DIXMIER ALGEBRAS ON COMPLEX CLASSICAL NILPOTENT ORBITS AND THEIR REPRESENTATION THEORIES

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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August 2013

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Cornell University 2013

For a nilpotent orbit $\mathcal{O}$ in a complex classical Lie group G, R. Brylinski in [7] constructed a Dixmier Algebra model of its Zariski closure, based on an earlier construction by Kraft and Procesi. On the other hand, Barbasch in [6] constructed another model on $\mathcal{O}$ itself. Treating G as a real Lie group with maximal compact subgroup K, both models can be seen as admissible $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-modules of finite length. We are interested in finding out the composition factors of both models.

We first list out all the possible factors that can appear in both models, and compute which of them appear in the Barbasch model. When the Zariski closure of $\mathcal{O}$ is normal, we prove the composition factors of the Brylinski model are the same as the Barbasch model. Also, we give a conjecture on the composition factors in the Brylinski model, irrespective of the normality of the orbit closure.

BIOGRAPHICAL SKETCH

Kayue (Daniel) Wong was born in Hong Kong in October 26, 1984. His interest in Mathematics was discovered in his late secondary school studies, when he met his Mathematics teacher Mr. Siu-sik Cheung. In 2003, he successfully applied for the Lee Shau Kee Scholarship to read Mathematics in Wadham College, Oxford University from 2003 to 2007. In his last year of undergraduate studies, he wrote a dissertation on symplectic geometry, under the supervision of Dr. Andrew Dancer. In August 2008, he began his Ph.D. studies in Mathematics in Cornell University.

To my family.

ACKNOWLEDGEMENTS

I would like to thank Dan Barbasch for proposing the problem, and his patience and kindness of teaching the theory of unipotent representations, which is the basic tool of my thesis. His perspectives on nilpotent orbits have certainly shaped my view on the subject. He totally understands the way I think and always gives me the best advice at the right times. His encouragement on my research has also been an essential part for my continuous pursuit of academic research over the years.

Birgit Speh has opened my eyes on the daunting subject of Automorphic Forms and the Langlands Program. I always wish to have understood this beautiful subject earlier so that I can benefit more from her immense knowledge of the subject.

My gratitude goes to Ravi Ramakrishna for his lectures and seminars on modular forms and the Local Langlands Conjecture, which gives me a perspective on where my research fits in the bigger picture of Mathematics.

I have also benefitted a lot from attending classes of different disciplines in Mathematics over the last five years. This includes Tara Holm for her lectures on symplectic geometry, equivariant cohomology and toric varieties, Allen Knutson for his lectures on Schubert Varieties and sheaf cohomology, Mike Stillman for his lectures on schemes and intersection theory.

I would also like to thank Chi-kwong Fok and Shisen Luo for their constant inspirations in both mathematics and life. It has been my pleasure to study the lengthy notes on Algebraic Geometry by Ravi Vakil with them, going through almost all the exercises in the notes for more than a year. I certainly learn more mathematics from both of them than the other way round, and their knowledge and enthusiasm in mathematics has certainly kept me pursuing my studies.

Outside Mathematics, my life in Cornell and in America is greatly enriched by my friends who traveled, skiied, lived, cooked, dined, etc. with me over the last five years. Thank you Adam, Alan, David, Harry, John, Josiah, Marco, Melanie, Patrick, Rex, Ryan, Trista, Yao, Yasemin along with many others.

Finally, I am greatly indebted to my brothers, Ka-ming and Kar-fai and my mother Cheung-hing, who wholeheartedly support my decision of pursuing a Ph.D. degree by working twice as hard to keep my family a healthy, loving one.

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CHAPTER 1
INTRODUCTION

Let G be a complex Lie group with Lie algebra $\mathfrak{g}$. Then the adjoint action of G on $\mathfrak{g}$ makes $\mathfrak{g}$ into a union of adjoint orbits. The idea of the Orbit Method, originally proposed by Kirillov, says that every (co)adjoint orbit in $\mathfrak{g}$ (or its dual $\mathfrak{g}^*$) is related to an irreducible, unitary representation of G. This idea is realized perfectly when $\mathfrak{g}$ is a nilpotent Lie algebra, and some generalizations are needed if $\mathfrak{g}$ is a solvable Lie algebra. However, the situation becomes much more complicated in the case of semisimple Lie algebras. One of the many difficulties arising from the semisimple case is, not all adjoint orbits in $\mathfrak{g}$ are closed. It is therefore suggested by Vogan and McGovern that in the case of semisimple Lie algebras, one should study the **orbit datum** of $\mathfrak{g}$ which is a generalization of the adjoint orbits in $\mathfrak{g}$, and the **Dixmier algebras** which is related to the irreducible unitary representations. More precisely, if we treat G as a real Lie group with maximal compact subgroup K (i.e. the complexification of K is $K_{\mathbb{C}} \cong G$), a Dixmier algebra X is a filtered algebra endowed with a $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-action, where the $K_{\mathbb{C}}$-orbits, i.e. G-orbits on X spans a finite-dimensional vector space and respects the grading.

Conjecture 1.1 (Vogan)**.** *Let G be a connected complex simple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathcal{O}$ be an adjoint orbit in $\mathfrak{g}$ and $\tilde{\mathcal{O}} \to \mathcal{O}$ be a connected covering of $\mathcal{O}$ so that G acts compatibly. Then there is a completely prime Dixmier algebra $A_{\tilde{\mathcal{O}}}$ corresponding to $\tilde{\mathcal{O}}$, such that $A_{\tilde{\mathcal{O}}} \cong R[\tilde{\mathcal{O}}]$ (the ring of regular functions of $\tilde{\mathcal{O}}$) as representations of G.*

By Jordan Decomposition, it is known that every element in the semisimple Lie algebra can be split into a semisimple part and a nilpotent part. Also, it is known that all adjoint orbits for the semisimple part are closed, and their quantization is known. It is therefore of interest to study how one can attach unitary representations to nilpotent orbits.

We focus on classical simple complex Lie algebras. In this case, R. Brylinski in [7] constructed a Dixmier algebra corresponding to the closure of a nilpotent orbit $\overline{\mathcal{O}}$. Her construction of the model $X_{\overline{\mathcal{O}}}$ is based on an earlier construction of the ring of regular functions of $\overline{\mathcal{O}}$ given by Kraft and Procesi. By construction, $gr(X_{\overline{\mathcal{O}}}) = R[\overline{\mathcal{O}}]$ as G-modules. However, the construction is highly geometrical, and one is unable to extract much representation theoretic data out of her construction. For instance, there is no direct way to find out the decomposition of $X_{\overline{\mathcal{O}}}$ as finite-dimensional G-modules with multiplicities. On the other hand, Barbasch in [6] constructed a $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-module $X_{\mathcal{O}}$ such that $X_{\mathcal{O}} \cong R[\mathcal{O}]$ as $K_{\mathbb{C}} \cong G$-modules. The building blocks of his construction are **unipotent representations**, whose representation theory is well studied by Barbasch and Vogan. One would hope that the Barbasch model can give some representation theoretic insight into the Brylinski model.

The normality of $\overline{\mathcal{O}}$ plays an important role in studying the relations between the two models. In fact, the ring of regular functions of $\overline{\mathcal{O}}$ and $\mathcal{O}$ are the same if and only if $\overline{\mathcal{O}}$ is normal. Consequently, if $\overline{\mathcal{O}}$ is normal, then $X_{\mathcal{O}}$, $R[\mathcal{O}]$ and $X_{\overline{\mathcal{O}}}$ are isomorphic as G-modules, and $X_{\overline{\mathcal{O}}}$ becomes a candidate of $A_{\mathcal{O}}$ in the above Conjecture. In fact, more is true in this case. We will see in Chapter 10 that as $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-modules, $X_{\mathcal{O}}$ and $X_{\overline{\mathcal{O}}}$ share the same composition factors (with multiplicities). Even if $\overline{\mathcal{O}}$ is not normal, the inclusion relation $R[\overline{\mathcal{O}}] \hookrightarrow R[\mathcal{O}]$ gives an upper bound on the multiplicities of irreducible, finite-dimensional G-representations of $X_{\overline{\mathcal{O}}}$. This also imposes a strong constraint on the representation theory of $X_{\overline{\mathcal{O}}}$.

The thesis is organized as follows. Chapter 2 gives the basic information for nilpotent orbits in classical, simple, complex Lie algebras. This includes the classification on all nilpotent orbits, and the closure relationships between the orbits. Chapter 3 gives the basic relations between $R[\overline{\mathcal{O}}]$ and $R[\mathcal{O}]$ when $\overline{\mathcal{O}}$ is normal, and Theorem 3.3 gives a combinatorial criterion on the normality of $\overline{\mathcal{O}}$, proved by Kraft-Procesi.

Chapter 4 focuses on the construction of the Dixmier algebra $X_{\overline{\mathcal{O}}}$ given by Brylinski. Proposition 4.11 provides the infinitesimal character of the model, which is the starting point of studying the representation theoretic aspects of $X_{\overline{\mathcal{O}}}$.

Chapter 5 gives an introduction to the theory of unipotent representations. This is essential in the construction of the Barbasch model on the orbit. Also, the theory provides the lower bound of the **associated variety** of the composition factors of $X_{\overline{\mathcal{O}}}$ given its infinitesimal character. Given a fixed infinitesimal character, we study the number of unipotent representations, their associated varieties and their character formulas. The construction of $X_{\mathcal{O}}$ and some covers of $\mathcal{O}$ given by Barbasch is in Chapter 6.

Chapter 7 exhausts all the possible candidates of the composition factors of $X_{\overline{\mathcal{O}}}$ and $X_{\mathcal{O}}$. It provides character formulas for all candidates. Chapter 8 determines which of the candidates appear in $X_{\mathcal{O}}$.

Chapter 9 gives an algorithm computing the K-types (i.e. finite-dimensional irreducible G-representations) of the Barbasch model $X_{\mathcal{O}}$. More precisely, Theorem 9.2 and Theorem 9.5 give the algorithms computing the multiplicities of fundamental representations of $X_{\mathcal{O}}$. This in turn gives an upper bound on the K-type multiplicities of $R[\overline{\mathcal{O}}]$, and gives another criterion on the normality of $\overline{\mathcal{O}}$ by comparing the multiplicities of the fundamental representations of $R[\overline{\mathcal{O}}]$ and $R[\mathcal{O}]$.

Chapter 10 starts with a proof of the case when $\overline{\mathcal{O}}$ is normal, the composition factors of $X_{\overline{\mathcal{O}}}$ is the same as that of $X_{\mathcal{O}}$. The remaining part of the Chapter is devoted to a conjecture on the possible composition factors of $X_{\overline{\mathcal{O}}}$ for any orbit $\mathcal{O}$, and a possible character formula for the model.

Chapter 11 discusses the role of reductive dual pairs in our construction. By a Theorem of Adams and Barbasch, for some mild conditions on n and $2m$, the composition factors of $X_{\mathcal{P}}$ in $O(n, \mathbb{C})$ correspond to that of $X_{\mathcal{O}}$ in $Sp(2m, \mathbb{C})$ via the dual pair correspondence. On the other hand, the study of harmonics in the dual pair correspondence gives another upper bound on K-type multiplicities of $X_{\overline{\mathcal{O}}}$. We compare this upper bound with the one given in Chapter 10, and draft some possible directions for future research.

CHAPTER 2

NILPOTENT ORBITS IN CLASSICAL LIE ALGEBRA

This Chapter gives some basic notions and theorems on nilpotent orbits. More details can be found in [16] and [8].

Definition 2.1. *Let V be a complex vector space. An element $\phi \in End(V)$ is* ***semisimple*** *if every ϕ-invariant subspace has a ϕ-invariant complement. An element $\phi \in End(V)$ is called* ***nilpotent*** *if $\phi^r = 0$ for some finite $r > 0$.*

Let $\mathfrak{g}$ be a complex Lie algebra. For every $X \in \mathfrak{g}$, the adjoint representation $ad : \mathfrak{g} \to End(\mathfrak{g})$ gives a Lie algebra homomorphism. Hence we have the following definition:

Definition 2.2. *$X \in \mathfrak{g}$ is* ***semisimple*** *if $ad(X)$ is semisimple in $End(\mathfrak{g})$. And $X \in \mathfrak{g}$ is* ***nilpotent*** *if $ad(X)$ is nilpotent in $End(\mathfrak{g})$.*

2.1 Jordan Decomposition

Theorem 2.3 (Jordan Decomposition)**.** *Any $\phi \in End(V)$ can be decomposed as $\phi = \phi_s + \phi_n$, where ϕ_s is semisimple, ϕ_n is nilpotent. Both ϕ_s and ϕ_n are polynomials of ϕ.*

Theorem 2.4. *Suppose $\mathfrak{g} \subset End(V)$ is semisimple, then the Jordan decomposition of $X \in \mathfrak{g}$ is the same as the Jordan decomposition of $X \in End(V)$. More generally, for any finite dimensional representation of $\mathfrak{g}$, $\rho : \mathfrak{g} \to End(W)$, if $X = X_s + X_n$ is the Jordan* *decomposition of $X \in \mathfrak{g}$, then $\rho(X) = \rho(X_s) + \rho(X_n)$ is the Jordan decomposition of $\rho(X) \in End(W)$*

In particular, for any matrix Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$, for example all complex simple Lie algebras, we can decompose $\mathfrak{g}$ into a sum of a semisimple element and a nilpotent element. And the decomposition is the same as that in $\mathfrak{gl}(n, \mathbb{C})$.
Now we start looking at the conjugates of nilpotent elements:

Example 2.5. *Consider the Lie algebra $\mathfrak{sl}(2, \mathbb{C}) = \{M \in M_{2\times 2}(\mathbb{C})|\ tr(M) = 0\}$. We know that for any $M \in \mathfrak{sl}(2, \mathbb{C})$, there exists $Q \in SL(2, \mathbb{C})$ such that QMQ^{-1} is the **Jordan normal form**. If M is semisimple, it can be diagonalized, hence it is of the form*

$$QMQ^{-1} = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}$$

If M is nilpotent, the only eigenvalue of M must be zero, hence

$$QMQ^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ or \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

*Hence both the semisimple and nilpotent orbits in $\mathfrak{sl}(2, \mathbb{C})$ are completely classified. In particular, **There are infinitely many semisimple orbits, and 2 nilpotent orbits**.*
More generally, we study the nilpotent orbits $\mathfrak{sl}(n, \mathbb{C})$. The Jordan normal form tells us that the non-conjugate representatives of the nilpotent elements are of the following:

$$X = \begin{pmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & J_k \end{pmatrix} \; with \; J_i = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

where each Jordan block J_i *is a* $r_i \times r_i$ *matrix. So the set of nilpotent orbits in* $SL(n, \mathbb{C})$ *can be parameterized by the parititions of* n*, i.e.* $\{[r_1, r_2, \ldots, r_i] |\; r_1 \geq r_2 \geq \cdots \geq r_k \geq 0,\; \sum_{j=1}^{k} r_j = n\}$.

2.2 Nilpotent Orbits in B_n, C_n and D_n

As we have seen in the previous Section, each nilpotent orbit of type A_n corresponds to a partition of $n+1$. This Section concerns about the classification of nilpotent orbits in $Sp(2m, \mathbb{C})$ and $O(n, \mathbb{C})$. The main Theorem is the following:

Theorem 2.6. *Let* $\epsilon = \pm 1$*, and consider a nondegenerate bilinear form* $\langle \cdot, \cdot \rangle_\epsilon$ *on* $\mathbb{C}^k$ *such that*

$$\langle A, B \rangle_\epsilon = \epsilon \langle B, A \rangle_\epsilon \;\; for\ all\ A, B \in \mathbb{C}^k$$

We write

$$I(\langle \cdot, \cdot \rangle_\epsilon) = \{g \in GL_k(\mathbb{C}) |\; \langle gA, gB \rangle_\epsilon = \langle A, B \rangle_\epsilon \;\; for\ all\ A, B \in \mathbb{C}^k\}$$

$$\mathfrak{g}_\epsilon = \{X \in \mathfrak{sl}(k, \mathbb{C}) |\; \langle XA, B \rangle_\epsilon = - \langle A, XB \rangle_\epsilon \;\; for\ all\ A, B \in \mathbb{C}^k\}$$

$$P_\epsilon(k) = \{[d_1, \ldots, d_k] \mid \#\{j|d_j = i\} \; is \; even \; for \; all \; i \; such \; that \; (-1)^i = \epsilon\}$$

Then the nilpotent $I(\langle\cdot,\cdot\rangle_\epsilon)$-orbits in $\mathfrak{g}_\epsilon$ are in one-to-one correspondence with partitions in $P_\epsilon(k)$.

With this theorem, we can conclude that

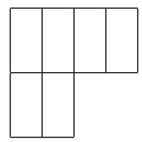
Corollary 2.7. *If $\epsilon = -1$, then $k = 2m$ must be even, and $I(\langle\cdot,\cdot\rangle_\epsilon) = Sp(2m, \mathbb{C})$, $\mathfrak{g}_\epsilon = \mathfrak{sp}(2m, \mathbb{C})$. Hence the $Sp(2m, \mathbb{C})$-orbits of nilpotent elements in $\mathfrak{sp}(2m, \mathbb{C})$ are identified with the paritions of $2m$ in which odd parts occur with even multiplicity.*
If $\epsilon = 1$, then $k = n$ can be any integer, and $I(\langle\cdot,\cdot\rangle_\epsilon) = O(n, \mathbb{C})$, $\mathfrak{g}_\epsilon = \mathfrak{o}(n, \mathbb{C})$. So the $O(n, \mathbb{C})$-orbits of nilpotent elements in $\mathfrak{o}(n, \mathbb{C})$ are identified with the partitions of n in which even parts occur with even multiplicity.

2.3 Another Characterization of Classical Nilpotent Orbits

In the last couple of Sections, all nilpotent orbits are characterized by partitions. And the partitions are often expressed as Young diagrams whose row sizes are determined by the corresponding partitions. In fact, in studying nilpotent orbits, it is sometimes more convenient to look at the column sizes of a Young diagram. The column sizes of the Young diagram corresponding to a partition is given by the **dual partition** of the original partition, which is defined by the following:

Definition 2.8. *Let $[r_1, r_2, \dots, r_i]$ be a partition of n, with $r_1 \geq r_2 \geq \cdots \geq r_i > 0$, then its* ***dual partition*** *is given by $(c_k, c_{k-1}, \dots, c_1)$, where $c_{k+1-j} = \#\{i | r_i \geq j\}$.*

Example 2.9. *Let $\mathcal{O} = [4, 2]$ in $Sp(6, \mathbb{C})$. Then the Young diagram corresponding to $\mathcal{O}$ is given by*



the dual partition of $\mathcal{O}$ is $(2, 2, 1, 1)$.

The dual partition of $\mathcal{O}$ has an algebraic intepretation on the rank of X^i, where X is any nilpotent element in the orbit.

Proposition 2.10. *Let X be any nilpotent element in the orbit $\mathcal{O}$ parametrized by the dual partition $(c_k, \dots, c_1)$, then*

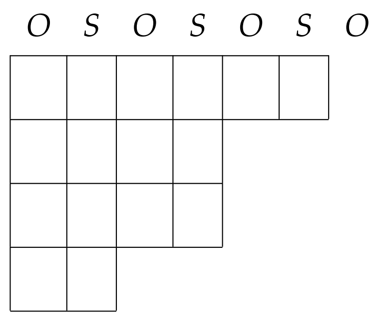
$$rank(X^j) = \sum_{i=1}^{k-j} c_i$$

From now on, we will determine a nilpotent orbit by its dual partition, or equivalently the column sizes of its corresponding Young diagram. Here is a restatement of Corollary 2.7 in terms of column sizes.

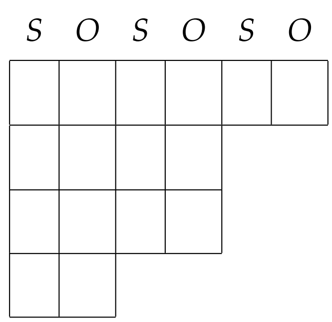
Corollary 2.11. *Any nilpotent orbit in $Sp(2m, \mathbb{C})$ can be parametrized by a partition of $2m$ with column sizes $(c_{2k}, c_{2k-1}, \dots, c_0)$, where $c_{2k} \geq c_{2k-1} \geq \cdots \geq c_0 \geq 0$ (by insisting c_{2k} is the longest column, we put $c_0 = 0$ if necessary), such that $c_{2i} + c_{2i-1}$ is even for all i (c_{-r} and $c_{2k+r} = 0$ for all $r > 0$).*
Any nilpotent orbit in $O(n, \mathbb{C})$ can be parametrized by a partition of n with column sizes

$(b_{2k+1}, b_{2k}, \ldots, b_0)$, *where* $b_{2k+1} \geq b_{2k} \geq \cdots \geq b_0 \geq 0$ *(putting* $b_0 = 0$ *if necessary), such that* $b_{2i} + b_{2i-1}$ *is even for all* i *(*b_{-r} *and* $c_{2k+1+r} = 0$ *for all* $r > 0$*).*

Example 2.12. *Consider the dual partition* $(4, 4, 3, 3, 1, 1)$*. To check whether it defines an orbit in* $O(16, \mathbb{C})$*, we name the longest column* 4 *by O, second and third longest column* S *and O and so on. We get*



In order for the partition to be a nilpotent orbit in $O(16, \mathbb{C})$*, we want the sum of each* S*-O column pair (which is different from O-*S *column) to be even. However, the first pair* $4+3$ *and the third pair* $1+0$ *are odd. So it does not define a nilpotent orbit in* $O(16, \mathbb{C})$*.*
To check whether $(4, 4, 3, 3, 1, 1)$ *defines an orbit in* $Sp(16, \mathbb{C})$*, name the longest and second longest columns* S *and O, third and fourth longest columns* S *and O and so on. We get*

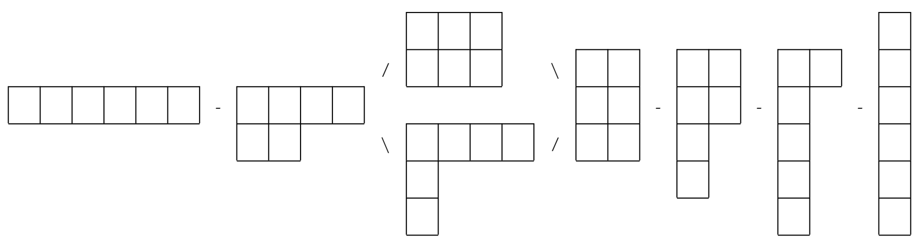


Note that the sum of the S-O column pairs are 8, 6, 2 respectively. So it defines an orbit in $Sp(16, \mathbb{C})$.

2.4 Closure Relations Between Orbits

In this Section, we study the Zariski closure of nilpotent orbits. In the classical Lie algebras, there is a nice combinatorial way of describing the orbit closures.

Definition 2.13. *Let $\mathcal{O}'$ and $\mathcal{O}$ be nilpotent orbits of a classical Lie algebra $\mathfrak{g}$, and let $X' \in \mathcal{O}'$, $X \in \mathcal{O}$. We say $\mathcal{O}' \leq \mathcal{O}$ iff $rank(X'^i) \leq rank(X^i)$ for all i. So $\mathcal{O}' \leq \mathcal{O}$ iff $\sum_{k=1}^{m'-i} c'_k \leq \sum_{k=1}^{l-i} c_k$. For example, the diagram below shows the case of $G = Sp(6, \mathbb{C})$:*



where the larger orbits appear on the left.

Definition 2.14. *Let $\overline{\mathcal{O}} := \bigcup_{\mathcal{O}' \leq \mathcal{O}} \mathcal{O}'$. Then any element X in $\overline{\mathcal{O}_{c_k,\ldots,c_1}}$ must satisfy the condition $rank(X^j) \leq \sum_{i=1}^{k-j} c_i =: p_j$. Note that the rank conditions defining the orbit closures, namely $rank(X^j) \leq p_j$, can be expressed as the vanishing set of some algebraic equations. Therefore, $\overline{\mathcal{O}}$ is closed in the Zariski topology, and set theoretically it is indeed the Zariski closure of the orbit $\mathcal{O}$.*

CHAPTER 3

NORMALITY OF ORBIT CLOSURES

As mentioned in the Introduction, it is suggested that the machinery of quantization works better with orbit closure than the orbit itself. In fact, they are closely related in the case of complex Lie algebras.

Theorem 3.1. *Let $\mathcal{O}$ be a G-orbit with Zariski closure $\overline{\mathcal{O}}$. If $\overline{\mathcal{O}} \setminus \mathcal{O}$ has codimension greater than or equal to 2, then $R[\mathcal{O}]$ is the integral closure of $R[\overline{\mathcal{O}}]$ in its field of fractions.*

Proof. First of all, $\mathcal{O}$ is smooth since it is a G-orbit. Therefore $\mathcal{O}$ is normal, i.e. $R[\mathcal{O}]$ is integrally closed. Now, take Y be the **normalization** of $\overline{\mathcal{O}}$ and let $\pi : Y \to \overline{\mathcal{O}}$ be the corresponding finite map. Also, let $Y' := \pi^{-1}(\mathcal{O})$. Then we have the commutative diagram

$$\begin{array}{ccccccc} Y' & \longrightarrow & Y & & R[Y'] & \longleftarrow & R[Y] \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow & \longrightarrow & {\scriptstyle\pi^*}\uparrow & & {\scriptstyle\pi^*}\uparrow \\ \mathcal{O} & \longrightarrow & \overline{\mathcal{O}} & & R[\mathcal{O}] & \longleftarrow & R[\overline{\mathcal{O}}] \end{array}$$

We want to show $R[Y'] \cong R[Y]$, since then

$$R[Y] \subset R[\mathcal{O}] \subset R[Y'] = R[Y]$$

and therefore $R[Y] = R[\mathcal{O}]$ as required.

Note that the first inclusion holds since (by definition of integral closure of $R[\overline{\mathcal{O}}]$) for any $x \in R[Y]$, there exists a monic polynomial $f \in R[\overline{\mathcal{O}}][t_1, \cdots, t_k]$ such that $f(x) = 0$. But $R[\overline{\mathcal{O}}] \subset R[\mathcal{O}]$, hence x is also in the integral closure of $R[\mathcal{O}]$, which is $R[\mathcal{O}]$ itself since $\mathcal{O}$ is normal.

The second inclusion holds because $\pi|_{Y'} : Y' \to \mathcal{O}$ is dominant (as is π itself), hence $\pi|_{Y'}^*$ gives the required inclusion.

To see why $R[Y] = R[Y']$, first note that $R[Y] \subset R[Y']$ by finiteness of π and the easy fact that $R[\overline{\mathcal{O}}] \subset R[\mathcal{O}]$. For the inverse inclusion, suppose $f \in R[Y']$. Then it extends to a rational function on Y (since Y' is dense in Y). Let X be the closed set of poles of f in Y, which is at least codimension 1 in Y. If it were of codimension 1, by the assumption in the Proposition and finiteness of π, $Y \backslash Y'$ is of codimension 2 in Y. Therefore X cannot lie completely inside $Y \backslash Y'$, and hence $Y' \cap X$ is dense in X.

However, f is regular on Y', hence regular on $Y' \cap X$, yet our setting says f has a pole along $Y' \cap X$, a contradiction.

Therefore, f cannot contain **any** pole along **any** hypersurface of Y, i.e. $f \in R[Y]_{\mathfrak{p}}$ for any height 1 prime ideal $\mathfrak{p}$. Consequently,

$$f \in \bigcap_{ht\ \mathfrak{p}=1} R[Y]_{\mathfrak{p}} = R[Y]$$

□

Note that all (real or complex) nilpotent orbits are symplectic manifolds with the **Kirilov-Kostant-Souriau symplectic form**, therefore they are all of even (real or complex) dimensions. In particular, the nilpotent orbit closure $\overline{\mathcal{O}}$ satisfies the hypothesis of the above Theorem. So we have the following:

Corollary 3.2. $R[\mathcal{O}] \cong R[\overline{\mathcal{O}}]$ *if and only if* $\overline{\mathcal{O}}$ *is normal.*

Therefore, it is fruitful to study quantization on both $\mathcal{O}$ and its closure, so that one can extract information from each other. In fact, this philosophy will be applied to the fullest extent in the later Chapters (See Chapter 10). Also, for classical nilpotent orbits, Kraft-Procesi [19] gave a criterion on normality:

Theorem 3.3 (Kraft-Procesi)**.**
(a) All nilpotent orbit closures in $SL(n,\mathbb{C})$ are normal.
(b) Let $\mathcal{O} = (c_{2k}, c_{2k-1}, \ldots, c_0)$ be a nilpotent orbit in $Sp(2m,\mathbb{C})$. If there is a chain of column lengths of the form

$$c_{2i} \neq c_{2i-1} = c_{2i-2} = \cdots = c_{2j-1} = c_{2j-2} \neq c_{2j-3}$$

then $\overline{\mathcal{O}}$ is not normal along $(c_{2k}, \ldots, c_{2i}, c_{2i-1}+2, c_{2i-2}, \ldots, c_{2j-1}, c_{2j-2}-2, c_{2j-3}, \ldots, c_0)$. Similarly, the closure of a nilpotent orbit $\mathcal{P} = (b_{2k+1}, \ldots, b_0)$ in $O(n,\mathbb{C})$ is not normal if there is a chain of column lengths of the form

$$b_{2i} \neq b_{2i-1} = b_{2i-2} = \cdots = b_{2j-1} = b_{2j-2} \neq b_{2j-3}$$

Remark 3.4.
(i) We will see in later Chapters that in the quantization model of $\overline{\mathcal{O}}$, the normality of nilpotent orbit closures plays an important role. More precisely, if the orbit closure is normal, the representation theoretic aspects of its corresponding model can be completely determined (e.g. an analog of Theorem 10.1 in the type A situation holds). Since every nilpotent orbit is normal in the type A situation, we focus on the type B, C, D cases.
(ii) Using the notation in Example 2.12, the normality criterion of nilpotent orbit closures of types B, C, D amounts to checking whether there are even number of columns, starting

with an O column, that have the same size. For example, $\overset{S\;O\;S\;O}{\ydiagram{4,4,2,2}}$*,* $\overset{S\;O\;S\;O}{\ydiagram{4,4,1,1}}$ *are normal, while* $\overset{S\;O\;S\;O}{\ydiagram{4,4,3,3,1,1}}$ *is not normal.*

Later on, we will come across another criterion of the normality of nilpotent orbit closures, by considering the multiplicities of some fundamental representations of G in $R[\mathcal{O}]$ and $R[\overline{\mathcal{O}}]$ (Theorem 9.12).

