

Chapter 2 Solutions

1. (a) The current account identity can be written as $B_{s+1} = (1+r)B_s + TB_s$. Now just plug in the assumed trade balance rule.

(b) Using the answer to part a, for any $\xi > 0$,

$$\begin{aligned} -\sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} \xi r B_s &= -\sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} \xi r [1 + (1-\xi)r]^{s-t} B_t \\ &= \frac{-\xi r B_t}{1 - \frac{1 + (1-\xi)r}{1+r}} = -(1+r)B_t. \end{aligned}$$

(c) Under the rule above, debt grows without bound if $\xi < 1$. But once the debt is as big as Y/r , the country can honor its foreign commitments only if debt stops growing and consumption is zero forever. Thus, the suggested rule must entail *negative* consumption levels at some point, which are inadmissible. To see directly why, consider the constant-output case, in which $TB_s = Y - C_s = -\xi r B_s$ so that the payback rule implies $C_s = Y + \xi r B_s$. Notice that since $B_s \rightarrow -\infty$, C_s must at some point become negative. The rule therefore is consistent with intertemporal solvency only if we counterfactually allow for negative consumption levels: the price of high consumption today would be infeasibly high trade surpluses later on. In general, suppose output grows at the gross rate $1 + g$, so that $Y_s = (1 + g)^{s-t} Y_t$. Unless $1 + g$

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is at least as great as the gross growth rate of debt, which was shown to be $1 + r(1 - \xi)$ in part a, the external debt-output ratio is unbounded. Thus the minimal payback fraction ξ consistent with intertemporal solvency and positive consumption is $\xi = 1 - (g/r)$ (which is positive if we assume that $g < r$).

2. (a) The expected utility $E_t U_t$ is a weighted average over different life spans, with weights equal to the survival probabilities:

$$\begin{aligned} E_t U_t = & (1 - \varphi) [u(C_t)] + \varphi(1 - \varphi) [u(C_t) + \beta u(C_{t+1})] + \\ & + \varphi^2(1 - \varphi) [u(C_t) + \beta u(C_{t+1}) + \beta^2 u(C_{t+2})] + \dots \end{aligned}$$

(b) The result follows simply by expanding the expression in part a and grouping terms together.

3. Recall that with isoelastic utility,

$$u(C) = \begin{cases} \frac{C^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, & \sigma > 0, \\ \log C, & \sigma = 1. \end{cases}$$

Using the intertemporal Euler equation, we thus obtain,

$$(1 + r)\beta = 1 = \frac{1}{E_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-1/\sigma} \right\}}. \quad (1)$$

Since consumption has a conditional lognormal distribution, the natural log of the gross consumption growth rate is conditionally normally distributed:

$$\log \frac{C_{t+1}}{C_t} \sim \mathcal{N} \left(E_t \left\{ \log \frac{C_{t+1}}{C_t} \right\}, \text{Var}_t \left\{ \log \frac{C_{t+1}}{C_t} \right\} \right).$$

Thus

$$\begin{aligned} E_t \left\{ \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \right\} &= E_t \left\{ \exp \left[-\frac{1}{\sigma} \log \left(\frac{C_{t+1}}{C_t} \right) \right] \right\} \\ &= \exp \left[-\frac{1}{\sigma} E_t \left\{ \log \frac{C_{t+1}}{C_t} \right\} + \frac{1}{2\sigma^2} \text{Var}_t \left\{ \log \frac{C_{t+1}}{C_t} \right\} \right]. \quad (2) \end{aligned}$$

