha t}$, and it is repeated in a cyclic manner as shown in Figure 15. This voltage wave form is fed to the analog circuit of Figure 7, and the voltage is measured across the capacitance to evaluate the electrical charge (Q) which is related to the displacement by prediction equation (11). Therefore, the voltage wave across the condenser should have the general shape shown in Figure 16. However, it is not possible for the voltage across a condenser to change instantaneously as indicated by the sharp angles at point A in Figure 16. The actual experimental wave appeared more



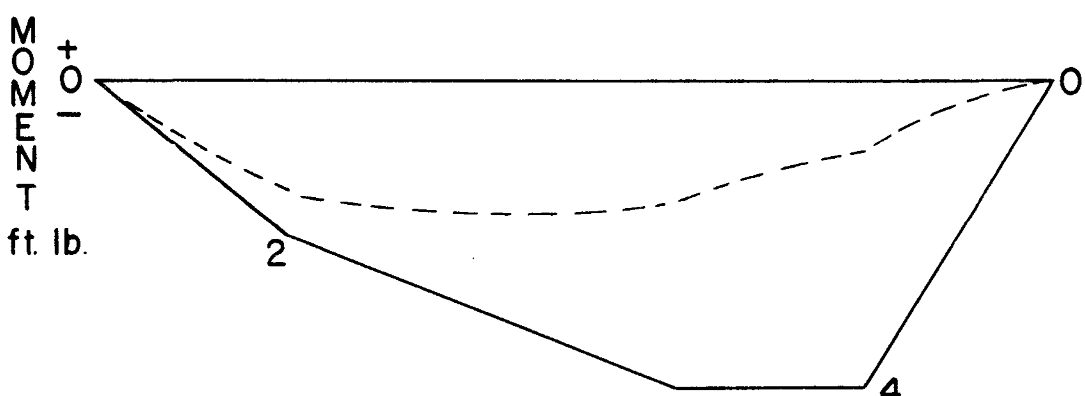
SKETCH OF LOADED BEAM COLUMN

FIG. 12



SHEAR DIAGRAM OF LATERAL LOADS FOR BEAM COLUMN SHOWN IN FIG. 12

FIG. 13



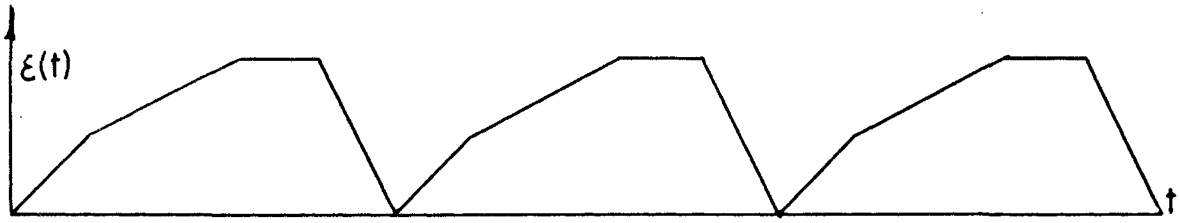
MOMENT DIAGRAM OF LATERAL LOADS FOR BEAM COLUMN SHOWN IN FIG. 12

FIG. 14

like the wave shown in Figure 17 with points of inflection at B and zero slopes at point A.

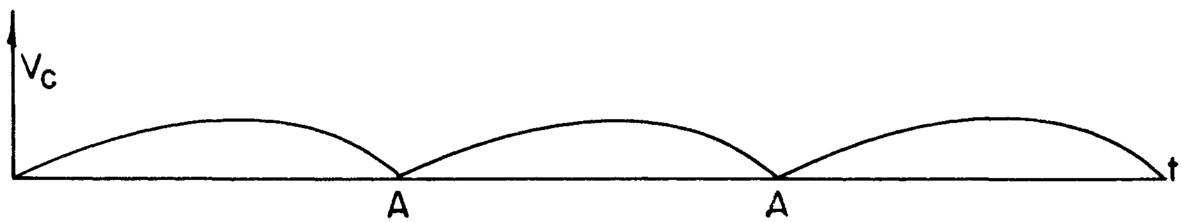
The method used to eliminate this source of error was to use a mask having two waveforms, with the second waveform inverted as shown in Figure 18. This reduced the sharp change of applied voltage at point A and the reintrant angles of the voltage waveform across the condenser as indicated in Figure 19. To further reduce the voltage change at points A, which represent the ends of the beam, the second waveform was reversed (right to left) giving a photoformer output wave as shown in Figure 20. This procedure establishes the boundary conditions at the time simultaneously representing the right end of the beam column for one cycle and the left end of the beam column for the next cycle.

Since the photoformer gives better reproduction of waveforms when the highest peaks are located at the ends of the mask, the mask was constructed from point D to point F in Figure 20 instead of from point A to point E. This also served to place one complete waveform (from B to E in Figure 20) near the center of the cathode ray tube where the reproduction by the photoformer is the best. Hence, it is possible to increase the accuracy by taking measurements on this wave and the corresponding voltage wave across the condenser. Comparison of the wave shape plotted accurately in Figure 21 and the actual output of the



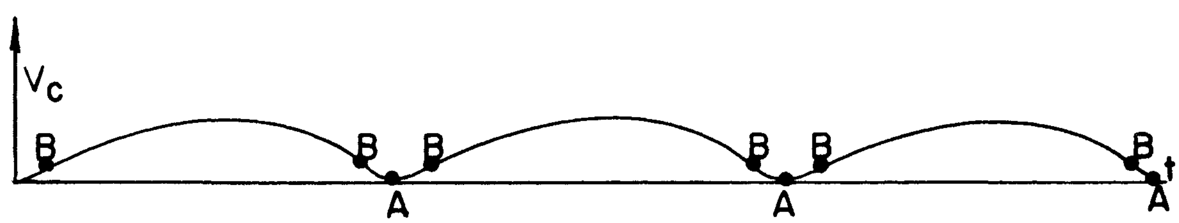
SKETCH OF OUTPUT VOLTAGE WAVE OF THE ARBITRARY FUNCTION GENERATOR

FIG. 15



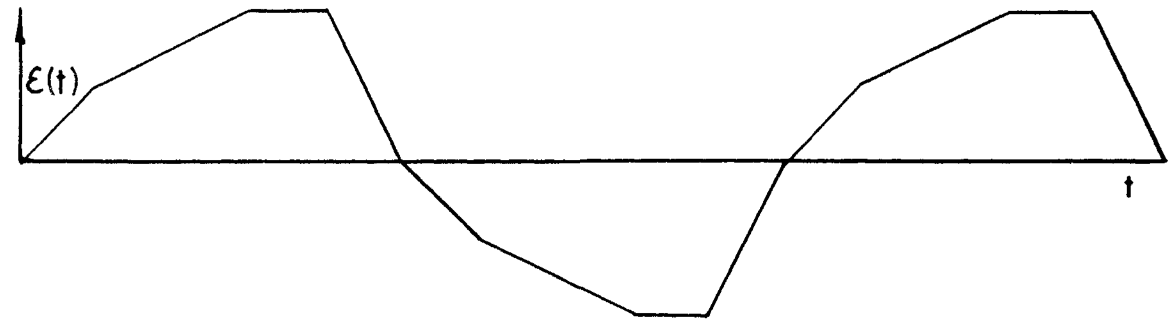
SKETCH OF THE THEORETICAL VOLTAGE ACROSS THE CONDENSER IN THE ANALOG CIRCUIT HAVING THE INPUT VOLTAGE WAVE OF FIG. 15

