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COMMON BEHAVIOUR OF SOLUTIONS OF SOME COLLECTIONS OF HILL'S EQUATIONS

1. Consider Hill's equation see [3], [8]

$$y'' + \left[\lambda^2 + Q(t)\right] y = 0, \tag{1}$$

where $Q(t+\pi) = Q(t)$, $Q \in L^2(0,\pi)$, λ is a real parameter and $\lambda \geq 1$.

Solutions of (1) are understood in the sense of Carathéodory (see [1]). If Q has the properties stated above, then (1) has continuously differentiable solutions y_1 and y_2 which are uniquely determined by the conditions:

$$y_1(0,\lambda) = 1, \quad y_1'(0,\lambda) = 0, \quad y_2(0,\lambda) = 0, \quad y_2'(0,\lambda) = 1. \tag{2}$$

These solutions are referred to as normalized solutions of (1).

As in [3] we have

$$y_1(t,\lambda) = \sum_{k=0}^{\infty} u_k(t,\lambda), \quad y_2(t,\lambda) = \sum_{k=0}^{\infty} v_k(t,\lambda), \tag{3}$$

where

$$\begin{cases} u_0(t,\lambda) = \cos \lambda t, \\ u_k(t,\lambda) = -\frac{1}{\lambda}\int_0^t \sin \lambda(t-s)\, Q(s)\, u_{k-1}(s,\lambda)\, ds, \quad k=1,2,\ldots \\ v_0(t,\lambda) = \frac{1}{\lambda}\sin \lambda t, \\ v_k(t,\lambda) = -\frac{1}{\lambda}\int_0^t \sin \lambda(t-s)\, Q(s)\, v_{k-1}(s,\lambda)\, ds, \quad k=1,2,\ldots \end{cases} \tag{4}$$

For simplicity of notations in the sequel we shall write $y_1(t)$, $y_2(t)$,... understanding that these functions are not only dependent on t but also on λ. Putting $M = \|Q\|_{L^2(0,\pi)}$ we have for $t \in [0,\pi]$ and $k = 0,1,2,\ldots$

$$(5)\qquad \begin{cases} |u_k(t)| \leqslant \dfrac{M^k t^{\frac{k}{2}}}{\sqrt{k!}\,\lambda^k}, & |u_k'(t)| \leqslant \dfrac{M^k t^{\frac{k}{2}}}{\sqrt{k!}\,\lambda^{k-1}}, \\ |v_k(t)| \leqslant \dfrac{M^k t^{\frac{k}{2}}}{\sqrt{k!}\,\lambda^{k+1}}, & |v_k'(t)| \leqslant \dfrac{M^k t^{\frac{k}{2}}}{\sqrt{k!}\,\lambda^k}. \end{cases}$$

Therefore

$$(6)\qquad \begin{cases} |y_1(t)| \leqslant D_1, & |y_1'(t)| \leqslant D_1\lambda, \\ |y_2(t)| \leqslant D_1\lambda^{-1}, & |y_2'(t)| \leqslant D_1, \end{cases}$$

where $D_1 = \sum_{k=0}^{\infty} \dfrac{M^k \pi^{\frac{k}{2}}}{\sqrt{k!}}$ and

$$(7)\qquad \begin{cases} |y_1(t) - \cos\lambda t| \leqslant D_2\lambda^{-1}, \\ |y_2'(t) - \cos\lambda t| \leqslant D_2\lambda^{-1}, \end{cases}$$

where $D_2 = D_1 M\sqrt{\pi}$. Let $\Delta(\lambda) = y_1(\pi) + y_2'(\pi)$. As above we have

$$(8)\qquad |\Delta(\lambda) - 2\cos\lambda\pi| \leqslant D_2\lambda^{-1}.$$

T h e o r e m 1. For every $\varepsilon \in (0,1)$ there exists $D > 0$ (depending on ε and M but not on λ) such that for each $\alpha \geqslant 0$ and each $\lambda \geqslant 1$, the inequality