[Consult footnote 41 on p. 313 of the book. Equation (2) follows from computing the mean and variance of the random variable $(-1/\sigma) \log(C_{t+1}/C_t)$, which is normally distributed when $\log(C_{t+1}/C_t)$ is.] Combining eqs. (1) and (2) above and taking natural logs of the result, we arrive at

$$E_t \left\{ \log \frac{C_{t+1}}{C_t} \right\} = \frac{1}{2\sigma} \text{Var}_t \left\{ \log \frac{C_{t+1}}{C_t} \right\}$$

or

$$\log C_{t+1} - \log C_t = \frac{1}{2\sigma} \text{Var}_t \{ \varepsilon_{t+1} \} + \varepsilon_{t+1},$$

where $\varepsilon_{t+1} \equiv \log C_{t+1} - E_t \{ \log C_{t+1} \}$. Since ε_{t+1} is a normal random variable that is uncorrelated with past information (because it is a pure forecast error), it is also statistically independent of that information on the assumption that the past information itself is generated by a jointly normal (i.e., Gaussian) stochastic process. In that case the conditional variance in the preceding equation actually is a time-invariant constant, so the natural log of consumption follows a random walk with a constant drift equal to $\frac{1}{2\sigma} \text{Var} \{ \varepsilon_{t+1} \}$.

4. (a) Using eq. (32) in Chapter 2, we can write

$$C_{t+1} - C_t = r(B_{t+1} - B_t) + \frac{r}{1+r} \left\{ \sum_{s=t+1}^{\infty} \left(\frac{1}{1+r} \right)^{s-(t+1)} E_{t+1} Y_s - \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} E_t Y_s \right\}.$$

The current account identity gives

$$B_{t+1} - B_t = Y_t + rB_t - C_t = Y_t - \frac{r}{1+r} \sum_{s=t}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} E_t Y_s,$$

which can be substituted into the previous equation for consumption to give the result that the change in consumption equals the present value of changes in expected future output levels.

(b) If the process for output follows

$$Y_{t+1} - Y_t = \rho(Y_t - Y_{t-1}) + \epsilon_{t+1},$$

then $(E_{t+1} - E_t)Y_{t+1} = \epsilon_{t+1}$, $(E_{t+1} - E_t)Y_{t+2} = (1 + \rho)\epsilon_{t+1}$, $(E_{t+1} - E_t)Y_{t+3} = (1 + \rho + \rho^2)\epsilon_{t+1}$, and so on. Therefore, for $s > t$,

$$(E_{t+1} - E_t)Y_s = \frac{1 - \rho^{s-t}}{1 - \rho}\epsilon_{t+1}.$$

(c) Substituting the last expression into the equation for the change in consumption derived in part a, we get the following

$$\begin{aligned} C_{t+1} - C_t &= \frac{r}{1+r}\epsilon_{t+1} \left[\left(\frac{1}{1-\rho} \right) + \left(\frac{1}{1-\rho} \right) \frac{1}{1+r} + \dots \right. \\ &\quad \left. - \frac{\rho}{1-\rho} - \frac{\rho^2}{(1+r)(1-\rho)} - \dots \right] \\ &= \frac{r}{1+r}\epsilon_{t+1} \left[\frac{(1+r)}{(1-\rho)r} - \frac{\rho}{1-\rho} \left(\frac{1+r}{1+r-\rho} \right) \right] \\ &= \frac{1+r}{1+r-\rho}\epsilon_{t+1}. \end{aligned} \tag{3}$$

As a result, provided that $0 < \rho < 1$, the desire to smooth consumption makes consumption innovations more variable than output innovations.

(d) The current account identity for date $t + 1$ is

$$CA_{t+1} = B_{t+2} - B_{t+1} = Y_{t+1} + rB_{t+1} - C_{t+1}.$$

Because $Y_{t+1} - E_t Y_{t+1} = \epsilon_{t+1}$ and, by eq. (3) from part c above,

$$C_{t+1} - E_t C_{t+1} = C_{t+1} - C_t = \frac{1+r}{1+r-\rho}\epsilon_{t+1},$$

the preceding current account identity gives a current account innovation of

$$\epsilon_{t+1} - \frac{1+r}{1+r-\rho}\epsilon_{t+1} = \frac{-\rho}{1+r-\rho}\epsilon_{t+1} < 0.$$

Thus, a positive output innovation leads to a current account deficit, as claimed at the end of section 2.3.3 in the book.