FIG. 16



SKETCH OF THE ACTUAL VOLTAGE ACROSS THE CONDENSER IN THE ANALOG CIRCUIT HAVING INPUT VOLTAGE WAVE OF FIG. 15

FIG. 17

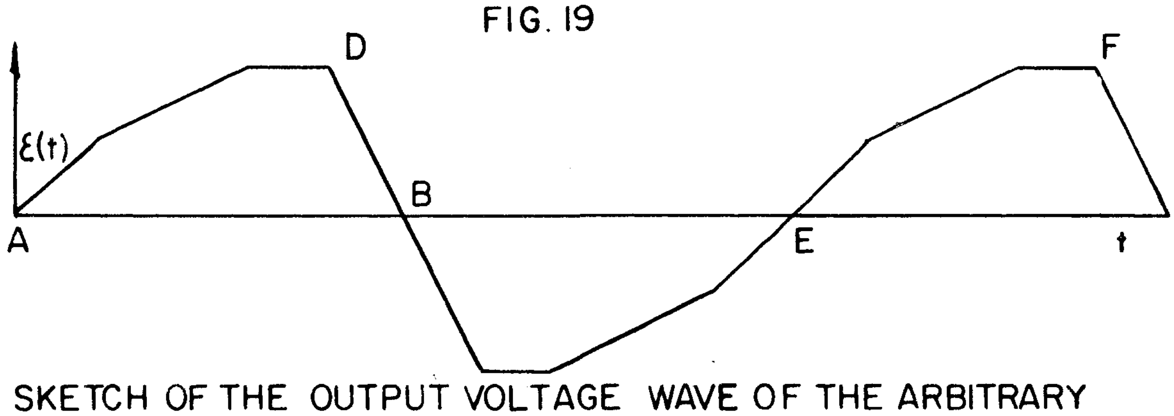


SKETCH OF OUTPUT VOLTAGE WAVE OF THE ARBITRARY FUNCTION GENERATOR WHEN EVERY OTHER WAVE IS INVERTED

FIG. 18

SKETCH OF THE THEORETICAL VOLTAGE ACROSS THE CONDENSER IN THE ANALOG CIRCUIT HAVING THE INPUT VOLTAGE WAVE OF FIG. 18

FIG. 19



SKETCH OF THE OUTPUT VOLTAGE WAVE OF THE ARBITRARY FUNCTION GENERATOR WHEN EVERY OTHER WAVE IS INVERTED AND REVERSED

FIG. 20

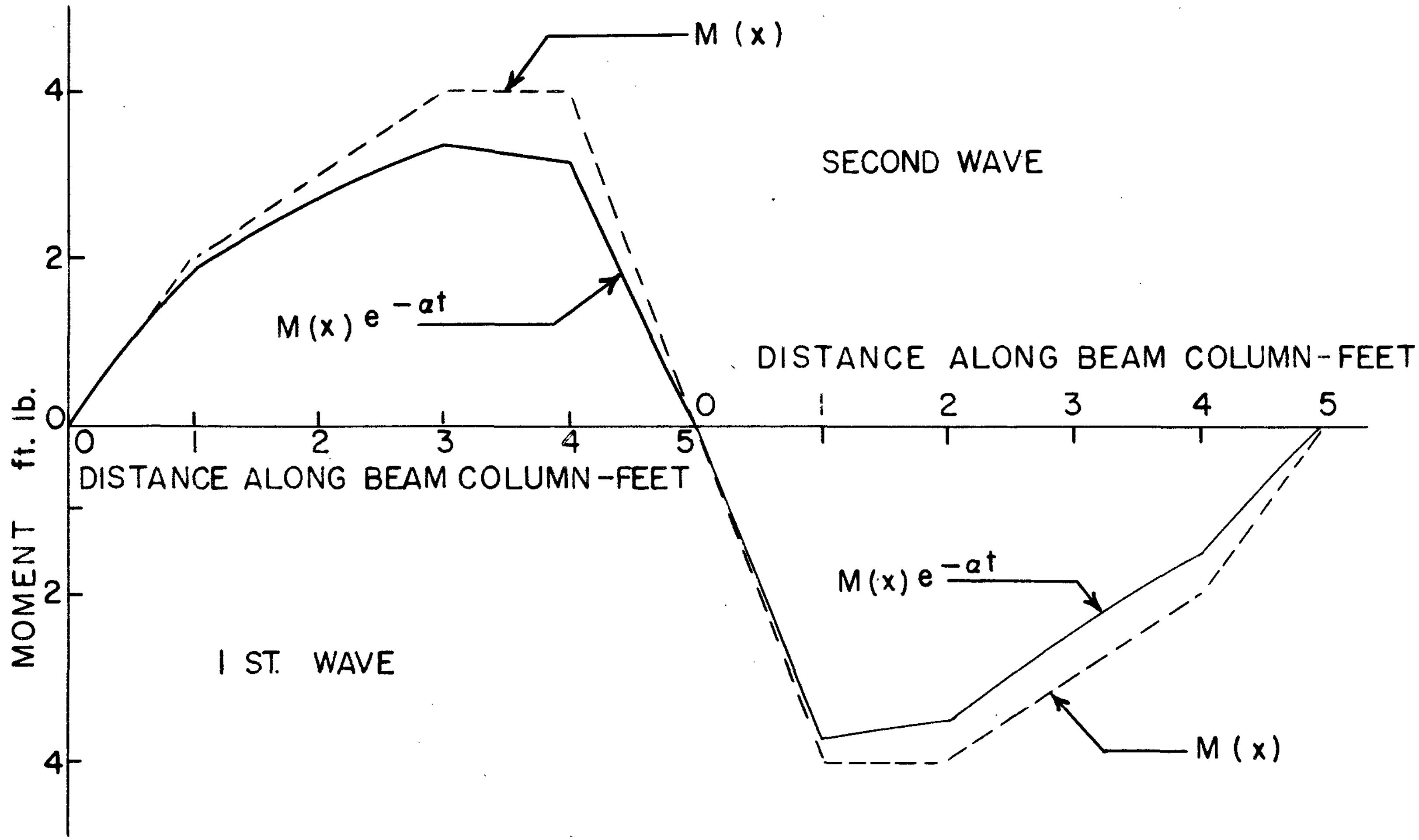


FIG. 21 PLOT OF MOMENT DIAGRAM AND OF ACTUAL WAVE FORM TO BE GENERATED BY PHOTOFORMER.

photoformer and cathode follower amplifier shown in Figure 22 gives a good indication of the accuracy of the wave reproduction.

