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COMMON BEHAVIOUR OF SOLUTIONS
OF SOME COLLECTIONS OF HILL'S EQUATIONS

1. Consider Hill's equation see [3], [8]

$$(1) \quad y'' + [\lambda^2 + Q(t)]y = 0,$$

where $Q(t+\pi) = Q(t)$, $Q \in L^2(0, \pi)$, λ is a real parameter and $\lambda \geq 1$.

Solutions of (1) are understood in the sense of Carathéodory (see [1]). If Q has the properties stated above, then (1) has continuously differentiable solutions y_1 and y_2 which are uniquely determined by the conditions:

$$(2) \quad y_1(0, \lambda) = 1, \quad y_1'(0, \lambda) = 0, \quad y_2(0, \lambda) = 0, \quad y_2'(0, \lambda) = 1.$$

These solutions are referred to as normalized solutions of (1).

As in [3] we have

$$(3) \quad y_1(t, \lambda) = \sum_{k=0}^{\infty} u_k(t, \lambda), \quad y_2(t, \lambda) = \sum_{k=0}^{\infty} v_k(t, \lambda),$$

where

$$(4) \quad \begin{cases} u_0(t, \lambda) = \cos \lambda t, \\ u_k(t, \lambda) = -\frac{1}{\lambda} \int_0^t \sin \lambda(t-s) Q(s) u_{k-1}(s, \lambda) ds, \\ v_0(t, \lambda) = \frac{1}{\lambda} \sin \lambda t, \\ v_k(t, \lambda) = -\frac{1}{\lambda} \int_0^t \sin \lambda(t-s) Q(s) v_{k-1}(s, \lambda) ds, \end{cases} \quad k=1, 2, \dots$$

For simplicity of notations in the sequel we shall write $y_1(t), y_2(t), \dots$ understanding that these functions are not only dependent on t but also on λ . Putting $M = \|q\|_{L^2(0,\pi)}$ we have for $t \in [0, \pi]$ and $k = 0, 1, 2, \dots$

$$(5) \quad \begin{cases} |u_k(t)| \leq \frac{M^k t^{\frac{k}{2}}}{\sqrt{k!} \lambda^k}, & |u'_k(t)| \leq \frac{M^k t^{\frac{k}{2}}}{\sqrt{k!} \lambda^{k-1}}, \\ |v_k(t)| \leq \frac{M^k t^{\frac{k}{2}}}{\sqrt{k!} \lambda^{k+1}}, & |v'_k(t)| \leq \frac{M^k t^{\frac{k}{2}}}{\sqrt{k!} \lambda^k}. \end{cases}$$

Therefore

$$(6) \quad \begin{cases} |y_1(t)| \leq D_1, & |y'_1(t)| \leq D_1 \lambda, \\ |y_2(t)| \leq D_1 \lambda^{-1}, & |y'_2(t)| \leq D_1, \end{cases}$$

where $D_1 = \sum_{k=0}^{\infty} \frac{M^k \pi^{\frac{k}{2}}}{\sqrt{k!}}$ and

$$(7) \quad \begin{cases} |y_1(t) - \cos \lambda t| \leq D_2 \lambda^{-1}, \\ |y_2(t) - \cos \lambda t| \leq D_2 \lambda^{-1}, \end{cases}$$

where $D_2 = D_1 M \sqrt{\pi}$. Let $\Delta(\lambda) = y_1(\pi) + y'_2(\pi)$. As above we have

$$(8) \quad |\Delta(\lambda) - 2 \cos \lambda \pi| \leq D_2 \lambda^{-1}.$$

Theorem 1. For every $\varepsilon \in (0, 1)$ there exists $D > 0$ (depending on ε and M but not on λ) such that for each $\alpha \geq 0$ and each $\lambda \geq 1$, the inequality

$$(9) \quad |\Delta(\lambda)| \leq 2(1 - \varepsilon^2 \lambda^{-\alpha})$$

implies, for all real values of t ,

$$(10) \quad \begin{aligned} |y_1(t)| &\leq D \lambda^\beta, & |y_1'(t)| &\leq D \lambda^{\beta+1} \\ |y_2(t)| &\leq D \lambda^{\beta-1}, & |y_2'(t)| &\leq D \lambda^\beta, \end{aligned}$$

where

$$\beta = \begin{cases} \alpha - 1 & \text{for } \alpha \geq 2, \\ \frac{\alpha}{2} & \text{for } 0 \leq \alpha < 2. \end{cases}$$

