Solutions to December and January Team Selection Tests

United States of America

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§1 Solution to December TST, Problem 1

This problem was proposed by Maria Monks Gillespie.

Let us denote $s(g) = n - c(g)$ for every permutation g. Thus, the problem is equivalent to showing that

$$s\left(f_1 \circ f_2 \circ \cdots \circ f_n\right) \le s(f_1) + \cdots + s(f_n).$$

Now, the critical claim is that:

Claim. For a permutation $g \in S_n$, $s(g)$ is the minimal number of transpositions t_1, t_2, $\dots$, t_k for which $g = t_1 \circ \cdots \circ t_k$.

Proof. This is a standard fact. Note $s(\mathrm{id}) = n - n = 0$. Now remark that applying a transposition to a permutation either joins two disjoint orbits or splits one orbit into two, so we certainly need at least $s(g)$ permutations. Moreover we can explicitly construct a cycle of length ℓ using $\ell - 1$ transpositions, which completes the proof. $\square$

The conclusion of the problem now follows as a "triangle inequality".

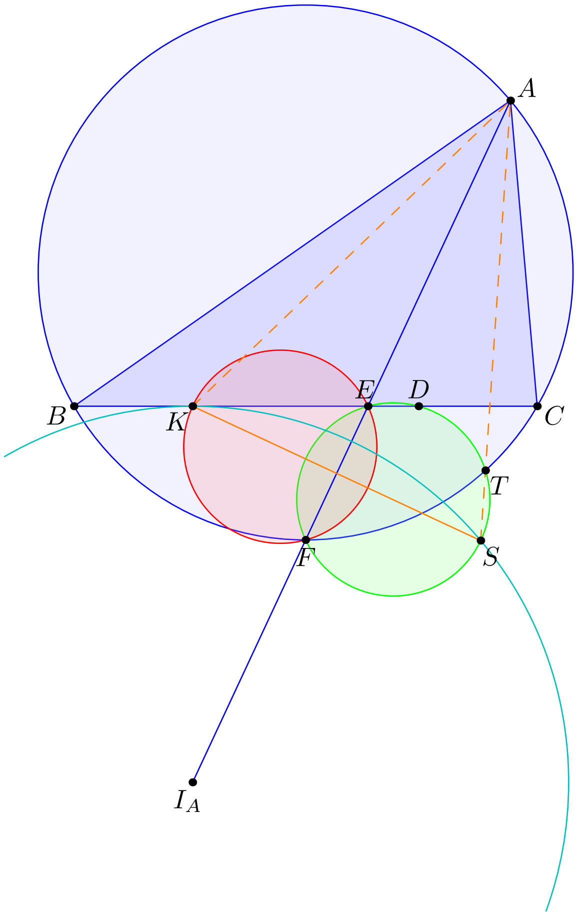
§2 Solution to December TST, Problem 2

This problem was proposed by Evan Chen.

First solution

Assume for simplicity $AB < AC$. Let K be the contact point of the A-excircle on BC; also let ray TD meet Ω again at L. From the fact that $\angle FTL = \angle FTD = 180^\circ - \angle FED$, we can deduce that $\angle FTL = \angle ACF$, meaning that F is the reflection of A across the perpendicular bisector ℓ of BC. If we reflect T, D, L over ℓ, we deduce A, K and the reflection of T across ℓ are collinear, which implies that $\angle BAT = \angle CAK$.

Now, consider the reflection point K across line AI, say S. Since ray AI passes through the A-excenter, S lies on the A-excircle. Since $\angle BAT = \angle CAK$, S also lies on ray AT. But the circumcircles of triangles DEF and EFK are congruent (from $DF = KF$), so S lies on the circumcircle of $\triangle DEF$ too. Hence S is the desired intersection point.