There are three separate factors that affect the selection of frequency. It is necessary that the voltage drop across the capacitance (which is proportional to the charge Q) be sufficiently large that it can be observed on an oscilloscope without becoming lost in the stray pickup which often accompanies high gain. This voltage drop is dependent upon the relative size of the capacitive impedance compared to the other impedances in the analog circuit. Since the capacitive impedance is inversely proportional to the frequency

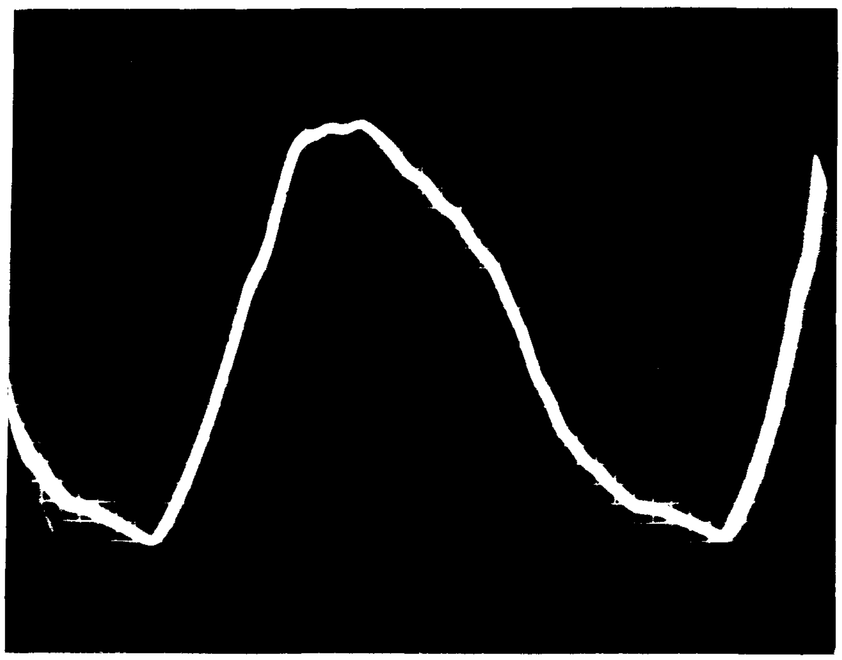
$$X_C = \frac{1}{2\pi f C} \qquad (197)$$

and the inductive impedance is directly proportional to the frequency

$$X_L = 2\pi f L, \qquad (198)$$

the choice of frequency must be compatible with the inductances and capacitances available.

It is also necessary that the exponential decay factor $e^{-\alpha t}$ not become small at $t = \tau$, i.e. the dotted curve of Figure 9 must not approach the horizontal axis at the right end. At $t = \tau$, it is desirable that $e^{-\alpha t}$ be no smaller than e^{-1}, i.e.



Photograph of Actual Output Waveform of the
Photoformer and Cathode Follower Amplifier

Fig. 22

$$\alpha \tau \leq 1 \quad . \tag{199}$$

At the right end of the beam column, the prediction equation

$$x = n_1 \, t \tag{12}$$

indicates that the period τ is related to the beam column length by

$$\ell = n_1 \tau \, . \tag{200}$$

The time necessary for the cathode ray tube trace to make one sweep across the face of the tube (which is the reciprocal of sweep frequency) is equal to two τ periods, because the mask on the face on the cathode ray tube represents two complete cycles. Hence, this relationship

$$\frac{1}{f} = 2 \, \tau \tag{201}$$

can be substituted into equation (200) to give

$$f = \frac{n_1}{2 \, \ell} \tag{202}$$

indicating that the frequency of the electrical analog circuit is dependent upon the length of the beam column and the scale n_1.

The third factor affecting the selection of frequency is the frequency response of the photoformer and cathode follower amplifier. The frequency response of the photoformer appeared to improve with frequency up to about

4000 cycles per second and then dropped quite rapidly. The frequency response of the cathode follower amplifier was good up to 10,000 cycles per second, but decreased to approximately one-half of its "flat range" value at 18,000 cycles per second. However, in amplifying voltage wave-forms having sharp corners, it is necessary to accurately amplify at least the third (and preferably the fourth) harmonic of the fundamental frequency. Therefore, a frequency of 3000 cycles per second was selected as the upper limit.

The design equations previously given for the analog circuit representing the beam column are

$$R/L = 2\alpha \qquad (17)$$

$$\frac{1}{L\,C} = \alpha^2 + \frac{P\,n_1^{\,2}}{E\,I} \qquad (18)$$

$$\frac{\mathcal{E}(t)}{L} = \frac{n_1^{\,2}\,e^{-\alpha t}\,M\,(n_1\,t)}{n\,E\,I}\,. \qquad (196)$$

A change in the lateral loading of the beam column causes a change in the moment diagram $M\,(n_1\,t)$, and hence requires the construction of a new mask for the photoformer. However, a change in end load P can be represented in the analog circuit by a change in capacitance as indicated by design equation (18).

The frequency used in the analog circuit was 3000 cycles per second. A value of 0.3 was chosen for $\alpha \tau$ giving a value of 1800/sec. for α, because it allowed sufficient resistance in the analog circuit to take care of inherent resistance of the inductances while introducing only a small damping factor in the electrical circuit. Two small wire wound inductors were connected in series to give a value of 0.1274 henry for inductance. The total resistance of these two inductors were measured to be 27.4 ohms. The total resistance of the analog circuit was calculated to be 459.0 ohms by using equation (17). The output impedance of the cathode follower amplifier was measured to be 12.0 ohms. The external resistance was calculated to be 419.6 ohms, and the variable resistor was adjusted to give this value. Equation (202) was used to calculate a value of 30,000 ft./sec. for n_1. The mask for the photoformer was cut to any convenient scale which gave a mask having a peak to peak dimension of approximately one and one-half inches, and a length of approximately two and one-half inches, which is the maximum size of mask that the photoformer can use effectively. The value of n was calculated by equation (196) using the peak value of voltage and the moment $M(n_1 t)$ corresponding to that peak voltage.

Figure 21 is a large scale plot of the wave that is fed to the analog circuit. It can be easily seen that the upper and lower peak values of voltage are not equal. This is due to the factor $e^{-\alpha t}$ decreasing as t varies between 0 and τ and the peak moment of the two waves plotted do not occur at corresponding points. This fact must be taken into account when locating the reference axis on the output wave of the cathode follower shown in Figure 22.