P r o o f . Since $|\Delta(\lambda)| < 2$, by Floquet's theorem, the characteristic equation $q^2 - \Delta(\lambda)q + 1 = 0$ corresponding to (1) have two solutions of modulus 1

$$q = \frac{\Delta(\lambda) + \sqrt{\Delta^2(\lambda) - 4}}{2} \quad \text{and} \quad \bar{q} = \frac{\Delta(\lambda) - \sqrt{\Delta^2(\lambda) - 4}}{2},$$

and the equation (1) have two linearly independent solutions of the type

$$(11) \quad \begin{cases} f_1(t) = \exp(i\omega t) q_1(t), \\ f_2(t) = \exp(-i\omega t) q_2(t), \end{cases}$$

where $q = \exp i\omega\pi$ and q_1, q_2 are periodic functions with period π . By the linear independence of y_1 and y_2 there exist c_{11}, c_{12}, c_{21} and c_{22} such that

$$(12) \quad \begin{cases} f_1(t) = c_{11} y_1(t) + c_{12} y_2(t), \\ f_2(t) = c_{21} y_1(t) + c_{22} y_2(t). \end{cases}$$

Put $\delta_1 = y_1(\pi)$, $\delta_1' = y_1'(\pi)$, $\delta_2 = y_2(\pi)$ and $\delta_2' = y_2'(\pi)$. From (2), (11), (12) it follows that c_{11}, c_{12} is a nontrivial solution of the system

$$(13) \quad \begin{cases} c_{11}(\delta_1 - q) + c_{12}\delta_2 = 0, \\ c_{11}\delta_1' + c_{12}(\delta_2' - q) = 0 \end{cases}$$

and c_{21}, c_{22} is a nontrivial solution of the system

$$(14) \quad \begin{cases} c_{21}(\delta_1 - \bar{q}) + c_{22}\delta_2 = 0, \\ c_{21}\delta_1' + c_{22}(\delta_2' - \bar{q}) = 0. \end{cases}$$

By (7), (8) we get

$$(15) \quad |\delta_1 - \varrho| \leq |\delta_1 - \cos \lambda \pi| + \frac{1}{2} |\Delta(\lambda) - 2 \cos \lambda \pi| + \\ + \frac{1}{2} \sqrt{4 - \Delta^2(\lambda)} \leq \frac{3}{2} D_2 \lambda^{-1} + \frac{1}{2} |\varrho - \bar{\varrho}|.$$

Similarly, we have

$$(16) \quad \begin{cases} |\delta_1 - \bar{\varrho}| \leq \frac{3}{2} D_2 \lambda^{-1} + \frac{1}{2} |\varrho - \bar{\varrho}|, \\ |\delta'_2 - \varrho| \leq \frac{3}{2} D_2 \lambda^{-1} + \frac{1}{2} |\varrho - \bar{\varrho}|, \\ |\delta'_2 - \bar{\varrho}| \leq \frac{3}{2} D_2 \lambda^{-1} + \frac{1}{2} |\varrho - \bar{\varrho}|. \end{cases}$$

By Liouville's theorem we obtain $\delta_1 \delta'_2 - \delta_2 \delta'_1 = 1$. If $\delta_1 \delta'_2 \leq 0$, then $|\delta'_1 \delta_2| \geq 1 \geq \frac{\varepsilon^2}{\lambda^\alpha}$. If $\delta_1 \delta'_2 > 0$, then by (9) $|\delta'_1 \delta_2| \geq$
 $\geq 1 - \frac{(\delta_1 + \delta'_2)^2}{4} \geq \frac{\varepsilon^2}{\lambda^\alpha}$. Hence $|\delta'_1 \delta_2| \geq \frac{\varepsilon^2}{\lambda^\alpha}$ always holds, and then

$$(17) \quad |\delta_2| \geq \varepsilon \lambda^{-(\frac{\alpha}{2}+1)} \quad \text{or} \quad |\delta'_1| \geq \varepsilon \lambda^{-(\frac{\alpha}{2}+1)}. \quad \text{Suppose}$$