CHAPTER 4

BRYLINSKI'S CONSTRUCTION OF $X_{\overline{\mathcal{O}}}$

4.1 Kraft-Procesi Construction of $R[\overline{\mathcal{O}}]$

In [19], Kraft and Procesi constructed a realization of $R[\overline{\mathcal{O}}]$ to prove their non-normality results in Theorem 3.3. Since the construction of the Brylinski model $X_{\overline{\mathcal{O}}}$ is based on their construction, we give a brief account of the Kraft-Procesi construction here.

Definition 4.1. *Let $(U, \langle\ ,\ \rangle)$, $(V, (\ ,\ ))$ be complex vector spaces equipped with symmetric (or anti-symmetric) and anti-symmetric (or symmetric) inner products respectively, with $\dim U = m < \dim V = n$. Let $X \in L(V,U) := Hom(V,U)$ be* ***surjective****. Define $\pi : L(V,U) \to End(U)$ and $\rho : L(V,U) \to End(V)$ by $\pi(X) = XX^*$, $\rho(X) = X^*X$, where * is the adjoint operator of the corresponding inner product spaces.*
It can be checked that the images $\pi(X)$ and $\rho(X)$ are invariant operators with respect to their inner products, therefore both are in $O(U)$(or $Sp(U)$) and $Sp(V)$(or $O(V)$) respectively.

Theorem 4.2 (Kraft-Procesi)**.** *Let $x \in \overline{\mathcal{O}} \subset \mathfrak{o}(U)$ (or $\mathfrak{sp}(U)$) be a nilpotent element. For large enough $n = \dim V$, $\rho(\pi^{-1}(\overline{\mathcal{O}})) = \overline{\mathcal{O}'}$, where $\mathcal{O}'$ is the nilpotent orbit in $Sp(V)$(or $O(V)$) by adding a column of length $(n-m)$ on the Young diagram corresponding to the nilpotent $\mathcal{O}$.*

(By 'large enough' we mean the adding of the column makes sense, i.e. the first column of $\mathcal{O}$ has to be of length shorter than $(n-m)$.)

Proof. Let $\mathcal{O} = (c_k, c_{k-1}, \cdots, c_1)$ with $c_1 \neq 0$. Then by the Definition 2.12, $\overline{\mathcal{O}}$ is the union of orbits with partitions equal to the 'toppling' of that of $\mathcal{O}$. Consider $X \in \pi^{-1}(\overline{\mathcal{O}})$, i.e. $XX^* \in \overline{\mathcal{O}}$ and hence by Definition 2.12, $rank(XX^*)^j \leq \sum_{i \leq k-j} c_i$ for all j and

$$rank(X^*X)^l = rankX^*(XX^*)^{l-1}X \leq rank(XX^*)^{l-1} \leq \sum_{i \leq k-(l-1)} c_i$$

Therefore, $X^*X \in \overline{\mathcal{O}'}$, i.e. $\rho(\pi^{-1}(\overline{\mathcal{O}})) \subset \overline{\mathcal{O}'}$.

Now, check that $\mathcal{O}' \subset \rho(\pi^{-1}(\overline{\mathcal{O}})) \subset \overline{\mathcal{O}'}$. Let $Y : V \to V$ be an element in $\mathcal{O}'$. By definition of $\mathcal{O}'$, $rank(Y) = \dim U$. So write $U = Y(V)$ and $Y|_U : U \to U$ can be treated as an element in $End(U)$ lying on the nilpotent orbit $\mathcal{O}$.

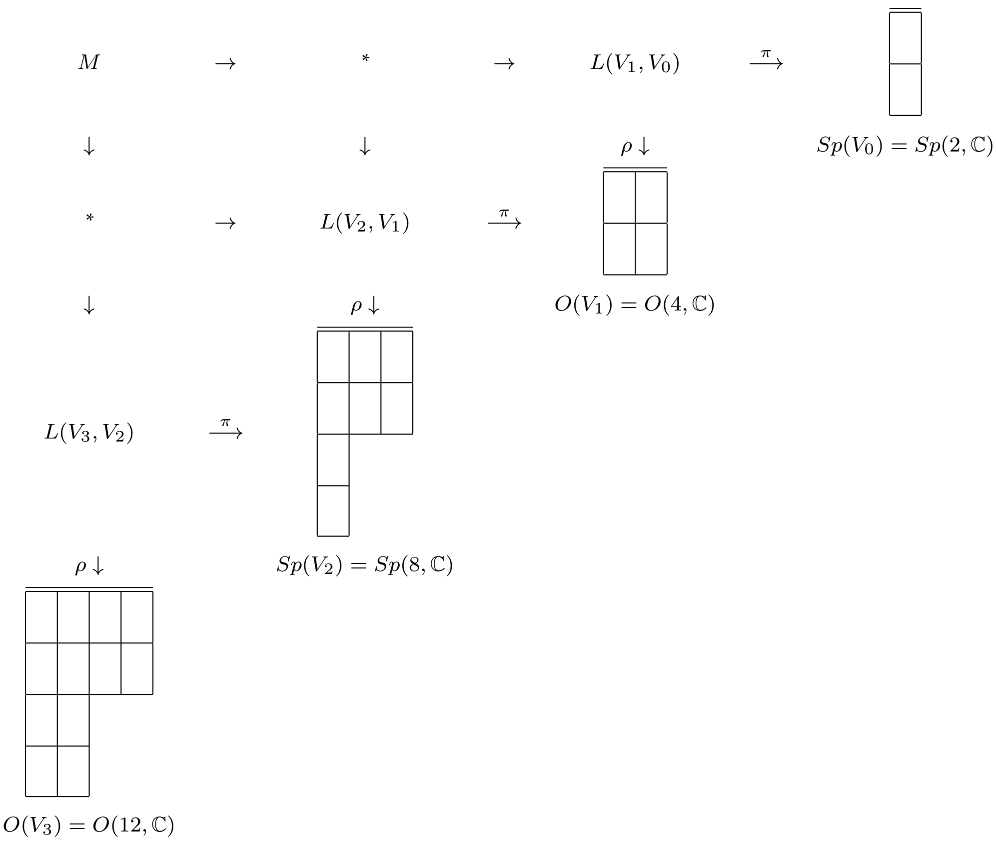
Consequently, if we denote $Z = Y : V \to U$, Z^* will simply be the inclusion map $U \hookrightarrow V$ and

$$Z^*Z = Y \quad , \quad ZZ^* = Y|_U \in \mathcal{O}$$

which means $Y \in \rho(\pi^{-1}(\overline{\mathcal{O}}))$.

On knowing $\mathcal{O}' \subset \rho(\pi^{-1}(\overline{\mathcal{O}})) \subset \overline{\mathcal{O}'}$, we just need to show the middle element is closed. But [19], Theorem 2 says ρ (and π) are quotient maps, hence they map closed sets to closed sets, which proves the theorem. $\square$

Example 4.3. *To construct the closure of the nilpotent orbit $(4,4,2,2)$ in $O(12,\mathbb{C})$, we* *need to add columns consecutively each time. This can be seen by the diagram below:*



where $M = L(V_1, V_0) \oplus L(V_2, V_1) \oplus L(V_3, V_2)$.

Note that each of the $L(V_i, V_{i-1})$ *has a natural symplectic structure given by the* (V_i, V_{i-1}) *pair. Write* $G := O(V_3) = O(12, \mathbb{C})$, $S := G_0 \times G_1 \times G_2 = Sp(V_0) \times O(V_1) \times Sp(V_2)$, *then* $G \times S$ *acts on* M *by*

$$(g, s_0, s_1, s_2) \cdot (X_1, X_2, X_3) := (s_0 X_1 s_1^{-1}, s_1 X_1 s_2^{-1}, s_2 X_3 g^{-1})$$

This action is Hamiltonian with moment maps $\mu_1 : M \to \mathfrak{s}^* \cong \mathfrak{s}$ *given by* $(X_1, X_2, X_3) \mapsto (X_1 X_1^*, X_1^* X_1 - X_2 X_2^*, X_2^* X_2 - X_3 X_3^*)$, *and* $\mu_2 : M \to \mathfrak{g}^* \cong \mathfrak{g}$ *given by* $(X_1, X_2, X_3) \mapsto X_3^* X_3$ *and by our construction, the equation of the closure of* $(4, 4, 2, 2)$ *is exactly given by* $\mu_2(\mu_1^{-1}(0))$.

Proposition 4.4 (Kraft-Procesi)**.**

$$R[\overline{\mathcal{O}}] = (\frac{\mathbb{C}[M]}{\langle \mu_1^x | x \in \mathfrak{s} \rangle})^S$$

where $M = L(V_1, V_0) \oplus \cdots \oplus L(V_n, V_{n-1})$, $S = G_0 \times G_1 \times \cdots \times G_{n-1}$.

Proof. These are precisely the algebro-geometric statement of the Kraft-Procesi construction (see [19], Theorem 5.3), namely:

• $\mu_1^{-1}(0)$ is the complete intersection with respect to the equations $X_1 X_1^* = 0, \cdots,$ $X_{n-1}^* X_{n-1} - X_n X_n^* = 0$.

• $\mu_2 : \mu_1^{-1}(0) \to \overline{\mathcal{O}}$ is a quotient map under S. □

4.2 Some Basic Notions on Infinite Dimensional Representations

Before constructing the Brylinski model, we give some basic notions of infinite dimensional representations which are essential for the construction.

Let $\mathfrak{g}$ be a classical complex simple Lie algebra, and $U(\mathfrak{g})$ be its **universal enveloping algebra**. Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Then

Theorem 4.5 (Harish-Chandra Isomorphism)**.** *There is an isomorphism between* $Z(\mathfrak{g})$ *and* $S(\mathfrak{h})^W$*, the Weyl group invariant of the symmetric algebra of a Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$*. Therefore, by the Nullstellensatz, every maximal ideal of* $Z(\mathfrak{g})$ *can be identified as* $Z(\lambda)$*, where* λ *is an element in* $\mathfrak{h}^*/W$*.*

Let I be a two-sided ideal of $U(\mathfrak{g})$, then I is said to have **infinitesimal character** λ if I contains the ideal $U(\mathfrak{g})Z(\lambda)$.

Definition 4.6.

(a) Let G_0 be a real reductive Lie group with Lie algebra $Lie(G_0) = \mathfrak{g}_0$ and maximal compact subgroup K_0. Then a $(\mathfrak{g}, K) := ((\mathfrak{g}_0)_{\mathbb{C}}, (K_0)_{\mathbb{C}})$-module V is a $U(\mathfrak{g})$-module with a K-action such that:

• *For any $X \in \mathfrak{g}$ and any $k \in G$, $k \cdot X \cdot v = (Ad(k)X) \cdot (k \cdot v)$ for all $v \in V$.*

• *For all $Y \in \mathfrak{k}$, $Y \cdot v = \frac{d}{dt}(exp(tY) \cdot v)|_{t=0}$.*

• *V is **admissible**. Namely the K-action on V is finite-dimensional, i.e. V can be decomposed as a direct sum of finite-dimensional, irreducible K-modules, and each irreducible K-module E appears in V with finite multiplicity.*

*(b) An admissible $(\mathfrak{g}, K)$-module V is **finitely generated** if there is a finite-dimensional vector subspace $V_0 \leq V$ such that $U(\mathfrak{g}) \cdot V_0 = V$.*

*(c) Furthermore, an admissible $(\mathfrak{g}, K)$-module V has an **infinitesimal character** $\lambda \in \mathfrak{h}^*/W$ if the $U(\mathfrak{g})$-annihilator of V contains the ideal $U(\mathfrak{g})Z(\lambda)$.*

*(d) Let G be a complex semisimple Lie group with compact real form K. Treat G as a real Lie group, then $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}}) = (\mathfrak{g} \oplus \mathfrak{g}, G)$, and the corresponding $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-module is called a **Harish-Chandra bimodule**.*

Given an admissible, finitely-generated $(\mathfrak{g}, K)$-module, there is an important invariant attached to it called the associated variety, which will be used again and again in the following Chapters.

Definition 4.7. *Let X be an admissible, finitely generated $(\mathfrak{g}, K)$-module. Let X_0 be a finite-dimensional K-invariant generating subspace of X, define a filtration on X by $X_n := U_n(\mathfrak{g}) \cdot X_0$. Then $gr(X)$ becomes an $gr(U(\mathfrak{g})) = S(\mathfrak{g})$-module, and (by the compatibility condition of $(\mathfrak{g}, K)$-module) every element in $\mathfrak{k}$ annihilates $gr(X)$. So $gr(X)$ can be treated as a $S(\mathfrak{g}/\mathfrak{k})$-module. Let $I = Ann_{S(\mathfrak{g}/\mathfrak{k})}(gr(X))$, then*

the vanishing set $\mathbb{V}(I)$ does not depend on our choice of generators X_0 $(though\ I\ does)$

*Therefore, we can define the **associated variety** of X to be*

$$AV(X) = \mathbb{V}(I) \subset (\mathfrak{g}/\mathfrak{k})^*$$

4.3 Construction of $X_{\overline{\mathcal{O}}}$

Retain the notations in the last proposition, set $\mathcal{W}$ = Weyl algebra of $M = L(V_1, V_0) \oplus \cdots \oplus L(V_n, V_{n-1})$, i.e.

$$\mathcal{W} = T(M)/\langle a \otimes b - b \otimes a - \{a, b\}\rangle$$

with $\{\cdot, \cdot\}$ being a symplectic form on M (Recall each $L(V_i, V_{i-1})$ has a symplectic structure, therefore so does M). There is a natural inclusion $\xi^{\cdot} : \mathfrak{sp}(M, \mathbb{C}) \cong S^2(M) \hookrightarrow \mathcal{W}$. Then $\mathcal{W}$ is a $(\mathfrak{sp}(M, \mathbb{C})_{\mathbb{C}}, Sp(M)_{\mathbb{C}})$-module given by the actions

$$(x, y) \cdot A := \xi^x m - m\xi^y$$

$$g \cdot m = gmg^{-1}$$

for $(x,y) \in \mathfrak{sp}(M,\mathbb{C})_{\mathbb{C}} = \mathfrak{sp}(M,\mathbb{C}) \oplus \mathfrak{sp}(M,\mathbb{C})$, $g \in Sp(M)_{\mathbb{C}} \cong Sp(M,\mathbb{C})$ and $m \in M$ (extend the above actions to $T(M)$).

As in the last section, $\mathfrak{g}$ and $\mathfrak{s} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ are embedded in $\mathfrak{sp}(M,\mathbb{C})$. Define

Definition 4.8.

$$X_{\overline{\mathcal{O}}} := \mathcal{W}^{even} / \left\langle (x,y) \cdot A,\ A - s \cdot A \middle| \, (x,y) \in \mathfrak{s}_{\mathbb{C}},\ s \in S,\ A \in \mathcal{W}^{even} \right\rangle$$

Theorem 4.9. *There is a filtration of algebras in $X_{\overline{\mathcal{O}}}$ inherited from the filtration in $\mathcal{W}^{even} \subset \mathcal{W}$. Under this filtration,*

$$gr(X_{\overline{\mathcal{O}}}) = R[\overline{\mathcal{O}}]$$

as $G \cong K_{\mathbb{C}}$-modules.

Proof. Here is a sketch of the proof. Recall the definition of Lie group homology,

$$X_{\overline{\mathcal{O}}} = H_0(\mathfrak{s}_{\mathbb{C}}, S; \mathcal{W}^{even})$$

and the Koszul complex evaluating $H_i(\mathfrak{s}_{\mathbb{C}}, S; \mathcal{W}^{even})$ is

$$0 \leftarrow \wedge^0 \mathfrak{p} \otimes_S \mathcal{W}^{even} \leftarrow \wedge^1 \mathfrak{p} \otimes_S \mathcal{W}^{even} \leftarrow \cdots \leftarrow \wedge^l \mathfrak{p} \otimes_S \mathcal{W}^{even} \leftarrow 0$$

where $\mathfrak{p} = \{(x,x) | x \in \mathfrak{s}\}$, the noncompact part of the Cartan decomposition of $\mathfrak{s}_{\mathbb{C}}$, and the boundary map is given by

$$\partial(x_1 \wedge \cdots \wedge x_t \otimes E) = \sum_{i=1}^{t} x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots x_t \otimes (\xi^{x_i} E + E \xi^{x_i})$$

where x_i is a shorthand of $(x_i, x_i) \in \mathfrak{p}$.

Use a spectral sequence to evalute $H_\bullet(\mathfrak{s}_{\mathbb{C}}, S; \mathcal{W}^{even})$: Pick $E_r^{p,q}$ so that

$E_\infty^d = gr_p(H_{p+q}(\mathfrak{s}_\mathbb{C}, S; \mathcal{W}^{even}))$. We will show:

• $E_1^0 = R[\overline{\mathcal{O}}]$, $E_1^d = 0$

• $E_1^{p,q} = E_\infty^{p,q}$, i.e. E_r^d stabilizes at $r = 1$.

Indeed,

$$E_0^{p,q} = \wedge^{p+q}\mathfrak{p} \otimes_S \mathbb{C}[M]^{-q}$$

where the nonzero values are on the $(-\pi/4, 0)$ octant $\{(p,q)|q \leq 0,\ p+q \geq 0\}$. So E_0^d is the complex

$$(0 \leftarrow \wedge^0\mathfrak{p} \otimes \mathbb{C}[M]^{even} \leftarrow \wedge^1\mathfrak{p} \otimes \mathbb{C}[M]^{even} \leftarrow \cdots \leftarrow \wedge^{\dim S}\mathfrak{p} \otimes \mathbb{C}[M]^{even} \leftarrow 0)_S$$

the boundary map are the downward arrows $\downarrow$: $E_0^{p,q} \to E_0^{p,q-1}$ on the 0^{th}-page, given by

$$\partial(x_1 \wedge \cdots \wedge x_t \otimes E) = \sum_{i=1}^{t} x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots x_t \otimes gr(\xi^{x_i}E + E\xi^{x_i})$$

but $gr(\xi^{x_i}) = \mu_1^{x_i}$, so the complex is simply the Koszul complex of $\{\mu_1^{x_i}\}$.

However, by [19], $\{\mu_1^{x_i}|x_i \in \mathfrak{s}\}$ is a complete intersection, hence they form a regular sequence. By a result in commutative algebra, the homology of the above complex is

$$E_1^d: \quad 0 \leftarrow (R[M]/\langle \mu_1^x | x \in \mathfrak{s}\rangle)^S \leftarrow 0 \leftarrow \cdots$$

and hence the first claim is done by the Proposition above.

For the second claim, note that the boundary maps on the 1^{st} page become $\leftarrow$, so the value remains unchanged and so does E_2 and so on. So we are done. □

Proposition 4.10.

(a) $X_{\overline{\mathcal{O}}}$ is admissible.

(b) $X_{\overline{\mathcal{O}}}$ is finitely generated.

(c) The left and right annihilators of $X_{\overline{\mathcal{O}}}$ coincide.

(d) $X_{\overline{\mathcal{O}}}$ has an infinitesimal character.

Proof.

(a) First of all, note that both $X_{\overline{\mathcal{O}}}$ and $gr(X_{\overline{\mathcal{O}}})$ are isomorphic as $K_{\mathbb{C}}$-modules. Also, by the Theorem, we know $gr(X_{\overline{\mathcal{O}}}) \cong R[\overline{\mathcal{O}}]$ as $K_{\mathbb{C}}$-modules. However, by a standard theorem for reductive group actions on varieties, $R[\overline{\mathcal{O}}]$ decomposes into a direct sum of finite-dimensional irreducible $K_{\mathbb{C}}$-representations. It now remains to show the multiplicities of each finite-dimensional irreducible $K_{\mathbb{C}}$-representation is finite.

There are many ways to see this is the case, we will present one here which will be relevant to the later chapters. Consider the variety of all nilpotent elements $\mathcal{N}$. It is indeed the closure of an orbit called the **regular orbit** $\mathcal{O}_{reg}$. It is a standard result (which is generalized by Theorem 6.2 and Corollary 3.2) that as $K_{\mathbb{C}}$-modules,

$$R[\overline{\mathcal{O}_{reg}}] \cong U(\mathfrak{g})/ann_{U(\mathfrak{g})}(M(\lambda))$$

where $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ is any Verma module. In particular, $R[\overline{\mathcal{O}_{reg}}]$ is isomorphic to a **principal series representation** $X(\lambda, \lambda) := Ind_B^G(\mathbb{C}_{\lambda,\lambda})$ and hence admissible. This shows $X_{\overline{\mathcal{O}_{reg}}}$ is admissible.

For any orbit closures $\overline{\mathcal{O}}$, note that there is a $K_{\mathbb{C}} = G$-equivariant morphism

$$R[\overline{\mathcal{O}_{reg}}] \twoheadrightarrow R[\overline{\mathcal{O}}]$$

so for any finite-dimensional irreducible G representations V_μ, the multiplicity of V_μ in $R[\overline{\mathcal{O}}]$ is no larger than that of $R[\overline{\mathcal{O}_{reg}}]$. But we know the multiplicities of the latter are always finite, hence we are done.

(b) Let $\mathcal{W}^{inv}$ be the algebra of S-invariants in $\mathcal{W}^{even}$, i.e. $\mathcal{W}^{inv} = \{A \in \mathcal{W}^{even} | A = s \cdot A\}$. Then by definition of $\xi : \mathfrak{sp}(M, \mathbb{C}) \hookrightarrow \mathcal{W}^2$ in the beginning of this Section, we have the map of algebras $\zeta : \mathfrak{g} \hookrightarrow \mathcal{W}^{inv}$, which induces a homomorphism of filtered algebras

$$\zeta : U(\mathfrak{g}) \to \mathcal{W}^{inv}$$

On the other hand, by the very definition of $X_{\overline{\mathcal{O}}}$ we have another surjective homomorphism of filtered algebras

$$\phi : \mathcal{W}^{inv} \twoheadrightarrow X_{\overline{\mathcal{O}}}$$

In Section 7 of [7], it is proved that ϕ is surjective in each filtration degree. Also, Theorem 4.9 says $gr(X_{\overline{\mathcal{O}}}) \cong R[\overline{\mathcal{O}}]$. So the composition of filtered algebras $\phi \circ \zeta : U(\mathfrak{g}) \to X_{\overline{\mathcal{O}}}$ gives a surjective homomorphism $S(\mathfrak{g}) \twoheadrightarrow R[\overline{\mathcal{O}}] \cong gr(X_{\overline{\mathcal{O}}})$ (the kernel is precisely the defining ideal of the variety $\overline{\mathcal{O}}$, $I(\overline{\mathcal{O}})$). This implies $\phi \circ \zeta$ must be surjective in each filtration degree. Let $\mathfrak{I} := ker(\phi \circ \zeta)$, we have

$$U(\mathfrak{g})/\mathfrak{I} \cong X_{\overline{\mathcal{O}}}$$

Now it is obvious that the $U(\mathfrak{g}_{\mathbb{C}})$-module $U(\mathfrak{g})/\mathfrak{I}$ is generated by $1 \in U(\mathfrak{g})/\mathfrak{I}$.

(c) This holds if $\mathfrak{I}$ is a 2-sided ideal in $U(\mathfrak{g})$. But it follows directly from $\mathfrak{I} = Ann_{U(\mathfrak{g})}(X_{\overline{\mathcal{O}}})$, if we treat $X_{\overline{\mathcal{O}}}$ as a $U(\mathfrak{g})$-module under $\phi \circ \zeta$.

(d) We want to check that

$$Z(\lambda)U(\mathfrak{g}) \subset \mathfrak{I}$$

for some λ. In fact, it suffices to check the inclusion holds after taking gr. Taking gr, $gr(Z(\lambda)) = S^+(\mathfrak{g})^G$, the G-invariant elements in $S(\mathfrak{g})$ of positive degree. So

$gr(Z(\lambda)U(\mathfrak{g})) = \langle S^+(\mathfrak{g})^G\rangle$. On the other hand, $gr(\mathfrak{I}) = I(\overline{\mathcal{O}}) \supset I(\mathcal{N})$ where $\mathcal{N}$ is the variety of all nilpotent elements. By a theorem of Kostant, $I(\mathcal{N}) = \langle S^+(\mathfrak{g})^G\rangle$. Therefore

$$gr(Z(\lambda)U(\mathfrak{g})) = \langle S^+(\mathfrak{g})^G\rangle \subset I(\overline{\mathcal{O}}) = gr(\mathfrak{I})$$

and hence the Theorem follows. □

Therefore, $X_{\overline{\mathcal{O}}}$ is a Harish-Chandra bimodule. Moreover, its infinitesimal character is known explicitly.

Proposition 4.11. *Let $\mathcal{O} = (c_{2k}, \ldots, c_0)$ be a nilpotent orbit in $Sp(2m, \mathbb{C})$. Then $X_{\overline{\mathcal{O}}}$ has an infintesimal character $\chi = (\chi_0, \chi_1, \ldots, \chi_k)$, where χ_i is defined by the following:*

- *For i between 1 and k, $\chi_i = (\frac{c_{2i}}{2}, \frac{c_{2i}-2}{2}, \ldots, \frac{-c_{2i-1}+2}{2})$.*
- *$\chi_0 = (\frac{c_0}{2}, \ldots, 1)$.*

Let $\mathcal{P} = (b_{2k+1}, b_{2k}, b_{2k-1}, \ldots, b_0)$ be any nilpotent orbit in $O(n, \mathbb{C})$. The infinitesimal character of $X_{\overline{\mathcal{P}}}$ is (χ_h, χ), where

- *χ is defined in the same way as the symplectic case for $\mathcal{P}' = (b_{2k}, b_{2k-1}, \ldots, b_0)$, and*
- *$\chi_h = (\frac{b_{2k+1}-2}{2}, \ldots, \frac{1}{2})$ if b_{2k+1} is odd, $(\frac{b_{2k+1}-2}{2}, \ldots, 0)$ if b_{2k+1} is even.*

Proof. This is given in [26], using some techniques on **reductive dual pair correspondence**. We will study the subject further in Chapter 11. □

Example 4.12. *Consider the orbit $(7, 7, 6, 4)$ in $Sp(24, \mathbb{C})$. Then the infinitesimal character of χ can be read from the following diagram:*

<table>
<tr><th>S</th><th>O</th><th>S</th><th>O</th></tr>
<tr><td colspan="2">$\frac{1}{2}$</td><td rowspan="2">1</td><td rowspan="2">0</td></tr>
<tr><td rowspan="2">$\frac{3}{2}$</td><td rowspan="2">$\frac{1}{2}$</td></tr>
<tr><td rowspan="2">2</td><td rowspan="2">1</td></tr>
<tr><td rowspan="2">$\frac{5}{2}$</td><td rowspan="2">$\frac{3}{2}$</td></tr>
<tr><td rowspan="2">3</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">$\frac{7}{2}$</td><td rowspan="2">$\frac{5}{2}$</td></tr>
<tr><td></td><td></td></tr>
</table>

i.e. $\chi = (\frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, 3, 2, 1, 1, 0)$ *(up to a Weyl group action on the coordinates, by the Harish-Chandra isomorphism). For the orbit* $(7, 7, 7, 6, 4)$ *in* $O(31, \mathbb{C})$*, the infinitesimal character is given below:*

<table>
<tr><th>O</th><th>S</th><th>O</th><th>S</th><th>O</th></tr>
<tr><td></td><td colspan="2">$\frac{1}{2}$</td><td rowspan="2">1</td><td rowspan="2">0</td></tr>
<tr><td rowspan="2">$\frac{1}{2}$</td><td rowspan="2">$\frac{3}{2}$</td><td rowspan="2">$\frac{1}{2}$</td></tr>
<tr><td rowspan="2">2</td><td rowspan="2">1</td></tr>
<tr><td rowspan="2">$\frac{3}{2}$</td><td rowspan="2">$\frac{5}{2}$</td><td rowspan="2">$\frac{3}{2}$</td></tr>
<tr><td rowspan="2">3</td><td rowspan="2"></td></tr>
<tr><td rowspan="2">$\frac{5}{2}$</td><td rowspan="2">$\frac{7}{2}$</td><td rowspan="2">$\frac{5}{2}$</td></tr>
<tr><td></td><td></td></tr>
</table>

Remark 4.13. *Note that as a Corollary of Proposition 4.11, the* $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$*-module* $X_{\overline{\mathcal{O}}}$ *has a* ***composition series*** *of irreducible* $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$*-modules. The argument is sketched as follows:*

By Langlands classification of irreducible $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$*-modules, the number of irreducible*

*representations having a fixed infinitesimal character is bounded by the size of the Weyl group of G. Let the **lowest K-types** of such irreducible representations be the irreducible, finite dimensional $K_{\mathbb{C}}$-representations $V_1, \ldots, V_n$. It follows that all $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-modules having the fixed infinitesimal character must contain one of the above K-types V_i.*
If $X_{\overline{\mathcal{O}}}$ is irreducible then we are done. Suppose not, i.e. there is a submodule $W \subset X_{\overline{\mathcal{O}}}$. Then W, $X_{\overline{\mathcal{O}}}/W$ are both admissible, finitely generated and have the same infinitesimal character as $X_{\overline{\mathcal{O}}}$. Since $K_{\mathbb{C}}$-representations can be completely decomposed,

$$[V_i : W] + [V_i : X_{\overline{\mathcal{O}}}/W] = [V_i : X_{\overline{\mathcal{O}}}](< \infty)$$

for all i, and there are some V_i such that $[V_i : W] > 0$. In particular, $\sum_i [V_i : W], \sum_i [V_i : X_{\overline{\mathcal{O}}}/W] < \sum_i [V_i : X_{\overline{\mathcal{O}}}]$. So we can use induction argument on sum of multiplicities of V_i to conclude that both W and $X_{\overline{\mathcal{O}}}/W$ have composition series, and hence so does $X_{\overline{\mathcal{O}}}$.