$$(9)\qquad |\Delta(\lambda)| \leqslant 2(1 - \varepsilon^2\lambda^{-\alpha})$$

implies, for all real values of t,

$$(10)\qquad \begin{aligned} &|y_1(t)| \leqslant D\,\lambda^{\beta}, && |y_1'(t)| \leqslant D\,\lambda^{\beta+1} \\ &|y_2(t)| \leqslant D\,\lambda^{\beta-1}, && |y_2'(t)| \leqslant D\,\lambda^{\beta}, \end{aligned}$$

where

$$\beta = \begin{cases} \alpha - 1 & \text{for} \quad \alpha \geqslant 2, \\ \dfrac{\alpha}{2} & \text{for} \quad 0 \leqslant \alpha < 2. \end{cases}$$

Proof. Since $|\Delta(\lambda)| < 2$, by Floquet's theorem, the characteristic equation $\varrho^2 - \Delta(\lambda)\varrho + 1 = 0$ corresponding to (1) heve two solutions of modulus 1

$$\varrho = \frac{\Delta(\lambda) + \sqrt{\Delta^2(\lambda) - 4}}{2} \quad \text{and} \quad \bar{\varrho} = \frac{\Delta(\lambda) - \sqrt{\Delta^2(\lambda) - 4}}{2},$$

and the equation (1) have two linearly independent solutions of the type

$$(11)\qquad \begin{cases} f_1(t) = \exp(i\omega t)\, q_1(t), \\ f_2(t) = \exp(-i\omega t)\, q_2(t), \end{cases}$$

where $\varrho = \exp i\omega\pi$ and q_1, q_2 are periodic functions with period π. By the linear independence of y_1 and y_2 there exist c_{11}, c_{12}, c_{21} and c_{22} such that

$$(12)\qquad \begin{cases} f_1(t) = c_{11}\, y_1(t) + c_{12}\, y_2(t), \\ f_2(t) = c_{21}\, y_1(t) + c_{22}\, y_2(t). \end{cases}$$

Put $\delta_1 = y_1(\pi)$, $\delta_1' = y_1'(\pi)$, $\delta_2 = y_2(\pi)$ and $\delta_2' = y_2'(\pi)$. From (2), (11), (12) it follows that c_{11}, c_{12} is a nontrivial solution of the system

$$(13)\qquad \begin{cases} c_{11}(\delta_1 - \varrho) + c_{12}\delta_2 = 0, \\ c_{11}\delta_1' + c_{12}(\delta_2' - \varrho) = 0 \end{cases}$$

and c_{21}, c_{22} is a nontrivial solution of the system

$$(14)\qquad \begin{cases} c_{21}(\delta_1 - \bar{\varrho}) + c_{22}\delta_2 = 0, \\ c_{21}\delta_1' + c_{22}(\delta_2' - \bar{\varrho}) = 0. \end{cases}$$

By (7), (8) we get

$$(15)\qquad |\delta_1 - \varrho| \leqslant |\delta_1 - \cos\lambda\pi| + \frac{1}{2}|\Delta(\lambda) - 2\cos\lambda\pi| + $$
$$+ \frac{1}{2}\sqrt{4 - \Delta^2}(\lambda) \leqslant \frac{3}{2} D_2\lambda^{-1} + \frac{1}{2}|\varrho - \bar\varrho|.$$

Similarly, we have

$$(16)\qquad \begin{cases} |\delta_1 - \bar\varrho| \leqslant \frac{3}{2} D_2\lambda^{-1} + \frac{1}{2}|\varrho - \bar\varrho|, \\ |\delta_2' - \varrho| \leqslant \frac{3}{2} D_2\lambda^{-1} + \frac{1}{2}|\varrho - \bar\varrho|, \\ |\delta_2' - \bar\varrho| \leqslant \frac{3}{2} D_2\lambda^{-1} + \frac{1}{2}|\varrho - \bar\varrho|. \end{cases}$$