5. Work backward from the equation

$$CA_{t+1} - \Delta Z_{t+1} - (1+r)CA_t = \epsilon_{t+1},$$

where ϵ_{t+1} is uncorrelated with date t or earlier information. Taking expectations with respect to date t information yields

$$E_t CA_{t+1} - E_t \Delta Z_{t+1} - (1+r)CA_t = 0.$$

The previous equation can be rearranged to express CA_t as

$$CA_t = \frac{1}{1+r} E_t CA_{t+1} - \frac{1}{1+r} E_t \Delta Z_{t+1}.$$

Through forward recursive substitution (and using the law of iterated conditional expectations) we obtain

$$CA_t = - \sum_{s=t+1}^{\infty} \left(\frac{1}{1+r} \right)^{s-t} E_t \Delta Z_s$$

[because as $j \rightarrow \infty$, $\left(\frac{1}{1+r} \right)^j E_t CA_{t+j} \rightarrow 0$]. This is Campbell's (1987) "saving for a rainy day" equation, eq. (43) in Chapter 2. The equation can alternatively be derived using the lag and lead operator methodology described in supplement C to Chapter 2. Start again with

$$CA_{t+1} - \Delta Z_{t+1} - (1+r)CA_t = \epsilon_{t+1}$$

and take expectations with respect to date t information to get

$$E_t CA_{t+1} - E_t \Delta Z_{t+1} - (1+r)CA_t = 0.$$

Using the lead operator we write this as

$$L^{-1}CA_t - L^{-1}\Delta Z_t - (1+r)CA_t = 0,$$

or, dividing by $1 + r$ and rearranging, as

$$\left(1 - \frac{1}{1+r}L^{-1}\right)CA_t = -\frac{1}{1+r}L^{-1}\Delta Z_t.$$

Inversion of the lag polynomial on the left-hand side above gives

$$\begin{aligned} CA_t &= -\frac{1}{1+r} \left(1 - \frac{1}{1+r}L^{-1}\right)^{-1} E_t \Delta Z_{t+1} \\ &= -\frac{1}{1+r} \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} E_t \Delta Z_{s+1} \\ &= -\sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} E_t \Delta Z_s. \end{aligned}$$

To derive the converse, that the last equation implies $CA_{t+1} - \Delta Z_{t+1} - (1+r)CA_t = \epsilon_{t+1}$, one can simply reverse the steps above.

6. Write the expression for the current account as follows

$$\begin{aligned} CA_t &= Z_t - E_t \tilde{Z}_t = Z_t - \frac{r}{1+r} E_t \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} Z_s \\ &= Z_t - \frac{r}{1+r} \left(1 - \frac{1}{1+r}L^{-1}\right)^{-1} E_t Z_t, \end{aligned}$$

where the last equality is suggested in the hint. Then, multiplying both sides by $1 - \frac{1}{1+r}L^{-1}$, we have

$$\left(1 - \frac{1}{1+r}L^{-1}\right)CA_t = -\frac{1}{1+r} (E_t Z_{t+1} - Z_t) = -\frac{1}{1+r} L^{-1} E_t \Delta Z_t$$

Therefore,

$$\begin{aligned} CA_t &= -\frac{1}{1+r} L^{-1} E_t \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} \Delta Z_s \\ &= -\sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} E_t \Delta Z_s. \end{aligned}$$

7. Equation (75) in appendix 2A of the book shows that as time T passes and as $B/Y \rightarrow \infty$ along its unstable increasing trajectory,

$$\frac{B_{t+T+1}/Y_{t+T+1}}{B_{t+T}/Y_{t+T}} \rightarrow \frac{(1+r)^\sigma \beta^\sigma}{1+g}.$$