Finally, the associated variety of $X_{\overline{\mathcal{O}}}$ is $\overline{\mathcal{O}}$ as expected.

Proposition 4.14. *The associated variety of $X_{\overline{\mathcal{O}}}$ is $AV(X_{\overline{\mathcal{O}}}) = \overline{\mathcal{O}}$.*

Proof. In fact, if $X_{\overline{\mathcal{O}}} \cong U(\mathfrak{g})/\mathfrak{I}$ is a Harish-Chandra bimodule, it follows immediately from the definition of the associated variety of a $(\mathfrak{g}, K)$-module that $AV(X_{\overline{\mathcal{O}}}) \subset (\mathfrak{g}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^* \cong (\mathfrak{g} \oplus \mathfrak{g}/\mathfrak{g})^* \cong \mathfrak{g}^* \cong \mathfrak{g}$, and it is equal to $Ann(gr(\mathfrak{I})) \subset \mathfrak{g}$. However, in the proof of Theorem 3.9(b), $gr(\mathfrak{I}) = I(\overline{\mathcal{O}})$. Hence the result follows. $\square$

CHAPTER 5

UNIPOTENT REPRESENTATIONS

Before constructing the quantization model for the nilpotent orbit $\mathcal{O}$, it is important to introduce the basic building blocks of the model - unipotent representations.

Let $\mathfrak{g}$ be a complex classical Lie algebra, and fix $\lambda \in \mathfrak{h}^*$. It is known by Dixmier that there exists a maximal ideal $J_{max}(\lambda) \subset U(\mathfrak{g})$ so that the infinitesimal character of $U(\mathfrak{g})/J_{max}(\lambda)$ is λ.

Definition 5.1. *A* ***unipotent representation*** *is an irreducible Harish-Chandra bimodule X such that the left and right $U(\mathfrak{g})$ annihilators of X are both equal to $J_{max}(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.*

These objects are well-studied by Barbasch and Vogan, as the following theorem shows.

Theorem 5.2 (Barbasch-Vogan)**.** *Fix $\lambda \in \mathfrak{h}^*$.*
(a) The associated variety of any unipotent representations X depends only on λ. In particular, for a fixed λ, all unipotent representations X have the same associated variety $AV(U(\mathfrak{g})/J_{max}(\lambda))$.
(b) The number of unipotent representations X given a fixed infinitesimal character λ can be computed. The character theory of all such X is also known.
(c) All representations having infinitesimal character λ must have associated variety bigger than or equal to $AV(U(\mathfrak{g})/J_{max}(\lambda))$.

The rest of this Chapter will focus on applying Part (b) of the above Theorem, given the infinitesimal character is of the form as in Proposition 4.11.

5.1 Character Theory of Unipotent Representations

5.1.1 Integral Infinitesimal Character

Recall Proposition 4.11 that if λ is integral, the infinitesimal character we are interested in is of the form

$$(a_1, a_1 - 1, \ldots, 1; b_1, \ldots, 0; a_2, \ldots, 1; b_2, \ldots, 0; \ldots)$$

where $a_i > b_i$ in $G = Sp(2m, \mathbb{C})$, or $(d_1, \ldots, 0; a_1 \ldots 1; b_1, \ldots, 0; \ldots)$ for $G = O(2n, \mathbb{C})$. Given such λ, we give an algorithm computing the number of unipotent representations and their associated varieties.

Proposition 5.3. *Let $G = Sp(2m, \mathbb{C})$, and let λ be an integral infinitesimal character of the form given in Proposition 4.11. Replace λ by $w\lambda$ for some Weyl group element $w \in W$ such that $w\lambda = (a_1, a_1 - 1, \ldots, a_1 - 1, a_1 - 2, \ldots, a_1 - 2, \ldots, 0, \ldots, 0)$. Then extract one coordinate from each entry, i.e. $(a_1, a_1 - 1, \ldots, 0)$, and then adjoin it with the negatives of coordinates. This forms the coordinates of F^1. Similarly, form F^2, F^3 and so on from* *the remainder until all zeros are extracted. Then collect all the remaining non-zero terms and form T [5, Section 6.2]. Write*

$$F^i = (\alpha_i, \ldots, 1, 0, -1, \ldots, -\beta_i)$$

$$T = (d, d-1, \ldots, 1)$$

(The construction of λ in Proposition 4.11 forces T to be of the form above) then by construction, $\alpha_i \geq \beta_i \geq \alpha_{i+1} - 1$ for all i. If $\alpha_i \neq \beta_i$, let $c_i := (2\alpha_i, 2(\beta_i + 1))$. If $\alpha_i = \beta_i$, let $c_i := (2\alpha_i + 1, 2\alpha_i + 1)$. Write $\mathcal{O}' = (c_1, c_2, \ldots, c_k, 2d)$, then

$$AV(U(\mathfrak{g})/J_{max}(\lambda)) = \overline{\mathcal{O}'}$$

Proof. The algorithm is essentially given in [4]. More generally, the λ we are dealing with are called **q-unipotent** by McGovern in [24]. Here is a brief description of the algorithm. Given $\lambda \in \mathfrak{h}^*$, first correspond $\mathfrak{h}^*$ with the Langlands L-group ${}^L\mathfrak{h}$. Next, find the Levi subalgebra ${}^L\mathfrak{m} \subset^L \mathfrak{h}$ such that $\langle \lambda, m \rangle = 0$ for all $m \in^L \mathfrak{m}$. Then the **left cell** corresponding to λ is given by $V(\lambda) = J^W_{W({}^L\mathfrak{m})}(sgn) \otimes sgn$, where J is the truncated induction defined by Lusztig. Let σ be the (unique) special Weyl group representation in that left cell. Then the orbit $\mathcal{O}(\sigma)$ corresponds to σ through **Springer correspondence** is the precisely the $\mathcal{O}'$ in the Proposition. $\square$

Example 5.4. *Let $\mathcal{O} = (6, 4, 4, 2, 2)$ in $Sp(18, \mathbb{C})$. Then by Proposition 4.11, $\lambda = (3, 2, 1; 1, 0;\ 2, 1; 0; 1)$. Rearrange the entries so that $w\lambda = (3, 2, 2, 1, 1, 1, 1, 0, 0)$. Now $F^1 = (3, 2, 1, 0, -1, -2)$, $F^2 = (1, 0, -1)$; so $c_1 = (6, 6)$, $c_2 = (3, 3)$ and hence $AV(U(\mathfrak{g})/J_{\max}(\lambda)) = \overline{\mathcal{O}'} = \overline{(6, 6, 3, 3)}$. Note that $\overline{\mathcal{O}'} \subset \overline{\mathcal{O}}$.*

Proposition 5.5. *Let $G = O(2n, \mathbb{C})$ and λ is an integral infinitesimal character of the form given in Proposition 4.11. Rearrange the entries of λ in non-increasing order as in the above Proposition. Then extract one coordinate from each entry to form E^0. For the remainder, extract F^i's and T as in the above Proposition. For $E_0 = (x, x-1, \ldots, 0)$, $T = (d, \ldots, 1)$ and c_i is defined as in the symplectic case,*

$$AV(U(\mathfrak{g})/J_{max}(\lambda)) = \overline{\mathcal{O}'}$$

where $\mathcal{O}' = (2x+2, c_1, c_2, \ldots, c_k, 2d)$.

The following Proposition gives the number of unipotent representations for a given infinitesimal character.

Proposition 5.6. *Let $G = Sp(2m, \mathbb{C})$ or $O(2n, \mathbb{C})$, and λ as above with $\mathcal{O}' = (x_1, x_2, \ldots, x_n)$. Let $s_{\mathcal{O}'}$ be the number of F^i's so that β_i exists and $\alpha_i \neq \beta_i$. Then for $G = Sp(2m, \mathbb{C})$, the number of unipotent representations having infinitesimal character λ is $2^{s_{\mathcal{O}'}}$. Similarly, for $G = O(2n, \mathbb{C})$, the number of unipotent representations having infinitesimal character λ is $2^{s_{\mathcal{O}'}+1}$.*

Proof. The number is precisely the number of irreducible Weyl group representations in the left cell $V(\lambda)$ in the above Propositions. More precisely, from [4, Proposition 5.28], the number is equal to the **Lusztig quotient** $\overline{A(\mathcal{O}')}$ of the nilpotent orbit $\mathcal{O}'$. □

Finally, the character formulas for all unipotent representations is given by the following Theorem:

Theorem 5.7. *Let λ, $\mathcal{O}'$ and $V(\lambda)$ as in the previous Propositions. For any $\sigma_i \in V(\lambda)$, $i = 1, \ldots 2^s$, there is a one-one correspondence between the abelian group $\overline{A(\mathcal{O}')} \cong (\mathbb{Z}/2\mathbb{Z})^s$*

$$x \in \overline{A(\mathcal{O}')} \longleftrightarrow \sigma_{i_x} \in V(\lambda)$$

such that the unipotent representations have character formulas

$$X_\chi = \frac{1}{2^s} \sum_x tr(\chi(x)) R_{\sigma_{i_x}}(\lambda)$$

where χ is any irreducible representation of $\overline{A(\mathcal{O}')} \cong (\mathbb{Z}/2\mathbb{Z})^s$, parametrized by $(\overbrace{\pm, \ldots, \pm}^{s\ terms})$, and

$$R_\sigma(\lambda) := \sum_{w \in W} tr(\sigma(w)) X \begin{pmatrix} \lambda \\ w\lambda \end{pmatrix}$$

where $X \begin{pmatrix} \lambda \\ \mu \end{pmatrix}$ is the K-finite part of the principal series representation $Ind_B^G(\mathbb{C}_{(\lambda,\mu)})$ ([4, Definition 1.7(d)])

Remark 5.8. *The Lusztig quotient $\overline{A(\mathcal{O}')}$ is a quotient of the fundamental group $\pi_1(\mathcal{O}')$ of $\mathcal{O}'$. Also, if $\chi = (+, +, \ldots, +)$, then $X_{+,\ldots,+}$ contains the trivial representation, i.e. $X_{+,\ldots,+}$ is **spherical**.*

Example 5.9. *Let $\lambda = (4, 3, 2, 2, 1, 1, 1, 0, 0)$ in $Sp(18, \mathbb{C})$. Then $F^1 = (4, 3, 2, 1, 0, -1, -2)$ and $F^2 = (1, 0)$. Then from the above Propositions, $\mathcal{O}' = (8, 6, 2, 2)$, the number of irreducible unipotent representations is four, having character formulas:*

$$\frac{1}{4} \sum_{W(C_4 \times D_3 \times C_1 \times D_1)} (-1)^{l(w)} X \begin{pmatrix} & 4321, 210 & 1, 0 \\ w(& 4321, 210 & 1, 0) \end{pmatrix}$$

$$+\frac{1}{4}\sum_{W(D_5\times C_2\times C_1\times D_1)}(-1)^{l(w)}X\begin{pmatrix} & 43210,21 & 1,0 \\ w(& 43210,21 & 1,0) \end{pmatrix}$$

$$+\frac{1}{4}\sum_{W(C_4\times D_3\times D_2\times C_0)}(-1)^{l(w)}X\begin{pmatrix} & 4321,210 & 10 \\ w(& 4321,210 & 10) \end{pmatrix}$$

$$+\frac{1}{4}\sum_{W(D_5\times C_2\times D_2\times C_0)}(-1)^{l(w)}X\begin{pmatrix} & 43210,21 & 10 \\ w(& 43210,21 & 10) \end{pmatrix}$$

and three others with signs $(+,-,+,-)$*,* $(+,+,-,-)$ *and* $(+,-,-,+)$*.*

For the future work, we denote the four unipotent representations by the following notation - $(\overbrace{8,6}^{+},\overbrace{2,2}^{+})$ *for the formula with sign* $(+,+,+,+)$*,* $(\overbrace{8,6}^{-},\overbrace{2,2}^{+})$ *for the formula with sign* $(+,-,+,-)$*,* $(\overbrace{8,6}^{+},\overbrace{2,2}^{-})$ *for the formula with sign* $(+,+,-,-)$*,* $(\overbrace{8,6}^{-},\overbrace{2,2}^{-})$ *for the formula with sign* $(+,-,-,+)$*.*

For another example, let $\lambda = (2,2,1,1,0)$ *in* $O(12,\mathbb{C})$*. Then* $E^0 = (2,1,0)$ *and* $T = (2,1)$*. Then the above Proposition says* $\mathcal{O}' = (6,4)$*, the number of irreducible unipotent representations is two, having character formulas:*

$$\sum_{w\in W(C_2)}(-1)^{l(w)}X\begin{pmatrix} & 210+ & 21 \\ w(& 210+ & 21) \end{pmatrix} \text{ and } \sum_{w\in W(C_2)}(-1)^{l(w)}X\begin{pmatrix} & 210- & 21 \\ w(& 210- & 21) \end{pmatrix}$$

Again, the first formula will be denoted $(\overbrace{6}^{+},4)$*, and the second formula will be denoted* $(\overbrace{6}^{-},4)$*.*

5.1.2 Half-Integral Infinitesimal Character

After dealing with the case of purely integral infinitesimal characters, we move to the case of purely half-integral infinitesimal characters. This is an important ingredient for such theory:

Theorem 5.10 (Kazhdan-Lusztig Conjecture). *For each $\lambda \in^L \mathfrak{h}$, consider the root system $\Delta(\lambda) := \{\alpha | \langle \check{\alpha}, \lambda \rangle \in \mathbb{Z}\}$. Then the character theory of unipotent representations in $\mathfrak{g}$ can be derived from that of $\Delta(\lambda)$.*

The conjecture was proved separately by Kashiwara-Vergne and Beilinson-Bernstein.

By the Kazhdan-Lusztig conjecture, we study the character theory of unipotent representations with half-integral infinitesimal characters in $W(D_m) \leq W(C_m)$ (note that the Kazhdan-Lusztig conjecture is vacuous for half-integral infinitesimal characters in the $W(D_n)$ and $W(B_n)$ cases). The following two Propositions are the analogues of the ones in last subsection:

Proposition 5.11. *Let $G = Sp(2m, \mathbb{C})$ and $\lambda = (\frac{a_1}{2}, \frac{a_1}{2} - 1, \dots \frac{1}{2}; \frac{a_2}{2} \dots, \frac{1}{2}; \dots)$ be of the form given in Proposition 4.11. Replace λ by $w\lambda$ for some Weyl group element $w \in W$ such that $w\lambda = (\frac{a_1}{2}, \frac{a_1}{2} - 1, \dots, \frac{a_1}{2} - 1, \frac{a_1}{2} - 2, \dots, \frac{a_1}{2} - 2, \dots, \frac{1}{2}, \dots, \frac{1}{2})$. Then extract one coordinate from each entry, i.e. $(\frac{a_1}{2}, \frac{a_1}{2} - 1, \dots, \frac{1}{2})$, and then adjoin it with the negatives of coordinates. This forms the coordinates of F^1. Similarly, form F^2, F^3 and so on from the remainder. Write*

$$F^i = (\frac{\alpha_i}{2}, \dots, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \dots, -\frac{\beta_i}{2})$$

then by construction, $\alpha_i \geq \beta_i \geq \alpha_{i+1} - 1$ for all i. If $\alpha_i \neq \beta_i$, let $c_i := (\alpha_i, \beta_i + 2)$ (if there are no negative entries in F^i, take $\beta_i = -1$). If $\alpha_i = \beta_i$, let $c_i := (\alpha_i + 1, \alpha_i + 1)$. Write $\mathcal{O}' = (c_1, c_2, \dots)$, then

$$AV(U(\mathfrak{g})/J_{max}(\lambda)) = \overline{\mathcal{O}'}.$$

Example 5.12. *Let $\mathcal{O} = (5, 3, 3, 1)$ in $Sp(12, \mathbb{C})$. Then by Proposition 4.11, $\lambda = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}; \frac{1}{2}; \frac{3}{2}, \frac{1}{2})$. Rearrange the entries so that $w\lambda = (\frac{5}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Now $F^1 = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2})$, $F^2 = (\frac{1}{2})$; so $c_1 = (5, 5)$, $c_2 = (1, 1)$ and hence $AV(U(\mathfrak{g})/J_{\max}(\lambda)) = \overline{\mathcal{O}'} = \overline{(5, 5, 1, 1)}$. Note again that $\overline{\mathcal{O}'} \subset \overline{\mathcal{O}}$.*

Proposition 5.13. *Let $G = O(n, \mathbb{C})$ and $\lambda = (\frac{b_0}{2}, \dots \frac{1}{2}, ; \frac{b_1}{2} \dots \frac{1}{2}; \frac{c_1}{2}, \dots, \frac{1}{2}; \dots)$ be of the form in Proposition 4.11. Rearrange the entries of λ in non-increasing order as in the above Proposition. Then extract one coordinate from each entry to form E^0. For the remainder, extract F^1, F^2 and so on as in the above Proposition. For $E^0 = (\frac{x}{2}, \frac{x}{2} - 1, \dots, \frac{1}{2})$, let $c_0 = x + 2$ and c_i is defined as above. Then*

$$AV(U(\mathfrak{g})/J_{max}(\lambda)) = \overline{\mathcal{O}'}$$

where $\mathcal{O}' = (c_0, c_1, c_2, \dots)$.

Proposition 5.14. *Let $G = Sp(2m, \mathbb{C})$ or $O(2n, \mathbb{C})$, and λ, and let λ and $\mathcal{O}' = (c_0, c_1, c_2, \dots)$ as above (omit c_0 if $G = Sp(2m, \mathbb{C})$). Denote $U \subset \{1, 2, \dots\}$ the subset satisfying $\alpha_i \neq \beta_i$ for all $i \in U$. Then the unipotent representations are parametrized by*

$$\{(\overbrace{c_0}^{\pm}, \overbrace{c_1}^{\pm}, \overbrace{c_2}^{\pm}, \dots)\}$$

where the sign of c_i for $i \notin U$ is always $+$ (and hence can be omitted). In particular, if $G = Sp(2m, \mathbb{C})$, the number of unipotent representations having infinitesimal character λ is $2^{s_{\mathcal{O}'}}$. Similarly, for $G = O(2n, \mathbb{C})$, the number of unipotent representations having infinitesimal character λ is $2^{s_{\mathcal{O}'}+1}$.

The character formula, however, is different from the integral case. It can be better expressed by examples:

Example 5.15. *Let $\lambda = (\frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$ in $Sp(12, \mathbb{C})$. Then $F^1 = (\frac{7}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2})$ and $c_1 = (7, 5)$. Therefore, $\mathcal{O}' = (7, 5)$ and there are two unipotent representations - $(\overbrace{7, 5}^{+})$ and $(\overbrace{7, 5}^{-})$. The character formulas are*

$$\sum_{w \in W(D_4 \times D_2)} (-1)^{l(w)} X \begin{pmatrix} & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} \\ w(& \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2}) \end{pmatrix} - \sum_{w \in W(D_4 \times D_2)} (-1)^{l(w)} X \begin{pmatrix} & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} \\ w(& \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{-1}{2} & \frac{3}{2}\frac{-1}{2}) \end{pmatrix}$$

and

$$\sum_{w \in W(D_4 \times D_2)} (-1)^{l(w)} X \begin{pmatrix} & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} \\ w(& \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{-1}{2} & \frac{3}{2}\frac{1}{2}) \end{pmatrix} - \sum_{w \in W(D_4 \times D_2)} (-1)^{l(w)} X \begin{pmatrix} & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} \\ w(& \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{-1}{2}) \end{pmatrix}$$

respectively.

For another example, consider $\lambda = (\frac{5}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in $Sp(16, \mathbb{C})$. Now $F^1 = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-3}{2})$ and $F^2 = (\frac{3}{2}, \frac{1}{2}, \frac{-1}{2})$. Therefore, $c_1 = (5, 5)$, $c_2 = (3, 3)$ and $\mathcal{O}' = (5, 5, 3, 3)$. There is a total of four unipotent representations, denoted by $(\overbrace{5, 5}^{\pm}, \overbrace{3, 3}^{\pm})$. For example, the character formula of $(\overbrace{5, 5}^{-}, \overbrace{3, 3}^{+})$ is given by

$$\sum_{w \in W(D_3 \times D_2 \times D_2 \times D_1)} (-1)^{l(w)} X \begin{pmatrix} & \frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} & ; & \frac{3}{2}\frac{1}{2} & \frac{1}{2} \\ w(& \frac{5}{2}\frac{3}{2}\frac{-1}{2} & \frac{3}{2}\frac{1}{2} & ; & \frac{3}{2}\frac{1}{2} & \frac{1}{2}) \end{pmatrix}$$

$$
- \sum_{w \in W(D_3 \times D_2 \times D_2 \times D_1)} (-1)^{l(w)} X \begin{pmatrix} & \frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} & ; & \frac{3}{2}\frac{1}{2} & \frac{1}{2} \\ w(& \frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{-1}{2} & ; & \frac{3}{2}\frac{1}{2} & \frac{1}{2}) \end{pmatrix}
$$

$$
- \sum_{w \in W(D_3 \times D_2 \times D_2 \times D_1)} (-1)^{l(w)} X \begin{pmatrix} & \frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} & ; & \frac{3}{2}\frac{1}{2} & \frac{1}{2} \\ w(& \frac{5}{2}\frac{3}{2}\frac{-1}{2} & \frac{3}{2}\frac{1}{2} & ; & \frac{3}{2}\frac{-1}{2} & \frac{-1}{2}) \end{pmatrix}
$$

$$
+ \sum_{w \in W(D_3 \times D_2 \times D_2 \times D_1)} (-1)^{l(w)} X \begin{pmatrix} & \frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} & ; & \frac{3}{2}\frac{1}{2} & \frac{1}{2} \\ w(& \frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{-1}{2} & ; & \frac{3}{2}\frac{-1}{2} & \frac{-1}{2}) \end{pmatrix}
$$

As the final example, consider $\lambda = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ *in* $O(9, \mathbb{C})$. *Then* $E^0 = (\frac{3}{2}, \frac{1}{2})$, $F^1 = (\frac{1}{2}, \frac{-1}{2})$, $c_0 = (5)$, $c_1 = (2,2)$ *and hence* $\mathcal{O}' = (5,2,2)$. *The unipotent representations can be expressed by* $(\overbrace{5}^{\pm}, 2, 2)$. *The character formulas are given by*

$$
\sum_{w \in W(D_1 \times D_1)} (-1)^{l(w)} X \begin{pmatrix} & \frac{3}{2}\frac{1}{2}\pm & ; & \frac{1}{2} & \frac{1}{2} \\ w(& \frac{3}{2}\frac{1}{2}\pm & ; & \frac{1}{2} & \frac{1}{2}) \end{pmatrix}
$$

$$
- \sum_{w \in W(D_1 \times D_1)} (-1)^{l(w)} X \begin{pmatrix} & \frac{3}{2}\frac{1}{2}\pm & ; & \frac{1}{2} & \frac{1}{2} \\ w(& \frac{3}{2}\frac{1}{2}\pm & ; & \frac{-1}{2} & \frac{-1}{2}) \end{pmatrix}
$$

5.1.3 General Case

In the previous subsections, we have seen the character theory of unipotent representations with purely integral or purely half-integral infinitesimal characters. In fact, by the Kazhdan-Lusztig Conjecture (Theorem 5.10), it is enough to derive the theory of unipotent representations out of these two cases. One just need to separate the integral and half-integral coordinates, apply the algorithms in the previous two sections, combine them together to get the result.

Example 5.16. *Let* $\mathcal{P} = (9,7,5,4,2)$ *in* $O(27,\mathbb{C})$. *Then* $\lambda = (\frac{7}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2};\frac{3}{2},\frac{1}{2};\frac{3}{2},\frac{1}{2};2,1;0)$. *The half integral infinitesimal character* $(\frac{7}{2},\frac{5}{2},\frac{3}{2},\frac{1}{2};\frac{3}{2},\frac{1}{2})$ *gives the orbit* $(9,7,5)$ *in* $O(21,\mathbb{C})$. *The four corresponding character formulas are*

$$(\overbrace{9}^{\pm},\overbrace{7,5}^{+}) = \sum_{W(D_4\times D_2)} (-1)^{l(w)} X \begin{pmatrix} & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\pm & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} \\ w(& \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\pm & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2}) \end{pmatrix}$$
$$- \sum_{W(D_4\times D_2)} (-1)^{l(w)} X \begin{pmatrix} & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\pm & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} \\ w(& \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\pm & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{-1}{2} & \frac{3}{2}\frac{-1}{2}) \end{pmatrix}$$

and

$$(\overbrace{9}^{\pm},\overbrace{7,5}^{-}) = \sum_{W(D_4\times D_2)} (-1)^{l(w)} X \begin{pmatrix} & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\pm & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} \\ w(& \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\pm & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{-1}{2} & \frac{3}{2}\frac{1}{2}) \end{pmatrix}$$
$$- \sum_{W(D_4\times D_2)} (-1)^{l(w)} X \begin{pmatrix} & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\pm & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} \\ w(& \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\pm & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{-1}{2}) \end{pmatrix}$$

For the integral part of the infinitesimal character, $(2,1,0)$, *the corresponding orbit is* $(4,2)$. *The two corresponding character formulas are* $(\overbrace{4,2}^{\pm})$, *of the form*

$$\frac{1}{2}\Big(\sum_{W(C_2\times D_1)} (-1)^{l(w)} X \begin{pmatrix} & 21 & 0 \\ w(& 21 & 0) \end{pmatrix} \pm \sum_{W(D_3\times C_0)} (-1)^{l(w)} X \begin{pmatrix} & 210 \\ w(& 210) \end{pmatrix}\Big)$$

Combining the two results, $\mathcal{O}'$ *for the infinitesimal character* λ *is* $(9,7,5,4,2)$ *(which happens to be the same as* $\mathcal{O}$*). The number of unipotent representations attached to* λ *is* $4\times 2 = 8$, *of the form*

$$(\overbrace{9}^{\pm},\overbrace{7,5}^{\pm},\overbrace{4,2}^{\pm})$$

and the eight character formulas are just the 'cocatenation' of the two character formulas above. For instance, the character of $(\overbrace{9}^{+}, \overbrace{7,5}^{+}, \overbrace{4,2}^{+})$ *is:*

$$\frac{1}{2}\sum_{W(D_4\times D_2\times C_2\times D_1)}(-1)^{l(w)}X\begin{pmatrix} & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}+ & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} & 21 & 0 \\ w(& \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}+ & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} & 21 & 0) \end{pmatrix}$$

$$-\frac{1}{2}\sum_{W(D_4\times D_2\times C_2\times D_1)}(-1)^{l(w)}X\begin{pmatrix} & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}+ & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} & 21 & 0 \\ w(& \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}+ & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{-1}{2} & \frac{3}{2}\frac{-1}{2} & 21 & 0) \end{pmatrix}$$

$$+\frac{1}{2}\sum_{W(D_4\times D_2\times D_3\times C_0)}(-1)^{l(w)}X\begin{pmatrix} & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}+ & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} & 210 \\ w(& \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}+ & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} & 210) \end{pmatrix}$$

$$-\frac{1}{2}\sum_{W(D_4\times D_2\times D_3\times C_0)}(-1)^{l(w)}X\begin{pmatrix} & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}+ & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2} & \frac{3}{2}\frac{1}{2} & 210 \\ w(& \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}+ & \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{-1}{2} & \frac{3}{2}\frac{-1}{2} & 210) \end{pmatrix}$$

CHAPTER 6

THE BARBASCH MODEL $X_{\mathcal{O}}$

In [6, Section 2], Barbasch constructed a $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-module of any nilpotent orbit of classical type, which is denoted as $X_{\mathcal{O}}$. The definition of $X_{\mathcal{O}}$ is given below:

Definition 6.1. *Let $\mathcal{O} = (d_k, \ldots, d_0)$ be any nilpotent orbit in G, where $G = Sp(2m, \mathbb{C})$ or $O(n, \mathbb{C})$. Define $\mathcal{E}$ and $\mathcal{Q}$ inductively as follows:*
Let $\mathcal{E}_0 = \phi$ and $\mathcal{Q}_0 = \mathcal{O}$. Suppose i is the smallest integer such that $d_i = d_{i+1}$, then $\mathcal{Q}_1 := (d_k, \ldots, d_{k+2}, d_{k-1}, \ldots, d_0)$ and $\mathcal{E}_1 := \mathcal{E}_0 \cup \{d_i\}$.
Continue the above process until we get $\mathcal{Q} = (e_l, \ldots, e_0)$, with $e_i \neq e_{i+1}$ for all i. Then $\mathcal{E}$ are the lengths of the removed columns with multiplicities. Let

$$X_{\mathcal{O}} := Ind_{G(\mathcal{Q},\mathbb{C}) \times GL(\mathcal{E})}^{G(2m,\mathbb{C})} (X_{\mathcal{Q}} \otimes |det|^{1/2})$$

where $G(\mathcal{Q}, \mathbb{C}) = G(\sum e_i, \mathbb{C})$, $GL(\mathcal{E}) = \Pi_{d_i \in \mathcal{E}} GL(d_i, \mathbb{C})$ and $X_{\mathcal{Q}}$ is the spherical unipotent representation attached to the nilpotent orbit $\mathcal{Q}$.