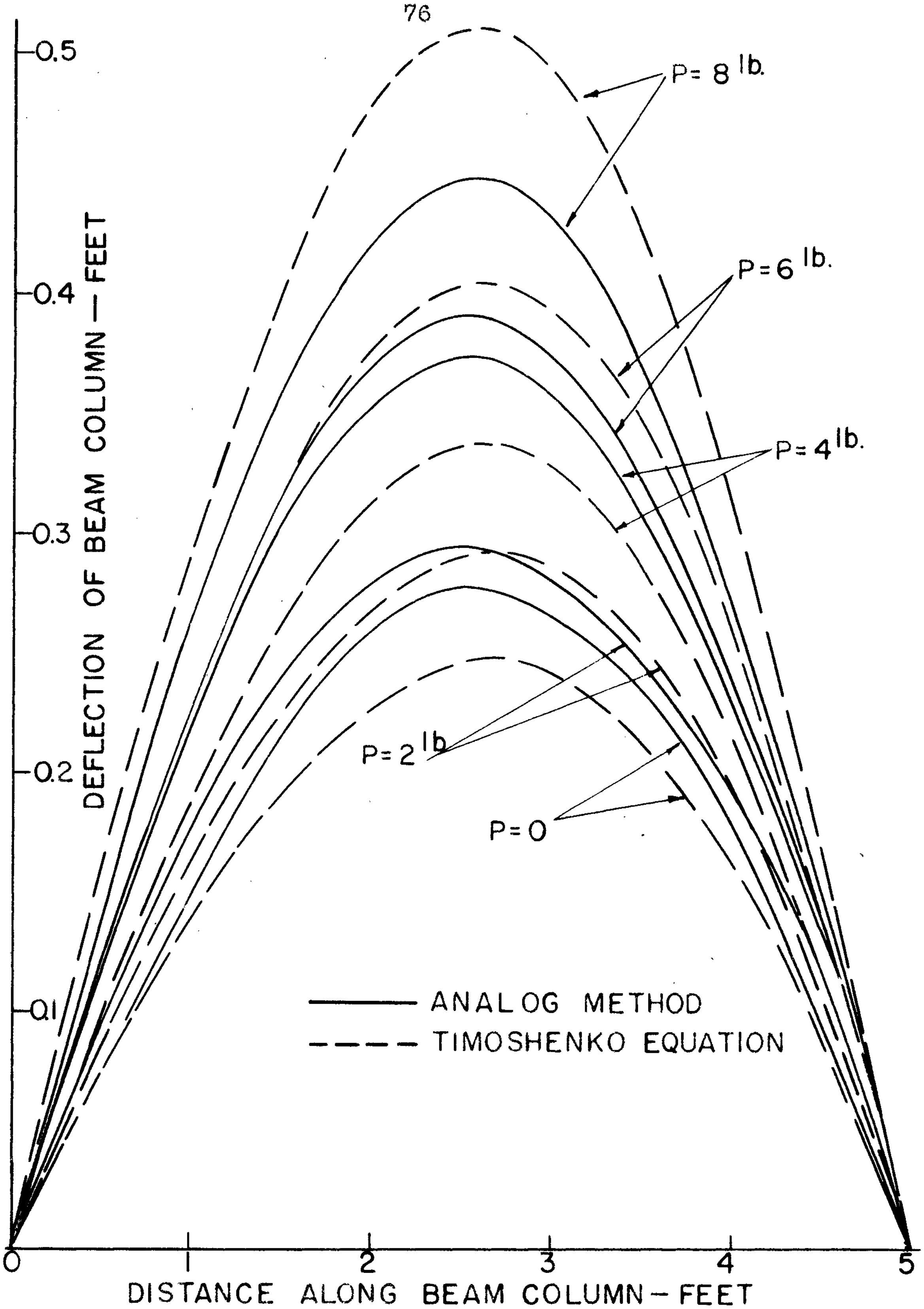
The axial load P was varied in two pound increments, up to 8 pounds, and equation (18) was used to give the values of capacitance in the following table.

Table I. Capacitance for Five Axial Loads

Axial Load P (lb.)	Capacitance (mfd.)
0	2.420
2	0.163
4	0.084
6	0.057
8	0.043

<u>Results</u>

<u>Data from analog circuit</u>. The results of this experiment are the deflection curves of Figure 23 for the beam column loaded as shown in Figure 5 for axial loads of 0, 2, 4, 6 and 8 pounds.



DEFLECTION -CURVES OF BEAM COLUMN

FIG. 23

Pictures were taken of the variation of voltage drop with time across the capacitance of the analog circuit as viewed on an oscilloscope. Since the charge Q in the analog circuit is related to the voltage drop across the capacitance by

$$Q = V_C \, C \tag{203}$$

the prediction equations

$$y = n \, e^{\alpha t} \, Q \tag{11}$$

$$x = n_1 \, t \tag{12}$$

could be used to obtain the deflection curves. The numerical values of deflection are tabulated for the five end loads in Table II.

<u>Calculations using theory of elastic stability.</u> The equation for the deflection of a beam column of Figure 12 between loads F_m and F_{m+1} is given by Timoshenko[25] to be

$$y = \frac{\sin kx}{P \, k \sin k} \sum_{i=m+1}^{i=b} F_i \sin k(\ell - a_i) + \frac{\sin k(\ell - x)}{P \, k \sin k} \sum_{i=1}^{i=m} F_i \sin k \, a_i - \frac{x}{P \ell} \sum_{i=m+1}^{i=b} F_i (\ell - a_i) - \frac{(\ell - x)}{P \ell} \sum_{i=1}^{i=m} F_i \, a_i \tag{204}$$