$$|\delta_2| \geq \varepsilon \lambda^{-(\frac{\alpha}{2}+1)}.$$

Since determinants of systems (13) and (14) equal zero, we have $c_{11} = R_1 \delta_2$, $c_{12} = R_1 (\varrho - \delta_1)$, $c_{21} = R_2 \delta_2$, $c_{22} = R_2 (\bar{\varrho} - \delta_1)$, where R_1, R_2 are some constants. Thus by (6), (12) we get for $t \in [0, \pi]$

$$(18) \quad \frac{1}{|R_1|} |f_1(t)| \leq D_3 \lambda^{-1}, \quad \frac{1}{|R_2|} |f_2(t)| \leq D_3 \lambda^{-1},$$

where $D_3 = D_1(2D_1+1)$. Up to now we have estimated values of the functions f_1 and f_2 only on the interval $[0, \pi]$, but by (11) we see that (18) is still valid for the whole real line.

By (12) we have

$$(19) \quad \begin{cases} y_1(t) = \frac{1}{\delta_2(\bar{q}-q)} \left[(\bar{q}-\delta_1) \frac{f_1(t)}{R_1} - (q-\delta_1) \frac{f_2(t)}{R_2} \right], \\ y_2(t) = \frac{1}{\bar{q}-q} \left[\frac{1}{R_2} f_2(t) - \frac{1}{R_1} f_1(t) \right]. \end{cases}$$

From (9) we get

$$(20) \quad |q - \bar{q}| = \sqrt{4 - \Delta^2(\lambda)} \geq 2 \varepsilon \lambda^{-\frac{\alpha}{2}},$$

and, from (6), (15)-(20)

$$|y_1(t)| \leq D \lambda^\beta, \quad |y_2(t)| \leq D \lambda^{\beta-1},$$

where $D = D_3 \varepsilon^{-1} (\frac{3}{2} D_2 \varepsilon^{-1} + 1)$. By analogous arguments we obtain

$$|y'_1(t)| \leq D \lambda^{\beta+1}, \quad |y'_2(t)| \leq D \lambda^\beta.$$

For the case $|\delta'_1| \geq \varepsilon \lambda^{(1-\frac{\alpha}{2})}$ proof is similar.

By $V[0, \pi]$ we denote the set of all functions of bounded variation and by $H^k(0, \pi)$ Sobolev space. Boundary values of functions in $H^k(0, \pi)$ are understood in the sense of trace (see [4]).

For $Q \in H^3(0, \pi)$ we define constants $c_{n,m}$ by

$$c_{n,m} = \int_0^\pi [Q^{(n)}(t)]^m dt, \quad n, m = 0, 1, 2, 3.$$

Theorem 2. Suppose that $\int_0^\pi Q(t) dt = 0$.

^{1°} If $Q \in V[0, \pi]$, then

$$(21) \quad \Delta(\lambda) = 2 \cos \lambda \pi + o(\lambda^{-3}).$$

2° If $q \in H^1(0, \pi)$ and $q(0) = q(\pi)$, then

$$(22) \quad \Delta(\lambda) = 2 \cos \lambda \pi + 2^{-2} \lambda^{-3} c_{0,2} \sin \lambda \pi + o(\lambda^{-4}) = \\ = 2 \cos(\lambda \pi - 2^{-3} \lambda^{-3} c_{0,2}) + o(\lambda^{-4}).$$

3° If $q \in H^2(0, \pi)$, $q'' \in V[0, \pi]$, $q(0) = q(\pi)$ and $q'(0) = q'(\pi)$, then

$$(23) \quad \Delta(\lambda) = (2 - 2^{-6} \lambda^{-6} c_{0,2}^2) \cos \lambda \pi + \\ + [2^{-2} \lambda^{-3} c_{0,2} + (2^{-4} c_{1,2} - 2^{-3} c_{0,3}) \lambda^{-5}] \sin \lambda \pi + o(\lambda^{-7}) = \\ = 2 \cos[\lambda \pi - 2^{-3} \lambda^{-3} c_{0,2} + (2^{-4} c_{0,3} - 2^{-5} c_{1,2}) \lambda^{-5}] + o(\lambda^{-7}).$$

P r o o f . The proof is not difficult, but very laborious - see [5], pp.54-84.