Theorem 6.2 (Barbasch)**.** *As $K_{\mathbb{C}} \cong G$-modules,*

$$X_{\mathcal{O}} \cong R[\mathcal{O}]$$

Example 6.3. *Suppose $\mathcal{O} = (d_{2l}, \ldots, d_0)$ is an orbit in $Sp(2m, \mathbb{C})$ such that $d_i \neq d_{i-1}$ for all i. Then $\mathcal{O} = \mathcal{Q}$, and $X_{\mathcal{Q}}$ is the spherical unipotent representation attached to $\mathcal{Q}$. Using the notations in Chapter 5,*

$$X_{\mathcal{Q}} = (\overbrace{d_{2l}, d_{2l-1}}^{+}, \ldots, \overbrace{d_2, d_1}^{+}, d_0)$$

where $\overbrace{p,q}^{+}$ *has character formula*

$$\frac{1}{2}\sum_{W(C_{p/2}\times D_{q/2+1})}(-1)^{l(w)}X\begin{pmatrix} & \frac{p}{2}\dots 1 \quad \frac{q}{2}\dots 0 \\ w(& \frac{p}{2}\dots 1 \quad \frac{q}{2}\dots 0)\end{pmatrix}$$

$$+\frac{1}{2}\sum_{W(D_{p/2+1}\times C_{q/2})}(-1)^{l(w)}X\begin{pmatrix} & \frac{p}{2}\dots 0 \quad \frac{q}{2}\dots 1 \\ w(& \frac{p}{2}\dots 0 \quad \frac{q}{2}\dots 1)\end{pmatrix}$$

if p, q *are both even, and*

$$\sum_{W(D_{p/2}\times D_{q/2-1})}(-1)^{l(w)}X\begin{pmatrix} & \frac{p}{2}\dots\frac{1}{2} \quad \frac{q-2}{2}\dots\frac{1}{2} \\ w(& \frac{p}{2}\dots\frac{1}{2} \quad \frac{q-2}{2}\dots\frac{1}{2})\end{pmatrix}$$

$$-\sum_{W(D_{p/2}\times D_{q/2-1})}(-1)^{l(w)}X\begin{pmatrix} & \frac{p}{2}\dots\frac{3}{2}\frac{1}{2} \quad \frac{q-2}{2}\dots\frac{3}{2}\frac{1}{2} \\ w(& \frac{p}{2}\dots\frac{3}{2}\frac{-1}{2} \quad \frac{q-2}{2}\dots\frac{3}{2}\frac{-1}{2})\end{pmatrix}$$

if p, q *are both odd. And* d_0 *has the character formula*

$$T := \sum_{w\in W(C_{d_0/2})}(-1)^{l(w)}X\begin{pmatrix} & \frac{d_0}{2},\dots,1 \\ w(& \frac{d_0}{2},\dots,1)\end{pmatrix}$$

Similarly, if $\mathcal{P} = (d_{2l+1}, d_{2l}, \dots, d_0)$ *is an orbit in* $O(n,\mathbb{C})$ *where all columns are of distinct lengths. Then the character of* $X_{\mathcal{P}}$ *is of the form*

$$X_{\mathcal{P}} \cong (\overbrace{d_{2l+1}}^{+}; \overbrace{d_{2l}, b_{2l-1}}^{+}; \dots; \overbrace{d_2, d_1}^{+}; d_0)$$

where the parts $(\overbrace{d_{2l}, d_{2l-1}}^{+}; \dots; \overbrace{d_2, d_1}^{+}; d_0)$ *is defined as in the* $Sp(2m,\mathbb{C})$ *case, and* $\overbrace{d_{2l+1}}^{+} = X\begin{pmatrix}\frac{d_{2l+1}}{2}-1,\dots,0+ \\ \frac{d_{2l+1}}{2}-1,\dots,0+\end{pmatrix}$ *if* b_{2d+1} *is even, and* $\overbrace{d_{2l+1}}^{+} = X\begin{pmatrix}\frac{d_{2l+1}}{2}-1,\dots,\frac{1}{2}+ \\ \frac{d_{2l+1}}{2}-1,\dots,\frac{1}{2}+\end{pmatrix}$ *if* d_{2l+1} *is odd.*

Remark 6.4. *The above construction also gives a $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-module of $\tilde{\mathcal{O}}$ for some covers $\tilde{\mathcal{O}} \to \mathcal{O}$. Recall in Remark 5.8 that for any orbit $\mathcal{Q}$, $\overline{A(\mathcal{Q})}$ is a quotient of $\pi_1(\mathcal{Q})$. However, $\pi_1(\mathcal{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^r$ and hence any quotient of it, in particular $\overline{A(\mathcal{Q})}$, can be treated as a subgroup. In fact, according [6], for any subgroup $S \leq (\overline{A(\mathcal{Q})})^\vee \cong \overline{A(\mathcal{Q})} \leq \pi_1(Q)$,*

$$X_{\tilde{\mathcal{Q}}} = \sum_{\chi \in S} X_\chi$$

corresponds to the model of a cover of the orbit $\tilde{\mathcal{Q}} \to \mathcal{Q}$. By replacing $X_{\mathcal{Q}}$ with $X_{\tilde{\mathcal{Q}}}$ in the induction formula in Definition 6.1, it gives the model of a cover of the orbit $\tilde{\mathcal{O}} \to \mathcal{O}$.

Theorem 6.5. *Let $\mathcal{O} = (c_{2k}, c_{2k-1}, c_{2k-2}, \dots c_0)$ be any nilpotent orbit in $Sp(2m, \mathbb{C})$ or $O(n, \mathbb{C})$. The infinitesimal character of $X_{\mathcal{O}}$ is the same as that of $X_{\overline{\mathcal{O}}}$.*

Proof. The infinitesimal character of $X_{\mathcal{O}}$ and $X_{\mathcal{P}}$ are precisely given in [6]. □

From last Chapter, the character of $X_{\mathcal{O}}$ is completely known. By Corollary 3.2, we know that $X_{\mathcal{O}}$ and $X_{\overline{\mathcal{O}}}$ are isomorphic as G-modules iff $\overline{\mathcal{O}}$ is normal. We therefore hope the following to be true:

- If $\overline{\mathcal{O}}$ is normal, then the composition factors of $X_{\overline{\mathcal{O}}}$ is the same as that of $X_{\mathcal{O}}$.
- If $\overline{\mathcal{O}}$ is not normal, then the composition factors of $X_{\overline{\mathcal{O}}}$ is strictly contained in the set of composition factors of $X_{\mathcal{O}}$.

Before we proceed, we must first of all investigate which irreducible representations can possibly be composition factors of $X_{\mathcal{O}}$ and $X_{\overline{\mathcal{O}}}$. And from the list of all possible irreducible representations, we need to determine which of them appear in $X_{\mathcal{O}}$ and $X_{\overline{\mathcal{O}}}$. This is essentially the work of the next couple of Chapters.

CHAPTER 7

EXHAUSTION OF COMPOSITION FACTORS

Recall in Chapter 4 and 6, we have quantization models of $\mathcal{O}$ and $\overline{\mathcal{O}}$. Their corresponding Harish-Chandra bimodules have the same infinitsimal characters. The natural question is, what are the composition factors of both models? In this Chapter, we will give a list of all possible composition factors for both models, along with their character formulas.

First of all, both quantization models $X_{\overline{\mathcal{O}}}$ and $X_{\mathcal{O}}$ have infinitesimal character $\lambda_{\mathcal{O}}$. From the calculations in Chapter 5, it is known that all irreducible representations with infinitesimal character $\lambda_{\mathcal{O}}$ must have associated variety bigger than or equal to $\overline{\mathcal{O}'}$. On the other hand, it is almost tautological that the associated varieties of $X_{\overline{\mathcal{O}}}$ and $X_{\mathcal{O}}$ are both $\overline{\mathcal{O}}$. Therefore the following observation comes at no surprise:

Proposition 7.1. *Let $\mathcal{O} = (c_{2k}, c_{2k-1}, \ldots, c_0)$ be a nilpotent orbit in $Sp(2m, \mathbb{C})$. Then the $\mathcal{O}'$ appearing in Chapter 5 is always contained in $\overline{\mathcal{O}}$. More precisely, $\mathcal{O}' = \mathcal{O}$ iff $c_{2i-1} \neq c_{2i-2}$ for all i.*
Similarly, let $\mathcal{P} = (c_{2k+1}, c_{2k}, \ldots, c_0)$ be a nilpotent orbit in $O(n, \mathbb{C})$. Then $\mathcal{P}'$ appearing in Chapter 5 is always contained in $\overline{\mathcal{P}}$. More precisely, $\mathcal{P}' = \mathcal{P}$ iff $c_{2i+1} \neq c_{2i}$ for all i.

Proof. It is just a direct consequence of the algorithm given in Chapter 5. □

Therefore, we are interested in all irreducible representations having associated varieties between $\overline{\mathcal{O}'}$ and $\overline{\mathcal{O}}$. For the case $\mathcal{O}' = \mathcal{O}$, all possible irreducible representations are unipotent representations given in Chapter 5, and their character theory is known. Hence we are interested in the case when $\overline{\mathcal{O}'} \subsetneq \overline{\mathcal{O}}$.

For the sake of keeping our computations and book-keeping clear, we would like to keep the focus on certain kinds of orbits in $Sp(2m,\mathbb{C})$ or $O(n,\mathbb{C})$. The condition is the following:

Let $\mathcal{O} = (d_k, d_{k-1}, \ldots, d_0)$ be an orbit in $Sp(2m,\mathbb{C})$ or $O(n,\mathbb{C})$. Then $\mathcal{O}$ satisfies condition (†) if

$$\text{Whenever } \; d_{i-1} = d_{i-2},\; d_i \neq d_{i-3} \qquad (\dagger)$$

7.1 A Special Case

In this Section, we focus ourselves on integral infinitesimal characters in $Sp(2m,\mathbb{C})$ satisfying (†). Then every orbit $\mathcal{O} = (c_{2k}, c_{2k-1}, \ldots, c_0)$ can be partitioned into

$$(c_{2k} = \overbrace{x_{k_1}, x_{k_1-1}, x_{k_1-1}, \ldots, x_1, x_1, x_0}, \overbrace{y_{k_2}, y_{k_2-1}, y_{k_2-1}, \ldots, y_0},$$

$$\ldots, \overbrace{z_{k_r}, z_{k_r-1}, z_{k_r-1}, \ldots,, z_0} = c_0)$$

where $x_i \neq x_j$ if $i \neq j$, and the 'tail' of each partition is not equal to the 'head' of its adjacent partition, for instance $x_0 \neq y_{k_2}$. Since we insist working on integral infinitesimal characters, all column sizes are even. And in particular we will work on the orbit

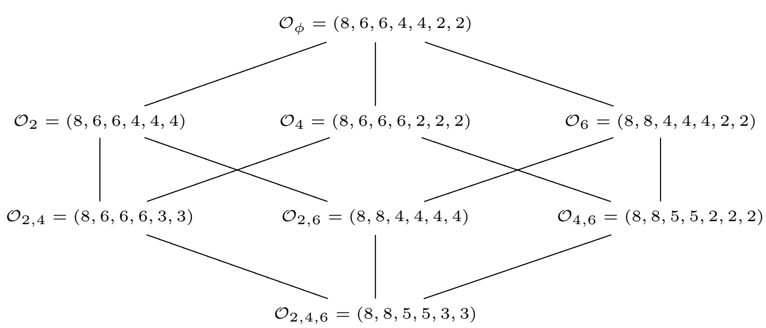
$$\mathcal{O} = (x_{k_1}, x_{k_1-1}, x_{k_1-1}, \ldots, x_1, x_1, x_0) = (2a_{n+1}, 2a_n, 2a_n, \ldots, 2a_1, 2a_1, 2a_0)$$

By Proposition 5.3, $\mathcal{O}' = (b'_{n+1}, b_n, b'_n\ , b_{n-1}, b'_{n-1}, \ldots, b_1, b'_1, b_0)$, where

- If $a_{n+1} > a_n$, then $b'_{n+1} = 2a_{n+1}, b_n = 2a_n + 2$; if $a_{n+1} = a_n$, then $b'_{n+1} = b_n = 2a_n + 1$;
- For i between 2 and $n-1$, if $a_{i+1} > a_i + 1$, then $b'_{i+1} = 2a_{i+1} - 2, b_i = 2a_i + 2$; if $a_{i+1} = a_i + 1$, then $b'_{i+1} = b_i = 2a_i + 1$; if $a_{i+1} = a_i$, then $b_{i+1} = b_i = 2a_i$.
- If $a_1 \neq a_0$, then $b_1 = 2a_1 + 2$, $b'_1 = 2a_1 - 2$, $b_0 = 2a_0$; if $a_1 = a_0$, then $b_1 = 2a_1 + 2$, $b'_1 = b_0 = a_0 - 1$.

which is the 'toppling' at columns of sizes $2a_1, 2a_2, \cdots, 2a_n$ of the partition $(2a_{n+1}, 2a_n, 2a_n, \cdots, 2a_1, 2a_1, 2a_0)$.

Example 7.2. *For $\mathcal{O} = (8,6,6,4,4,2,2)$, then $\mathcal{O}' = (8,8,5,5,3,3)$ and we have the following 'toppling' of nilpotent orbits:*



Note that each 'toppled' nilpotent orbit can be expressed as a subset of $\{2,4,6\}$, for instance $\{2,6\}$ corresponds to the orbit $\mathcal{O}_{2,6} = (8,8,4,4,4,4)$.

Therefore, all possible composition factors of $X_{\mathcal{O}}$ and $X_{\overline{\mathcal{O}}}$ must have associated variety equal to one of the above diagrams. They are all parametrized by a subset $S \subset \{2a_1, 2a_2, \cdots, 2a_n\}$, corresponding to the parts of partition to be 'toppled'. From now on, we will denote a nilpotent orbit $\mathcal{O}_S$ by specifying its 'toppled subset' S.

Proposition 7.3. *Let $\mathcal{O}$ be a nilpotent orbit in $Sp(2m,\mathbb{C})$ satisfying (†), and let $\lambda_{\mathcal{O}}$, $\mathcal{O}'$ as before. Suppose $\mathcal{O}_S = (d_k,\dots,d_0)$. Then the number of composition factors having infinitesimal character $\lambda_{\mathcal{O}}$ and associated variety $\overline{\mathcal{O}_S}$ is equal to 2^{α_S}, where α_S is the number of segments in $(d_k,\dots,d_0)$ of the form $(2p_r, 2p_{r-1}, 2p_{r-1},\dots,2p_1,2p_1,2p_0 \neq 0)$.*

Example 7.4. *Let $\mathcal{O} = (10,10,10,8,8,6,4,2,2)$ be a nilpotent orbit in Sp. Then $\mathcal{O}' = (11,11,9,9,6,6,4,4)$. The correspondence between $S \subset \{2,8,10\}$ and the orbits are given as follows:*

$$\begin{tikzcd}[column sep=small]
& \mathcal{O}_\phi = (10,10,10,8,8,6,4,2,2) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathcal{O}_2 = (10,10,10,8,8,6,4,4) \arrow[d, dash] \arrow[dr, dash] & \mathcal{O}_8 = (10,10,10,10,6,6,4,2,2) \arrow[dl, dash] \arrow[dr, dash] & \mathcal{O}_{10} = (11,11,8,8,8,6,4,2,2) \arrow[dl, dash] \arrow[d, dash] \\
\mathcal{O}_{2,8} = (10,10,10,10,6,6,4,4) \arrow[dr, dash] & \mathcal{O}_{2,10} = (11,11,8,8,8,6,4,4) \arrow[d, dash] & \mathcal{O}_{8,10} = (11,11,9,9,6,6,4,2,2) \arrow[dl, dash] \\
& \mathcal{O}_{2,8,10} = (11,11,9,9,6,6,4,4) &
\end{tikzcd}$$

The composition factors of each $\mathcal{O}_S$ are parametrized as follows:

$$\begin{tikzcd}[column sep=small]
& \mathcal{O}_\phi = (\overbrace{10,10,10,8,8,6}^{\pm},4,2,2) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathcal{O}_2 = (\overbrace{10,10,10,8,8,6}^{\pm},\overbrace{4,4}^{\pm}) \arrow[d, dash] \arrow[dr, dash] & \mathcal{O}_8 = (\overbrace{10,10,10,10}^{\pm},\overbrace{6,6}^{\pm},4,2,2) \arrow[dl, dash] \arrow[dr, dash] & \mathcal{O}_{10} = (11,11,\overbrace{8,8,8,6}^{\pm},4,2,2) \arrow[dl, dash] \arrow[d, dash] \\
\mathcal{O}_{2,8} = (\overbrace{10,10,10,10}^{\pm},\overbrace{6,6}^{\pm},\overbrace{4,4}^{\pm}) \arrow[dr, dash] & \mathcal{O}_{2,10} = (11,11,\overbrace{8,8,8,6}^{\pm},\overbrace{4,4}^{\pm}) \arrow[d, dash] & \mathcal{O}_{8,10} = (11,11,9,9,\overbrace{6,6}^{\pm},4,2,2) \arrow[dl, dash] \\
& \mathcal{O}_{2,8,10} = (11,11,9,9,\overbrace{6,6}^{\pm},\overbrace{4,4}^{\pm}) &
\end{tikzcd}$$

For example, $\alpha_{2,8} = 3$ and hence there are $2^3 = 8$ composition factors with infinitesimal character $\lambda_{\mathcal{O}}$ and associated variety $\overline{\mathcal{O}_{2,8}}$.

Proof. The number of irreducible representations having infinitesimal character $\lambda_{\mathcal{O}}$ and associated variety $\overline{\mathcal{O}_S}$ is given precisely by the multiplicity of the Weyl group representation

$$[Ind_{W(^L\mathfrak{m})}^{W}(triv) : V_{\sigma(\mathcal{O}_S)}]$$

where $^L\mathfrak{m}$ is defined in Proposition 5.3, $V_{\sigma(\mathcal{O}_S)}$ is the left cell containing the special Weyl group representation $\sigma(\mathcal{O}_S)$ ($\mathcal{O}_S$ and $\sigma(\mathcal{O}_S)$ are related to each other by the Springer correspondence). It is easy to check that $Ind_{W(^L\mathfrak{m})}^{W}(triv)$ has multiplicity one for each of the irreducible component in $V_{\sigma(\mathcal{O}_S)}$. Therefore, the number of irreducible representation is precisely the size of the left cell $V_{\sigma(\mathcal{O}_S)}$. And now the result follows from Proposition 4.14 in [4]. $\square$

What are the possible composition factors with infinitesimal character $\lambda_{\mathcal{O}}$ and associated variety equal to $\overline{\mathcal{O}_S}$? First of all, for all $2a_i \in \{2a_1, \ldots, 2a_n\}\backslash S$, remove the coordinates $(a_i, a_i - 1, a_i - 1, \ldots, 1, 1, 0)$ from $\lambda_{\mathcal{O}}$, and let $\lambda_{\mathcal{O}'_S}$ be the remaining entries. Now consider all unipotent representations X_j with infinitesimal character $\lambda_{\mathcal{O}'_S}$, and then take all induced representation of the form

$$\pi_{S,j} = Ind_{Sp \times GL(2(\{a_1, \cdots, a_n\}\backslash S))}^{Sp(2m)}(X_j \otimes \overbrace{det \otimes \cdots \otimes det}^{n-|S|\ terms})$$

(Note: In writing $GL(2S)$ for a set $S = \{a_{s_1}, \cdots, a_{s_k}\}$, we mean $GL(2a_{s_1}) \times \cdots \times GL(2a_{s_k})$.) Then by the calculations in Chapter 5, X_j will automatically have associated variety $\mathcal{O}'_S$, where $\mathcal{O}'_S$ is the orbit after removing the column pairs $\{(2a_j, 2a_j)|2a_j \in S\}$ in $\mathcal{O}_S$.

Example 7.5. *Let $\mathcal{O} = (10, 10, 10, 8, 8, 6, 4, 2, 2)$ and $\mathcal{O}_{10} = (11, 11, 8, 8, 8, 8, 6, 4, 2, 2)$ as in the above example. Then $\mathcal{O}'_{10} = (11, 11, 8, 6, 4)$, and the representations are of the form*

$$\pi_{10;\pm} = Ind_{Sp\times GL(2,\mathbb{C})\times GL(8,\mathbb{C})}^{Sp(2m)}((11, 11, \overbrace{8, 6}^{\pm}, 4) \otimes det \otimes det)$$

By construction, all the $\pi_{S,j}$'s have the required infinitesimal character and associated variety, and the number of such representations is equal to that in the Proposition. Therefore, it remains to show that they are irreducible.

Theorem 7.6. *The induced modules*

$$\pi_{S,j} := Ind_{Sp\times GL(2(\{a_1,\cdots,a_n\}\backslash S))}^{Sp(2m)}(X_j \otimes \overbrace{det \otimes \cdots \otimes det}^{n-|S|\ terms})$$

are irreducible. Consequently, they exhaust all the irreducible representions with infinitesimal character $\lambda_{\mathcal{O}}$ and associated variety $\overline{\mathcal{O}_S}$.

Proof. Before proving the Theorem, it is important to look at the lowest K-types of the induced modules $\pi_{S,j}$. They are of the form

$$(lowest\ K-type\ of\ X_j, \underbrace{1, 1, \ldots, 1}_{2\sum_{a_s\notin S} a_s}, 0, \ldots, 0)$$

To find out the lowest K-types of X_j as j runs through $\overline{A(\mathcal{O}'_S)}$, it is easy to see that $\lambda_{\mathcal{O}'_S}$ is **special unipotent** in the sense of [5, Definition 6.5], which means the number of $(x+1)$'s is greater than the number of x's in $\lambda_{\mathcal{O}'_S}$. Using the algorithm in [4, Section 9] or [5, 2.8], the lowest K-types of the special unipotent representations X_j must be **small** [B 1989 Definition 3.1], i.e. consisting of even number of 1's.

To conclude, the lowest K-type of $\pi_{S,j}$, and hence $\overline{X}$, must be

$$(\overbrace{1,\ldots,1}^{2x},\overbrace{1,\ldots,1}^{2y},0,\ldots,0)$$

where the lowest K-type of X_j is $(\overbrace{1,\ldots,1}^{2x},0,\ldots,0)$, and $y = \sum_{a_s \notin S} a_s$.

Consider the infinitesimal character of $\pi_{S,j}$: It must be of the form (λ, λ'), where $\lambda - \lambda' = (\underbrace{1,\ldots,1}_{2x+2y},0,\ldots,0)$. And in terms of Langlands parameter (which will be described in greater details in Chapter 11), the irreducible subquotient in $\pi_{S,j}$ can be written as $\overline{X} = \overline{X}(\lambda, \lambda')$. Let

$$\mu_z = (\overbrace{1,\ldots,1}^{2x},\overbrace{1,\ldots,1}^{2y},\overbrace{1,\ldots,1}^{2z}0,\ldots,0)$$

then μ_z is a **bottom-layer** K**-type** of $\overline{X}(\lambda, \lambda')$. There is a recipe in [5] computing the bottom layer K-type multiplicities of $\overline{X}(\lambda, \lambda')$, which we describe below: Let $(\lambda, \lambda') = (\mu + \mu', \mu + \mu'')$, where $\mu' - \mu'' = (\underbrace{1,\ldots,1}_{2x+2y},0,\ldots,0)$. Then

$$[\overline{X}(\lambda, \lambda') : \mu_z] = [\overline{X}(\mu, \mu) : (\overbrace{1,\ldots,1}^{2z},0,\ldots,0)]$$

The right hand side is the K-type multiplicity of a spherical representation, whose character formula is known (in fact, it is just $U(\mathfrak{g})/J_{max}(\mu)$, so the techniques in

Chapter 5 carry over). On the other hand, we know by Frobenius reciprocity the K-type multiplicity $[\pi_{S,j} : \mu_z]$, and it is easy to check the following holds:

$$[\overline{X}(\lambda, \lambda') : \mu_z] = [\pi_{S,j} : \mu_z]$$

Now we start to prove the Theorem by induction. Consider the smallest orbit $\mathcal{O}_S$, where $S = \{2a_1, \ldots, 2a_n\}$. All the representations in this case are special unipotent representations, and hence irreducible.

Suppose the hypothesis is true for all smaller orbits, i.e. $\pi_{S,j}$ are irreducible for $|S| > k$ for some integer k. For the case when $|S| = k$, suppose on the contrary that there is an irreducible Y with $AV(Y) = \overline{\mathcal{O}_{S'}} \subsetneq \overline{\mathcal{O}_S}$ such that

$$Y \oplus \overline{X} \subset \pi_{S,j}$$

then Y must be one of the smaller induced representations, having lowest K-type $(\underbrace{1, \ldots, 1}_{2w}, 0, \ldots, 0)$. On the other hand, suppose $\pi_{S,j}$ has lowest K-type $(\underbrace{1, \ldots, 1}_{2x+2y}, 0, \ldots, 0)$. By the inclusion relation among the modules, $w \geq x + y$. Note that $(\underbrace{1, \ldots, 1}_{2w}, 0, \ldots, 0)$ is a bottom layer K-type of $\overline{X} \subset \pi_{S,j}$. However, the formula above says

$$
\begin{aligned}
[\overline{X} : (\overbrace{1,\ldots,1}^{2w},0,\ldots,0)] &= [\pi_{S,j} : (\overbrace{1,\ldots,1}^{2w},0,\ldots,0)] \\
&\geq [\overline{X} : (\overbrace{1,\ldots,1}^{2w},0,\ldots,0)] + [Y : (\overbrace{1,\ldots,1}^{2w},0,\ldots,0)] \\
&= [\overline{X} : (\overbrace{1,\ldots,1}^{2w},0,\ldots,0)] + 1 \\
&> [\overline{X} : (\overbrace{1,\ldots,1}^{2w},0,\ldots,0)]
\end{aligned}
$$

which gives a contradiction. Consequently, such Y does not exist at all, and the induced module $\pi_{S,j}$ is equal to $\overline{X}$ exactly, i.e. it is irreducible. $\square$

Therefore, they exhaust all composition factors lying between $\mathcal{O}'$ and $\mathcal{O}$. More generally, suppose Condition (†) is lifted, then the orbits between $\mathcal{O}'$ and $\mathcal{O}$ can no longer be parametrized by a subset $S \subset \mathcal{S}$. However, the computation is precisely the same as before:

Proposition 7.7. *Let $\mathcal{O}$ be a nilpotent orbit in $Sp(2m,\mathbb{C})$ with even column sizes, and let $\lambda_{\mathcal{O}}$ and $\mathcal{O}'$ as before. For any $\mathcal{D} = (d_k,\ldots,d_0)$ between $\mathcal{O}'$ and $\mathcal{O}$, the number of composition factors having infinitesimal character $\lambda_{\mathcal{O}}$ and associated variety $\overline{\mathcal{D}}$ is equal to $2^{\alpha_{\mathcal{D}}}$, where $\alpha_{\mathcal{D}}$ is the number of segments in $(d_k,\ldots,d_0)$ of the form $(2p_r, 2p_{r-1}, 2p_{r-1},\ldots, 2p_1, 2p_1, 2p_0 \neq 0)$.*

Example 7.8. *Let $\mathcal{O} = (6,4,4,4,4,4,4,4,4,2,2)$. Then $\mathcal{O}' = (6,6,5,5,4,4,3,3,3,3)$*

and we have a diagram showing all composition factors between $\mathcal{O}'$ and $\mathcal{O}$:

$$
\begin{tikzcd}[column sep=tiny]
 & \mathcal{O}_\phi = (6,4,4,4,4,4,4,4,4,2,2) \arrow[dl, dash] \arrow[dr, dash] & \\
(\overbrace{6,6}^{\pm}, \overbrace{4,4,4,4,4,4}^{\pm}, 2,2,2) \arrow[d, dash] \arrow[drr, dash] & & (\overbrace{6,4,4,4,4,4,4,4,4,4}^{\pm}) \arrow[d, dash] \\
(\overbrace{6,6}^{\pm}, 5,5, \overbrace{4,4}^{\pm}, 3,3,2,2,2) \arrow[dr, dash] & & (\overbrace{6,6}^{\pm}, \overbrace{4,4,4,4,4,4}^{\pm}, 3,3) \arrow[dl, dash] \\
 & (\overbrace{6,6}^{\pm}, 5,5, \overbrace{4,4}^{\pm}, 3,3,3,3) &
\end{tikzcd}
$$

For instance, the irreducible representation $(\overbrace{6,6}^{+}, \overbrace{4,4,4,4,4,4}^{-}, 3,3)$ *is*

$$Ind_{Sp \times GL(4,\mathbb{C}) \times GL(4,\mathbb{C})}^{Sp(2m,\mathbb{C})}((\overbrace{6,6}^{+}, \overbrace{4,4}^{-}, 3,3) \otimes det \otimes det)$$

7.2 Half-integral Characters

The case of half-integral characters are exactly the same as that of integral characters in $Sp(2m, \mathbb{C})$. The computation on the number of irreducible representations is done by computing Weyl group representations of type D_m instead of type C_m. We therefore state the results without going into the computations:

Proposition 7.9. *Let $\mathcal{O}$ be a nilpotent orbit in $Sp(2m, \mathbb{C})$ with odd column sizes, and let $\lambda_{\mathcal{O}}$, $\mathcal{O}'$ as before. Suppose $\mathcal{D} = (d_k, \ldots, d_0)$. Then the number of composition factors having infinitesimal character $\lambda_{\mathcal{O}}$ and associated variety $\overline{\mathcal{D}}$ is equal to $2^{\alpha_{\mathcal{D}}}$, where $\alpha_{\mathcal{D}}$ is the number of segments in $(d_k, \ldots, d_0)$ of the form $(2p_r + 1, 2p_{r-1} + 1, 2p_{r-1} + 1, \ldots,$ $2p_1 + 1, 2p_1 + 1, 2p_0 + 1)$.*

Example 7.10. *Let $\mathcal{O} = (9, 9, 9, 7, 5, 3, 3, 1)$, the composition factors are given by:*

$$\begin{tikzcd}
 & \mathcal{O}_\phi = (\overbrace{9,9,9,7}^{\pm}, \overbrace{5,3,3,1}^{\pm}) \arrow[dl, dash] \arrow[dr, dash] & \\
(10, 10, \overbrace{7,7}^{\pm}, \overbrace{5,3,3,1}^{\pm}) \arrow[dr, dash] & & (\overbrace{9,9,9,7}^{\pm}, \overbrace{5,5}^{\pm}, \overbrace{1,1}^{\pm}) \arrow[dl, dash] \\
 & (10, 10, \overbrace{7,7}^{\pm}, \overbrace{5,5}^{\pm}, \overbrace{1,1}^{\pm}) &
\end{tikzcd}$$

7.3 The Case in the Orthogonal Group

As in last Chapter, the arguments are exactly the same for the orthogonal case. However, the statement of the Proposition is more involved.