Table II. Deflection of Beam Column

x in.	Deflection of Beam Column in Feet — Timoshenko Equation P=0	P= 2 lb	P= 4 lb	P= 6 lb	P= 8 lb	Analog Method P=0	P= 2 lb	P= 4 lb	P= 6 lb	P= 8
0	0	0	0	0	0	0	0	0	0	0
2	0.024	0.028	0.034	0.040	0.051	0.021	0.034	0.041	0.037	0.038
4	0.048	0.058	0.066	0.080	0.102	0.047	0.067	0.082	0.075	0.082
6	0.071	0.083	0.098	0.119	0.153	0.069	0.095	0.115	0.129	0.133
8	0.094	0.110	0.130	0.157	0.201	0.097	0.124	0.156	0.169	0.181
10	0.116	0.135	0.161	0.193	0.248	0.120	0.152	0.193	0.195	0.227
12	0.137	0.157	0.187	0.227	0.290	0.148	0.174	0.221	0.226	0.264
14	0.156	0.184	0.215	0.260	0.330	0.172	0.199	0.253	0.252	0.303
16	0.169	0.206	0.240	0.288	0.359	0.191	0.221	0.278	0.291	0.330
18	0.188	0.221	0.261	0.315	0.394	0.216	0.236	0.302	0.313	0.364
20	0.205	0.238	0.279	0.338	0.431	0.232	0.254	0.324	0.337	0.381
22	0.217	0.253	0.299	0.358	0.456	0.248	0.268	0.339	0.355	0.402
24	0.228	0.267	0.312	0.378	0.476	0.261	0.278	0.354	0.370	0.422
26	0.234	0.274	0.324	0.390	0.495	0.268	0.286	0.365	0.379	0.434
28	0.242	0.283	0.331	0.398	0.504	0.275	0.292	0.372	0.388	0.444
30	0.246	0.288	0.335	0.403	0.511	0.278	0.296	0.374	0.392	0.450
32	0.248	0.292	0.338	0.406	0.512	0.276	0.292	0.369	0.385	0.448
34	0.246	0.290	0.337	0.400	0.506	0.274	0.288	0.364	0.384	0.443
36	0.243	0.287	0.330	0.394	0.495	0.266	0.281	0.359	0.376	0.435
38	0.236	0.280	0.319	0.384	0.478	0.259	0.276	0.346	0.363	0.417
40	0.226	0.270	0.303	0.368	0.455	0.247	0.264	0.329	0.345	0.396
42	0.215	0.255	0.287	0.343	0.428	0.234	0.245	0.311	0.326	0.370
44	0.199	0.238	0.269	0.323	0.396	0.220	0.230	0.289	0.299	0.351
46	0.183	0.215	0.242	0.294	0.361	0.201	0.212	0.267	0.280	0.316
48	0.163	0.190	0.214	0.262	0.317	0.177	0.192	0.239	0.254	0.275
50	0.141	0.164	0.181	0.216	0.273	0.157	0.173	0.211	0.213	0.242
52	0.114	0.132	0.149	0.181	0.229	0.126	0.148	0.178	0.180	0.202
54	0.086	0.100	0.110	0.136	0.175	0.110	0.114	0.143	0.152	0.151
56	0.055	0.066	0.077	0.080	0.107	0.066	0.077	0.095	0.105	0.097
58	0.023	0.031	0.041	0.043	0.060	0.033	0.043	0.051	0.062	0.047
60	0	0	0	0	0	0	0	0	0	0

where

$$k = \left(\frac{P}{E\ I}\right)^{1/2} . \qquad (205)$$

Equation (204) was used to calculate values for the deflection curves given in Figure 23 for loads of 2, 4, 6 and 8 pounds. For zero axial load, the deflection curve was calculated using the equation for the elastic curve of a beam as given by Murphy.[12] The numerical values of deflections are tabulated for the five axial loads in Table II.

Discussion of experimental results

The agreement between the experimental deflection curves for the beam column obtained from the analog circuit and the theoretical deflection curves obtained from the Timoshenko equation was only fair. The experimental deflections were smaller than the theoretical deflections for large values of axial load, P, and greater for small values of axial load. The error varies from about 3 per cent for an axial load of 2 pounds up to about 13 per cent for the 8 pound load. This general trend tends to indicate that the resistance in the analog circuit might be higher than the proper value. This would cause excess damping of the movement of the electrical charge in the circuit and would cause the analog circuit to indicate deflections that were too small.

The most obvious source of error in this experiment is the poor operation of the photoformer. It was found that the photomultiplier tube circuit loaded the power supply so heavily that the negative 1000 volt direct voltage was decreased to approximately 350 volts. This dropped the accelerating voltage of the photomultiplier tube to approximately 35 volts per dynode stage which is not sufficient for proper operation. The poor operation may also be partially due to the emissive material on dynodes of the photomultiplier tube, which gives the secondary emission of electrons, being destroyed or greatly deteriorated by repeated exposure to strong light sources while the high operating voltage was applied to the photomultiplier tube.

The output of the photoformer is taken from one side of the "Y" amplifier of the 304-H oscilloscope. This amplifier is a push-pull type amplifier having two output terminals, a "high" and a "low", which are connected to the vertical deflection plates of the cathode ray tube. It is not uncommon for one side of a push-pull amplifier to behave quite differently than the other side, even though the composite output of the amplifier is a good amplification of the input waveform. This source of error could be eliminated by using a difference amplifier between the output of the push-pull amplifier and the cathode follower amplifier. If a conventional difference amplifier loaded the output of the photoformer and distorted the waveform, a difference amplifier of the cathode follower type could be built. A cathode follower difference amplifier could be used in series with the present cathode follower amplifier, or if properly designed, it could replace the cathode follower presently used.

CONCLUSIONS

A new method of establishing analogies between phenomena having equations that are not identical has been presented, and several mathematical and one experimental examples have been presented to illustrate the use of this new method.

The results of the theoretical examples agree with the results indicated in the literature for the particular example.

The results of the experimental example agree reasonably well with the analytical solution of the problem. The error varies from 3 per cent to 13 per cent, but a major portion of this discrepancy can be attributed to the improper operation of the experimental equipment.

The method of establishing analogies presented in this dissertation can be used to establish many analogies which would not be otherwise possible. In several examples the functional prediction equation was used to introduce a first derivative into the transformed equation so that an electrical circuit containing resistance could be used as a model. In another example the functional prediction equation was used to change the coefficient of the first derivative term from a function of time to a constant so that a constant resistance could be used in the electrical circuit.

SUGGESTED TOPICS FOR FURTHER INVESTIGATION

The study of the behavior of beam columns by the analog method developed in this dissertation could be extended by investigating beam columns with fixed moments or deflections or both at the ends, or investigating cantilever beam columns. It appears that the only additional work necessary is the construction of new masks for the photoformer.

It is the author's belief that some method of predicting the condition of instability in the beam column can be introduced into the analogy. This would not necessarily mean the introduction of an unstable condition in the electrical model as such an unstable condition in the model would be highly undesirable.

The photoformer operation during this investigation was somewhat less than satisfactory. Modification of the photoformer should be undertaken before any additional investigations are begun which require this unit. The design and construction of a cathode follower type difference amplifier that could be used with the photoformer would improve the output waveshape of the arbitrary function. It is often highly desirable to change the shape of the mask while the photoformer is in operation. Several methods of doing this have been tried, but none has been entirely successful. An improved method of continuously varying the shape of the mask during operation would greatly increase the usefulness of the photoformer.

A mask having two waveforms with the second waveform inverted and reversed was used to eliminate the sharp changes in the voltage applied to the analog circuit at the times representing the ends of the beam column. A single waveform might be used if some electronic device could be used to discharge the condenser and set the initial conditions in the circuit at the end of each cycle. This would make each cycle completely independent of the previous cycle and might give more accurate results.