By Liouville's theorem we obtain $\delta_1\delta_2' - \delta_2\delta_1' = 1$. If $\delta_1\delta_2' \leqslant 0$, then $|\delta_1'\delta_2| \geqslant 1 \geqslant \frac{\varepsilon^2}{\lambda^\alpha}$. If $\delta_1\delta_2' > 0$, then by (9) $|\delta_1'\delta_2| \geqslant$ $\geqslant 1 - \frac{(\delta_1 + \delta_2')^2}{4} \geqslant \frac{\varepsilon^2}{\lambda^\alpha}$. Hence $|\delta_1'\delta_2| \geqslant \frac{\varepsilon^2}{\lambda^\alpha}$ always holds, and then $|\delta_2| \geqslant \varepsilon\lambda^{-(\frac{\alpha}{2}+1)}$ or $|\delta_1'| \geqslant \varepsilon\lambda^{(-\frac{\alpha}{2}+1)}$. Suppose

$$(17)\qquad |\delta_2| \geqslant \varepsilon\lambda^{-(\frac{\alpha}{2}+1)}.$$

Since determinants of systems (13) and (14) equal zero, we have $c_{11} = R_1\delta_2$, $c_{12} = R_1(\varrho - \delta_1)$, $c_{21} = R_2\delta_2$, $c_{22} = R_2(\bar\varrho - \delta_1)$, where R_1, R_2 are some constants. Thus by (6), (12) we get for $t \in [0,\pi]$

$$(18)\qquad \frac{1}{|R_1|}|f_1(t)| \leqslant D_3\lambda^{-1}, \quad \frac{1}{|R_2|}|f_2(t)| \leqslant D_3\lambda^{-1},$$

where $D_3 = D_1(2D_1+1)$. Up to now we have estimated values of the functions f_1 and f_2 only on the interval $[0,\pi]$, but by (11) we see that (18) is still valid for the whole real line.

By (12) we have

$$(19)\quad \begin{cases} y_1(t) = \dfrac{1}{\delta_2(\bar{\varrho}-\varrho)}\left[(\bar{\varrho}-\delta_1)\dfrac{f_1(t)}{R_1} - (\varrho-\delta_1)\dfrac{f_2(t)}{R_2}\right], \\ y_2(t) = \dfrac{1}{\bar{\varrho}-\varrho}\left[\dfrac{1}{R_2} f_2(t) - \dfrac{1}{R_1} f_1(t)\right]. \end{cases}$$

From (9) we get

$$(20)\qquad |\varrho-\bar{\varrho}| = \sqrt{4-\Delta^2(\lambda)} \geqslant 2\,\varepsilon\,\lambda^{-\frac{\alpha}{2}},$$

and, from (6), (15)-(20)

$$|y_1(t)| \leqslant D\,\lambda^{\beta}, \qquad |y_2(t)| \leqslant D\,\lambda^{\beta-1},$$

where $D = D_3\varepsilon^{-1}\left(\frac{3}{2} D_2\varepsilon^{-1} + 1\right)$. By analogous arguments we obtain

$$|y_1'(t)| \leqslant D\,\lambda^{\beta+1}, \qquad |y_2'(t)| \leqslant D\,\lambda^{\beta}.$$

For the case $|\delta_1'| \geqslant \varepsilon\,\lambda^{(1-\frac{\alpha}{2})}$ proof is similar.

By $V[0,\pi]$ we denote the set of all functions of bounded variation and by $H^k(0,\pi)$ Sobolev space. Boundary values of functions in $H^k(0,\pi)$ are understood in the sense of trace (see [4]).

For $Q \in H^3(0,\pi)$ we define constants $c_{n,m}$ by

$$c_{n,m} = \int_0^{\pi}\left[Q^{(n)}(t)\right]^m dt, \qquad n,m = 0,\ 1,\ 2,\ 3.$$

T h e o r e m 2. Suppose that $\int_0^{\pi} Q(t)dt = 0$.