Proposition 7.11. *Let $\mathcal{P}$ be a nilpotent orbit with even column sizes in $O(n, \mathbb{C})$, and let $\lambda_{\mathcal{P}}$, $\mathcal{P}'$ as before. For any orbits $\mathcal{D} = (d_{2k+1}, d_{2k}, \ldots, d_0)$ between $\mathcal{P}'$ and $\mathcal{P}$, the number of irreducible representations with infinitesimal character $\lambda_{\mathcal{P}}$ and associated variety $\overline{\mathcal{D}}$ is $2^{\alpha_{\mathcal{D}}}$, where $\alpha_{\mathcal{D}}$ is the number of segments in $(d_k, \ldots, d_0)$ of the form $(d_{2k+1} = 2q_r, 2q_r, 2q_{r-1}, 2q_{r-1}\ , \ldots, 2q_1, 2q_1, 2q_0)$ or $(d_{2i} = 2p_r, 2p_{r-1}, 2p_{r-1}, \ldots, 2p_1, 2p_1, 2p_0 \neq 0)$.*

Similarly, if $\mathcal{P}$ is a nilpotent orbit with even column sizes in $O(n, \mathbb{C})$. Then the number of irreducible representations with infinitesimal character $\lambda_{\mathcal{P}}$ and associated variety $\overline{\mathcal{D}}$ is $2^{\alpha_{\mathcal{D}}}$, where $\alpha_{\mathcal{D}}$ is the number of segments in $(d_k, \ldots, d_0)$ of the form $(d_{2k+1} = 2q_r + 1, 2q_r + 1, 2q_{r-1} + 1, 2q_{r-1} + 1\ , \ldots, 2q_1 + 1, 2q_1 + 1, 2q_0 + 1)$ or $(d_{2i} = 2p_r + 1, 2p_{r-1} + 1, 2p_{r-1} + 1, \ldots, 2p_1 + 1, 2p_1 + 1, 2p_0 + 1 \neq 0)$.

Example 7.12. *Let $\mathcal{P} = (4,4,2,2,2)$ in $O(12,\mathbb{C})$. Then $\mathcal{P}' = (6,3,3,1,1)$ and the composition factors are as follows:*

$$\begin{tikzcd}[column sep=small]
 & \mathcal{O}_\phi = (\overbrace{4,4,2,2,2}^{\pm}) \arrow[dl, dash] \arrow[dr, dash] & \\
(\overbrace{4,4,4}^{\pm},1,1) \arrow[dr, dash] & & (\overbrace{6}^{\pm},\overbrace{2,2,2,2}^{\pm}) \arrow[dl, dash] \\
 & (\overbrace{6}^{\pm},3,3,1,1) &
\end{tikzcd}$$

As the final example, suppose $\mathcal{P} = (7,7,7,6,4,4,2)$ be a nilpotent orbit in $O(37,\mathbb{C})$ with a mixture of odd and even columns. Then the composition factors are as follows:

$$\begin{tikzcd}[column sep=small]
 & \mathcal{O}_\phi = (\overbrace{7,7,7}^{\pm},\overbrace{6,4,4,2}^{\pm}) \arrow[dl, dash] \arrow[dr, dash] & \\
(\overbrace{9}^{\pm},6,6,\overbrace{6,4,4,2}^{\pm}) \arrow[dr, dash] & & (\overbrace{7,7,7}^{\pm},\overbrace{6,6}^{\pm}\overbrace{2,2}^{\pm}) \arrow[dl, dash] \\
 & (\overbrace{9}^{\pm},6,6,\overbrace{6,6}^{\pm},\overbrace{2,2}^{\pm}) &
\end{tikzcd}$$

Note that there are unbracketed $(6,6)$ on the left and at the bottom, since the unbracketed pair $(6,6)$ comes from collapsing the odd columns $(7,7,7)$.

CHAPTER 8

DECOMPOSITION OF $X_{\mathcal{O}}$

From last Chapter, we know all the possible composition factors of $X_{\mathcal{O}}$ and $X_{\overline{\mathcal{O}}}$. More precisely, we know the character formulas of all such factors. On the other hand, from Chapter 6, we know the character formula of $X_{\mathcal{O}}$. Now it remains an exercise to compute which composition factors appear in $X_{\mathcal{O}}$. As in last Chapter, we begin with the special case of $\mathcal{O}$, and then generalize it to all orbits of classical type.

8.1 A Special Case Revisited

Throughout this Section, we consider orbits in $Sp(2m,\mathbb{C})$ having integral infinitesimal character and satisfying Condition (†). Here is the Decomposition Theorem for all such orbits:

Theorem 8.1. *Let $\mathcal{O} = (d_k, d_{k-1}, \ldots, d_0)$ be a nilpotent orbit in $Sp(2m,\mathbb{C})$ with columns of even sizes and satisfies (†). The composition factors $\pi_{S,j}$ appearing in $X_{\mathcal{O}}$ can be computed as follows:*
(0) *There is exactly one composition factor π_ϕ appearing in $X_{\mathcal{O}}$. This is determined by taking out the parts $(2a_{n+1}, 2a_n, 2a_n, \ldots, 2a_1, 2a_1, 2a_0)$ of $\mathcal{O} = \mathcal{O}_\phi$, where a_i is allowed to be equal to a_{i-1}, and we put $a_0 = 0$ if necessary. If $a_0 \neq 0$, we assign the sign $(-1)^n$ on each part.*

(1) *Suppose* $\overline{\mathcal{O}_S} \supset \overline{\mathcal{O}_{S'}}$ *is not normal in codimension two, then for the toppled part*

$$(2a_{n+1}, 2a_n, 2a_n, \ldots, 2a_1, 2a_1, 2a_0) \to (2a_{n+1}, \ldots, 2a_{i+1}, 2a_i + 2, 2a_i - 2, 2a_{i-1}, \ldots)$$

(a) If the sign assigned to $(2a_{n+1}, 2a_n, 2a_n, \ldots, 2a_1, 2a_1, 2a_0)$ *in* $\mathcal{O}_S$ *is* $+$, *then assign* $+, -$ *and* $-, +$ *to the parts* $(2a_{n+1}, \ldots, 2a_{i+1}, 2a_{i+1}, 2a_i + 2)$ *and* $(2a_i - 2, 2a_{i-1}, 2a_{i-1}, \ldots, 2a_0)$ *in* $\mathcal{O}_{S'}$ *respectively.*

(b) If the sign assigned to $(2a_{n+1}, 2a_n, 2a_n, \ldots, 2a_1, 2a_1, 2a_0)$ *in* $\mathcal{O}_S$ *is* $-$, *then assign* $+, +$ *and* $-, -$ *to the parts* $(2a_{n+1}, \ldots, 2a_{i+1}, 2a_{i+1}, 2a_i + 2)$ *and* $(2a_i - 2, 2a_{i-1}, 2a_{i-1}, \ldots, 2a_0)$ *in* $\mathcal{O}_{S'}$ *respectively.*

(c) If no sign is assigned to $(2a_{n+1}, 2a_n, 2a_n, \ldots, 2a_1, 2a_1, 2a_0)$ *in* $\mathcal{O}_S$, *i.e.* $a_0 = 0$, *then assign* $\pm$ *to the part* $(2a_{n+1}, \ldots, 2a_{i+1}, 2a_{i+1}, 2a_i + 2)$ *in* $\mathcal{O}_{S'}$.

(2) *Suppose* $\overline{\mathcal{O}_S} \supset \overline{\mathcal{O}_{S'}}$ *is normal in codimension two, and* $\alpha_S = \alpha_{S'}$. *Then it is either of the form*

$$(2a_{n+1}, 2a_n, 2a_n, \ldots, 2a_1, 2a_1, 2a_1) \to (2a_{n+1}, \ldots, 2a_2, 2a_2, 2a_1 + 2, 2a_1 - 1, 2a_1 - 1)$$

or

$$(2a_n, 2a_n, 2a_n, \ldots, 2a_1, 2a_1, 2a_0) \to (2a_n + 1, 2a_n + 1, 2a_n - 2, 2a_{n-1}, 2a_{n-1}, \ldots, 2a_0)$$

In both cases, we assign a sign to parts $(2a_{n+1}, \ldots, 2a_2, 2a_2, a_1 + 2)$ *or* $(2a_n - 2, 2a_{n-1}, 2a_{n-1}, \ldots, 2a_0)$ *in* $\mathcal{O}_{S'}$ *that is opposite to that assigned to* $(2a_{n+1}, 2a_n, 2a_n, \ldots, 2a_1, 2a_1, 2a_1)$.

(3) *Suppose* $\overline{\mathcal{O}_S} \supset \overline{\mathcal{O}_{S'}}$ *is normal in codimension two, and* $2\alpha_S = \alpha_{S'}$. *Then it must be of the form*

$$(2a_1, 2a_1, 2a_1, 2a_1) \to (2a_1 + 1, 2a_1 + 1, 2a_1 - 1, 2a_1 - 1)$$

Then we cancel the sign assigned to $(2a_1, 2a_1, 2a_1, 2a_1)$ *of the larger orbit. However, if the part* $(2a_1, 2a_1, 2a_1, 2a_1)$ *comes from the toppling of a non-normal orbit, e.g.* $(8,6,6,4,4,4) \xrightarrow{1} (8,8)(4,4,4,4) \xrightarrow{3} (8,8)(5,5,3,3)$*, then assign a sign to the remaining part* $(8,8)$ *that is the same as that appearing in* $(8,6,6,4,4,4)$.

Example 8.2. *Let* $\mathcal{O} = (10,10,10,8,8,6,4,2,2)$. *According to the Theorem, the composition factors of* $X_{\mathcal{O}}$ *are listed below:*

$$
\begin{tikzcd}[column sep=tiny]
 & \mathcal{O}_\phi = (\overbrace{10,10,10,8,8,6}^{+},4,2,2) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathcal{O}_2 = (\overbrace{10,10,10,8,8,6}^{+},\overbrace{4,4}^{\pm}) \arrow[d, dash] \arrow[dr, dash] & \mathcal{O}_8 = (\overbrace{10,10,10,10,6,6}^{+-,-+},4,2,2) \arrow[dl, dash] \arrow[dr, dash] & \mathcal{O}_{10} = (11,11,\overbrace{8,8,8,6}^{-},4,2,2) \arrow[dl, dash] \arrow[d, dash] \\
\mathcal{O}_{2,8} = (\overbrace{10,10,10,10,6,6}^{+-,-+},\overbrace{4,4}^{\pm}) \arrow[dr, dash] & \mathcal{O}_{2,10} = (11,11,\overbrace{8,8,8,6}^{-},\overbrace{4,4}^{\pm}) \arrow[d, dash] & \mathcal{O}_{8,10} = (11,11,9,9,\overbrace{6,6}^{+},4,2,2) \arrow[dl, dash] \\
 & \mathcal{O}_{2,8,10} = (11,11,9,9,\overbrace{6,6}^{+},\overbrace{4,4}^{\pm}) &
\end{tikzcd}
$$

We do not give a proof of the Theorem here, but instead illustrate the computations with an example. The following Lemma is the key of the computations:

Lemma 8.3.
(a) $$\sum_{w \in Sp \times O} (-1)^{l(w)} X \left(\begin{array}{cccc} & n \ldots 1 & ; & (n-1) \ldots 0 \\ w(& n \ldots 1 & ; & (n-1) \ldots 0 \quad) \end{array} \right) =$$

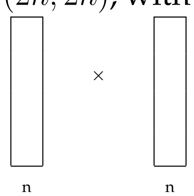
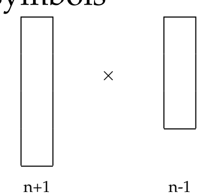
$$\sum_{w \in GL} (-1)^{l(w)} X \left(\begin{array}{cc} & n \ldots -(n-1) \\ w(& n \ldots -(n-1) \quad) \end{array} \right) + \sum_{w \in GL} (-1)^{l(w)} X \left(\begin{array}{cc} & n \ldots -(n-1) \\ w(& (n-1) \ldots -n \quad) \end{array} \right)$$

(b) $\displaystyle\sum_{w\in O\times Sp}(-1)^{l(w)}X\begin{pmatrix} & n\dots 0 & ; & (n-1)\dots 1 & \\ w(& n\dots 0 & ; & (n-1)\dots 1 &) \end{pmatrix} =$

$$\sum_{w\in GL}(-1)^{l(w)}X\begin{pmatrix} & n\dots -(n-1) & \\ w(& n\dots -(n-1) &) \end{pmatrix} - \sum_{w\in GL}(-1)^{l(w)}X\begin{pmatrix} & n\dots -(n-1) & \\ w(& (n-1)\dots -n &) \end{pmatrix}$$

(c) $\displaystyle\sum_{w\in O\times Sp}(-1)^{l(w)}X\begin{pmatrix} & m\dots 0 & ; & m\dots 1 & \\ w(& m\dots 0 & ; & m\dots 1 &) \end{pmatrix} = \sum_{w\in GL}(-1)^{l(w)}X\begin{pmatrix} & m\dots -m & \\ w(& m\dots -m &) \end{pmatrix}$

Proof. One can either do it combinatorially, or use the idea in [4]. For (a), note that $Ind_{GL(2n)}^{Sp(4n)}(triv)$, $Ind_{GL(2n)}^{Sp(4n)}(det)$ are special unipotent with associated variety equal to $(2n, 2n)$, with Lusztig symbols

() and (). By [4, Theorem III (a)]

$$\sum_{w\in Sp\times O}(-1)^{l(w)}X\begin{pmatrix} & n\dots 1 & ; & (n-1)\dots 0 & \\ w(& n\dots 1 & ; & (n-1)\dots 0 &) \end{pmatrix} = Ind_{GL(2n)}^{Sp(4n)}(triv)+Ind_{GL(2n)}^{Sp(4n)}(det)$$

$$\sum_{w\in O\times Sp}(-1)^{l(w)}X\begin{pmatrix} & n\dots 0 & ; & (n-1)\dots 1 & \\ w(& n\dots 0 & ; & (n-1)\dots 1 &) \end{pmatrix} = Ind_{GL(2n)}^{Sp(4n)}(triv)-Ind_{GL(2n)}^{Sp(4n)}(det)$$

Hence (a), (b) follows. For (c), use $Ind_{GL(2n+1)}^{Sp(4n+2)}(triv)$ instead. □

Example 8.4. *Consider* $\mathcal{O} = (8, 6, 6, 4, 4)$*. Then* $\lambda_{\mathcal{O}} = (4, 3, 2, 1; 3, 2, 1, 0, 1, 2; 2, 1, 0, 1)$

and $\mathcal{O}' = (8,8,5,5,2)$*. By the Theorem, the composition factors of* $X_{\mathcal{O}}$ *is given by*

$$\begin{tikzcd}[column sep=tiny]
 & \mathcal{O}_\phi = (8,6,6,4,4) \arrow[dl, no head] \arrow[dr, no head] & \\
\mathcal{O}_6 = (\overbrace{8,8}^{\pm},4,4,4) \arrow[dr, no head] & & \mathcal{O}_4 = (\overbrace{8,6,6,6}^{\pm},2) \arrow[dl, no head] \\
 & \mathcal{O}_{4,6} = (\overbrace{8,8}^{\pm},5,5,2) &
\end{tikzcd}$$

Now, the formula in the previous chapters gives

$$(8,6,6,4,4) = \sum_{w\in W(C_4\times A_5\times A_3)} (-1)^{l(w)} X \begin{pmatrix} & 4321 & 3210-1-2 & 210-1 \\ w(& 4321 & 210-1-2-3 & 10-1-2) \end{pmatrix}$$

$$(\overbrace{8,8}^{+},4,4,4)+(\overbrace{8,8}^{-},4,4,4) = \sum_{w\in W(C_4\times D_4\times A_3\times C_2)} (-1)^{l(w)} X \begin{pmatrix} & 4321 & 3210 & 210-1 & 21 \\ w(& 4321 & 3210 & 10-1-2 & 21) \end{pmatrix}$$

$$(\overbrace{8,6,6,6}^{+},2)+(\overbrace{8,6,6,6}^{-},2) = \sum_{w\in W(C_4\times D_3\times A_5\times C_1)} (-1)^{l(w)} X \begin{pmatrix} & 4321 & 210 & 3210-1-2 & 1 \\ w(& 4321 & 210 & 210-1-2-3 & 1) \end{pmatrix}$$

$$(\overbrace{8,8}^{+},5,5,2)+(\overbrace{8,8}^{-},5,5,2) = \sum_{w\in W(C_4\times D_4\times C_2\times D_3\times C_1)} (-1)^{l(w)} X \begin{pmatrix} & 4321 & 3210 & 21 & 210 & 1 \\ w(& 4321 & 3210 & 21 & 210 & 1) \end{pmatrix}$$

Using the above Lemma, the sum of the first two lines gives

$(8,6,6,4,4) + (\overbrace{8,8}^{+},4,4,4) + (\overbrace{8,8}^{-},4,4,4)$

$= \sum_{w\in W(C_4\times A_5\times A_3)} (-1)^{l(w)} X \begin{pmatrix} & 4321 & 3210-1-2 & 210-1 \\ w(& 4321 & 3210-1-2 & 10-1-2) \end{pmatrix}$

and the sum of the last two lines gives

$(\overbrace{8,6,6,6}^{+},2) + (\overbrace{8,6,6,6}^{-},2) + (\overbrace{8,8}^{+},5,5,2) + (\overbrace{8,8}^{-},5,5,2)$
$= \sum_{w\in W(C_4\times D_3\times A_5\times C_1)} (-1)^{l(w)} X \begin{pmatrix} & 4321 & 210 & 3210-1-2 & 1 \\ w(& 4321 & 210 & 3210-1-2 & 1) \end{pmatrix}$

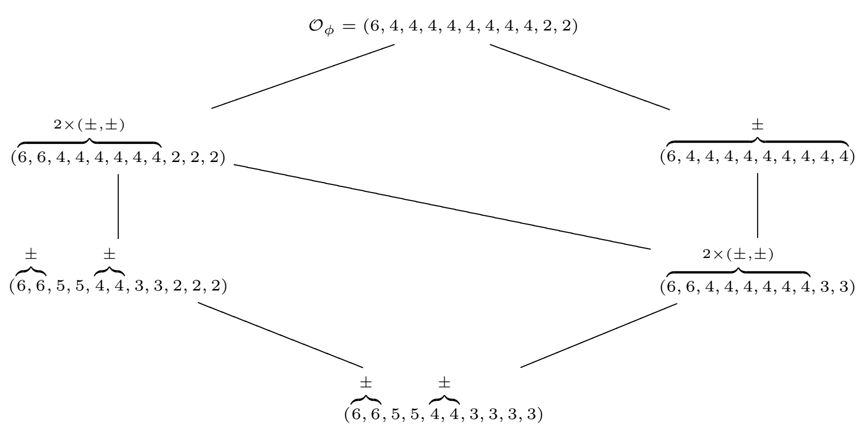
Now use the Lemma again to see the sum of the seven terms is equal to

$$\sum_{w\in W(C_4\times A_5\times A_3)} (-1)^{l(w)} X \begin{pmatrix} & 4321 & 3210-1-2 & 210-1 \\ w(& 4321 & 3210-1-2 & 210-1) \end{pmatrix}$$

which is precisely the character formula of $X_{\mathcal{O}}$.

Note that the composition factors of $X_{\mathcal{O}}$ is always of multiplicity one in the above description. This is no longer true if (†) does not hold. Also, there are no nice description of the composition factors of $X_{\mathcal{O}}$ in the general case.

Example 8.5. *Let $\mathcal{O} = (6,4,4,4,4,4,4,4,4,2,2)$. The composition factors of $X_{\mathcal{O}}$ are given by*



Note that, for instance, the irreducible representation $(\overbrace{6,6}^{+},\overbrace{4,4,4,4,4,4}^{+},2,2,2)$ appears twice in the composition factors of $X_{\mathcal{O}}$.

8.2 Half-integral Characters

The computations for the case of half-integral infinitesimal characters is totally different, due to the difference in the character formulas. However, the description of the composition factors of $X_{\mathcal{O}}$ is completely analogous to that of the integral character case. We give an example below to give its resemblance to the integral case:

Example 8.6. *Let $\mathcal{O} = (9,9,9,7,5,3,3,1)$, the composition factors are given by:*

$$\begin{tikzcd}[column sep=tiny]
 & \mathcal{O}_\phi = (\overbrace{9,9,9,7}^{-},\overbrace{5,3,3,1}^{-}) \arrow[dl, dash] \arrow[dr, dash] & \\
(10,10,\overbrace{7,7}^{+},\overbrace{5,3,3,1}^{-}) \arrow[dr, dash] & & (\overbrace{9,9,9,7}^{-},\overbrace{5,5,1,1}^{++,--}) \arrow[dl, dash] \\
 & (10,10,\overbrace{7,7}^{+},\overbrace{5,5,1,1}^{++,--}) &
\end{tikzcd}$$

Notice that the assignment of signs is the same as that of $\mathcal{Q} = (10,10,10,8,6,4,4,2)$, which is obtained by adding one extra block on each column. This phenomenon holds for all orbits in $Sp(2m,\mathbb{C})$ with odd column sizes.

8.3 The Case in the Orthogonal Group

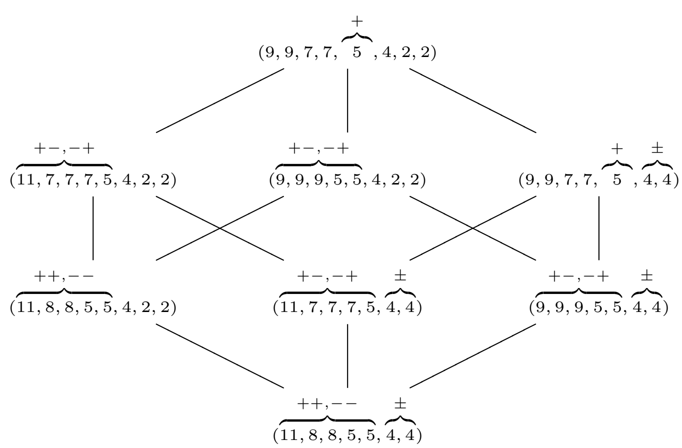
The computations for the symplectic group case can carry over in the same fashion to the orthogonal case. Here is the algorithm on which composition factors appear in $X_{\mathcal{P}}$ for any nilpotent orbit $\mathcal{P}$ in $O(n,\mathbb{C})$.

Theorem 8.7. *Let $\mathcal{P} = (b_{2k+1}, b_{2k}, \dots, b_0)$ be a nilpotent orbit in $O(n, \mathbb{C})$ satisfying (†). Consider the 'head' of the orbit*

$$(b_{2k+1} = q_r = b_{2k}, b_{2k-1} = q_{r-1} = b_{2k-2}, \dots, b_{2k-2r+3} = q_1 = b_{2k-2r+2}, b_{2k-2r+1} = q_0 \neq b_{2k-2r})$$

Define $\mathcal{Q} = (q_r + 2, q_r, q_r, q_{r-1}, \dots, q_0)$ be an orbit in $Sp(2m, \mathbb{C})$. Then the assignment of signs of the head of the orbit $\mathcal{P}$ is the same as that of $\mathcal{Q}$. For the 'tail' of the orbit $(b_{2k-2r}, \dots, b_0)$, use the same rule as in the symplectic case (Theorem 8.1) for the sign assignment.

Example 8.8. *Let $\mathcal{P} = (9, 9, 7, 7, 5, 4, 2, 2)$ in $O(45, \mathbb{C})$. Then $\mathcal{P}' = (11, 8, 8, 5, 5, 4, 4)$. The composition factors of $X_{\mathcal{P}}$ is as follows:*



Compare the above with the composition factors of $\mathcal{Q} = (11, 9, 9, 7, 7, 5, 4, 2, 2)$ below:

$$
\begin{tikzcd}[column sep=small]
& \overbrace{(11,9,9,7,7,5}^{+},4,2,2) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\overbrace{(11,11,7,7,7,5}^{+-,-+},4,2,2) \arrow[d, dash] \arrow[dr, dash] & \overbrace{(11,9,9,9,5,5}^{+-,-+},4,2,2) \arrow[dl, dash] \arrow[dr, dash] & \overbrace{(11,9,9,7,7,5}^{+},\overbrace{4,4)}^{\pm} \arrow[dl, dash] \arrow[d, dash] \\
\overbrace{(11,11,8,8,5,5}^{++,--},4,2,2) \arrow[dr, dash] & \overbrace{(11,11,7,7,7,5}^{+-,-+},\overbrace{4,4)}^{\pm} \arrow[d, dash] & \overbrace{(11,9,9,9,5,5}^{+-,-+},\overbrace{4,4)}^{\pm} \arrow[dl, dash] \\
& \overbrace{(11,11,8,8,5,5}^{++,--},\overbrace{4,4)}^{\pm} &
\end{tikzcd}
$$

CHAPTER 9

COMPUTATION OF K-TYPE MULTIPLICITIES

In this Chapter, we describe how one can compute K-type multiplicities of a representation given its character formula. In particular, we will focus on the multiplicities of the fundamental representations of $G = Sp(2m, \mathbb{C})$ and $O(n, \mathbb{C})$.

Let μ be a finite-dimensional representation of G. Then μ is parametrized by its highest weight. By Frobenius reciprocity:

$$[X \begin{pmatrix} \lambda \\ \lambda' \end{pmatrix} : \mu] = [(\lambda - \lambda') : \mu|_T]$$

Note that by a Theorem of Parasarathy-Rao-Varadarajan [4, Proposition 1.8], one can replace (λ, λ') by $(w\lambda, w\lambda')$ for some $w \in W$ such that $\lambda - \lambda'$ is dominant. Therefore, the multiplicity is known once we apply Weyl character formula on μ. In our situation, all irreducible representations we are dealing with have character formula of the form $\pi = \sum (-1)^{l(w)} X \begin{pmatrix} \lambda \\ w\lambda \end{pmatrix}$. Hence we can compute $[\pi : \mu]$ in theory.

9.1 Computations in $Sp(2m, \mathbb{C})$

In general, one can find out the character formula of $\mu|_T$ directly from `LiE`. In this Section, we focus ourselves on the fundamental representations of $G = Sp(2m, \mathbb{C})$, namely $\mu = \mu_i = \wedge^i \mathbb{C}^{2m} / \wedge^{i-2} \mathbb{C}^{2m}$ for $i = 0, \dots, m$ (by convention, we let negative wedge powers be the trivial representation). In this case, the weights of μ_i lying on the dominant Weyl chamber and their multiplicities are known completely.