2. Now we consider the nonhomogeneous equation corresponding to (1)

$$(24) \quad z'' + [\lambda^2 + q(t)]z = p_\lambda(t),$$

where $p_\lambda(t+\pi) = p_\lambda(t)$, $p_\lambda \in L^2(0, \pi)$. We shall use the notations introduced in section 1. In particular, by y_1 and y_2 we denote the normalized solutions of (1). Let z_0 be the solution of (24) satisfying

$$(25) \quad z_0(0) = z'_0(0) = 0,$$

and let

$$Y(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{bmatrix}, \quad p_\lambda^*(t) = \begin{bmatrix} 0 \\ p_\lambda(t) \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$P_\lambda = \int_0^\pi |p_\lambda(t)| dt.$$

Theorem 3. For every $\varepsilon \in (0, 1)$, there exists $K > 0$ independent of λ such that for each $\alpha \geq 0$, for each $\lambda \geq 1$ and for all $t \in (-\infty, +\infty)$ we have

$$(26) \quad |z_0(t)| < K P_\lambda \lambda^\gamma, \quad |z'_0(t)| \leq K P_\lambda \lambda^{\gamma+1}$$

if $\Delta(\lambda)$ satisfies (9), where

$$\gamma = \begin{cases} 2(\alpha-1) & \text{for } \alpha \geq 2, \\ \frac{3}{2}\alpha - 1 & \text{for } 0 \leq \alpha < 2. \end{cases}$$

Proof. By (9) the corresponding homogeneous equation has no periodic solutions with period π . Therefore (24) has the unique periodic solution with period π (see [2], p.251). Denoting by z this periodic solution we have

$$(27) \quad \begin{bmatrix} z(t) \\ z'(t) \end{bmatrix} = Y(t) \begin{bmatrix} z(0) \\ z'(0) \end{bmatrix} + \int_0^t Y(t) Y^{-1}(s) p_\lambda^*(s) ds,$$

where

$$\begin{bmatrix} z(0) \\ z'(0) \end{bmatrix} = [E - Y(\pi)]^{-1} Y(\pi) \int_0^\pi Y^{-1}(t) p_\lambda^*(t) dt.$$

Consequently

$$z(0) = \frac{1}{2-\Delta(\lambda)} \left[(1-\delta_1) \int_0^\pi p_\lambda(t) y_2(t) dt + \delta_2 \int_0^\pi p_\lambda(t) y_1(t) dt \right],$$

$$z'(0) = \frac{1}{2-\Delta(\lambda)} \left[-\delta'_1 \int_0^\pi p_\lambda(t) y_2(t) dt + (\delta'_2-1) \int_0^\pi p_\lambda(t) y_1(t) dt \right].$$

By (6), (9) we get

$$(28) \quad |z(0)| \leq \frac{1}{2} \varepsilon^{-2} D_3 P_\lambda \lambda^{(\alpha-1)}, \quad |z'(0)| \leq \frac{1}{2} \varepsilon^{-2} D_3 P_\lambda \lambda^\alpha.$$

Using the periodicity of z and (6), (27), (28), we have

$$(29) \quad \begin{cases} |z(t)| \leq D_1(D_3\varepsilon^{-2} + 2D_1) P_\lambda \lambda^{(\alpha-1)}, \\ |z'(t)| \leq D_1(D_3\varepsilon^{-2} + 2D_1) P_\lambda \lambda^\alpha. \end{cases}$$

It is easy to see that

$$z_0(t) = z(t) - z(0) y_1(t) - z'(0) y_2(t)$$

is the solution of (24) satisfying (25). By (10), (28), (29) we get (26) with $K = D_1(D_3\varepsilon^{-2} + 2D_1) + D D_3\varepsilon^{-2}$.

3. Now we consider the following collection of equations

$$(30) \quad y'' + [\lambda_n^2 + Q(t)y] = 0, \quad n = 1, 2, \dots$$

with $\lambda_n = nh$, where h is a positive constant. Putting

$$y_{n1}(t) = y_1(t, \lambda_n), \quad y_{n2}(t) = y_2(t, \lambda_n),$$

$$\Delta_k(\lambda_n) = u_k(\pi, \lambda_n) + v'_k(\pi, \lambda_n)$$

we have $\Delta(\lambda_n) = y_{n1}(\pi) + y'_{n2}(\pi) = \sum_{k=0}^{\infty} \Delta_k(\lambda_n)$.

By (5) it is easy to see that

$$(31) \quad \sum_{k=1}^{\infty} \Delta_k(\lambda_n) = O(\lambda_n^{-1}).$$

The following four theorems establish estimations for $\Delta(\lambda_n)$. These estimations are not only dependent on Q but also on property of number h .