1° If $Q \in V[0,\pi]$, then

$$(21)\qquad \Delta(\lambda) = 2\cos\lambda\pi + O(\lambda^{-3}).$$

2^0 If $Q \in H^1(0,\pi)$ and $Q(0) = Q(\pi)$, then

(22) $$\Delta(\lambda) = 2\cos\lambda\pi + 2^{-2}\lambda^{-3} c_{0,2} \sin\lambda\pi + O(\lambda^{-4}) =$$

$$= 2\cos(\lambda\pi - 2^{-3}\lambda^{-3} c_{0,2}) + O(\lambda^{-4}).$$

3^0 If $Q \in H^2(0,\pi)$, $Q'' \in V[0,\pi]$, $Q(0) = Q(\pi)$ and $Q'(0) = Q'(\pi)$, then

(23) $$\Delta(\lambda) = (2 - 2^{-6}\lambda^{-6} c_{0,2}^2)\cos\lambda\pi +$$

$$+\left[2^{-2}\lambda^{-3} c_{0,2} + (2^{-4} c_{1,2} - 2^{-3} c_{0,3})\lambda^{-5}\right]\sin\lambda\pi + O(\lambda^{-7}) =$$

$$= 2\cos\left[\lambda\pi - 2^{-3}\lambda^{-3} c_{0,2} + (2^{-4} c_{0,3} - 2^{-5} c_{1,2})\lambda^{-5}\right] + O(\lambda^{-7}).$$

P r o o f . The proof is not difficult, but very laborius - see [5], pp.54-84.

2. Now we consider the nonhomogeneous equation corresponding to (1)

(24) $$z'' + [\lambda^2 + Q(t)]z = p_\lambda(t),$$

where $p_\lambda(t+\pi) = p_\lambda(t)$, $p_\lambda \in L^2(0,\pi)$. We shall use the notations introduced in section 1. In particular, by y_1 and y_2 we denote the normalized solutions of (1). Let z_0 be the solution of (24) satisfying

(25) $$z_0(0) = z_0'(0) = 0,$$

and let

$$Y(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}, \quad p_\lambda^*(t) = \begin{bmatrix} 0 \\ p_\lambda(t) \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$P_\lambda = \int_0^\pi |p_\lambda(t)|\,dt.$$

T h e o r e m 3. For every $\varepsilon \in (0,1)$, there exists $K>0$ independent of λ such that for each $\alpha \geqslant 0$, for each $\lambda \geqslant 1$ and for all $t \in (-\infty, +\infty)$ we have

$$|z_0(t)| < K\, P_\lambda \lambda^{\gamma}, \quad |z_0'(t)| \leqslant K\, P_\lambda\, \lambda^{(\gamma+1)} \tag{26}$$

if $\Delta(\lambda)$ satisfies (9), where

$$\gamma = \begin{cases} 2(\alpha-1) & \text{for} \quad \alpha \geqslant 2, \\ \frac{3}{2}\alpha - 1 & \text{for} \quad 0 \leqslant \alpha < 2. \end{cases}$$

P r o o f . By (9) the corresponding homogeneous equation has no periodic solutions with period π. Therefore (24) has the unique periodic solution with period π (see [2], p.251). Denoting by z this periodic solution we have

$$\begin{bmatrix} z(t) \\ z'(t) \end{bmatrix} = Y(t) \begin{bmatrix} z(0) \\ z'(0) \end{bmatrix} + \int_0^t Y(t)\, Y^{-1}(s)\, p_\lambda^*(s)ds, \tag{27}$$

where

$$\begin{bmatrix} z(0) \\ z'(0) \end{bmatrix} = \left[E - Y(\pi)\right]^{-1} Y(\pi) \int_0^\pi Y^{-1}(t)\, p_\lambda^*(t)dt.$$

Consequently

$$z(0) = \frac{1}{2-\Delta(\lambda)} \left[(1-\delta_1) \int_0^\pi p_\lambda(t)\, y_2(t)dt + \delta_2 \int_0^\pi p_\lambda(t)\, y_1(t)dt \right],$$

$$z'(0) = \frac{1}{2-\Delta(\lambda)} \left[-\delta_1' \int_0^\pi p_\lambda(t)\, y_2(t)dt + (\delta_2'-1) \int_0^\pi p_\lambda(t)\, y_1(t)dt \right].$$