Lemma 9.1.

$$[\wedge^{2i}\mathbb{C}^{2m}|_T : (\overbrace{1, \dots, 1}^{j}, \overbrace{0, \dots, 0}^{m-j})] = \begin{cases} 0 & \textit{if } 2i < j \textit{ or } j \textit{ is odd} \\ \binom{m-j}{\frac{2i-j}{2}} & \textit{otherwise} \end{cases}$$

$$[\wedge^{2i+1}\mathbb{C}^{2m}|_T : (\overbrace{1, \dots, 1}^{j}, \overbrace{0, \dots, 0}^{m-j})] = \begin{cases} 0 & \textit{if } 2i+1 < j \textit{ or } j \textit{ is even} \\ \binom{m-1-j}{\frac{2i+1-j}{2}} & \textit{otherwise} \end{cases}$$

Proof. This can easily be seen by looking at the weights of $\wedge^i \mathbb{C}^{2m}$. $\square$

Example 9.2. *We can now compute the K-type multiplicities of some unipotent representations. Let $\pi = (\overbrace{8,4}^{+}) + (\overbrace{8,4}^{-})$ be a representation in $Sp(12, \mathbb{C})$. Then*

$$(\overbrace{8,4}^{+}) + (\overbrace{8,4}^{-}) = \sum_{w \in W(C_4 \times D_2)} (-1)^{l(w)} X \begin{pmatrix} & 4321 & 10 \\ w(& 4321 & 10) \end{pmatrix}$$

To find out $[\pi : \wedge^2 \mathbb{C}^{12}]$, for example, one needs to find out which $w \in W(C_4 \times D_2)$ so that $(4321, 10) - w(4321, 10)$ can be W-conjugated to have weight (110000). The list of all such $w(4321, 10)$ is given below:

$w\lambda$	$\lambda - w\lambda$
3421,10	*1-10000*
4231,10	*01-1000*
4312,10	*001-100*
4321,01	*00001-1*
4321,0-1	*000011*

Therefore,

$$\begin{aligned}
[\pi : \wedge^2\mathbb{C}^{12}] &= [\sum_w (-1)^{l(w)} X \begin{pmatrix} & 4321 & 10 \\ w(& 4321 & 10) \end{pmatrix} : \wedge^2\mathbb{C}^{12}] \\
&= \sum_w (-1)^{l(w)} [X \begin{pmatrix} & 4321 & 10 \\ w(& 4321 & 10) \end{pmatrix} : \wedge^2\mathbb{C}^{12}] \\
&= [1 \times (000000) - 5 \times (110000) + \cdots : \wedge^2\mathbb{C}^{12}|_T]
\end{aligned}$$

by the above Lemma,

$$\wedge^2\mathbb{C}^{12}|_T = 1 \times (110000) + 6 \times (000000)$$

Hence,

$$[\pi : \wedge^2\mathbb{C}^{12}] = [1 \times (000000) - 5 \times (110000) : 1 \times (110000) + 6 \times (000000)] = 6 - 5 = 1$$

Also, it is known that $[\pi : \wedge^0\mathbb{C}^{12}] = [\pi : triv] = 1$. *This can be seen by either the argument above, or noting that* $R[\widetilde{(8,4)}] \cong \pi$. *Consequently, the constant function in the ring* *of regular functions $R[\widetilde{(8,4)}]$ constitutes the multiplicity one of the trivial representation in π. Now,*

$$\begin{aligned}[\pi : \mu_2] &= [\pi : \wedge^2\mathbb{C}^{12} / \wedge^0 \mathbb{C}^{12}] \\ &= [\pi : \wedge^2\mathbb{C}^{12}] - [\pi : \wedge^0\mathbb{C}^{12}] \\ &= 1 - 1 = 0\end{aligned}$$

We now give an algorithm computing the multiplicities of the fundamental K-types of $X_{\mathcal{O}}$ for any orbit $\mathcal{O}$ in $Sp(2m, \mathbb{C})$:

Theorem 9.3. *Suppose $\mathcal{O} = (c_{2k}, c_{2k-1}, \ldots, c_0)$ is a nilpotent orbit in $Sp(2m, \mathbb{C})$. First remove all column pairs of same size, leaving the orbit $(d_{2l}, d_{2l-1}, \ldots, d_0)$. For each of the removed column pair $c_i = c_{i-1} = y$, let $\mathcal{E} = \{c_i | c_i = c_{i-1}$ are removed from $\mathcal{O}\}$ with multiplicities. Also, let $Z = \{z_j = \frac{d_{2j}+d_{2j-1}}{2} | j = 0, \ldots, l\}$. Then rearrange elements in $\mathcal{E} \cup Z$ in non-decreasing order to get $W := \mathcal{E} \cup Z = \{w_i | i = 0, 1, \ldots, k\}$, with $w_i \leq w_j$ if $i < j$.*

Now define a sequence of sequences $\beta_i = (\beta_{i0}, \beta_{i1}, \ldots)$ recursively by:

- *Begin with the sequence $\beta_0 = (\beta_{00}, \beta_{01}, \beta_{02}, \ldots) = (\binom{k-1}{0}, \binom{k}{1}, \binom{k+1}{2}, \ldots)$.*
- *Define the i-th sequence α_i recursively by $\beta_{i+1} = \beta_i - (\overbrace{0, \ldots, 0}^{w_i+1}, \beta_{i0}, \beta_{i1}, \beta_{i2}, \ldots)$.*

Then $[R[\mathcal{O}] : \mu_{2i}] = [X_{\mathcal{O}} : \mu_{2i}] = \beta_{(k+1)i}$ for $i \leq \frac{m}{2}$, and $[R[\mathcal{O}] : \mu_{2i+1}] = [X_{\mathcal{O}} : \mu_{2i+1}] = 0$ for all i.

Example 9.4. *Let $\mathcal{O} = (8, 6, 6, 4, 4, 2, 2)$ in $Sp(32, \mathbb{C})$. Then the w_i's are $\{2, 4, 4, 6\}$. We therefore have the multiplicities as follows:*

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$[R[\mathcal{O}] : \mu_i]$	1	0	3	0	6	0	9	0	12	0	13	0	12	0	8	0	3

We first prove the Theorem holds for orbits with no columns of the same size. In this case, Example 6.3 gives the character formula for $X_{\mathcal{O}}$.

Lemma 9.5. *In $Sp(2m, \mathbb{C})$, let $\overbrace{p, q}^{+}$ and T as in Example 6.3. Then*

$$[\overbrace{(p, q)}^{+} : \mu_i] = [Ind_{GL(\frac{p+q}{2}, \mathbb{C})}^{Sp(p+q, \mathbb{C})}(triv) : \mu_i] = \delta_{i0}$$

$$[T : \mu_i] = [Ind_{GL(\frac{d_0}{2}, \mathbb{C})}^{Sp(d_0, \mathbb{C})}(triv) : \mu_i] = \delta_{i0}$$

Proof. The computations of $[\overbrace{(p, q)}^{+} : \mu_i]$ is essentially given in the beginning of this Chapter (more precisely, we did the example of $[\overbrace{(8, 4)}^{+} \oplus \overbrace{(8, 4)}^{-} : \mu_i]$). For T, note that it is just the character formula of the trivial representation in $Sp(d_0, \mathbb{C})$. $\square$

Proposition 9.6. *Theorem 9.4 holds for $\mathcal{O}$ having no columns of the same size, i.e. $\mathcal{O} = (d_{2l}, \ldots, d_0)$ with $d_i \neq d_{i-1}$ for all i.*

Proof. Recall the character formula of $X_{\mathcal{O}}$ in Example 6.3,

$$X_{\mathcal{O}} = (\overbrace{d_{2l}, d_{2l-1}}^{+}, \overbrace{d_{2l-2}, d_{2l-3}}^{+}, \ldots, \overbrace{d_2, d_1}^{+}, d_0)$$

$X_{\mathcal{O}}$ has the same virtual character as

$$Ind_{Sp(d_{2l}+d_{2l-1}, \mathbb{C}) \times \cdots \times Sp(d_2+d_1, \mathbb{C}) \times Sp(d_0, \mathbb{C})}^{Sp(2m, \mathbb{C})}(\overbrace{d_{2l}, d_{2l-1}}^{+} \otimes \cdots \otimes \overbrace{d_2, d_1}^{+} \otimes T)$$

Therefore, to compute $[X_{\mathcal{Q}} : \mu_i]$, we need to know:

$$
\begin{aligned}
[X_{\mathcal{Q}} : \mu_i] &= [Ind^{Sp(2m,\mathbb{C})}_{Sp(d_{2l}+d_{2l-1},\mathbb{C})\times\cdots\times Sp(d_2+d_1,\mathbb{C})\times Sp(d_0,\mathbb{C})}(\overbrace{d_{2l}, c_{2l-1}}^{+} \otimes \cdots \otimes \overbrace{d_2, d_1}^{+} \otimes triv) : \mu_i] \\
&= [\overbrace{d_{2l}, d_{2l-1}}^{+} \otimes \cdots \otimes \overbrace{d_2, d_1}^{+} \otimes T : \wedge^i \mathbb{C}^{2m}|_{Sp(d_{2l}+d_{2l-1},\mathbb{C})\times\cdots\times Sp(d_2+d_1,\mathbb{C})\times Sp(d_0,\mathbb{C})}] \\
&\quad - [\overbrace{d_{2l}, d_{2l-1}}^{+} \otimes \cdots \otimes \overbrace{d_2, d_1}^{+} \otimes T : \wedge^{i-2} \mathbb{C}^{2m}|_{Sp(d_{2l}+d_{2l-1},\mathbb{C})\times\cdots\times Sp(d_2+d_1,\mathbb{C})\times Sp(d_0,\mathbb{C})}] \\
&= [\overbrace{d_{2l}, d_{2l-1}}^{+} \otimes \cdots \otimes \overbrace{d_2, d_1}^{+} \otimes T : \wedge^i \mathbb{C}^{2m}|_{Sp(2z_l,\mathbb{C})\times\cdots\times Sp(2z_1,\mathbb{C})\times Sp(2z_0,\mathbb{C})}] \\
&\quad - [\overbrace{d_{2l}, d_{2l-1}}^{+} \otimes \cdots \otimes \overbrace{d_2, d_1}^{+} \otimes T : \wedge^{i-2} \mathbb{C}^{2m}|_{Sp(2z_l,\mathbb{C})\times\cdots\times Sp(2z_1,\mathbb{C})\times Sp(2z_0,\mathbb{C})}]
\end{aligned}
$$

Also, the restriction decomposes as

$$
\wedge^i \mathbb{C}^{2m}|_{Sp(2z_l,\mathbb{C})\times\cdots\times Sp(2z_1,\mathbb{C})\times Sp(2z_0,\mathbb{C})} = \bigoplus_{i_0+\cdots+i_l=i} \bigotimes_{p=0}^{l} \wedge^{i_p} \mathbb{C}^{2z_p}
$$

So

$$
\begin{aligned}
[X_{\mathcal{Q}} : \mu_i] &= [\overbrace{d_{2l}, d_{2l-1}}^{+} \otimes \cdots \otimes \overbrace{d_2, d_1}^{+} \otimes T : \bigoplus_{i_0+\cdots+i_l=i} \bigotimes_{p=0}^{l} \wedge^{i_p} \mathbb{C}^{2z_p}] \\
&\quad - [\overbrace{d_{2l}, d_{2l-1}}^{+} \otimes \cdots \otimes \overbrace{d_2, d_1}^{+} \otimes T : \bigoplus_{i_0+\cdots+i_l=i-2} \bigotimes_{p=0}^{l} \wedge^{i_p} \mathbb{C}^{2z_p}] \\
&= \sum_{i_0\cdots+i_l=i} [T : \wedge^{i_0} \mathbb{C}^{2z_0}] \prod_{p=1}^{l} [\overbrace{d_{2p}, d_{2p-1}}^{+} : \wedge^{i_p} \mathbb{C}^{2z_p}] \\
&\quad - \sum_{i_0\cdots+i_l=i-2} [T : \wedge^{i_0} \mathbb{C}^{2z_0}] \prod_{p=1}^{l} [\overbrace{d_{2p}, d_{2p-1}}^{+} : \wedge^{i_p} \mathbb{C}^{2z_p}]
\end{aligned}
$$

On the other hand, by Lemma 9.5,

$$
\begin{aligned}
[\overbrace{d_{2p}, d_{2p-1}}^{+} : \wedge^i \mathbb{C}^{2z_p}] &= [\overbrace{d_{2p}, d_{2p-1}}^{+} : \mu_i \oplus \mu_{i-2} \oplus \mu_{i-4} \oplus \dots] \\
&= [Ind_{GL(z_p,\mathbb{C})}^{Sp(2z_p,\mathbb{C})}(triv) : \mu_i \oplus \mu_{i-2} \oplus \mu_{i-4} \oplus \dots] \\
&= [Ind_{GL(z_p,\mathbb{C})}^{Sp(2z_p,\mathbb{C})}(triv) : \wedge^i \mathbb{C}^{2z_p}]
\end{aligned}
$$

and

$$[T : \wedge^i \mathbb{C}^{2z_0}] = [Ind_{GL(z_0,\mathbb{C})}^{Sp(2z_0,\mathbb{C})}(triv) : \wedge^i \mathbb{C}^{2z_0}]$$

Consequently,

$$
\begin{aligned}
[X_{\mathcal{Q}} : \mu_i] = &\sum_{i_0 \cdots + i_l = i} \prod_{p=0}^{l} [Ind_{GL(z_p,\mathbb{C})}^{Sp(2z_p,\mathbb{C})}(triv) : \wedge^{i_p} \mathbb{C}^{2z_p}] \\
&- \sum_{i_0 \cdots + i_l = i-2} \prod_{p=0}^{l} [Ind_{GL(z_p,\mathbb{C})}^{Sp(2z_p,\mathbb{C})}(triv) : \wedge^{i_p} \mathbb{C}^{2z_p}]
\end{aligned}
$$

Reversing the process, we get

$$[X_{\mathcal{Q}} : \mu_i] = [Ind_{GL(z_l,\mathbb{C}) \times \cdots \times GL(z_0,\mathbb{C})}^{Sp(2m,\mathbb{C})}(triv \otimes \cdots \otimes triv) : \mu_i]$$

Then the result follows by Frobenius reciprocity and induction on the number of columns. $\square$

Proof of Theorem 9.4

Recall that

$$X_{\mathcal{O}} := Ind_{G(\mathcal{Q},\mathbb{C}) \times GL(\mathcal{E})}^{G(2m,\mathbb{C})}(X_{\mathcal{Q}} \otimes triv \otimes \cdots \otimes triv)$$

By Proposition 9.6 and induction in stages, for any fundamental representations μ,

$$[X_{\mathcal{O}} : \mu] = [Ind_{GL(\mathcal{E})\times GL(Z)}^{Sp(2m,\mathbb{C})}(triv \otimes \cdots \otimes triv) : \mu]$$

Consequently the result follows by induction on the number of columns. $\square$

9.2 Computations in $O(n, \mathbb{C})$

The case of $O(n, \mathbb{C})$ is more complicated, mainly because of the $\pm$ sign in the character formula. By Proposition 7.11, the character formulas of all representations we are dealing with has the term

$$X\begin{pmatrix} & a,\ldots 1,0\pm & b_1,\ldots,1 & b_2,\ldots,0 & c_1,\ldots,1 & \ldots \\ w(& a,\ldots 1,0\pm & b_1,\ldots,1 & b_2,\ldots,0 & c_1,\ldots,1 & \ldots) \end{pmatrix}$$

The following Lemma gives an algorithm computing their K-type multiplicities:

Lemma 9.7.

$$X\begin{pmatrix} & a,\ldots 1,0+ & b_1,\ldots,1 & b_2,\ldots,0 & \ldots \\ w(& a,\ldots 1,0+ & b_1,\ldots,1 & b_2,\ldots,0 & \ldots) \end{pmatrix} = Ind_{O(2a+2,\mathbb{C})\times SO(2n-2a-2,\mathbb{C})}^{O(2n,\mathbb{C})}(triv \otimes X')$$

where X' is a representation of $SO(2n-2a-2, \mathbb{C})$ having virtual character

$$X' = X\begin{pmatrix} & b_1,\ldots,1 & b_2,\ldots,0 & \ldots \\ w(& b_1,\ldots,1 & b_2,\ldots,0 & \ldots) \end{pmatrix}$$

$$X\begin{pmatrix} & a,\ldots 1,0- & b_1,\ldots,1 & b_2,\ldots,0 & \ldots \\ w(& a,\ldots 1,0- & b_1,\ldots,1 & b_2,\ldots,0 & \ldots) \end{pmatrix} = Ind_{O(2a+2,\mathbb{C})\times SO(2n-2a-2,\mathbb{C})}^{O(2n,\mathbb{C})}(det \otimes X')$$

with the same X' as above.

With this Lemma, we can give an algorithm computing the fundamental K-type multiplicities of $X_{\mathcal{P}}$:

Theorem 9.8. *Suppose $\mathcal{P} = (c_{2k+1}, c_{2k}, c_{2k-1}, \dots, c_0)$ is a nilpotent orbit in $O(n, \mathbb{C})$. First remove all column pairs of same size, leaving the orbit $(d_{2l+1}, d_{2l}, d_{2l-1}, \dots, d_0)$. For each of the removed column pair $c_i = c_{i-1} = y$, let $\mathcal{E} = \{c_i | c_i = c_{i-1}$ are removed from $\mathcal{P}\}$ with multiplicities. Also, let $Z = \{z_j = \frac{d_{2j}+d_{2j-1}}{2} | j = 0, \dots, l\}$ (note that d_{2l+1} is not used in the algorithm). Then rearrange elements in $Y \cup Z$ in non-decreasing order to get $\mathcal{E} \cup Z = \{w_i | i = 0, 1, \dots, k\}$, with $w_i \leq w_j$ if $i < j$.*

Now define a sequence of sequences $\beta_i = (\beta_{i0}, \beta_{i1}, \dots)$ recursively by:

- *Begin with the sequence $\beta_0 = (\beta_{00}, \beta_{01}, \beta_{02}, \dots) = (\binom{k}{0}, \binom{k+1}{1}, \binom{k+2}{2}, \dots)$.*
- *Define the i-th sequence α_i recursively by $\beta_{i+1} = \beta_i - (\overbrace{0, \dots, 0}^{w_i+1}, \beta_{i0}, \beta_{i1}, \beta_{i2}, \dots)$.*

Then for the fundamental representations $\mu'_j := \wedge^j \mathbb{C}^n$ in $O(n, \mathbb{C})$,

$[R[\mathcal{P}] : \mu'_{2i}] = [X_{\mathcal{P}} : \mu'_{2i}] = \beta_{(k+1)i}$ for $2i \leq n$, and $[R[\mathcal{P}] : \mu'_{2i+1}] = [X_{\mathcal{P}} : \mu'_{2i+1}] = 0$ for all i.

Example 9.9. *Let $\mathcal{P} = (7, 5, 3, 3, 1)$ in $O(19, \mathbb{C})$. Then Then the w_i's are $\{6, 6\}$. We therefore have the multiplicities of μ'_{2i} as follows:*

i	0	2	4	6	8	10	12	14	16	18
$[R[\mathcal{P}] : \mu'_i]$	1	2	3	4	3	2	1	0	0	0

There is a Lemma analogous to Lemma 9.6 in the orthogonal case:

Lemma 9.10. *In $SO(n,\mathbb{C})$, let $\overbrace{p,q}^{+}$ and T as in Example 6.3. Then*

$$[\overbrace{p,q}^{+} : \mu_i'] = [Ind_{GL(\frac{p+q}{2},\mathbb{C})}^{SO(p+q,\mathbb{C})}(triv) : \mu_i'] = \begin{cases} 1 & \textit{if } i \textit{ is even} \\ 0 & \textit{otherwise} \end{cases}$$

$$[T : \mu_i'] = [Ind_{GL(\frac{d_0}{2},\mathbb{C})}^{SO(d_0,\mathbb{C})}(triv) : \mu_i'] = \begin{cases} 1 & \textit{if } i \textit{ is even} \\ 0 & \textit{otherwise} \end{cases}$$

Proof of Theorem 9.8

As in the proof of Theorem 9.4, we first look at the orbit $\mathcal{P} = (d_{2l+1}, \ldots, d_0)$ with $d_i \neq d_{i-1}$ for all i. In this case, we have

$$X_{\mathcal{P}} = (\overbrace{d_{2l+1}}^{+}, \overbrace{d_{2l}, d_{2l-1}}^{+}, \ldots, \overbrace{d_2, d_1}^{+}, d_0)$$

which has the same virtual character as

$$Ind_{O(d_{2l+1}) \times SO(d_{2l}+d_{2l-1},\mathbb{C}) \times \cdots \times SO(d_2+d_1,\mathbb{C}) \times SO(d_0,\mathbb{C})}^{O(n,\mathbb{C})}(triv \otimes \overbrace{d_{2l}, d_{2l-1}}^{+} \otimes \cdots \otimes \overbrace{d_2, d_1}^{+} \otimes T)$$

by the same technique as in Theorem 9.4,

$$[X_{\mathcal{P}} : \mu_i'] = [Ind_{O(d_{2l+1}) \times GL(z_l,\mathbb{C}) \times \cdots \times GL(z_0,\mathbb{C})}^{O(n,\mathbb{C})}(triv \otimes \cdots \otimes triv) : \mu_i']$$

Therefore, by Frobenius reciprocity and induction on the number of columns, the Theorem follows for $\mathcal{P}$ with no columns of the same size.

For the general case, the argument works in exactly the same way as in the symplectic one, which gives

$$[X_{\mathcal{P}} : \mu_i'] = [Ind_{O(d_{2l+1}) \times GL(\mathcal{E}) \times GL(Z)}^{O(n,\mathbb{C})}(triv \otimes \cdots \otimes \ triv) : \mu_i']$$

So the result follows. $\square$

9.3 A Criterion of Normality of Orbit Closures

With the algoritms above, we can now state and prove another criterion of the normality of $\overline{\mathcal{O}}$, by computing the K-type multiplicities of the representations $X_{\mathcal{O}}$. Note that the computations above does not involve the longest column of the orbit in $O(n, \mathbb{C})$, we can just focus ourselves for the case of $Sp(2m, \mathbb{C})$.

Lemma 9.11. *Let $\mathcal{O} = (c_{2k}, c_{2k-1}, \ldots, c_0)$ be a nilpotent orbit in $G = Sp(2m, \mathbb{C})$, and μ be any finite dimensional irreducible representation of G, then*

$$[R[\overline{\mathcal{O}}] : \mu] = [X_{\overline{\mathcal{O}}} : \mu] \leq [X_{\mathcal{O}^\sharp} : \mu] = [R[\mathcal{O}^\sharp] : \mu]$$

where $\mathcal{O}^\sharp = (\frac{c_{2k}+c_{2k-1}}{2}, \frac{c_{2k}+c_{2k-1}}{2}, \frac{c_{2k-2}+c_{2k-3}}{2}, \frac{c_{2k-2}+c_{2k-3}}{2}, \ldots, \frac{c_2+c_1}{2}, \frac{c_2+c_1}{2}, c_0)$

Proof. Note that $\overline{\mathcal{O}^\sharp}$ is normal and $\overline{\mathcal{O}^\sharp} \supset \overline{\mathcal{O}}$. Consequently, we have a G-module surjection

$$R[\mathcal{O}^\sharp] = R[\overline{\mathcal{O}^\sharp}] \twoheadrightarrow R[\overline{\mathcal{O}}]$$

and hence $[R[\overline{\mathcal{O}}] : \mu] = [X_{\overline{\mathcal{O}}} : \mu] \leq [R[\mathcal{O}^\sharp] : \mu]$ for any finite dimensional G-representations μ. Hence the result follows. $\square$

From results in the last Section (or from [22] directly), one can find out the K-type multiplicities of $X_{\mathcal{O}^\sharp}$. And hence we come to the Theorem below:

Theorem 9.12. *Let $\mathcal{O}$ be a nilpotent orbit in $Sp(2m, \mathbb{C})$. Then $\overline{\mathcal{O}}$ is not normal iff*

$$[R[\overline{\mathcal{O}}] : \mu_i] < [R[\mathcal{O}] : \mu_i]$$

for some $i > 0$.

Let $\mathcal{P} = (b_{2k+1}, \ldots, b_0)$ be a nilpotent orbit in $O(n, \mathbb{C})$ such that $b_{2k+1} \neq b_{2k}$. Then $\overline{\mathcal{O}}$ is not normal iff

$$[R[\overline{\mathcal{P}}] : \mu_i'] < [R[\mathcal{P}] : \mu_i']$$

for some $i > 0$.

Proof. One direction is easy - if $\overline{\mathcal{O}}$ is normal, then $R[\overline{\mathcal{O}}] = R[\mathcal{O}]$ as G-modules, hence $[R[\overline{\mathcal{O}}] : \mu_i] = [R[\mathcal{O}] : \mu_i]$ for all i. Now suppose $\overline{\mathcal{O}}$ is not normal, and let $\mathcal{O}^\sharp$ as in last Section. Then we obtain a new set of integers $\{x_i\}$ computing the K-type multiplicities of $X_{\mathcal{O}^\sharp}$. By the Kraft-Procesi criterion (Theorem 3.3), the two sets of integers $\{x_i\}$ and $\{w_i\}$ are different, and therefore there exists an $i > 0$ such that $[R[\mathcal{O}^\sharp] : \mu_i] < [R[\mathcal{O}] : \mu_i]$. Now Lemma 9.11 says $[R[\overline{\mathcal{O}}] : \mu_i] \leq [R[\mathcal{O}^\sharp] : \mu_i]$ for all i, and consequently the theorem follows. $\square$

Remark 9.13. *The Theorem holds even in the case when $b_{2k+1} = b_{2k}$ for nilpotent orbit $\mathcal{P}$ in $O(n, \mathbb{C})$. We omit the proof here.*

Example 9.14. *Let $\mathcal{O} = (8, 6, 6, 4, 4, 2, 2)$ in $Sp(32, \mathbb{C})$. From the previous Proposition, the w_i's are $\{2, 4, 4, 6\}$. Now $\mathcal{O}^\sharp = (7, 7, 5, 5, 3, 3, 2)$, the x_i's are $\{1, 3, 5, 7\}$. We therefore have the multiplicities as follows:*

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$[R[\mathcal{O}^\sharp] : \mu_i]$	1	0	3	0	5	0	7	0	8	0	8	0	7	0	5	0	2
$[R[\mathcal{O}] : \mu_i]$	1	0	3	0	6	0	9	0	12	0	13	0	12	0	8	0	3

Let $\mathcal{P} = (7,5,3,3,1)$ in $O(19,\mathbb{C})$. Then $\mathcal{P}^\sharp = (7,4,4,2,2)$, the w_i's are $\{6,6\}$, and the x_i's are $\{4,8\}$. We therefore have the multiplicities of μ'_{2i} as follows:

i	0	2	4	6	8	10	12	14	16	18
$[R[\mathcal{P}^\sharp] : \mu'_i]$	1	2	3	3	3	2	1	0	0	0
$[R[\mathcal{P}] : \mu'_i]$	1	2	3	4	3	2	1	0	0	0

CHAPTER 10

SOME RESULTS ON $X_{\overline{\mathcal{O}}}$

10.1 Composition Factors of $X_{\overline{\mathcal{O}}}$ when $\overline{\mathcal{O}}$ is Normal

We now investigate the composition factors of $X_{\overline{\mathcal{O}}}$. As seen in Chapter 4, the construction of $X_{\overline{\mathcal{O}}}$ is more geometrical than algebraic. One does not have a character formula of $X_{\overline{\mathcal{O}}}$ as in the case of $X_{\mathcal{O}}$. However, by Corollary 3.2, the K-type multiplicities of $X_{\mathcal{O}}$ is the same as $X_{\overline{\mathcal{O}}}$ if $\overline{\mathcal{O}}$ is normal. It is therefore hoped that the composition factors of both models are the same (note that the equality of K-type multiplicities of both models does not guarantee their equality as $(\mathfrak{g}, K)$-modules). And in fact, it is true for all orbits $\mathcal{O}$ with normal closure.

Theorem 10.1. *Let $\mathcal{O}$ be an orbit in $Sp(2m, \mathbb{C})$ or $O(n, \mathbb{C})$ satisfying $(\dagger)$ such that $\overline{\mathcal{O}}$ is normal. Then the composition factors of $X_{\overline{\mathcal{O}}}$ is the same as that of $X_{\mathcal{O}}$.*

Remark 10.2. *In fact, the above Theorem is true without Condition $(\dagger)$. However the book-keeping tools we use below will be too cumbersome to be presented. For instance, the list in Lemma 10.3 will become infinite.*

Before proving the Theorem, recall that all orbits in $Sp(2m, \mathbb{C})$ satisfying $(\dagger)$ can be partitioned into

$$(x_k, x_{k-1}, x_{k-1}, \ldots, x_1, x_1, x_0)$$

and all orbits $\mathcal{P} = (b_{2k+1}, \ldots, b_0)$ in $O(n, \mathbb{C})$ satisfying $(\dagger)$ can be partitioned into the 'head'

$$(b_{2k+1} = h_i, b_{2k} = h_i, h_{i-1}, h_{i-1}, \ldots h_1, h_1, h_0 = b_{2k-2i+1})$$

and the 'remaining' part $(b_{2k-2i}, b_{2k-2i-1}, \ldots, b_0)$ is done in the same fashion as in the $Sp(2m, \mathbb{C})$ case. The following Lemma characterizes all orbits $\mathcal{O}$ satisfying $(\dagger)$ with normal orbit closures.

Lemma 10.3. *Suppose $\mathcal{O}$ is a nilpotent orbit in $Sp(2m, \mathbb{C})$ satisfying $(\dagger)$ with normal orbit closure. Then $\mathcal{O}$ must be composed of the following three fundamental types of partitions:*

(1) (a, a, a, b, b, b), $a \neq b$

(2) (a, a, a, b), $a \neq b$

(3) (a, b), *a can be equal to b*

Suppose $\mathcal{P}$ is a nilpotent orbit in $O(n, \mathbb{C})$ satisfying $(\dagger)$ with normal orbit closures. Then its head must be of the types of partitions:

(0) (a, a, a)

(0') (a)

and the remaining part of $\mathcal{P}$ must be composed of types (1), (2), (3) *as above.*

Proof. This can easily be seen by the Kraft-Procesi criterion of normality. $\square$

The next Definition is to record the lowest K-types of the orbits corresponding to the fundamental partitions above.

Definition 10.4. *Suppose (a,a,a,b,b,b) is a nilpotent orbit in $Sp(2m,\mathbb{C})$. Then the possible composition factors of $X_{(a,a,a,b,b,b)}$ are of the form*

$$\begin{tikzcd}[column sep=tiny]
& \mathcal{O}_\phi = (\overbrace{a,a,a,b,b,b}^{\pm}) \arrow[dl, dash] \arrow[dr, dash] & \\
(a+1,a+1,\overbrace{a-2,b,b,b}^{\pm}) \arrow[dr, dash] & & (\overbrace{a,a,a,b+2}^{\pm},b-1,b-1) \arrow[dl, dash] \\
& (a+1,a+1,\overbrace{a-2,b+2}^{\pm},b-1,b-1) &
\end{tikzcd}$$

(the bottom orbit is equal to $(a+1,a+1,b+1,b+1,b-1,b-1)$ if $a=b+2$). In either cases, record all the possible lowest K-types of the above irreducible representations. For instance, $\mathcal{O}_\phi$ has two irreducible components,

$$Ind_{GL(a,\mathbb{C})\times GL(b,\mathbb{C})\times Sp(a+b,\mathbb{C})}^{Sp(2m,\mathbb{C})}(det \otimes det \otimes \overbrace{a,b}^{\pm})$$

*having lowest K-types $(\overbrace{1,\ldots,1}^{a+b},0,\ldots,0)$ and $(\overbrace{1,\ldots,1}^{a+b+b},0,\ldots,0)$. For future purpose, keep track on the induced representations from $GL(a,\mathbb{C})$ and $GL(b,\mathbb{C})$ by defining the **lowest K-type set** to be $\{a\oplus b, a\oplus b\oplus b\}$. Then define the **lowest K-type set of fundamental type (1) orbit** to be*

$$\mathcal{K}_{(a,a,a,b,b,b)} := \{0,b,(b+2),a,a\oplus b,a\oplus(b+2),a\oplus b\oplus b\}$$

or

$$\mathcal{K}_{(b+2,b+2,b+2,b,b,b)} := \{0,b,(b+2),(b+2)\oplus b,(b+2)\oplus(b+2),(b+2)\oplus b\oplus b\}$$

if $a=b+2$. Similarly, the lowest K-type set of fundemental type (2) orbit (a,a,a,b) is

$$\mathcal{K}_{(a,a,a,b)} := \{0,a,b,a\oplus b\}$$

The lowest K-type set of fundemental type (3) orbit (a, b) is

$$\mathcal{K}_{(a,b)} := \{0, b\}$$

The lowest K-type set of fundemental type (0) orbit (a, a, a) is

$$\mathcal{K}_{(a,a,a)} := \{0, a, (a+2), a \oplus a\}$$

The lowest K-type set of fundemental type (0′) orbit (a) is

$$\mathcal{K}_{(a)} := \{0, a\}$$

With this on hand, we can further define the fundamental K-type set of any normal orbit $\mathcal{O}$ satisfying $(\dagger)$. This records the lowest K-types of all the possible composition factors of $X_{\mathcal{O}}$.