The construction of an analog circuit for a Bessel's equation system of any order N, as previously outlined in this dissertation would provide a valuable research tool because of its wide application. The rotary condenser suggested is limited to one particular application because of design equation (178), and a new rotary condenser would probably have to be built for each application. However, it might be possible to design a bank of capacitances in which an electronic or mechanical switching system varied the capacitance in finite increments according to the design equation (178). If each of the capacitances in the unit were adjustable, it would be possible to vary the capacitance of the unit in any desired manner.

The usefulness of the procedure for establishing analogies presented in this dissertation is dependent upon the change in form of a differential equation when it is transformed by a functional prediction equation. Therefore, a cataloging of the more important characteristic equations and the change in the form of these equations introduced by the use of many different prediction equations would be an excellent aid in establishing analogies.

Another type of transformation that might be useful in establishing analogies is the conformal mapping transformation. Because of its nature, the conformal mapping transformation is limited to two dimensions, but there are a large number of two dimensional problems, as well as many three dimensional problems that can be dealt with as two dimensional problems, for which this type of transformation might be useful.

SUMMARY

A method of establishing analogies between phenomena in which the characteristic equations are not identical has been presented. It differs from the "Indirect Procedure" given by Murphy[14] in that at least one of the prediction equations is a functional type transformation instead of a linear transformation. The usefulness of this modified procedure is dependent upon a change in the form of a characteristic equation when functional prediction equations are used.

Three familiar examples were investigated theoretically using the modified system of establishing analogies, and the results were shown to agree with those given in the literature.

(a) An analog circuit was designed to predict the deflection of a column. This use of a functional prediction equation made it possible to introduce resistance into the circuit even though the characteristic equation of a column does not contain a first derivative.

(b) The analog circuit for predicting the behavior of a simple vibrating system was investigated. The theoretical solution showed that it was possible to predict an infinite deflection for

an undamped system vibrating at resonance frequency even though the electrical system contained resistance and the electrical charge did not become infinite.

(c) A third analog circuit was designed for any phenomena having Bessel's equation as its characteristic equation. The theoretical solution was shown to agree with the known solution to Bessel's equations of the plus and minus one-half orders.

In addition to studying these three examples theoretically, an electrical analog circuit for a beam column was designed, built and used to predict deflections of the beam column for five different end loads. A photoformer was used in conjunction with a specially designed cathode follower amplifier having a low output impedance to supply an arbitrary voltage waveform to the analog circuit. The results of this investigation agree reasonably well with the analytical solution of the beam column problem using the theory of elastic stability. The error varied from 3 to 13 per cent, but a major portion of this discrepancy can be attributed to the improper operation of the experimental equipment.

The method of establishing analogies presented in this dissertation can be used to establish many analogies which would not be otherwise possible. In several examples, the functional prediction equation has been used to introduce a first derivative into the transformed equation so that an electrical circuit containing resistance could be used as a model. In another example, the functional prediction equation has been used to change the coefficient of the first derivative term from a function of time to a constant so that a constant resistance can be used in the electrical circuit. There are many other types of transformations that can be used to change the form of a characteristic equation so that the phenomena may be represented directly by some other phenomena rather than resort to the use of feedback amplifiers, computing circuits or negative resistances.

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APPENDICES

APPENDIX A

Solution of the Differential Equation of the Electrical Circuit Shown in Figure 2

The equation of the electrical circuit shown in Figure 2 is

$$\frac{d^2Q}{dt_m^{\ 2}} + K_1 \frac{dQ}{dt_m} + K_2\, Q \;=\; K_0\, e^{-\alpha t_m} \cos \omega\, n_1 t_m \qquad \text{(A-1)}$$

where K_1, K_2 and K_0 are defined as

$$K_1 \;=\; 2\,\alpha + \frac{g\ c\ n_1}{W} \qquad \text{(A-2)}$$

$$K_2 \;=\; \alpha^{\ 2} + \frac{c\ n_1\,\alpha\ g}{W} + \frac{K\ n_1^{\ 2}\ g}{W} \qquad \text{(A-3)}$$

$$K_0 \;=\; \frac{F\ n_1^{\ 2}\ g}{W\ n} \qquad \text{(A-4)}$$

The motion of the mechanical system is oscillatory and hence the solution of equation (A1) will be based upon an assumption of oscillatory motion. The complimentary solution of equation (A1) can be obtained by using the operator method. Equation (A1) expressed in operator form is

$$(D^2 + K_1\, D + K_2)Q = 0 \qquad \text{(A-5)}$$

The auxiliary equation is

$$m^2 + K_1 m + K_2 = 0 \tag{A-6}$$

$$m = -\frac{K_1}{2} \pm \sqrt{\left(\frac{K_1}{2}\right)^2 - K_2} \tag{A-7}$$

For an oscillatory form of solution

$$K_2 > \left(\frac{K_1}{2}\right)^2 \tag{A-8}$$

Hence

$$m = -\frac{K_1}{2} \pm i\sqrt{K_2 - \left(\frac{K_1}{2}\right)^2} \tag{A-9}$$

The solution of equation (A5) is

$$Q_C = e^{-\left(\frac{K_1}{2}\right)t_m} \left\{K_3 \cos\left[K_2 - \left(\frac{K_1}{2}\right)^2\right]^{\frac{1}{2}} t_m + K_4 \sin\left[K_2 - \left(\frac{K_1}{2}\right)^2\right]^{\frac{1}{2}} t_m\right\} \tag{A-10}$$

where K_3 and K_4 are arbitrary constants. Substitution of (A2) and (A3) for K_1 and K_2 gives

$$Q_C = e^{-\left(\alpha + \frac{g\, c\, n_1}{2W}\right)t_m} \left\{K_3 \cos\left[\frac{g\, n_1^2}{W}\left(K - \frac{g\, c^2}{4W}\right)\right]^{\frac{1}{2}} t_m + K_4 \sin\left[\frac{g\, n_1^2}{W}\left(K - \frac{g\, c^2}{4W}\right)\right]^{\frac{1}{2}} t_m\right\}. \tag{A-11}$$

The particular solution to equation (A1) can be obtained by the method of undetermined coefficients. If either the exponent or the frequency of the terms in equation (A12) are different than those in equation (A11), the form of the particular solution can be assumed to be

$$Q_p = e^{-\alpha t_m}\left[K_5 \sin \omega n_1 t_m + K_6 \cos \omega n_1 t_m\right] . \quad \text{(A-12)}$$