Let

$$Q_m = \frac{1}{\pi} \int_0^\pi Q(t) dt.$$

Theorem 4. If

(i) h is a rational number,

(ii) $Q_m \neq 0$,

(iii) $Q \in V[0, \pi]$,

then there exist $\varepsilon > 0$ and n_0 such that for $n > n_0$ we have

$$|\Delta(\lambda_n)| \leq 2 \left(1 - \frac{\varepsilon^2}{\lambda_n^2} \right).$$

Proof. Let $h = p/q$ be a fraction in its lowest terms. For $n \neq mq$, $m \in \mathbb{N}$, i.e. λ_n is not integral, we have

$$|\Delta_0(\lambda_n)| = 2 \left| \cos \frac{np\pi}{q} \right| \leq 2 \cos \frac{\pi}{q}.$$

In view of (31) for $\varepsilon < 1 - \cos \frac{\pi}{q}$ and for sufficiently large n we get

$$|\Delta(\lambda_n)| \leq 2(1 - \varepsilon^2) \leq 2(1 - \varepsilon^2 \lambda_n^{-2}).$$

For $n = mq$, $m \in \mathbb{N}$, i.e. λ_n is an integer, we have

$$\Delta_0(\lambda_n) = 2 \cos \lambda_n \pi = (-1)^{mp} 2,$$

$$\Delta_1(\lambda_n) = -\lambda_n^{-1} \sin \lambda_n \pi \int_0^\pi Q(t) dt = 0,$$

$$\Delta_2(\lambda_n) =$$

$$\begin{aligned} &= \lambda_n^{-2} \int_0^\pi Q(t_1) dt_1 \int_0^{t_1} Q(t_2) \sin \lambda_n(t_1 - t_2) \sin \lambda_n(\pi - t_1 + t_2) dt_2 = \\ &= \frac{1}{2} \lambda_n^{-2} \int_0^\pi Q(t_1) dt_1 \int_0^{t_1} Q(t_2) \cos \lambda_n(\pi - 2t_1 + t_2) dt_2 + \\ &\quad - \frac{1}{2} \lambda_n^{-2} \cos \lambda_n \pi \int_0^\pi Q(t_1) dt_1 \int_0^{t_1} Q(t_2) dt_2. \end{aligned}$$

Since $A \in V[0, \pi]$, then (see [7], p.168)

$$\int_0^\pi Q(t_1) dt_1 \int_0^{t_1} Q(t_2) \cos \lambda_n(\pi - 2t_1 + 2t_2) dt_2 = O(\lambda_n^{-1}).$$

Putting $F(t) = \int_0^t Q(s) ds$ yields

$$\int_0^\pi Q(t_1) dt_1 \int_0^{t_1} Q(t_2) dt_2 = \int_0^\pi Q(t) F(t) dt = \frac{1}{2} Q_m^2 \pi^2.$$

So

$$\Delta_2(\lambda_n) = \frac{(-1)^{mp+1}}{4} \lambda_n^{-2} \pi^2 Q_m^2 + O(\lambda_n^{-3}).$$

and then

$$|\Delta(\lambda_n)| \leq 2(1 - \varepsilon^2 \lambda_n^{-2})$$

for $\varepsilon < \frac{\pi Q_m}{\sqrt{8}}$ and for sufficiently large n .

Theorem 5. If

- (i) h is a rational number,
- (ii) $Q_m = 0$,
- (iii) $Q \in H^2(0, \pi)$, $Q'' \in V[0, \pi]$, $Q(0) = Q(\pi)$ and $Q'(0) = Q'(\pi)$,
then for $n > n_0$

$$(32) \quad |\Delta(\lambda_n)| \leq 2(1 - \varepsilon^2 \lambda_n^{-6}).$$

Proof. Let $h = p/q$ be a fraction in its lowest terms. For $n \neq mq$, $m \in \mathbb{N}$ the proof is the same as in Theorem 4. Let $n = mq$, then $\sin \lambda_n \pi = 0$. By Theorem 2 we have

$$\begin{aligned} \Delta(\lambda_n) &= (2 - 2^{-6} \lambda_n^{-6} c_{0,2}) \cos \lambda_n \pi + O(\lambda_n^{-7}) = \\ &= (-1)^{mp} 2(1 - 2^{-7} c_{0,2} \lambda_n^{-6}) + O(\lambda_n^{-7}). \end{aligned}$$

So, for $\varepsilon < 2^{-7} c_{0,2}$ and for sufficiently large n we get (32).