Definition 10.5. *Let $\mathcal{O}$ be a nilpotent orbit in $Sp(2m, \mathbb{C})$ or $O(n, \mathbb{C})$. So it can be partitioned into fundamental types $(0) - (3)$. Suppose $\mathcal{O} = (\mathcal{T}_0, \mathcal{T}_1, \dots \mathcal{T}_r)$ where $\mathcal{T}_i$ are of the fundamental types, and $\mathcal{K}_i$ is the corresponding lowest K-type set of $\mathcal{T}_i$. Define the lowest K-type set of $\mathcal{O}$ to be*

$$\mathcal{K}_{\mathcal{O}} = \mathcal{T}_0 \oplus \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_r$$

where $\mathcal{M} \oplus \mathcal{N} = \{m \oplus n | m \in \mathcal{M}, n \in \mathcal{N}\}$ (with the usual abelian rules $a \oplus b = b \oplus a$, $0 \oplus a = a$).

Example 10.6. *Let $\mathcal{O} = (8, 8, 8, 4, 3, 3)$ in $Sp(34, \mathbb{C})$. Then $\mathcal{T}_0 = \phi$, $\mathcal{T}_1 = (8, 8, 8, 4)$, $\mathcal{T}_2 = (3, 3)$. From above, $\mathcal{K}_1 = \{0, 4, 8, 8 \oplus 4\}$, $\mathcal{K}_2 = \{0, 3\}$. So*

$$\mathcal{K}_{\mathcal{O}} = \{0, 3, 4, 3 \oplus 4, 8, 3 \oplus 8, 4 \oplus 8, 3 \oplus 4 \oplus 8\}$$

The 8 possible composition factors of $X_{\mathcal{O}}$ are $(\overbrace{8,8,8,4}^{\pm}, \overbrace{3,3}^{\pm})$, $(9,9,\overbrace{6,4}^{\pm}, \overbrace{3,3}^{\pm})$. *By changing $\oplus$ in $\mathcal{K}_{\mathcal{O}}$ into $+$, it gives the lowest K-types of these 8 irreducible representations.*

The following observation is crucial for the proof of the Theorem:

Lemma 10.7. *Let $\mathcal{O}$ be any nilpotent orbit in Sp or O satisfying $(\dagger)$. Suppose there exists $a_1 \oplus \cdots \oplus a_m$, $b_1 \oplus \cdots \oplus b_n$ in $\mathcal{K}_{\mathcal{O}}$ such that $\sum_i a_i = \sum_j b_j$. Then $\{a_1, \ldots, a_m\} \neq \{b_1, \ldots, b_n\}$ as sets.*
In other words, all elements in $\mathcal{K}_{\mathcal{O}}$ are distinct.

Proof. First of all, it is easy to check the Lemma holds if $\mathcal{O}$ is of the fundamental types. Now use induction on the number of fundamental types in $\mathcal{O}$: Suppose the hypothesis holds for $\mathcal{O} = (\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_r)$. Then for $\mathcal{O}' = (\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_{r+1})$,

$$\mathcal{K}_{\mathcal{O}'} = \mathcal{K}_{\mathcal{O}} \oplus \mathcal{K}_{r+1}$$

where every element in $\mathcal{K}_{\mathcal{O}}$ are distinct and $\mathcal{K}_{r+1}$ belongs to one of the fundamental types $(0)-(3)$. If there exists a repetition of in $\mathcal{K}_{\mathcal{O}'}$, that means there must be some $a_{i_1} \oplus \cdots \oplus a_{i_p} \in \mathcal{K}_{\mathcal{O}}$, $a_{j_1} \oplus \cdots \oplus a_{j_{m-p}} \in \mathcal{K}_{r+1}$; $b_{k_1} \oplus \cdots \oplus b_{k_q} \in \mathcal{K}_{\mathcal{O}}$, $b_{l_1} \oplus \cdots \oplus b_{l_{m-q}} \in \mathcal{K}_{r+1}$ such that $a_{i_1} \oplus \cdots \oplus a_{i_p} \neq b_{k_1} \oplus \cdots \oplus b_{k_q}$, $a_{j_1} \oplus \cdots \oplus a_{j_{m-p}} \neq b_{l_1} \oplus \cdots \oplus b_{l_{m-q}}$ and

$$\bigoplus_i a_i \oplus \bigoplus_j a_j = \bigoplus_k b_k \oplus \bigoplus_l b_l$$

But note that all a_j and b_l are smaler than or equal to a, the longest column of $\mathcal{T}_{r+1}$, while all a_i and b_k are greater than or equal to b, the shortest column in $\mathcal{T}_r$. By construction, $b > a$ and hence this forces

$$\bigoplus_i a_i = \bigoplus_k b_k$$

$$\bigoplus_j a_j = \bigoplus_l b_l$$

which is impossible by our hypothesis. $\square$

Proof of Theorem 10.1

For any nilpotent orbit $\mathcal{O}$ with normal closure, we know the following holds:

• The composition factors of $X_{\mathcal{O}}$ give the correct K-type multiplicities of $X_{\overline{\mathcal{O}}}$.

• All the possible composition factors of $X_{\overline{\mathcal{O}}}$ are also known.

On the other hand, Definition 10.5 collects the lowest K-types of all the possible composition factors of $X_{\overline{\mathcal{O}}}$. Let

$$\mathcal{K}_{\mathcal{O}} = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\}$$

where α_i's are the lowest K-types of the composition factors of $X_{\mathcal{O}}$. By Lemma 10.7, since all elements in $\mathcal{K}_{\mathcal{O}}$ are distinct, write X_{α_i}, X_{β_i} be their corresponding irreducible representations (so that the composition factors of $X_{\mathcal{O}}$ are X_{α_i}, all appearing once), and let $(\overbrace{1, \ldots, 1}^{A_i}, 0, \ldots, 0)$, $(\overbrace{1, \ldots, 1}^{B_i}, 0, \ldots, 0)$ be the lowest K-type of X_{α_i}, X_{β_i} respectively.

We now begin to determine which of them appears in $X_{\overline{\mathcal{O}}}$. First of all, since $X_{\overline{\mathcal{O}}}$ is spherical, it contains a irreducible component with lowest weight $(0, \ldots, 0)$. From the results in Chapter 7 and 8, there is only one such irreducible component, and is of the form X_{α_i} for some i. Without loss of generality, let X_{α_1} be the irreducible, spherical component.

Next, consider the lowest K-type $(\overbrace{1,\ldots,1}^{2l},0,\ldots,0)$ of the virtual character $X_{\overline{\mathcal{O}}} - X_{\alpha_1}$ with multiplicity m_{2l} (Recall from Chapter 9 that the K-types of $X_{\overline{\mathcal{O}}} = X_{\mathcal{O}}$ must have even number of 1's). Since we explicitly know the K-type multiplicities of $X_{\overline{\mathcal{O}}}$,

$$2l = \min\{A_i | i \neq 1\}$$

$$m_{2l} = \#\{i | A_i = 2l\}$$

It is hoped that all the X_{α_i}'s with $A_i = 2l$ appear in the composition factors of $X_{\overline{\mathcal{O}}}$ exactly once, as in the case of $X_{\mathcal{O}}$. However, it is possible that some of such X_{α_i} appears more than once and some others does not appear in the composition factors of $X_{\overline{\mathcal{O}}}$. Or even worse, some X_{β_j} have $B_j = 2l$ and they appear in the composition factors of $X_{\overline{\mathcal{O}}}$. The Lemma below, to be proved later, is to rule out these possibilities.

Lemma 10.8. *Suppose $\gamma = \gamma_1 \oplus \cdots \oplus \gamma_r \in \mathcal{K}_{\mathcal{O}}$, with X_γ having lowest K-type $(\overbrace{1,\ldots,1}^{2l},0,\ldots,0)$ (so $\gamma_1 + \cdots + \gamma_r = 2l$). Considering γ as a partition P_γ of $2l$, then for any partition P_δ of $2l$,*

$$[X_\gamma : V(P_\delta, 0, \ldots, 0)] \neq 0 \textit{ iff } P_\delta \preccurlyeq P_\gamma^*$$

where P_γ^* is the dual partition of P_γ (Chapter 2.3), and $V(P_\delta, 0, \ldots, 0)$ is the K-type with highest weight $(\delta_1, \ldots, \delta_s, 0, \ldots, 0)$.

Suppose $Q := \{X_{\alpha_2}, X_{\alpha_3}, \ldots, X_{\alpha_s}, X_{\beta_1}, \ldots, X_{\beta_t}\}$ are the irreducible factors with lowest K-type $(\overbrace{1, \ldots, 1}^{2l}, 0, \ldots, 0)$. Then since X_{α_i} are the composition factors of $X_{\mathcal{O}}$, for all partitions δ of $2l$,

$$[X_{\overline{\mathcal{O}}} : V(P_\delta, 0, \ldots, 0)] = \sum_{i=1}^{n} [X_{\alpha_i} : V(P_\delta, 0, \ldots, 0)]$$

also by lowest K-type argument, if $X_\gamma \notin Q \cup \{X_{\alpha_1}\}$,

$$[X_\gamma : V(P_\delta, 0, \ldots, 0)] = 0$$

Now consider the $|$partitions of $2l| \times (s-1+t)$ matrix $M = [\mathbf{c}_{\alpha_2}| \ldots |\mathbf{c}_{\alpha_s}|\mathbf{c}_{\beta_1}| \ldots |\mathbf{c}_{\beta_t}]$, where the columns $\mathbf{c}_{\alpha_i}$, $\mathbf{c}_{\beta_j}$ of M are given by the multiplicities $[X_{\alpha_i} : V(P_\delta, 0 \ldots, 0)]$, $[X_{\beta_j} : V(P_\delta, 0 \ldots, 0)]$ for all partitions δ of $2l$. Then by Lemma 10.7, no two partitions α_i, β_j are the same, and by Lemma 10.8, all the columns of M are linearly independent.

Finally, suppose X_{α_i} appears in the composition factors of $X_{\overline{\mathcal{O}}}$ with multiplicity p_i, X_{β_j} appears in the composition factors of $X_{\overline{\mathcal{O}}}$ with multiplicity q_j, by the above two conditions of multiplicities of K-types of form $(P_\delta, 0, \ldots, 0)$, we get

$$[X_{\alpha_1} \oplus \bigoplus_{i=2}^{s} p_i X_{\alpha_i} \oplus \bigoplus_{j=1}^{t} q_j X_{\beta_j} : V(P, 0, \ldots, 0)] = \sum_{i=1}^{s} [X_{\alpha_i} : V(P, 0, \ldots, 0)]$$

grouping the equations together, we get

$$p_2 \mathbf{c}_{\alpha_2} + \cdots + p_s \mathbf{c}_{\alpha_s} + q_1 \mathbf{c}_{\beta_1} + \cdots + q_t \mathbf{c}_{\beta_t} = \mathbf{c}_{\alpha_2} + \cdots + \mathbf{c}_{\alpha_s}$$

Obviously, $p_i = 1$, $q_j = 0$ is a solution of the above system of linear equations. However, we have already seen that the columns of M are all linearly independent. This forces the solution to be unique, i.e. $X_{\alpha_2}, \ldots, X_{\alpha_s}$ appears in the composition factors of $X_{\overline{\mathcal{O}}}$ with multiplicity 1, X_{β_j} does not appear in the composition factors of $X_{\overline{\mathcal{O}}}$, which is the same as the case of the composition factors of $X_{\mathcal{O}}$. Continue the argument inductively on the size of $2l$, the result follows. $\square$

10.1.1 Proof of Lemma 10.8

Recall that the set $Q = \{X_{\alpha_2}, \ldots, X_{\alpha_s}, X_{\beta_1}, \ldots, X_{\beta_t}\}$ be the irreducible representations with lowest K-type $(\overbrace{1, \ldots, 1}^{2l}, 0, \ldots, 0)$. Using the notation in Section 2.6 of [5], $\mathfrak{m} = \mathfrak{gl}(2l, \mathbb{C}) \oplus \mathfrak{sp}(2m - 4l, \mathbb{C})$. Therefore, all the K-types of the form $(P, 0, \ldots, 0)$, where P is a partition of $2l$ is a bottom layer K-type of the irreducible representations in Q. By Proposition 2.6 in [5], for all $X_\gamma \in Q$,

$$[X_\gamma : (P, 0, \ldots, 0)] = [\overline{X}_{\gamma, \mathfrak{gl}(2l,\mathbb{C})} : (P, 0, \ldots, 0)]$$

where $\overline{X}_{\gamma, \mathfrak{gl}(2l,\mathbb{C})}$ is the Langlands quotient of a principal series representation in $GL(2l, \mathbb{C})$, dependent on γ. In fact, a result of Vogan says

$$\overline{X}_{\gamma, \mathfrak{gl}(2l,\mathbb{C})} = Ind_{GL(P_\gamma)}^{GL(2l,\mathbb{C})}(det \otimes \cdots \otimes det)$$

where $GL(P_\gamma) = GL(\gamma_1, \mathbb{C}) \times \cdots \times GL(\gamma_r, \mathbb{C})$ if γ is the partition $[\gamma_1, \ldots, \gamma_r]$ of $2l$. So the Lemma can be rephrased as

$$[Ind_{GL(P_\gamma)}^{GL(2l,\mathbb{C})}(det \otimes \cdots \otimes det) : V(P, 0, \ldots, 0)] \neq 0 \text{ iff } P \preccurlyeq P_\gamma^*$$

By Frobenius reciprocity, it suffices to understand $V(P, 0, \ldots, 0)|_{GL(P_\gamma)}$, and check if the det representation appears in all $GL(\gamma_i, \mathbb{C})$. But the restriction of K-types in GL is known to be related to the Littlewood-Richardson rule, for example [18, Proposition 2.6], and our problem can be reduced to:

Given columns of sizes $\gamma_1, \gamma_2, \ldots, \gamma_r$, add the columns up using Littlewood-Richardson rule. Suppose the partition P appears in the sum m_P times, then

$$[Ind_{GL(P_\gamma)}^{GL(2l,\mathbb{C})}(det \otimes \cdots \otimes det) : V(P, 0, \ldots, 0)] = m_P$$

And it is easy to see that $m_P \neq 0$ iff $P \preccurlyeq P_\gamma^*$. Hence the Lemma is proved.

10.2 A Conjecture for all Orbits

In this Section, we give a Conjecture on the character formula for $X_{\overline{\mathcal{O}}}$, and from the character formula we can derive which composition factors appear in $X_{\overline{\mathcal{O}}}$. Before we proceed, it is helpful to look at the constraints of the conjecutured composition factors:

• They must match our results in the last Section, namely when $\overline{\mathcal{O}}$ is normal, the composition factors of $X_{\overline{\mathcal{O}}}$ must be the same as that of $X_{\mathcal{O}}$.

• The K-type multiplicities of $X_{\overline{\mathcal{O}}}$ must be smaller than that of $X_{\mathcal{O}^\sharp}$.

In fact, the conjecutured composition factors will give $[X_{\overline{\mathcal{O}}} : \mu_i] = [X_{\mathcal{O}^\sharp} : \mu_i]$ for all fundamental representations μ_i.

10.2.1 A Special Case

Throughout this section, we assume $\mathcal{O}$ satisfies condition $(\dagger)$. We will pick up **one** distinguished composition factor π_{S,j_S} for each subset $S \subset \mathcal{S}$ such that the collection of such composition factors, $\{\pi_{S,j_S} | S \subset \mathcal{S}\}$, are conjecturally the composition factors of $X_{\overline{\mathcal{O}}}$. As a corollary of the conjecture, the inequality in the above Proposition is an equality for small K-types μ_i.

Recall Section 8.1 in determining composition factors of $X_{\mathcal{O}}$. We set up a new rule in assigning the signs, so that for any subset $S \subset \mathcal{S}$, there is only one sign assigned to $\mathcal{O}_S$. In fact, Rule (1) is the only rule doubling the possible sign assignments, so we replace Rule (1) by the following:

$(1')$ Suppose $\overline{\mathcal{O}_S} \supset \overline{\mathcal{O}_{S'}}$ is not normal in codimension two, for the toppled part

$$(a_{n+1}, a_n, a_n, \ldots, a_1, a_1, a_0) \to (a_{n+1}, \ldots, a_{i+1}, a_{i+1}, a_i + 2, a_i - 2, a_{i-1}, a_{i-1}, \ldots)$$

(a) If $a_0 \neq 0$, use Rule (0) to determine the signs of $(a_{n+1}, \ldots, a_{i+1}, a_{i+1}, a_i + 2)$ and $(a_i - 2, a_{i-1}, a_{i-1}, \ldots, a_0)$ in $\mathcal{O}_{S'}$ respectively.

(b) If $a_0 = 0$, use Rule (0) to determine the sign of $(a_{n+1}, \ldots, a_{i+1}, a_{i+1}, a_i + 2)$ in $\mathcal{O}_{S'}$.

In this case, we always get one set of signs assigned to any orbit $\mathcal{O}_S$, and the corresponding composition factor is denoted π_{S,j_S}. By construction, the collection $\{\pi_{S,j_S} | S \subset \mathcal{S}\}$ is always a subset of the composition factors of $X_{\mathcal{O}}$.

Example 10.9. *Back to Example 8.2, where $\mathcal{O} = (10, 10, 10, 8, 8, 6, 5, 5, 4, 2, 2)$. We apply Rule (0) to every orbit below $\mathcal{O} = \mathcal{O}_\phi$ to get all π_{S,j_S}. The list of π_{S,j_S} is stated below:*

$$\mathcal{O}_\phi = (\overbrace{10, 10, 10, 8, 8, 6}^{+}, \overbrace{5, 5}^{+}, 4, 2, 2)$$

$$\mathcal{O}_2 = (\overbrace{10, 10, 10, 8, 8, 6}^{+}, \overbrace{5, 5}^{+}, \overbrace{4, 4}^{+}) \qquad \mathcal{O}_8 = (\overbrace{10, 10, 10, 10}^{-}, \overbrace{6, 6}^{+}, \overbrace{5, 5}^{+}, 4, 2, 2) \qquad \mathcal{O}_{10} = (11, 11, \overbrace{8, 8, 8, 6}^{-}, \overbrace{5, 5}^{+}, 4, 2, 2)$$

$$\mathcal{O}_{2,8} = (\overbrace{10, 10, 10, 10}^{-}, \overbrace{6, 6}^{+}, \overbrace{5, 5}^{+}, \overbrace{4, 4}^{+}) \qquad \mathcal{O}_{2,10} = (11, 11, \overbrace{8, 8, 8, 6}^{-}, \overbrace{5, 5}^{+}, \overbrace{4, 4}^{+}) \qquad \mathcal{O}_{8,10} = (11, 11, 9, 9, \overbrace{6, 6}^{+}, \overbrace{5, 5}^{+}, 4, 2, 2)$$

$$\mathcal{O}_{2,8,10} = (11, 11, 9, 9, \overbrace{6, 6}^{+}, \overbrace{5, 5}^{+}, \overbrace{4, 4}^{+})$$

Lemma 10.10. *Let $\mathcal{Q} = (a_{n+1}, a_n, a_n, \ldots, a_1, a_1, a_0)$ be a nilpotent orbit in $Sp(2m, \mathbb{C})$ (putting $a_0 = 0$ if necessary), and $\mathcal{Q}^* = (d_2, d_1, a_{n+1}, a_n, a_n, \ldots, a_1, a_1, a_0)$ a nilpotent orbit in $Sp(2m', \mathbb{C})$ (with $m' > m$), both satisfying $(\dagger)$. Let $\pi_{\mathcal{Q}}$ and $\pi_{\mathcal{Q}^*}$ be the representations corresponding to $\mathcal{Q}$ and $\mathcal{Q}^*$ using Rule (0). Consider*

$$\mathcal{I} = Ind_{Sp(2m) \times Sp(d_1+d_2)}^{Sp(2m')}(\pi_{\mathcal{Q}} \otimes \overbrace{d_2, d_1}^{+})$$

where X_{d_2,d_1} is the spherical unipotent representation attached to the orbit (d_2, d_1), then

(a) If $d_1 \neq a_{n+1}$, then $\mathcal{I} = \pi_{\mathcal{Q}^*}$

(b) If $d_1 = a_{n+1} = r$, then $\mathcal{I} = \pi_{\mathcal{Q}^*} \oplus \pi_{\mathcal{Q}^*_n}$, where

(i) for $d_2 \neq d_1 = a_{n+1} = r \neq a_n$, $\pi_{\mathcal{Q}^*_n} = (\overbrace{d_2, r+2}^{+}, \overbrace{r-2, a_n, a_n \ldots, a_0}^{(-1)^n})$.

(ii) for $d_2 = d_1 = a_{n+1} = r \neq a_n$, $\pi_{\mathcal{Q}^*_n} = (r+1, r+1, \overbrace{r-2, a_n, \ldots, a_0}^{(-1)^n})$.

(iii) for $d_2 \neq d_1 = a_{n+1} = r = a_n$*, then by* $(\dagger)$, $n = 1$ *and* $\pi_{\mathcal{Q}_n^*} = (\overbrace{d_{k+2}, r+2}^{+}, r-1, r-1)$.

(iv) for $d_2 = d_1 = a_{n+1} = r = a_n$*, then by* $(\dagger)$, $n = 1$ *and* $\pi_{\mathcal{Q}_n^*} = (r+1, r+1, r-1, r-1)$.

Proof. Part (a) is trivial, by the definition of $\pi_{\mathcal{Q}}$. For Part (b), we do (iii) as an example and the other parts follows in exactly the same fashion. To simplify our computations, let $\mathcal{Q} = (2r, 2r)$ and $\mathcal{Q}^* = (2s, 2r, 2r, 2r)$. Then $\pi_{\mathcal{Q}} = (\overbrace{2r, 2r}^{+})$ and $\pi_{\mathcal{Q}^*} = (\overbrace{2s, 2r, 2r, 2r}^{-})$. The character formula of $\pi_{\mathcal{Q}}$ is

$$\frac{1}{2}[\sum_w (-1)^{l(w)} \begin{pmatrix} & (r\ldots 1; (r-1)\ldots 0) \\ w & (r\ldots 1; (r-1)\ldots 0) \end{pmatrix} + \sum_{w'} (-1)^{l(w')} \begin{pmatrix} & (r\ldots 0; (r-1)\ldots 1) \\ w' & (r\ldots 0; (r-1)\ldots 1) \end{pmatrix}]$$

Now for $(d_{k+2}, d_{k+1}) = (2s, 2r)$, $X_{d_{k+2}, d_{k+1}}$ has character formula

$$\frac{1}{2}[\sum_w (-1)^{l(w)} \begin{pmatrix} & (s\ldots 1; (r-1)\ldots 0) \\ w & (s\ldots 1; (r-1)\ldots 0) \end{pmatrix} + \sum_{w'} (-1)^{l(w')} \begin{pmatrix} & (s\ldots 0; (r-1)\ldots 1) \\ w' & (s\ldots 0; (r-1)\ldots 1) \end{pmatrix}]$$

Now inducing means concatenating the character formulas, which gives

$$\mathcal{I} = \frac{1}{4}[\begin{pmatrix} r\ldots 1; (r-1)\ldots 0 \\ r\ldots 1; (r-1)\ldots 0 \end{pmatrix} + \begin{pmatrix} r\ldots 0; (r-1)\ldots 1 \\ r\ldots 0; (r-1)\ldots 1 \end{pmatrix}][\begin{pmatrix} s\ldots 1; (r-1)\ldots 0 \\ s\ldots 1; (r-1)\ldots 0 \end{pmatrix} + \begin{pmatrix} s\ldots 0; (r-1)\ldots 1 \\ s\ldots 0; (r-1)\ldots 1 \end{pmatrix}]$$

(for simplicity, the summations were hidden from the calculations)

The character formula for $\pi_{\mathcal{Q}^*}$ is

$$\pi_{\mathcal{Q}^*} = \frac{1}{4}[\begin{pmatrix} r\ldots 1; (r-1)\ldots 0 \\ r\ldots 1; (r-1)\ldots 0 \end{pmatrix} - \begin{pmatrix} r\ldots 0; (r-1)\ldots 1 \\ r\ldots 0; (r-1)\ldots 1 \end{pmatrix}][\begin{pmatrix} s\ldots 1; (r-1)\ldots 0 \\ s\ldots 1; (r-1)\ldots 0 \end{pmatrix} - \begin{pmatrix} s\ldots 0; (r-1)\ldots 1 \\ s\ldots 0; (r-1)\ldots 1 \end{pmatrix}]$$

So $\mathcal{I} - \pi_{\mathcal{Q}^*}$ is equal to

$$\frac{1}{2}[\begin{pmatrix} r\dots 1; (r-1)\dots 0; s\dots 0; (r-1)\dots 1 \\ r\dots 1; (r-1)\dots 0; s\dots 0; (r-1)\dots 1 \end{pmatrix} + \begin{pmatrix} r\dots 0; (r-1)\dots 1; s\dots 1; (r-1)\dots 0 \\ r\dots 0; (r-1)\dots 1; s\dots 1; (r-1)\dots 0 \end{pmatrix}]$$

$$= \frac{1}{2}[\begin{pmatrix} r\dots 1; s\dots 0; (r-1)\dots 0; (r-1)\dots 1 \\ r\dots 1; s\dots 0; (r-1)\dots 0; (r-1)\dots 1 \end{pmatrix} + \begin{pmatrix} r\dots 0; s\dots 1; (r-1)\dots 0; (r-1)\dots 1 \\ r\dots 0; s\dots 1; (r-1)\dots 0; (r-1)\dots 1 \end{pmatrix}]$$

$$= \frac{1}{2}[\begin{pmatrix} s\dots 1; r\dots 0 \\ s\dots 1; r\dots 0 \end{pmatrix} + \begin{pmatrix} s\dots 0; r\dots 1 \\ s\dots 0; r\dots 1 \end{pmatrix}] \begin{pmatrix} r-1\dots 1; r-1\dots 0 \\ r-1\dots 1; r-1\dots 0 \end{pmatrix}$$

which is preceisely $\pi_{\mathcal{Q}_r^*}$, where $\mathcal{Q}_r^* = (2s, 2r+2, 2r-1, 2r-1)$. $\square$

Theorem 10.11. *Let $\mathcal{O} = (c_{2k}, c_{2k-1}, \dots, c_0)$ be a nilpotent orbit in $Sp(2m, \mathbb{C})$ satisfying (†). Then as G representations,*

$$Ind_{Sp(c_{2k}+c_{2k-1})\times\cdots\times Sp(c_1+c_2)\times Sp(c_0)}^{Sp(2m)}(\overbrace{c_{2k}, c_{2k-1}}^{+} \otimes \cdots \otimes \overbrace{c_2, c_1}^{+} \otimes triv) \cong \bigoplus_{S\subset\mathcal{S}} \pi_{S, j_S}$$

Proof. We prove by induction on the number of columns of $\mathcal{O}$. The result is obviously true when there are one or two columns. Suppose the Theorem holds for $\mathcal{O} = (c_{2i}, c_{2i-1}, \dots, c_0)$ (c_0 can be zero), let $\mathcal{O}^* = (c_{2i+2}, c_{2i+1}, \dots, c_0)$. From now on, we denote the 'toppling set' $\mathcal{S}$ of any nilpotent orbit $\mathcal{P}$ by $\mathcal{S}_\mathcal{P}$, and any subset S in $\mathcal{S}_\mathcal{P}$ by $S_\mathcal{P}$.