By (6), (9) we get

$$|z(0)| \leqslant \frac{1}{2}\, \varepsilon^{-2}\, D_3\, P_\lambda \lambda^{(\alpha-1)}, \quad |z'(0)| \leqslant \frac{1}{2}\, \varepsilon^{-2}\, D_3\, P_\lambda \lambda^{\alpha}. \tag{28}$$

~~Using~~ the periodicity of z and (6), (27), (28), we have

$$(29)\qquad \begin{cases} |z(t)| \leqslant D_1(D_3\varepsilon^{-2} + 2D_1)\ P_\lambda\ \lambda^{(\alpha-1)}, \\ |z'(t)| \leqslant D_1(D_3\varepsilon^{-2} + 2D_1)\ P_\lambda\ \lambda^{\alpha}. \end{cases}$$

It is easy to see that

$$z_0(t) = z(t) - z(0)\ y_1(t) - z'(0)\ y_2(t)$$

is the solution of (24) satisfying (25). By (10), (28), (29) we get (26) with $K = D_1(D_3\varepsilon^{-2} + 2D_1) + D\ D_3\varepsilon^{-2}$.

3. Now we consider the following collection of equations

$$(30)\qquad y'' + \left[\lambda_n^2 + Q(t)y\right] = 0, \quad n = 1,2,\ldots$$

with $\lambda_n = nh$, where h is a positive constant. Putting

$$y_{n1}(t) = y_1(t,\lambda_n),\quad y_{n2}(t) = y_2(t,\lambda_n),$$

$$\Delta_k(\lambda_n) = u_k(\pi,\lambda_n) + v_k'(\pi,\lambda_n)$$

we have $\Delta(\lambda_n) = y_{n1}(\pi) + y_{n2}'(\pi) = \sum\limits_{k=0}^{\infty} \Delta_k(\lambda_n)$.

By (5) it is easy to see that

$$(31)\qquad \sum_{k=i}^{\infty} \Delta_k(\lambda_n) = O(\lambda_n^{-i}).$$

The following four theorems establish estimations for $\Delta(\lambda_n)$. These estimations are not only dependent on Q but also on property of number h.

Let

$$Q_m = \frac{1}{\pi}\int_0^{\pi} Q(t)\ dt.$$

T h e o r e m 4. If

(i) h is a rational number,

(ii) $Q_m \neq 0$,

(iii) $Q \in V[0,\pi]$,

then there exist $\varepsilon > 0$ and n_0 such that for $n > n_0$ we have

$$|\Delta(\lambda_n)| \leqslant 2\left(1 - \frac{\varepsilon^2}{\lambda_n^2}\right).$$

P r o o f . Let $h = p/q$ be a fraction in its lowest terms. For $n \neq mq$, $m \in N$, i.e. λ_n is not integral, we have

$$\left|\Delta_0(\lambda_n)\right| = 2\left|\cos\frac{np\pi}{q}\right| \leqslant 2\cos\frac{\pi}{q}.$$

In view of (31) for $\varepsilon < 1 - \cos\frac{\pi}{q}$ and for sufficiently large n we get

$$|\Delta(\lambda_n)| \leqslant 2(1 - \varepsilon^2) \leqslant 2(1 - \varepsilon^2\lambda_n^{-2}).$$

For $n = mq$, $m \in N$, i.e. λ_n is an integer, we have

$$\Delta_0(\lambda_n) = 2\cos\lambda_n\pi = (-1)^{mp}\, 2,$$

$$\Delta_1(\lambda_n) = -\lambda_n^{-1}\sin\lambda_n\pi\int_0^\pi Q(t)dt = 0,$$

$$\Delta_2(\lambda_n) =$$

$$= \lambda_n^{-2}\int_0^\pi Q(t_1)dt_1\int_0^{t_1} Q(t_2)\sin\lambda_n(t_1-t_2)\sin\lambda_n(\pi-t_1+t_2)dt_2 =$$

$$= \frac{1}{2}\lambda_n^{-2}\int_0^\pi Q(t_1)dt_1\int_0^{t_1} Q(t_2)\cos\lambda_n(\pi-2t_1+t_2)dt_2 +$$

$$- \frac{1}{2}\lambda_n^{-2}\cos\lambda_n\pi\int_0^\pi Q(t_1)dt_1\int_0^{t_1} Q(t_2)dt_2.$$