Theorem 6. If

- (i) h is an irrational number such that there exist $r > 0$ and $\gamma \in (0, 3]$ satisfying

$$|nh - m| > r n^{-\gamma} \quad \text{for all } n, m \in \mathbb{N},$$

(ii) $Q_m = 0,$

(iii) $Q \in V[0, \pi]$ for $\gamma < \frac{3}{2},$

$Q \in H^1(0, \pi)$ and $Q(0) = Q(\pi)$ for $\frac{3}{2} \leq \gamma < 2,$

$Q \in H^2(0, \pi), \quad Q'' \in V[0, \pi], \quad Q(0) = Q(\pi)$ and $Q'(0) = Q'(\pi)$ for $2 \leq \gamma \leq 3,$ with $c_{0,2} < 8rh^3$ for $\gamma = 3,$

then there exist $\varepsilon > 0$ and n_0 such that for $n > n_0$

$$(33) \quad |\Delta(\lambda_n)| \leq 2(1 - \varepsilon^2 \lambda_n^{-2\gamma}),$$

Proof. Let $2 \leq \gamma \leq 3.$ Put $nh = m_n + r_n,$ where m_n is an integer and r_n satisfies $r n^{-\gamma} \leq |r_n| \leq \frac{1}{2}.$ By (23) we have

$$\begin{aligned} \Delta(\lambda_n) &= 2\cos[\pi nh - 2^{-3}c_{0,2}\lambda_n^{-3} + o(\lambda_n^{-5})] + o(\lambda_n^{-7}) = \\ &= 2\cos[\pi m_n + \pi r_n - 2^{-3}c_{0,2}\lambda_n^{-3} + o(\lambda_n^{-5})] + o(\lambda_n^{-7}). \end{aligned}$$

Therefore

$$|\Delta(\lambda_n)| \leq 2 \left[1 - \frac{1}{4} (\pi r n^{-\gamma} - 2^{-3}c_{0,2}\lambda_n^{-3})^2 \right] + o(\lambda_n^{-7}).$$

Thus, with $\varepsilon < \frac{1}{2}\pi h^\gamma r$ for $2 \leq \gamma < 3$ and with $\varepsilon < \frac{1}{2}(\pi h^3 r - 2^{-3}c_{0,2})$ for $\gamma = 3$ we have (33) for sufficiently large $n.$

For the cases $\gamma < \frac{3}{2}$ and $\frac{3}{2} \leq \gamma < 2$ the proofs are similar to the above, using (21) and (22) instead of (23).

Remark. The set of irrational numbers which do not satisfy condition (i) has measure zero (see [6], p.12).

Theorem 7. If

- (i) h is an irrational number such that

$$|nh - m| > \frac{r}{n} \quad \text{for all } n, m \in \mathbb{N},$$

(ii) $Q < |Q_m| < 2rh,$

(iii) $Q \in V[0, \pi],$

then there exist $\varepsilon > 0$ and n_0 such that for $n > n_0$

$$(34) \quad |\Delta(\lambda_n)| \leq 2(1 - \varepsilon^3 \lambda_n^{-2}).$$

P r o o f . Putting $q_1(t) = q(t) - q_m$ gives $\int_0^\pi q_1(t) dt = 0$. We shall write (30) in the form

$$(35) \quad y'' + [\lambda_n'^2 + q_1(t)]y = 0, \quad n = 1, 2, \dots,$$

where $\lambda_n'^2 = \lambda_n^2 + q_m$. The values $\Delta(\lambda_n')$ of equations (35) equal the values $\Delta(\lambda_n)$ of (30). We have

$$\lambda_n' = \lambda_n + \frac{1}{2} q_m \lambda_n^{-1} + o(\lambda_n^{-3}).$$

Put $nh = m_n + r_n$, where m_n is integer and r_n satisfy $nr_n^{-1} \leq |r_n| < \frac{1}{2}$. By (21) we have

$$\begin{aligned} \Delta(\lambda_n') &= 2 \cos \lambda_n' \pi + o(\lambda_n^{-3}) = \\ &= 2 \cos(\pi m_n + \pi r_n + \frac{1}{2} \pi q_m \lambda_n^{-1}) + o(\lambda_n^{-3}). \end{aligned}$$

From this we get

$$|\Delta(\lambda_n)| \leq 2 \left[1 - \frac{1}{4} (\pi r_n^{-1} - \frac{1}{2} \pi |q_m| \lambda_n^{-1}) \right] + o(\lambda_n^{-3}).$$

Thus, for $\varepsilon < \frac{\pi}{2} (r_h - \frac{1}{2} |q_m|)$ and for sufficiently large n we have (34).