Suppose $c_{2i+1} \neq c_{2i}$, then the places where $\mathcal{O}^*$ can topple is the same as that of $\mathcal{O}$, i.e. $\mathcal{S}_\mathcal{O} = \mathcal{S}_{\mathcal{O}^*}$. And for any $S \subset \mathcal{S}_\mathcal{O} = \mathcal{S}_{\mathcal{O}^*}$, $\mathcal{O}_S^* = (c_{2i+2}, c_{2i+1}, \mathcal{O}_S)$. Lemma 10.10(a) says for a fixed $S \subset \mathcal{S}_\mathcal{O} = \mathcal{S}_{\mathcal{O}^*}$,

$$Ind(\pi_{S_\mathcal{O}, j_{S_\mathcal{O}}} \otimes \overbrace{c_{2i+2}, c_{2i+1}}^{+}) = \pi_{S_{\mathcal{O}^*}, j_{S_{\mathcal{O}^*}}}$$

and hence

$$\begin{aligned}
Ind(\overbrace{c_{2i+2}, c_{2i+1}}^{+} \otimes \cdots \otimes triv) &= Ind[\overbrace{c_{2i+2}, c_{2i+1}}^{+} \otimes Ind(\overbrace{c_{2i}, c_{2i-1}}^{+} \otimes \cdots \otimes triv)] \\
&= Ind(\overbrace{c_{2i+2}, c_{2i+1}}^{+} \otimes \bigoplus_{S_{\mathcal{O}} \subset \mathcal{S}_{\mathcal{O}}} \pi_{S_{\mathcal{O}}, j_{S_{\mathcal{O}}}}) \\
&= \bigoplus_{S_{\mathcal{O}} \subset \mathcal{S}_{\mathcal{O}}} Ind(\overbrace{c_{2i+2}, c_{2i+1}}^{+} \otimes \pi_{S_{\mathcal{O}}, j_{S_{\mathcal{O}}}}) \\
&= \bigoplus_{S_{\mathcal{O}^*} \subset \mathcal{S}_{\mathcal{O}^*}} \pi_{S_{\mathcal{O}^*}, j_{S_{\mathcal{O}^*}}}
\end{aligned}$$

This finishes the proof for Condition (a). Now suppose $c_{2i+1} = c_{2i} = r$, and $\mathcal{S}_{\mathcal{O}^*} = \mathcal{S}_{\mathcal{O}} \cup \{r\}$. For any $S \subset \mathcal{S}_{\mathcal{O}}$, Lemma 10.10(b) says

$$Ind(\pi_{S_{\mathcal{O}}, j_{S_{\mathcal{O}}}} \otimes \overbrace{c_{2i+2}, c_{2i+1}}^{+}) = \pi_{S_{\mathcal{O}^*}, j_{S_{\mathcal{O}^*}}} \oplus \pi_{S_{\mathcal{O}^*} \cup \{r\}, j_{S_{\mathcal{O}^*} \cup \{r\}}}$$

by adding both sides for all subsets $S \subset \mathcal{S}_{\mathcal{O}}$, we get our desired result. □

Corollary 10.12. *Let* $\mathcal{O} = (c_{2k}, \ldots, c_0)$ *be any orbit in* $Sp(2m, \mathbb{C})$ *satisfying* (†)*, and* $\mathcal{O}^\sharp = (\frac{c_{2k}+c_{2k-1}}{2}, \frac{c_{2k}+c_{2k-1}}{2}, \frac{c_{2k-2}+c_{2k-3}}{2}, \frac{c_{2k-2}+c_{2k-3}}{2}, \ldots, \frac{c_2+c_1}{2}, \frac{c_2+c_1}{2}, c_0)$*. Then*

$$[R[\mathcal{O}^\sharp] : \mu_i] = [\bigoplus_{S \subset \mathcal{S}} \pi_{S, j_S} : \mu_i]$$

Proof. By Theorem 10.11, $\bigoplus_{all\ S} \pi_{S, j_S}$ has virtual character formula equal to

$$(\overbrace{c_{2k}, c_{2k-1}}^{+}, \ldots, \overbrace{c_2, c_1}^{+}, c_0)$$

(note that the character formula is the same as the case when all columns of $\mathcal{O}$ are distinct). By the techniques in Chapter 9, the multiplicities of the fundamental representations of the above character is given by

$$[Ind^{Sp(2m,\mathbb{C})}_{GL(\frac{c_{2k}+c_{2k-1}}{2}, \mathbb{C}) \times \cdots \times GL(\frac{c_2+c_1}{2}, \mathbb{C}) \times Sp(c_0, \mathbb{C})}(triv) : \mu_i]$$

which is precisely $[R[\mathcal{O}^\sharp] : \mu_i]$ by Theorem 9.4 (or [22] directly). $\square$

Conjecture 10.13. *Let $\mathcal{O}$ be a nilpotent orbit in $Sp(2m, \mathbb{C})$ satisfying $(\dagger)$. Then the set*

$$\{\pi_{S,j_S} | S \subset \mathcal{S}\}$$

is the set of composition factors of $X_{\overline{\mathcal{O}}}$. Consequently, by Lemma 9.11 and Corollary 10.12,

$$[R[\mathcal{O}^\sharp] : \mu_i] \geq [X_{\overline{\mathcal{O}}} : \mu_i] = [\bigoplus_{S \subset \mathcal{S}} \pi_{S,j_S} : \mu_i] = [R[\mathcal{O}^\sharp] : \mu_i]$$

and the inequality will become an equality. More generally, for any nilpotent orbit $\mathcal{O}$ in $Sp(2m, \mathbb{C})$,

$$[R[\mathcal{O}^\sharp] : \mu_i] = [X_{\overline{\mathcal{O}}} : \mu_i] = [R[\overline{\mathcal{O}}] : \mu_i]$$

Similarly, suppose $\mathcal{P} = (b_{2k+1}, \ldots, b_0)$ is an orbit in $O(n, \mathbb{C})$ satisfying $(\dagger)$ and $b_{2k+1} \neq b_{2k}$. Then

$$[R[\mathcal{P}^\sharp] : \mu_i'] = [X_{\overline{\mathcal{P}}} : \mu_i'] = [R[\overline{\mathcal{P}}] : \mu_i']$$

10.2.2 General Case

Here is the statement for the general case:

Conjecture 10.14. *Let $\mathcal{O} = (c_{2k}, \ldots, c_0)$ be an orbit in $Sp(2m, \mathbb{C})$. Then the character of $X_{\overline{\mathcal{O}}}$ is of the form*

$$X_{\overline{\mathcal{O}}} \cong (\overbrace{c_{2k}, c_{2k-1}}^{+}; \overbrace{c_{2k-2}, c_{2k-3}}^{+}; \ldots; \overbrace{c_2, c_1}^{+}; c_0)$$

Let $\mathcal{P} = (b_{2k+1}, \ldots, c_0)$ be an orbit in $O(n, \mathbb{C})$. Then the character of $X_{\overline{\mathcal{P}}}$ is of the form

$$X_{\overline{\mathcal{P}}} \cong (\overbrace{b_{2k+1}}^{+}; \overbrace{b_{2k}, b_{2k-1}}^{+}; \ldots; \overbrace{b_2, b_1}^{+}; b_0)$$

As a consequence, the second part of Conjecture 10.13 holds, i.e. $[R[\mathcal{O}^\sharp] : \mu_i] = [R[\overline{\mathcal{O}}] : \mu_i]$ *for all orbits in* $Sp(2m, \mathbb{C})$*, and* $[R[\mathcal{P}^\sharp] : \mu_i'] = [R[\overline{\mathcal{O}}] : \mu_i']$ *for all orbits* $\mathcal{P} = (b_{2k+1}, \ldots, b_0)$ *in* $O(n, \mathbb{C})$ *such that* $b_{2k+1} \neq b_{2k}$.

To check the validity of the conjecture if $\overline{\mathcal{O}}$ or $\overline{\mathcal{P}}$ are normal, note that Lemma 8.3 says $\overbrace{c, c}^{+}$ and $Ind_{GL(c,\mathbb{C})}^{G}(|det|^{1/2})$ have the same character formula. So the conjectured character formula for $X_{\overline{\mathcal{O}}}$,

$$(\overbrace{c_{2k}, c_{2k-1}}^{+}; \overbrace{c_{2k-2}, c_{2k-3}}^{+}; \ldots; \overbrace{c_2, c_1}^{+}; c_0)$$

will have the same character as $X_{\mathcal{O}}$ and similarly for $X_{\mathcal{P}}$ (Check Definition 6.1). Therefore the Conjecture is consistent with Theorem 10.1.

CHAPTER 11

LINKS TO DUAL PAIR CORRESPONDENCE

11.1 Langlands Parameters of Irreducible Modules

In the previous Chapters, we have seen that for a fixed infinitesimal character $\lambda_{\mathcal{O}}$, there is a list of all candidates of the composition factors of $X_{\mathcal{O}}$ or $X_{\overline{\mathcal{O}}}$. Since they are all irreducible $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-modules, they all appear as **Langlands quotients** of a principal series representation. Since a principal series representation can be parametrized by its infinitesimal character, we can parametrize all irreducible $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-modules by their infinitesimal characters, and they are called **Langlands parameters**.

Example 11.1. *Consider the orbit $\mathcal{O} = (6,4,4,2,2)$ in $Sp(32,\mathbb{C})$. The composition factors of $X_{\mathcal{O}}$, written as the Langlands quotient of a principal series, is given below:*

$$\pi_{\phi} = \bar{X}\begin{pmatrix} 321, & 210-1, & 10 \\ 321, & 10-1-2, & 0-1 \end{pmatrix}$$

$$\pi_{2,+} = \bar{X}\begin{pmatrix} 3210-1, & 210-1 \\ 3210-1, & 10-1-2 \end{pmatrix}; \pi_{2,-} = \bar{X}\begin{pmatrix} 3210-1, & 210-1 \\ 310-1-2, & 10-1-2 \end{pmatrix}$$

$$\pi_{4,+} = \bar{X}\begin{pmatrix} 3210-1-2, & 10, & 1 \\ 3210-1-2, & 0-1, & 1 \end{pmatrix}; \pi_{4,-} = \bar{X}\begin{pmatrix} 3210-1-2, & 10, & 1 \\ 210-1-2-3, & 0-1, & 1 \end{pmatrix}$$

$$\pi_{2,4,+} = \bar{X}\begin{pmatrix} 3210-1-2, & 10-1 \\ 3210-1-2, & 10-1 \end{pmatrix}, \pi_{2,4,-} = \bar{X}\begin{pmatrix} 3210-1-2 & 10-1 \\ 210-1-2-3 & 10-1 \end{pmatrix}$$

The reason why Langlands parameter is of interest in our work is the following Theorem:

Theorem 11.2. *Let $\mathcal{P} = (b_{2k+1}, \ldots, b_0)$ be a nilpotent orbit in $O(n, \mathbb{C})$ so that $\mathcal{P}' = (b_{2k+1} + 2, b_{2k+1}, \ldots, b_0)$ is a nilpotent orbit in $Sp(2m, \mathbb{C})$ satisfying $(\dagger)$. As $O(n, \mathbb{C})$-modules,*

$$X_{\mathcal{P}} \cong R[\mathcal{P}] \cong \bigoplus_j \theta(\pi_j)$$

where the π_j's are the composition factors of $X_{\mathcal{P}'}$, and θ is the dual pair correspondence given in [1, Theorem 2.8].

Proof. This is just a direct computation of the formulas given in [6] and [1]. $\square$

The above Theorem shows a connection between the composition factors of nilpotent orbits in $O(n, \mathbb{C})$ and $Sp(2m, \mathbb{C})$. It shows that there are some links between the construction of $X_{\mathcal{O}}$ (or $X_{\overline{\mathcal{O}}}$) and the dual pair correspondence between irreducible representations of $O(n, \mathbb{C})$ and $Sp(2m, \mathbb{C})$.

11.2 Lowest Harmonics and K-type Multiplicities

In this Section, we explore another link between the construction of $R[\overline{\mathcal{O}}]$ and the dual pair correspondence. Here is a brief recap of the basic idea of the theory:

Theorem 11.3. *Let $R[\mathbb{C}^{2mn}]$ be the ring of polynomial functions on a $2m \times n$ matrix X, and let $GL(2m, \mathbb{C}) \times O(n, \mathbb{C})$ act on $R[X]$ by the following:*

$$(g_1, g_2) \cdot f(X) := f(g_1^{-1} X g_2)$$

where $g_1 \in GL(2m, \mathbb{C})$, $g_2 \in O(n, \mathbb{C})$ and $X \in M_{2m\times n}$. Then

$$R[X] = \sum_{\tau} H_\tau \cdot \mathbb{C}[r_{ij}^2]$$

where r_{ij}^2 are the coordinate entries of the XX^*, where X^* is defined in Definition 4.1, and $H_\tau \cong \tau \otimes \tau'$ as an irreducible, finite-dimensional $O(n, \mathbb{C}) \times GL(2m, \mathbb{C})$ module. Moreover, τ' can be determined by τ.

Similarly, let $R[\mathbb{C}^{2mn}]$ be the ring of polynomial functions on a $n \times 2m$ matrix X', and let $GL(n, \mathbb{C}) \times Sp(2m, \mathbb{C})$ acts on $R[X']$ by the following:

$$(g_1, g_2) \cdot f(X') := f(g_1^{-1} X' g_2)$$

where $g_1 \in GL(n, \mathbb{C})$, $g_2 \in Sp(2m, \mathbb{C})$ and $X' \in M_{n\times 2m}$. Then

$$R[X'] = \sum_{\sigma} H_\sigma \cdot \mathbb{C}[s_{ij}^2]$$

where s_{ij}^2 are the coordinate entries of the X'^*X', where X'^* is defined in Definition 3.1, and $H_\sigma \cong \sigma \otimes \sigma'$ as an irreducible, finite-dimensional $Sp(2m, \mathbb{C}) \times GL(n, \mathbb{C})$ module.

Moreover, σ' can be determined by σ.

*The functions in $R[X]$ or $R[X']$ represented by H_τ or H_σ above are called **lowest harmonics**, since they are solutions of some Laplacian equations by construction.*

Proof. This is given in [13] or [20]. $\square$

As a Corollary, the functions appearing in $R[\overline{\mathcal{O}}]$ can all be represented by the lowest harmonics.

Corollary 11.4. *Let M and μ as in Proposition 4.4. Then the elements in the space $\frac{R[M]}{\langle \mu_1^x | x\in\mathfrak{s}\rangle}$ are generated by the lowest harmonics.*

Proof. Use induction on the number of matrices in the space M. If M only contains $L(V_1, V_0)$, then the above Theorem says

$$R[M] = \sum H_\tau \cdot \mathbb{C}[r_{ij}^2]$$

(or $H_\sigma \cdot \mathbb{C}[s_{ij}^2]$). Therefore every element in M can be written as a sum of elements of the form $h \cdot (c^0 + \sum_{i,j} c_{ij}^1 r_{ij}^2 + h.o.t.)$. All the non-zero order terms are in the ideal $\langle \mu_1^x | x \in \mathfrak{s}\rangle$, and hence the result follows in this case.

Suppose $M = L(V_1, V_0) \oplus L(V_2, V_1) \oplus \cdots \oplus L(V_{k+1}, V_k)$, and $M_i \in L(V_i, V_{i-1})$. Then

$$\frac{R[M]}{\langle \mu_1^x | x \in \mathfrak{s}\rangle} = \frac{R[M_1, \ldots, M_{k+1}]}{\langle M_1 M_1^* = 0, M_1^* M_1 = M_2 M_2^*, \ldots M_k^* M_k = M_{k+1} M_{k+1}^* \rangle}$$

(or $\frac{R[M_1,\ldots,M_{k+1}]}{\langle M_1^* M_1 = 0, M_1 M_1^* = M_2^* M_2, \ldots, M_k M_k^* = M_{k+1}^* M_{k+1}\rangle}$). For simplicity, assume the former is true.

By induction hypothesis, $\frac{R[M_1,\ldots,M_k]}{\langle M_1 M_1^* = 0, M_1^* M_1 = M_2 M_2^*, \ldots M_{k-1}^* M_{k-1} = M_k M_k^* \rangle}$ can be represented by lowest harmonics. Suppose $f \in R[M]/\langle\mu_1^x|x \in \mathfrak{s}\rangle$, then we just need to consider summands of f with coordinates in M_{k+1}. In particular, we only consider $f \in R[M_{k+1}] \cap R[M]/\langle\mu_1^x|x \in \mathfrak{s}\rangle$.

Suppose f is not a lowest harmonic, then by lowest weight representation theory, it can be written as $f = \sum m_i f_i$, where m_i are the coordinate entries of $M_{k+1}M_{k+1}^*$, and $f_i \in R[M_{k+1}]$ have smaller degree than f. However, the relation $M_k^* M_k = M_{k+1}M_{k+1}^*$ means each m_i can be represented as an element in $R[M_k]$, and hence we can inductively reduce the degree of f until it cannot be further reduced. The final f_i's must be lowest harmonics, and hence f can be represented by a sum of product of an element in $R[M_k]$ and a lowest harmonic, and by induction hypothesis we are done. $\square$

With the above Corollary, we can now give another upper bound on the K-type multiplicities of $R[\overline{\mathcal{O}}]$. The key is to understand the relation between σ and σ' in Theorem 11.3, as we will see in the following Corollary:

Corollary 11.5. *Let $\mathcal{O} = (c_{2k}, c_{2k-1} \dots c_0)$ be a nilpotent orbit in $Sp(2m,\mathbb{C})$, with $c_0 \neq 0$. Let $d_r = \sum_0^r c_r$, then $[R[\overline{\mathcal{O}}] : \mu_i] \leq [Y_k : \mu_i]$, where $[Y_k : \mu_i]$ is defined inductively by*

$$[Y_k : \mu_i] = 0 \quad \textit{if } i \textit{ is odd}$$
$$[Y_{j+1} : \mu_i] = \sum_{l=0}^{min\{i, \frac{d_{2j}}{2} - i\}} [Y_j : \mu_{2l}] \quad \textit{if } i \textit{ is even}$$

with $[Y_0 : \mu_i] = \delta_{i0}$ for all i. If $c_0 = 0$ then begin with Y_1 instead of Y_0.

Let $\mathcal{P} = (b_{2k+1}, b_{2k}, \dots, b_0)$ be a nilpotent orbit in $O(n,\mathbb{C})$. Then $[R[\overline{\mathcal{P}}] : \mu_i'] < [Z_k :$

$\mu_i']$, *where* $[Z_k : \mu_i'] := \sum_{i-2r\geq 0}[Y_k : \mu_{i-2r}]$ *and* Y_k *is defined by taking the orbit* $(b_{2k}, b_{2k-1}, \ldots, b_0)$ *in* Sp.

Example 11.6. *Let* $\mathcal{O} = (8, 6, 5, 5, 4, 2, 2)$ *in* $Sp(32, \mathbb{C})$. *Then the algorithm above gives the multiplicities* $[Y_3 : \mu_i]$ *as listed below:*

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$[R[\overline{\mathcal{O}}] : \mu_i]$	1	0	3	0	5	0	7	0	8	0	8	0	7	0	5	0	2
$[Y : \mu_i]$	1	0	3	0	5	0	7	0	8	0	8	0	7	0	5	0	3

In this case, $[R[\overline{\mathcal{O}}] : \mu_i] \leq [Y : \mu_i]$ *as expected.*
Let $\mathcal{P} = (8, 8, 6, 5, 5, 4, 2, 2)$ *in* $O(40, \mathbb{C})$, *then the multiplicities are as follows:*

i	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34-40
$[Z : \mu_i']$	1	4	9	16	24	32	39	44	47	44	39	32	24	16	9	4	1	0

Proof. In Corollary, we know $\frac{R[M]}{\langle \mu_1^x | x \in \mathfrak{s}\rangle}$ is generated by lowest harmonics, i.e. every element in the ring can be written as a sum of products of lowest harmonics. In terms of representation theory, the summand of elements in the ring must be of the form

$$\{f_1 \otimes f_2 \otimes \cdots \otimes f_k | f_i \in H_{\sigma_i} \cong \sigma_i \otimes \sigma_i'\}$$

Recall the Kraft-Procesi construction of $R[\overline{\mathcal{O}}]$:

$$R[\overline{\mathcal{O}}] \cong (\frac{R[M]}{\langle \mu_1^x | x \in \mathfrak{s}\rangle})^S$$

for instance, if $f_1 \otimes \cdots \otimes f_k$ is an element in $R[\overline{\mathcal{O}}]$, with $f_i \in \sigma_i \otimes \sigma_i'$ a representation of $GL \times O$ or $GL \times Sp$, then $\sigma_i' \otimes \sigma_{i+1}$, as an irreducible representation of $Sp \times GL$

or $O \times GL$, must contain a trivial representation of $Sp \overset{\Delta}{\hookrightarrow} Sp \times Sp \subset Sp \times GL$ (or $O \overset{\Delta}{\hookrightarrow} O \times O \subset O \times GL$). Also it forces $\sigma_1 \cong triv$ as a representation of O or $Sp \subset GL$.

We now start proving the Theorem for the case of $Sp(2m, \mathbb{C})$ by induction on the number of columns. The Theorem holds when there is only one column (the zero orbit). For the case of two columns (so that $c_0 = 0$), then elements in $R[\overline{\mathcal{O}}]$ can be represented by $f \in \sigma \otimes \sigma'$, where

• $\sigma \otimes \sigma'$ is an irreducible, finite dimensional representation of $GL \otimes Sp$, and

• σ and σ' corresponds to each other by the dual pair correspondence, and

• $\sigma|_{O(c_1,\mathbb{C})}$ is the trivial representation.

Therefore, σ itself must also be trivial, and by [20] σ' is a trivial representation of Sp. Hence the Theorem is true for nilpotent orbits with two columns.

Now suppose the Theorem is true for orbits with $2k-1$ or $2k-2$ columns, and suppose $\mathcal{O} = (c_{2k}, \ldots, c_0)$ with $c_0 \neq 0$ (the case is the same if $c_0 = 0$). Let $f_1 \otimes \cdots \otimes f_{2k}$ be a representative of $R[\overline{\mathcal{O}}]$, with $f_i \in \sigma_i \otimes \sigma_i'$, with $\sigma_{2k} \otimes \sigma_{2k}'$ an irreducible representation of $GL(d_{2k-1}, \mathbb{C}) \times Sp(2m, \mathbb{C})$ and $\sigma_{2k-1} \otimes \sigma_{2k-1}'$ an irreducible representation of $GL(d_{2k-2}, \mathbb{C}) \times O(d_{2k-1}, \mathbb{C})$, satisfying:

• $\sigma_{2k} \otimes \sigma_{2k}'$ and $\sigma_{2k-1} \otimes \sigma_{2k-1}'$ correspond to each other by dual pair correspondence.

• $\sigma_{2k-1}' \otimes \sigma_{2k}$ contains a trivial representation of $O(d_{2k-1}, \mathbb{C})$.

By [20], if $\sigma_{2k}' = \mu_i$, then $\sigma_{2k} = \wedge^i \mathbb{C}^{d_{2k-1}}$. By the second condition above and the self-duality of representations orthogonal groups, $\sigma_{2k-1}' = \wedge^i \mathbb{C}^{d_{2k-1}}$. By [20] again, $\sigma_{2k-1} = \wedge^i \mathbb{C}^{d_{2k-2}}$ as a representation of $GL(d_{2k-2}, \mathbb{C})$. But upon restricted as a representation of $Sp(d_{2k-2}, \mathbb{C})$, $\wedge^i \mathbb{C}^{d_{2k-2}}$ is decomposed as $\mu_0 \oplus \mu_2 \oplus \cdots \oplus \mu_{\min\{i, d_{2k-2}-i\}}$.

Therefore the Theorem holds by induction hypothesis. $\square$

Obviously, the above argument can be extended to all irreducible K-types other than fundamental representations. The statement for the general case is not included for two reasons: The branching of non-fundamental representations from GL to Sp or O is more involved, which we do not have nice recursive formula as above. Also, the limitations of this upper bound can already be reflected for fundamental representations.

Recall the construction of $\mathcal{O}^\sharp$ and the upper bound on K-type multiplicities of $R[\overline{\mathcal{O}}]$ in Chapter 9. It is natural to ask whether the upper bound we just attained is as good as, or even better than the one we have in Lemma 9.11. Unfortunately, the following Theorem says the bound we just obtained can never be a better bound for the fundamental representations.

Theorem 11.7. *Retain the notations in Lemma 9.11 and Corollary 11.5, then* $[R[\mathcal{O}^\sharp] : \mu_i] \leq [Y_k : \mu_i]$ *for any* i.

Proof. Let $X_j := Ind_{GL(\frac{d_{2j}-d_{2j-2}}{2})\times GL(\frac{d_{2j-2}-d_{2j-4}}{2})\times\cdots\times Sp(d_0)}^{Sp(d_{2j})}(triv \otimes \cdots \otimes triv)$. Then, by induction in stages, $X_{j+1} = Ind_{Sp(d_{2j})\times GL(\frac{d_{2j+2}-d_{2j}}{2})}^{Sp(d_{2j+2})}(X_j \otimes triv)$, and $X_k \cong R[\mathcal{O}^\sharp]$ as in Lemma 9.11. We argue by induction on j that $[X_j : \mu_i] \leq [Y_j : \mu_i]$ for all i.

It is obviously true when $j = 0$, since $[X_0 : \mu_i] = [triv_{Sp(d_0)} : \mu_i] = \delta_{i0} = [Y_0 : \mu_i]$. Suppose that for a fixed j, $[X_j : \mu_i] \leq [Y_j : \mu_i]$ for all i, then we want the inequality holds for $j + 1$. Without loss of generality, assume $i \leq \frac{d_{2j+2}}{2}$, since $\mu_i = \mu_{d_{2j+2}-i}$ in

$Sp(d_{2j+2}, \mathbb{C})$. Then

$$\begin{aligned}
[X_{j+1} : \mu_i] &= [Ind^{Sp(d_{2j+2})}_{Sp(d_{2j}) \times GL(\frac{d_{2j+2}-d_{2j}}{2})}(X_j \otimes triv) : \mu_i] \\
&= [X_j \otimes triv : Res^{Sp(d_{2j+2})}_{Sp(d_{2j}) \times GL(\frac{d_{2j+2}-d_{2j}}{2})}(\wedge^i \mathbb{C}^{d_{2j+2}} / \wedge^{i-2} \mathbb{C}^{d_{2j+2}})] \\
&= \sum_{p+q=i} [X_j \otimes triv : \wedge^p \mathbb{C}^{d_{2j}} \otimes \wedge^q \mathbb{C}^{d_{2j+2}-d_{2j}}] \\
&\quad - \sum_{p'+q'=i-2} [X_j \otimes triv : \wedge^{p'} \mathbb{C}^{d_{2j}} \otimes \wedge^{q'} \mathbb{C}^{d_{2j+2}-d_{2j}}]
\end{aligned}$$

where the second equality is from Frobenius reciprocity.

Note that $[triv : \wedge^q \mathbb{C}^{d_{2j+2}-d_{2j}}] = 1$ iff $q = i - p \leq d_{2j+2} - d_{2j}$ and q is even, or zero otherwise, hence

$$[X_{j+1} : \mu_i] = \sum_{\substack{i \geq p \geq i-(d_{2j+2}-d_{2j}) \\ p \equiv i (mod 2)}} [X_j : \wedge^p \mathbb{C}^{d_{2j}}] - \sum_{\substack{i-2 \geq p' \geq i-2-(d_{2j+2}-d_{2j}) \\ p' \equiv i (mod 2)}} [X_j : \wedge^{p'} \mathbb{C}^{d_{2j}}]$$

Therefore, if i is odd, the right hand side of the equation is zero by inductive hypothesis, and hence $[X_{j+1} : \mu_i] = 0 = [Y_{j+1} : \mu_i]$. If i is even, then

$$\begin{aligned}
[X_{j+1} : \mu_i] = [X_j : \wedge^i \mathbb{C}^{d_{2j}} - \wedge^{i-2-(d_{2j+2}-d_{2j})} \mathbb{C}^{d_{2j}}] &\leq [X_j : \wedge^i \mathbb{C}^{d_{2j}}] \\
&= [X_j : \sum_{l=0}^{min\{i, \frac{d_{2j}}{2} - i\}} \mu_{2l}] \\
&\leq [Y_j : \sum_{l=0}^{min\{i, \frac{d_{2j}}{2} - i\}} \mu_{2l}] \\
&= [Y_{j+1} : \mu_i]
\end{aligned}$$

Hence the equality holds for any j, and we are done. $\square$

Remark 11.8. *In fact, using the Kraft-Procesi coordinates for $R[\overline{\mathcal{O}}]$, one can write down the functions on $\overline{\mathcal{O}}$ corresponding to a particular K-type. For instance, in Example 11.6, one can write down the three highest weight functions of μ_{16} corresponding to the multiplicity $[Y : \mu_{16}] = 3$. In this particular example, one can even check that one of the functions is precisely zero in $R[\overline{\mathcal{O}}]$, accounting for the discrepancy between the two rows in the Example.*

11.3 Some Untied Ends

According to our Conjecture in Chapter 10, the fundamental K-type multiplicities of $R[\overline{\mathcal{O}}]$ is given by that of $R[\mathcal{O}^\sharp]$. On the other hand, we have another upper bound in the above Section. Here is a Conjecture on the fundamental K-type multiplicities mixing the two algorithms:

Conjecture 11.9. *The fundamental K-type multiplicities of the ring of regular functions of nilpotent orbit closures can be computed recursively by:*
Let $\mathcal{P} = (b_{2k+1}, b_{2k}, \dots, b_0)$ be a nilpotent orbit in $O(n,\mathbb{C})$, then

$$[R[\overline{\mathcal{P}}] : \mu'_{2i+1}] = 0$$

$$[R[\overline{\mathcal{P}}] : \mu'_{2i}] = \sum_{j=0}^{\min\{2i, d_{2k}-2i\}} [R[\overline{\mathcal{P}'}] : \mu_j]$$

where $\mathcal{P}' = (b_{2k}, \dots, b_0)$, and $d_{2k} = \sum_0^{2k} b_i$ (and is equal to 0 if $\min\{2i, d_{2k} - 2i\} < 0$).

Let $\mathcal{O} = (c_{2k}, \ldots, c_0)$ be a nilpotent orbit in $Sp(2m, \mathbb{C})$, then

$$[R[\overline{\mathcal{O}}] : \mu_{2i+1}] = 0$$

$$[R[\overline{\mathcal{O}}] : \mu_{2i}] = [R[\overline{\mathcal{O}'}] : \mu'_{2i}] - [R[\overline{\mathcal{O}'}] : \mu'_{2d_{2k-1}-2i-2}]$$

where $\mathcal{O}' = (c_{2k-1}, \ldots, c_0)$, and $d_{2k-1} = \sum_0^{2k-1} c_i$. Note that $[R[\overline{\mathcal{O}'}] : \mu'_{2d_{2k-1}-2i-2}] = 0$ if $2d_{2k-1} - 2i - 2 > d_{2k-1}$.

This algorithm gives the same result as Conjecture 10.13.

Remark 11.10. *The algorithm in the above Conjecture is the same as that in Corollary 11.5, except the extra negative term $-[R[\overline{\mathcal{O}'}] : \mu'_{2d_{2k-1}-2i-2}]$ in the last equation.*

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