The derivatives of Q with respect to t_m are

$$\frac{d\,Q_p}{d\,t_m} = e^{-\alpha t_m}\left[\sin \omega n_1 t_m \left(-K_6 \omega n_1 - \alpha K_5\right)\right.$$

$$\left. + \cos \omega n_1 t_m \left(K_5 \omega n_1 - K_6 \alpha\right)\right] \quad \text{(A-13)}$$

$$\frac{d^2 Q}{d\,t_m^{\ 2}} = e^{-\alpha t}\left\{\sin \omega n_1 t_m \left[K_6 (2\alpha \omega n_1) + K_5(\alpha^2 - \omega^2 n_1^{\ 2})\right]\right.$$

$$\left. + \cos \omega n_1 t_m \left[K_6(\alpha^2 - \omega^2 n_1^{\ 2}) + K_5(-2\alpha\omega n_1)\right]\right\} .$$

$$\text{(A-14)}$$

Equations (A12), (A13) and (A14) can be substituted into equation (A1) to give after simplification

$$e^{-\alpha t_m}\Big\{\sin \omega n_1 t_m\Big[K_6(2\alpha\omega n_1 - K_1\omega n_1)$$

$$+ K_5(\alpha^2 - \omega^2 n_1^2 - K_1\alpha + K_2)\Big]$$

$$+ \cos\omega n_1 t_m\Big[K_6(\alpha^2 - \omega^2 n_1^2 - K_1\alpha + K_2)$$

$$+ K_5(-2\alpha\omega n_1 + K_1\omega n_1)\Big]\Big\}$$

$$= K_0 e^{-\alpha t_m}\cos\omega n_1 t_m \quad . \qquad \text{(A-15)}$$

If K_7 and K_8 are defined as

$$K_7 = (\alpha^2 - \omega^2 n_1^2 - K_1\alpha + K_2) \qquad \text{(A-16)}$$

$$K_8 = (2\alpha\omega n_1 - K_1\omega n_1) \qquad \text{(A-17)}$$

equation (A15) can be reduced to

$$\sin \omega n_1 t_m \left(K_6 K_8 + K_5 K_7\right) + \cos \omega n_1 t_m \left(K_6 K_7 + K_5 K_8\right) = K_0 \cos \omega n_1 t_m . \quad \text{(A-18)}$$

Equating coefficients of like terms gives

$$K_6 K_7 - K_5 K_8 = K_0 \quad \text{(A-19)}$$

$$K_6 K_8 + K_5 K_7 = 0 \quad \text{(A-20)}$$

Equations (A19) and (A20) can be solved simultaneously for K_5 and K_6

$$K_5 = \frac{- K_0 K_8}{K_7^2 + K_8^2} \quad \text{(A-21)}$$

$$K_6 = \frac{K_0 K_7}{K_7^2 + K_8^2} \quad \text{(A-22)}$$

Substitution of the defined values for K_0, K_7 and K_8 into equations (A21) and (A22) will give

$$K_5 = \frac{-\dfrac{F g^2 \omega c}{n W^2}}{\left(\dfrac{K g}{W} - \omega^2\right)^2 + \left(\dfrac{\omega g c}{W}\right)^2} \quad \text{(A-23)}$$

$$K_6 = \frac{\dfrac{F g}{W n}\left(\dfrac{K g}{W} - \omega^2\right)}{\left(\dfrac{K g}{W} - \omega^2\right)^2 + \left(\dfrac{\omega g c}{W}\right)^2} \quad \text{(A-24)}$$

Substitution of (A23) and (A24) into equation (A12) gives the particular solution

$$Q_p = e^{-\alpha t_m}\left[\frac{-\dfrac{F\,g^2\omega\,c}{n\,W^2}\sin\omega n_1\,t_m + \dfrac{F\,g}{W\,n}\left(\dfrac{K\,g}{W} - \omega^2\right)\cos\omega n_1\,t_m}{\left(\dfrac{K\,g}{W} - \omega^2\right)^2 + \left(\dfrac{\omega\,g\,c}{W}\right)^2}\right] \qquad \text{(A-25)}$$

The general solution is

$$Q = Q_C + Q_p$$

Therefore

$$Q = e^{-\left(\alpha + \frac{g\,c\,n_1}{2W}\right)t_m}\left\{K_3\cos\left[\frac{g\,n_1^2}{W}\left(K - \frac{g\,c^2}{4W}\right)\right]^{1/2} t_m + K_4\sin\left[\frac{g\,n_1^2}{W}\left(K - \frac{g\,c^2}{4W}\right)\right]^{1/2} t_m\right\}$$

$$+\ e^{-\alpha t_m}\left[\frac{-\dfrac{F\,g^2\omega\,c}{n\,W^2}\sin\omega n_1\,t_m + \dfrac{F\,g}{W\,n}\left(\dfrac{K\,g}{W} - \omega^2\right)\cos\omega n_1\,t_m}{\left(\dfrac{K\,g}{W} - \omega^2\right)^2 + \left(\dfrac{\omega\,g\,c}{W}\right)^2}\right] \qquad \text{(A-26)}$$

where K_3 and K_4 are arbitrary constants.

For the special case of zero damping and vibration at the natural frequency of the system where

$$c = 0 \qquad \text{(A-27)}$$

$$\omega = \omega_n = \left(\frac{K\ g}{W}\right)^{1/2}, \qquad \text{(A-28)}$$

the complementary solution given in equation (A11) becomes

$$Q_c = e^{-\alpha t_m}\left[K_3 \cos \omega_n n_1 t_m + K_4 \sin \omega_n n_1 t_m\right]. \qquad \text{(A-29)}$$

It can easily be seen that both the exponent and the frequency of the terms in equation (A29) are the same as those in equation (A12). Therefore, the form of the particular solution must be assumed to be

$$Q_p = e^{-\alpha t_m}\left[K_9 t_m \sin \omega n_1 t_m + K_{10} t_m \cos \omega n_1 t_m\right]. \qquad \text{(A-30)}$$

The derivatives of Q with respect to t_m are

$$\frac{d\ Q_p}{d\ t_m} = e^{-\alpha t_m}\Big[\sin \omega n_1 t_m (K_9) + \cos \omega n_1 t_m (K_{10})$$

$$+ t_m \sin \omega n_1 t_m (-K_9 \alpha - K_{10} \omega n_1) + t_m \cos \omega n_1 t_m (K_9 \omega n_1 - \alpha K_{10})\Big]$$

(A-31)

$$\frac{d^2 Q_p}{d\, t_m^2} = e^{-\alpha t_m}\Big\{\Big[\sin \omega n_1 t_m \;\; K_9(-2\alpha) + K_{10}(-2\omega n_1)\Big]$$

$$+ \cos \omega n_1 t_m \;\Big[K_9(2\omega n_1) + K_{10}(-2\alpha)\Big]$$

$$+ t_m \sin \omega n_1 t_m \Big[K_9(\alpha^2 - \omega^2 n_1^2) + K_{10}(2\alpha\omega n_1)\Big]$$

$$+ t_m \cos \omega n_1 t_m \Big[K_9(-2\alpha\omega n_1) + K_{10}(\alpha^2 - \omega^2 n_1^2)\Big]\Big\} .$$

(A-32)