R e m a r k . In Theorems 4-7 we proved that for the collection of equations (30) with certain assumptions, for sufficiently large n we have inequalities of the form (9) and so we get estimations of the solutions of the form (10). To get (10) for all $n \in \mathbb{N}$ we have only to assume that the equations (30) are stable for all $n \in \mathbb{N}$, for example using stability criterion of V.A. Jakubovič (see [8]) we have only to assume that $|q(t)|$ is sufficiently small.

The following examples show the existence of irrational numbers h such that not all equations (30) are stable even if $|Q(t)|$ is arbitrarily small.

Example 1. Let h be a positive irrational number such that the inequality

$$(36) \quad |nh - m| \leq n^{-(1+\gamma)}, \quad \gamma > 0,$$

has infinitely many solutions n, m in the set of natural numbers. Then, for each $\varepsilon > 0$ there exists a function Q satisfying $|Q(t)| < \varepsilon$ and $Q_m \neq 0$ such that not all equations (30) are stable.

Select

$$Q(t) = \begin{cases} 0 & \text{for } 0 < t < \frac{\pi}{2}, \\ q & \text{for } \frac{\pi}{2} < t < \pi, \end{cases}$$

and $Q(t+\pi) = Q(t)$.

We have (see [2], p.239)

$$\begin{aligned} (37) \quad \Delta(\lambda_n) &= 2 \cos nh \frac{\pi}{2} \cos \frac{\pi}{2} \sqrt{n^2 h^2 + q} + \\ &\quad - \frac{2n^2 h^2 + q}{nh \sqrt{n^2 h^2 + q}} \sin nh \frac{\pi}{2} \sin \frac{\pi}{2} \sqrt{n^2 h^2 + q} = \\ &= 2 \cos \left(nh + \sqrt{n^2 h^2 + q} \right) \frac{\pi}{2} + \\ &\quad - \frac{(\sqrt{n^2 h^2 + q} - nh)^2}{nh \sqrt{n^2 h^2 + q}} \sin nh \frac{\pi}{2} \sin \frac{\pi}{2} \sqrt{n^2 h^2 + q}. \end{aligned}$$

Choosing q so that

$$(38) \quad \frac{\pi}{2} \left(nh + \sqrt{n^2 h^2 + q} \right) = m\pi, \quad m \in \mathbb{N},$$

we have $q = 4m(m-nh)$. By (36) we can choose n_0, m_0 so that

$$q = 4m_0(m_0 - n_0 h) \quad \text{and} \quad |q| < \varepsilon.$$

Since h is irrational, by (38) we have

$$(39) \begin{cases} \sin n_0 h \frac{\pi}{2} \sin \frac{\pi}{2} \sqrt{n_0^2 h^2 + q} > 0 & \text{for } m_0 = 2k-1, \quad k \in \mathbb{N}, \\ \sin n_0 h \frac{\pi}{2} \sin \frac{\pi}{2} \sqrt{n_0^2 h^2 + q} < 0 & \text{for } m_0 = 2k, \quad k \in \mathbb{N}. \end{cases}$$

Now the desired conclusion follows from (37)-(39).

Example 2. Let h be a positive irrational number such that the inequality

$$0 < nh - m < n^{-3-\gamma}, \quad \gamma > 0,$$

has infinitely many solutions n, m among natural numbers. Then for each $\varepsilon > 0$ there exists a function Q satisfying $|Q(t)| < \varepsilon$ and $Q_m = 0$ such that not all equations (30) are stable.

Choose

$$Q(t) = \begin{cases} q & \text{for } 0 < t < \frac{\pi}{2}, \\ -q & \text{for } \frac{\pi}{2} < t < \pi, \end{cases}$$

and $Q(t+\pi) = Q(t)$. The proof is similar to that of Example 1.

In a similar way we can consider the collection of solutions

$$y'' + \left[\left(\frac{2n-1}{2} h \right)^2 + Q(t) \right] y = 0, \quad n=1, 2, \dots$$

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