Since $A \in V[0,\pi]$, then (see [7], p.168)

$$\int_0^{\pi} Q(t_1)dt_1 \int_0^{t_1} Q(t_2)\cos\lambda_n(\pi-2t_1+2t_2)dt_2 = O(\lambda_n^{-1}).$$

Putting $F(t) = \int_0^t Q(s)ds$ yields

$$\int_0^{\pi} Q(t_1)dt_1 \int_0^{t_1} Q(t_2)dt_2 = \int_0^{\pi} Q(t)\ F(t)\ dt = \frac{1}{2} Q_m^2\pi^2.$$

So

$$\Delta_2(\lambda_n) = \frac{(-1)^{mp+1}}{4} \lambda_n^{-2}\pi^2 Q_m^2 + O(\lambda_n^{-3}),$$

and then

$$|\Delta(\lambda_n)| \leqslant 2(1 - \varepsilon^2\lambda_n^{-2})$$

for $\varepsilon < \frac{\pi Q_m}{\sqrt{8}}$ and for sufficiently large n.

Theorem 5. If

(i) h is a rational number,

(ii) $Q_m = 0$,

(iii) $Q \in H^2(0,\pi)$, $Q'' \in V[0,\pi]$, $Q(0) = Q(\pi)$ and $Q'(0) = Q'(\pi)$, then for $n > n_0$

(32) $$|\Delta(\lambda_n)| \leqslant 2(1 - \varepsilon^2\lambda_n^{-6}).$$

Proof. Let $h = p/q$ be a fraction in its lowest terms. For $n \neq mq$, $m \in N$ the proof is the same as in Theorem 4. Let $n = mq$, then $\sin\lambda_n\pi = 0$. By Theorem 2 we have

$$\Delta(\lambda_n) = (2 - 2^{-6}\lambda_n^{-6}c_{0,2})\cos\lambda_n\pi + O(\lambda_n^{-7}) =$$

$$= (-1)^{mp}\, 2(1 - 2^{-7}c_{0,2}\lambda_n^{-6}) + O(\lambda_n^{-7}).$$

So, for $\varepsilon < 2^{-7}c_{0,2}$ and for sufficiently large n we get (32).

T h e o r e m 6. If

(i) h is an irrational number such that there exist $r > 0$ and $\gamma \in (0,3]$ satisfying

$$|nh - m| > r\, n^{-\gamma} \quad \text{for all} \quad n,m \in N,$$

(ii) $Q_m = 0$,

(iii) $Q \in V[0,\pi]$ for $\gamma < \frac{3}{2}$,

$Q \in H^1(0,\pi)$ and $Q(0) = Q(\pi)$ for $\frac{3}{2} \leqslant \gamma < 2$,

$Q \in H^2(0,\pi)$, $Q'' \in V[0,\pi]$, $Q(0) = Q(\pi)$ and $Q'(0) = Q'(\pi)$ for $2 \leqslant \gamma \leqslant 3$, with $c_{0,2} < 8rh^3$ for $\gamma = 3$,

then there exist $\varepsilon > 0$ and n_0 such that for $n > n_0$

$$|\Delta(\lambda_n)| \leqslant 2(1 - \varepsilon^2 \lambda_n^{-2\gamma}), \tag{33}$$