Second solution

It's known that T is the touch-point of the A-mixtilinear incircle. Let K be contact point of A-excircle on BC. Now the circumcircles of $\triangle DEF$ and $\triangle EFK$ are congruent, since $DF = FK$ and the angles at E are supplementary. Let S be the reflection of K across line EF, which by the above the above comment lies on the circumcircle of $\triangle DEF$. Since EF passes through the A-excenter, S also lies on the A-excircle. But S also lies on line AT, since lines AT and AK are isogonal (the mixtilinear cevian is isogonal to the Nagel line). Thus S is the desired intersection point.

§3 Solution to December TST, Problem 3

This problem was proposed by Mark Sellke.

Let $\daleth \subseteq \mathbb{F}_p[x]$ denote the set of polynomials in the image of Ψ. Thus $\Psi : \mathbb{F}_p[x] \to \daleth$ is a bijection on the level of sets.

Claim. If $A, B \in \daleth$ then $\gcd(A, B) \in \daleth$.

Proof. It suffices to show that if A and B are monic, and $\deg A > \deg B$, then the remainder when A is divided by B is in $\daleth$. Suppose $\deg A = p^k$ and $B = x^{p^{k-1}} - c_2 x^{p^{k-2}} - \cdots - c_k$. Then

$$\begin{aligned} x^{p^k} &\equiv \left(c_2 x^{p^{k-2}} + c_3 x^{p^{k-3}} + \cdots + c_k \right)^p \pmod{B} \\ &\equiv c_2 x^{p^{k-1}} + c_3 x^{p^{k-2}} \cdots + c_k \pmod{B} \end{aligned}$$

since exponentiation by p commutes with addition in $\mathbb{F}_p$. This is enough to imply the conclusion. The proof if $\deg B$ is smaller less than p^{k-1} is similar. $\square$

Thus, if we view $\mathbb{F}_p[x]$ and $\beth$ as partially ordered sets under polynomial division, then gcd is the "greatest lower bound" or "meet" in both partially ordered sets. We will now prove that Ψ is an *isomorphism* of the posets. This requires two parts:

Claim. If $P \mid Q$ then $\Psi(P) \mid \Psi(Q)$.

Proof. Observe that Ψ is also a linear map of $\mathbb{F}_p$ vector spaces, and that $\Psi(xP) = \Psi(P)^p$ for any $P \in \mathbb{F}_p[x]$.

Set $Q = PR$, where $R = \sum_{i=0}^{k} r_i x^i$. Then

$$\begin{aligned}\Psi(Q) &= \Psi\left(P\sum_{i=0}^{k} r_i x^i\right) \\ &= \sum_{i=0}^{k} \Psi\left(P \cdot r_i x^i\right) \\ &= \sum_{i=0}^{k} r_i \Psi(P)^{p^i}.\end{aligned}$$

which is divisible by $\Psi(P)$. $\square$

Claim. If $\Psi(P) \mid \Psi(Q)$ then $P \mid Q$.

Proof. Suppose $\Psi(P) \mid \Psi(Q)$, but $Q = PA + B$ where $\deg B < \deg P$. Thus $\Psi(P) \mid \Psi(PA) + \Psi(B)$, hence $\Psi(P) \mid \Psi(B)$, but $\deg \Psi(P) > \deg \Psi(B)$ hence $\Psi(B) = 0 \implies B = 0$. $\square$

This completes the proof.

Remark. In fact $\psi : \mathbb{F}_p[x] \to \beth$ is a ring isomorphism if we equip $\beth$ with function composition as the ring multiplication.

§4 Solution to January TST, Problem 1

This problem was proposed by Iurie Boreico.

Assume the contrary, so that for some integer k we have

$$k < 2^{n-1}\sqrt{3} < k + \frac{1}{2^{n-1}}.$$

Squaring gives

$$\begin{aligned}k^2 < 3 \cdot 2^{2n-2} &< k^2 + \frac{k}{2^n} + \frac{1}{2^{2n+2}} \\ &\le k^2 + \frac{2^{n-1}\sqrt{3}}{2^n} + \frac{1}{2^{2n+2}} \\ &= k^2 + \frac{\sqrt{3}}{2} + \frac{1}{2^{2n+2}} \\ &\le k^2 + \frac{\sqrt{3}}{2} + \frac{1}{16} \\ &< k^2 + 1\end{aligned}$$

and this is a contradiction.