Equations (A30), (A31) and (A32) can be substituted into equation (A1) to give after simplification

$$e^{-\alpha t} \Big\{\sin \omega n_1 t_m \Big[K_9(-2\alpha + K_1) + K_{10}(-2\omega n_1)\Big]$$

$$+ \cos \omega n_1 t_m \Big[K_9(2\omega n_1) + K_{10}(-2\alpha + K_1)\Big]$$

$$+ t_m \sin \omega n_1 t_m \;\; K_9(\alpha^2 - \omega^2 n_1^2 - K_1\alpha + K_2)$$

$$+ K_{10}(2\alpha\omega n_1 - K_1 \omega n_1)\Big]$$

$$+ t_m \cos \omega n_1 t_m \Big[K_9(-2\alpha\omega n_1 + K_1\omega n_1)$$

$$+ K_{10}(\alpha^2 - \omega^2 n_1^2 - K_1\alpha + K_2)\Big]\Big\}$$

$$= K_0 \;\; e^{-\alpha t} \;\; \cos \omega n_1 t_m \quad . \tag{A-33}$$

If K_{11} and K_{12} are defined as

$$K_{11} = +2\omega n_1 \tag{A-34}$$

$$K_{12} = (-2\alpha + K_1) \tag{A-35}$$

equation (A33) can be reduced to

$$\sin \omega n_1 t_m (K_9 K_{12} - K_{10} K_{11}) + \cos \omega n_1 t_m (K_9 K_{11} + K_{10} K_{12})$$

$$+ t_m \sin \omega n_1 t_m (K_9 K_7 + K_{10} K_8)$$

$$+ t_m \cos \omega n_1 t_m (-K_9 K_8 + K_{10} K_7) = K_0 \cos \omega n_1 t_m \,. \quad \text{(A-36)}$$

Equating coefficients of like terms gives

$$K_9 K_{12} - K_{10} K_{11} = 0 \quad \text{(A-37)}$$

$$K_9 K_{11} + K_{10} K_{12} = K_0 \quad \text{(A-38)}$$

$$K_9 K_7 + K_{10} K_8 = 0 \quad \text{(A-39)}$$

$$- K_9 K_8 + K_{10} K_7 = 0 \,. \quad \text{(A-40)}$$

However, substitution of conditions (A27) and (A28) into equations (A16), (A17), (A34) and (A35) gives

$$K_7 = 0 \quad \text{(A-41)}$$

$$K_8 = 0 \quad \text{(A-42)}$$

$$K_{11} = 2 \omega n_1 = 2 \omega_n n_1 \quad \text{(A-43)}$$

$$K_{12} = 0 \,. \quad \text{(A-44)}$$

Substitution of (A41) through (A44) into equations (A37) through (A40) gives

$$K_9 K_{11} = K_0 \quad \text{(A-45)}$$

$$K_{10} = 0 \,. \quad \text{(A-46)}$$

Substitution of (A4), (A28) and (A43) into equation (A45) gives

$$K_9 = \frac{F\ n_1\ g}{2W\ n\omega_n} = \frac{F\ \omega_n\ n_1}{2K\ n} \ . \qquad \text{(A-47)}$$

Substitution of (A46) and (A47) into equation (A30) gives the particular solution

$$Q_p = e^{-\alpha t_m} \ \frac{F\ \omega_n\ n_1}{2K\ n}\ t_m\ \sin \omega_n n_1\ t_m \ . \qquad \text{(A-48)}$$

Therefore, the general solution for the special case of zero damping and vibration at the natural frequency of the system is

$$Q = Q_c + Q_p$$

$$= e^{-\alpha t_m} \Big[K_3 \cos\ n_1\ \omega_n\ t_m + K_4 \sin\ n_1\ \omega_n\ t_m$$

$$+ \frac{F\ \omega_n\ n_1}{2K\ n}\ t_m\ \sin\ \omega_n\ n_1\ t_m \Big] \ . \qquad \text{(A-49)}$$

where K_3 and K_4 are arbitrary constants.

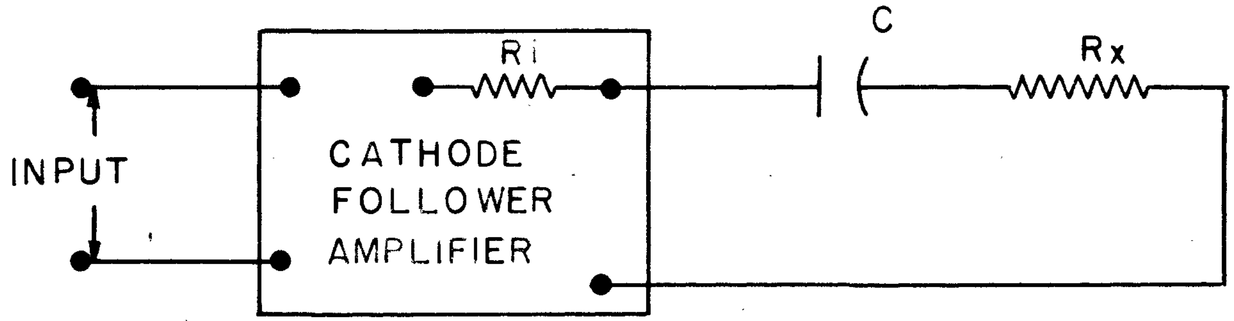
APPENDIX B

Measurement of the Output Impedance of the Cathode Follower Amplifier

The output impedance (pure resistance in this case) of the cathode follower amplifier was measured experimentally using the circuit indicated in Figure 24. The capacitor has capacitive reactance numerically less than one-tenth of the external resistance for the frequency used. For best results the external resistance R_x was chosen approximately equal to the output impedance (resistance) R_1 of the cathode follower amplifier. The resistance and capacitance were connected to the output of the cathode follower amplifier and the Hewlett-Packard audio frequency sine wave generator was connected to the input. The voltage across the external resistance (V_x) and the voltage across the output terminals of the cathode follower amplifier with no external load ($V_{no\ load}$) were measured using the 303-AH oscilloscope which has an internal voltage calibrator. The output impedance R_1 can then be calculated using the formula

$$\frac{V_x}{V_{no\ load}} = \frac{R_1}{R_1 + R_x} . \qquad \text{(A-50)}$$

For the cathode follower amplifier used in this investigation



Ri — OUTPUT IMPEDANCE

Rx — EXTERNAL RESISTANCE

C — CAPACITANCE ($X_C \leq 0.1 R_X$)

CIRCUIT USED TO MEASURE OUTPUT IMPEDANCE
OF THE CATHODE AMPLIFIER

FIG. 24

$$\frac{1.07}{2.00} \qquad \frac{R_1}{R_1 + 10.4} \quad .$$

Hence

$$R_1 = 12.0 \text{ ohms output impedance.}$$