Because $Y_{s+1}/Y_s = 1+g$, however, it follows that $B_{t+T+1}/B_{t+T} \rightarrow (1+r)^\sigma \beta^\sigma$ as $T \rightarrow \infty$, that is, net foreign assets grow asymptotically at the growth rate of consumption. But the gross rate of consumption growth, $(1+r)^\sigma \beta^\sigma$, must be strictly below $1+r$ if, as we have assumed, an individual optimal plan exists (see the discussion on p. 117 of the book). Thus the asymptotic gross growth rate of foreign assets is below $1+r$ and the transversality condition

$$\lim_{T \rightarrow \infty} (1+r)^{-T} B_{t+T+1} = 0$$

therefore is satisfied.

8. If permanent output fluctuates more than current output, as is the case when output is a nonstationary random variable, then, as shown in exercise 4(d), a positive output innovation implies a decline in the current account: the intertemporal approach can therefore yield a countercyclical current account. Also, in the presence of investment which enters the current account with a negative sign, the current account can worsen following a positive output innovation if ρ is sufficiently large. Refer to pp. 86-87 in the book for details.

9. (a) Differentiating the firm's objective function with respect to I_s and K_s , we get, respectively,

$$q_s = 1 + \chi I_s$$

and

$$q_s = \frac{1}{1+r} [A_{s+1} F'(K_{s+1}) + q_{s+1}].$$

(b) Combining the preceding two first-order conditions, we get the following dynamic system for q and K :

$$K_{t+1} - K_t = \frac{q_t - 1}{\chi},$$

$$q_{t+1} - q_t = rq_t - A_{t+1}F' \left(K_t + \frac{q_t - 1}{\chi} \right).$$

(c) The phase diagram looks qualitatively the same as figure 2.9 in the book, which is based on eqs. (66) and (67) in Chapter 2. The dynamics of q and K and the slopes of the corresponding schedules are, however, quantitatively different. For example, the slope of the $\Delta q = 0$ schedule now is

$$\left. \frac{dq}{dK} \right|_{\Delta q=0} = \frac{AF_{KK} \left(K + \frac{q-1}{\chi} \right)}{r - \left(\frac{1}{\chi} \right) AF_{KK} \left(K + \frac{q-1}{\chi} \right)} < 0,$$

not just in a neighborhood of the steady state, but globally. (Compare with the slope given on p. 109 of the book.) As in the slightly more complex q model of Chapter 2, the steady state is given by $\bar{q} = 1$, $AF_K(\bar{K}, L) = r$. The steady state is independent of the adjustment costs, which determine only the speed of transition to the steady state.

(d) Figure 2.9 can be used for the exercise. An unanticipated permanent rise in A shifts the $\Delta q = 0$ schedule immediately and permanently to the right, raising the steady state capital stock to \bar{K}' . The unique convergent saddle-path **SS** also shifts to the right, becoming **S'S'**. Because the initial capital stock is given as $\bar{K} < \bar{K}'$, q rises in the short run (to place the economy on its new saddle-path) and investment surges. Over time, however, q falls back to 1 and investment decreases as $K \rightarrow \bar{K}'$.

(e) The principle for analyzing anticipated shocks is the same as that applied in figure 2.11. The firm learns on date t that productivity A will rise permanently to A' at a known future date T . Thus, the shifts described in

schedules described in part d occur only on date T . Nonetheless, the firm will adjust in anticipation so as to smooth its investment costs; and between dates t and T , q and K therefore will follow the original equations of motion (those involving A rather than A'). Thus, q jumps up initially and, until date T , continues rising as capital is accumulated. The firm reaches the new saddle-path $\mathbf{S'S'}$ precisely on date T , and thereafter q falls toward 1 and K rises to its new steady state. (An anticipated future *fall* in A would induce a path qualitatively similar to the path shown in figure 2.11.)

(f) Marginal q does not equal average q because the installation cost function assumed in this exercise is not linear homogeneous in K and I .