§5 Solution to January TST, Problem 2

This problem was proposed by Zilin Jiang.

Of course, $W(k,k)$ is arbitrary for $k \in [n]$. We claim that $W(a,b) = \pm 1$ for any $a \neq b$, with the sign fixed. (These are all solutions.)

First, let $X_{abc} = W(a,b)W(b,c)$ for all distinct a, b, c, so the given condition is

$$\sum_{a,b,c \in A \times B \times C} X_{abc} = |A||B||C|.$$

Consider the given equation with the particular choices

- $A = \{1\}$, $B = \{3\}$, $C = \{2, 4, \dots, n\}$.
- $A = \{2\}$, $B = \{3\}$, $C = \{1, 4, \dots, n\}$.
- $A = \{1, 2\}$, $B = \{3\}$, $C = \{4, \dots, n\}$.

Adding the first two and subtracting the second gives $X_{132} + X_{231} = 2$. Similarly, $X_{132} + X_{312} = 2$, and in this way, we get that $X_{231} = X_{312}$. Then, $W(2,3)W(3,1) = W(3,1)W(1,2)$, Clearly, $W(3,1) \neq 0$, or else take $A = \{3\}$, $B = \{1\}$ in the original given to get a contradiction. Thus, $W(1,2) = W(2,3)$.

Analogously, for any distinct a, b, c we have $W(a,b) = W(b,c)$. For $n \geq 4$ this is enough to imply $W(a,b) = \pm 1$ for $a \neq b$ where the choice of sign is the same for all a and b.

Surprisingly, the $n = 3$ case has "extra" solutions for $W(1,2) = W(2,3) = W(3,1) = \pm 1$, $W(2,1) = W(3,2) = W(1,3) = \mp 1$.

§6 Solution to January TST, Problem 3

This problem was proposed by Ivan Borsenco.

The locus of points is three points: the incenter I, the circumcenter O and the orthocenter H of triangle ABC, and they clearly satisfy the given conditions. So we show they are the only ones.

First solution

In complex numbers with ABC the unit circle, it is equivalent to solving the following two cubic equations in p and $q = \overline{p}$:

$$\begin{aligned}(p-a)(p-b)(p-c) &= (abc)^2(q-1/a)(q-1/b)(q-1/c) \\ 0 &= \prod_{\text{cyc}}(p+c-b-bcq) + \prod_{\text{cyc}}(p+b-c-bcq).\end{aligned}$$

Viewing this as two cubic curves in $(p,q) \in \mathbb{C}^2$, by *Bézout's Theorem* it follows there are at most nine solutions (unless both curves are not irreducible, but it's easy to check the first one cannot be factored). Moreover it is easy to name nine solutions (for ABC scalene): the three vertices, the three excenters, and the aforementioned I, O, H. Hence the answer is just those three triangle centers I, O and H.

Second solution

Set

$$x_1 = \frac{1}{2}\angle PAB,\ y_1 = \frac{1}{2}\angle PBC,\ z_1 = \frac{1}{2}\angle PCA,$$

and

$$x_2 = \frac{1}{2}\angle PAC,\ y_2 = \frac{1}{2}\angle PBA,\ z_2 = \frac{1}{2}\angle PCB.$$

Because AP, BP, CP are concurrent at point P, from trigonometric version of Ceva's Theorem, we have

$$\sin\frac{x_1}{2}\sin\frac{y_1}{2}\sin\frac{z_1}{2} = \sin\frac{x_2}{2}\sin\frac{y_2}{2}\sin\frac{z_2}{2}. \tag{1}$$