P r o o f . Let $2 \leqslant \gamma \leqslant 3$. Put $nh = m_n + r_n$, where m_n is an integer and r_n satisfies $r\, n^{-\gamma} \leqslant |r_n| \leqslant \frac{1}{2}$. By (23) we have

$$\Delta(\lambda_n) = 2\cos\left[\pi nh - 2^{-3} c_{0,2} \lambda_n^{-3} + O(\lambda_n^{-5})\right] + O(\lambda_n^{-7}) =$$

$$= 2\cos\left[\pi m_n + \pi r_n - 2^{-3} c_{0,2} \lambda_n^{-3} + O(\lambda_n^{-5})\right] + O(\lambda_n^{-7}).$$

Therefore

$$|\Delta(\lambda_n)| \leqslant 2 \left[1 - \frac{1}{4} (\pi r\, n^{-\gamma} - 2^{-3} c_{0,2} \lambda_n^{-3})^2\right] + O(\lambda_n^{-7}).$$

Thus, with $\varepsilon < \frac{1}{2} \pi h^{\gamma} r$ for $2 \leqslant \gamma < 3$ and with $\varepsilon < \frac{1}{2} (\pi h^3 r - 2^{-3} c_{0,2})$ for $\gamma = 3$ we have (33) for sufficiently large n.

For the cases $\gamma < \frac{3}{2}$ and $\frac{3}{2} \leqslant \gamma < 2$ the proofs are similar to the above, using (21) and (22) instead of (23).

R e m a r k . The set of irrational numbers which do not satisfy condition (i) has measure zero (see [6], p.12).

T h e o r e m 7. If

(i) h is an irrational number such that

$$|nh - m| > \frac{r}{n} \quad \text{for all} \quad n,m \in N,$$

(ii) $$Q < |Q_m| < 2rh,$$

(iii) $$Q \in V[0,\pi],$$

then there exist $\varepsilon > 0$ and n_0 such that for $n > n_0$

(34) $$|\Delta(\lambda_n)| \leqslant 2(1 - \varepsilon^3 \lambda_n^{-2}).$$

P r o o f . Putting $Q_1(t) = Q(t) - Q_m$ gives $\int_0^\pi Q_1(t)dt = 0$. We shall write (30) in the form

(35) $$y'' + \left[\lambda_n'^2 + Q_1(t)\right] y = 0, \quad n = 1,2,\ldots,$$

where $\lambda_n'^2 = \lambda_n^2 + Q_m$. The values $\Delta(\lambda_n')$ of equations (35) equal the values $\Delta(\lambda_n)$ of (30). We have

$$\lambda_n' = \lambda_n + \frac{1}{2} Q_m \lambda_n^{-1} + O(\lambda_n^{-3}).$$

Put $nh = m_n + r_n$, where m_n is integer and r_n satisfy $rn^{-1} \leqslant \leqslant |r_n| < \frac{1}{2}$. By (21) we have

$$\Delta(\lambda_n') = 2\cos \lambda_n' \pi + O(\lambda_n^{-3}) =$$

$$= 2\cos(\pi m_n + \pi r_n + \frac{1}{2} \pi Q_m \lambda_n^{-1}) + O(\lambda_n^{-3}).$$

From this we get

$$|\Delta(\lambda_n)| \leqslant 2 \left[1 - \frac{1}{4} (\pi r n^{-1} - \frac{1}{2}\pi |Q_m| \lambda_n^{-1})\right] + O(\lambda_n^{-3}).$$

Thus, for $\varepsilon < \frac{\pi}{2} (rh - \frac{1}{2} |Q_m|)$ and for sufficiently large n we have (34).

R e m a r k . In Theorems 4-7 we proved that for the collection of equations (30) with certain assumptions, for sufficiently large n we have inequalities of the form (9) and so we get estimations of the solutions of the form (10). To get (10) for all $n \in N$ we have only to assume that the equations (30) are stable for all $n \in N$, for example using stability criterion of V.A. Jakubovič (see [8]) we have only to assume that $|Q(t)|$ is sufficiently small.

The following examples show the existence of irrational numbers h such that not all equations (30) are stable even if $|Q(t)|$ is arbitrarily small.