Using Ceva's Theorem for cevians AA_1, BB_1, CC_1, we get

$$1 = \frac{BA_1}{CA_1}\cdot\frac{CB_1}{AB_1}\cdot\frac{AC_1}{BC_1} = \frac{\tan\frac{y_1}{2}}{\tan\frac{z_2}{2}}\cdot\frac{\tan\frac{z_1}{2}}{\tan\frac{x_2}{2}}\cdot\frac{\tan\frac{x_1}{2}}{\tan\frac{y_2}{2}}.$$

from which we combine with (1) to get

$$\cos\frac{x_1}{2}\cos\frac{y_1}{2}\cos\frac{z_1}{2} = \cos\frac{x_2}{2}\cos\frac{y_2}{2}\cos\frac{z_2}{2}. \tag{2}$$

If we square (1) and (2) while multiplying both sides by 8, we then obtain

$$(1-\cos x_1)(1-\cos y_1)(1-\cos z_1) = (1-\cos x_2)(1-\cos y_2)(1-\cos z_2), \tag{3}$$
$$(1+\cos x_1)(1+\cos y_1)(1+\cos z_1) = (1+\cos x_2)(1+\cos y_2)(1+\cos z_2).. \tag{4}$$

From now on, define

$$\begin{aligned} p_i &= \cos x_i + \cos y_i + \cos z_i \\ q_i &= \cos x_i\cos y_i + \cos y_i\cos z_i + \cos z_i\cos x_i \\ r_i &= \cos x_i\cos y_i\cos z_i \end{aligned}$$

for $i = 1, 2$. The sum and difference of (3) and (4) then gives

$$q_1 = q_2 \tag{5}$$
$$p_1 + r_1 = p_2 + r_2. \tag{6}$$

From the fact that $x_1 + y_1 + z_1 = x_2 + y_2 + z_2 = 180°$, we find one more relation between angles x_1, y_1, z_1 and x_2, y_2, z_2:

$$\begin{aligned} 1 = \cos 180° &= \cos^2 x_1 + \cos^2 y_1 + \cos^2 z_1 + 2\cos x_1\cos y_1\cos z_1 \\ &= \cos^2 x_2 + \cos^2 y_2 + \cos^2 z_2 + 2\cos x_2\cos y_2\cos z_2 \\ \implies 1 &= p_1^2 - 2q_1 + 2r_1 = p_2^2 - 2q_2 + 2r_2. \end{aligned} \tag{7}$$

If we combine (5), (6), (7) we easily obtain $p_1^2 - 2p_1 = p_2^2 - 2p_2 \implies (p_1-1)^2 = (p_2-1)^2$. But it is well known that $p_i > 1$, and hence $p_1 = p_2$. Then $q_1 = q_2$ and $r_1 = r_2$.

From this it follows that $(\cos x_1, \cos y_1, \cos z_1)$ and $(\cos x_2, \cos y_2, \cos z_2)$ are permutations of each other, by considering the polynomial $t^3 - p_i t^2 + q_i t - r_i$. Because $0 < x_1, y_1, z_1, x_2, y_2, z_2 < 180°$, we conclude that angles (x_1, y_1, z_1) are a permutation of angles (x_2, y_2, z_2).

Consider the following cases:

- Suppose $x_1 = x_2$. Then if $y_1 = z_2$ and $z_1 = y_2$, we obtain triangle ABC is isosceles. Hence $y_1 = y_2$, $z_1 = z_2$ and $P = I$.

- Suppose $x_1 = y_2$. Then if $y_1 = x_2$, we obtain triangle ABC is isosceles. Hence $y_1 = z_2$, $z_1 = x_2$ and $P = O$.
- Suppose $x_1 = z_2$. Then if $z_1 = x_2$, we obtain triangle ABC is isosceles. Hence $z_1 = y_2$, $y_1 = x_2$ and $P = H$.

This completes the proof.