Example 1. Let h be a positive irrational number such that the inequality

$$(36)\qquad |nh - m| \leqslant n^{-(1+\gamma)}, \quad \gamma > 0,$$

has infinitely many solutions n,m in the set of natural numbers. Then, for each $\varepsilon > 0$ there exists a function Q satisfying $|Q(t)| < \varepsilon$ and $Q_m \neq 0$ such that not all equations (30) are stable.

Select

$$Q(t) = \begin{cases} 0 & \text{for } 0 < t < \frac{\pi}{2}, \\ q & \text{for } \frac{\pi}{2} < t < \pi, \end{cases}$$

and $Q(t+\pi) = Q(t)$.

We have (see [2], p.239)

$$(37)\qquad \Delta(\lambda_n) = 2\cos nh\frac{\pi}{2}\cos\frac{\pi}{2}\sqrt{n^2h^2+q} + $$

$$- \frac{2n^2h^2+q}{nh\sqrt{n^2h^2+q}}\sin nh\frac{\pi}{2}\sin\frac{\pi}{2}\sqrt{n^2h^2+q} =$$

$$= 2\cos\left(nh + \sqrt{n^2h^2+q}\right)\frac{\pi}{2} +$$

$$- \frac{(\sqrt{n^2h^2+q} - nh)^2}{nh\sqrt{n^2h^2+q}}\sin nh\frac{\pi}{2}\sin\frac{\pi}{2}\sqrt{n^2h^2+q}.$$

Choosing q so that

$$(38)\qquad \frac{\pi}{2}\left(nh + \sqrt{n^2h^2+q}\right) = m\pi, \quad m \in N,$$

we have $q = 4m(m-nh)$. By (36) we can choose n_0, m_0 so that

$$q = 4m_0(m_0 - n_0h) \quad \text{and} \quad |q| < \varepsilon.$$

Since h is irrational, by (38) we have

$$(39)\quad \begin{cases} \sin n_0 h \frac{\pi}{2} \sin \frac{\pi}{2}\sqrt{n_0^2h^2+q} > 0 & \text{for } m_0 = 2k-1, \quad k \in N, \\ \sin n_0 h \frac{\pi}{2} \sin \frac{\pi}{2}\sqrt{n_0^2h^2+q} < 0 & \text{for } m_0 = 2k, \quad k \in N. \end{cases}$$

Now the desired conclusion follows from (37)-(39).

Example 2. Let h be a positive irrational number such that the inequality

$$0 < nh-m < n^{-3-\gamma}, \quad \gamma > 0,$$

has infinitely many solutions n,m among natural numbers. Then for each $\varepsilon > 0$ there exists a function Q satisfying $|Q(t)| < \varepsilon$ and $Q_m = 0$ such that not all equations (30) are stable.

Choose

$$Q(t) = \begin{cases} q & \text{for} \quad 0 < t < \frac{\pi}{2}, \\ -q & \text{for} \quad \frac{\pi}{2} < t < \pi, \end{cases}$$

and $Q(t+\pi) = Q(t)$. The proof is similar to that of Example 1.

In a similar way we can consider the collection of solutions

$$y'' + \left[\left(\frac{2n-1}{2}\, h\right)^2 + Q(t)\right] y = 0, \quad n=1,2,\ldots$$

REFERENCES

[1] E.A. Coddington, N. Levinson: Theory of ordinary differential equations. New York 1955.

[2] B.P. Demidowicz: Matematyczna teoria stabilności. Warszawa 1972.

[3] W. Magnus, S. Winkler: Hill's equation. New York 1966.

[4] В.П. Михайлов: Дифференциальные уравнения в частных производных. Москва 1983.

[5] Pham ba Trung: Behaviour of solutions of some hiperbolic equations of Hill's type (Doctoral thesis, Technical University of Warsaw). Warszawa 1986.

[6] Б.И. Пташник: Некорректные граничные задачи для дифференциальных уравнений с частными производными. Киев 1980.

[7] F.G. Tricomi: Differential equations. London 1961.

[8] В.А. Якубович, В.М. Старжинский: Линейные дифференциальные уравнения с периодическими коэффициентами и их приложения. Москва 1972.

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